Contributions to Ramanujan's Schläfli-type Modular Equations, Class Invariants, Theta-Functions, and Continued Fractions

BY

NAYANDEEP DEKA BARUAH

Lecturer,

Department of Mathematical Sciences,

Tezpur University

THESIS

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तेजपूर विश्वविद्यालय
TEZPUR UNIVERSITY

Naapam, Tezpur-784 025.
ASSAM, INDIA
Tel : (Off:) 03712-67003, 67115
(Resi:) 03712-55685
Fax : 03712-67006
E-mail : pb@agnigarh.tezu.ernet.in
prodipbhatta@hotmail.com

Prof. P. BHATTACHARYYA, Ph.D. (London), D.I.C
Vice Chancellor

Certificate

I certify that the thesis titled "**Contributions to Ramanujan's Schläfli-type Modular Equations, Class Invariants, Theta-Functions, and Continued Fractions**," is the bonafide record of research work done by **Nayandeep Deka Baruah** under my supervision.

To the best of my knowledge no part of this work has been submitted to any other university or institute for any degree.

P. Bhattacharyya

P. Bhattacharyya

Dedication

Dedicated to my parents

Sjt. Satya Deka Baruah

and

Smti. Nirupama Deka Baruah

Abstract

In his notebooks [48] and lost notebook [49], Ramanujan listed many amazing results, most of them without any proof. It is now a remarkable chapter in the history of mathematics that most of Ramanujan's claims have been found to be true by several great mathematicians. Berndt ([11], [14], [15], [17], and [18]), Agarwal[1], and Andrews and Berndt [3] systematically discussed the claims made by Ramanujan. Many of the proofs given by later mathematicians used ideas or theorems not known to Ramanujan. That is, it was possible to find the technique used to establish the truth of these results because the end results were already known. We call such proofs "verifications." The main aim of this thesis is to give proofs of many results by using methods known to Ramanujan. In the process, many new results are also derived. We deal with Ramanujan's Schläfli-type "mixed" modular equations, class invariants, eta-function identities, explicit evaluations of theta-functions, Rogers-Ramanujan continued fraction, and Ramanujan's cubic continued fraction.

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Chapter 1

Introduction

1.1 Introduction

not enough credit to Andrews
Agarwal's work is rehashing of work done by others

The Indian mathematical genius Srinivasa Ramanujan Ayenger (1887-1920) recorded many spectacular mathematical results in his notebooks [48] and his lost notebook [49]. It is well known that Ramanujan rarely provided any proof for his stated results. Berndt ([11], [14], [15], [17], and [18]), Agarwal[1], and Andrews and Berndt [3] systematically discussed the claims made by Ramanujan and provided proofs for the results stated by Ramanujan. Some of their proofs are based on modern ideas and some of them are verified being knowing the result in advance. In this thesis, we prove some of those results regarding modular equations, class invariants, theta-functions, and continued fractions. In the course of our study, we have also discovered many new results.

1.2 Scope of the Thesis

The thesis has seven chapters including the introductory Chapter 1.

In Chapter 2, we deal with Ramanujan's Schläfli-type "mixed" modular equations. On pages 86 and 88 of his first notebook [48], Ramanujan recorded 12 Schläfli-type "mixed" modular equations. 11 of these were not recorded in his second notebook [48]. One of these 11 equations follows from a modular equation recorded by Ramanujan in Chapter 20 of his second notebook. This was first observed by K. G. Ramanathan [41, pp. 419-420]. Berndt [18] proved the other 10 equations by modular forms, a method with which Ramanujan was not familiar. We give alternate proofs for 8 of these equations. Two are proved by deriving some theta-function identities using Schröter's formulae, and the rest are proved by employing Ramanujan's Schläfli-type modular equations of prime degrees and some other modular equations. In the process, we also find two new Schläfli-type "mixed" modular equations [(2.3.19) and (2.3.60)]. For example, in Lemma 2.3.1 of Section 2.3, we find that, if

$$Q = \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{\frac{1}{18}}$$

and

$$R = \left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta}\right)^{\frac{1}{48}},$$

then,

$$R^2 + \frac{1}{R^2} = Q^4 + \frac{1}{Q^4} - 3,$$

where β, γ, and δ are of degrees 3, 7, and 21, respectively, over α.

In Chapter 3, we deal with Weber-Ramanujan's class invariants.

Let

$$(a;q)_\infty = \Pi_{n=0}^{\infty}(1-aq^n), \qquad |q|<1,$$

and, after Ramanujan, we set

$$\chi(q) = (-q;q^2)_\infty.$$

If $q=\exp(-\pi\sqrt{n})$, where n is any positive rational number, then Weber-Ramanujan's class invariant G_n is defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q).$$

In section 3.3, we derive

$$G_{217} = \left(\sqrt{\frac{11+4\sqrt{7}}{2}}+\sqrt{\frac{9+4\sqrt{7}}{2}}\right)^{1/2}\left(\sqrt{\frac{12+5\sqrt{7}}{4}}+\sqrt{\frac{16+5\sqrt{7}}{4}}\right)^{1/2}$$

by using Ramanujan's modular equations of degrees 7 and 31. Berndt, Chan, and Zhang [26] (Also see [18]) could not utilize the modular equations of degrees 7 and 31 recorded by Ramanujan to effect a proof for G_{217}. In Section 3.4, we employ some of the Schläfli-type "mixed" modular equations discussed in Chapter 2, along with some other Schläfli-type modular equations of prime degrees to evaluate Ramanujan's class invariants G_{15}, G_{21}, G_{33}, G_{39}, G_{55}, and G_{65}. It is worthwhile to note that our evaluation of G_{65} is much more easier than that of Berndt, Chan, and Zhang [26]. *The most important feature of our method is that we can also simultaneously get the values of* $G_{5/3}$, $G_{7/3}$, $G_{11/3}$, $G_{13/3}$, $G_{11/5}$, *and* $G_{13/5}$. Previously, these values were found by verifications. We also note that, these class invariants can be utilized to find some of the explicit values of certain q- continued fractions [25], certain values of Ramanujan's product of theta-functions [27], and some values of the quotient of eta-functions [30].

In Chapter 4, we deal with Ramanujan's eta-function identities.

If $q = \exp(2\pi i z)$, then Ramanujan's eta-function $f(-q)$ is defined by

$$f(-q) := q^{-1/24}\eta(z), \tag{1.2.1}$$

where $\eta(z)$ is classical Dedekind eta-function defined by

$$\eta(z) := e^{\pi i z/12}\prod_{n=1}^{\infty}(1 - e^{2\pi i n z}), \quad \operatorname{Im} z > 0. \tag{1.2.2}$$

In the unorganized portions of his second notebook, Ramanujan [48] recorded without proofs 25 beautiful identities involving quotients of only eta-functions and no other theta-functions. Berndt and Zhang [23] proved some of these identities. Proofs of all the 25 identities recorded by Ramanujan are given in Chapter 25 of Berndt's book [17]. Of these identities 19 were proved by employing modular equations and parameterizations and 6 were proved by invoking the theory of modular forms. But in many of their proofs via parameterizations, they used heavy amount of tedious algebra and the identities must be known beforehand. So those proofs may be merely called verifications. In Chapter 4, we deduce five of these identities [see Theorems 4.2.1-4.2.5] by using Ramanujan's other eta-functions identities and one of our newly derived identities [see Lemma 4.5.1]. The main advantage of our method is that one can find other identities of the same kind. For example, in Section 7.6 of our last chapter, we find three new identities of the same kind in connection with Ramanujan's cubic continued fraction.

In chapter 5, we deal with explicit evaluations of Ramanujan's theta-function $\phi(q)$, defined by

$$\phi(q) := 1 + 2\sum_{k=1}^{\infty} q^{k^2}, \tag{1.2.3}$$

where $|q| < 1$.

At different places of his notebooks [48], Ramanujan recorded several explicit values $\phi(q)$. Borwein and Borwein [31] first observed that Ramanujan's class invariants could be used to calculate certain explicit values of $\phi(e^{-n\pi})$. Berndt and Chan [21] verified all of Ramanujan's non-elementary values of $\phi(e^{-n\pi})$. They also derived some new values by combining Ramanujan's class invariants with his modular equations. We give simpler proofs for some of these evaluations and calculate some new values of $\phi(e^{-n\pi})$. We also find some new theorems for finding explicit values of quotients of theta-functions by deriving some theta-function identities

In Chapter 6, we deal with the famous Rogers-Ramanujan continued fraction $R(q)$, defined by

$$R(q) := \frac{q^{1/5}}{1}\,\frac{q}{+1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\cdots}, \qquad |q| < 1. \tag{1.2.4}$$

In his first and second letters to Hardy [22], Ramanujan communicated several explicit values of $R(q)$ and $S(q)$, where $S(q) = -R(-q)$. Watson [52]-[53] proved some of the results claimed by Ramanujan in those letters. In both his first [48] and lost notebooks [49], Ramanujan recorded several other evaluations. In particular, on page 210 of his lost notebook [49], Ramanujan provided a list of evaluations and intended evaluations. Ramanathan [42]-[46] made the first attempt to find a uniform method to evaluate $R(q)$ by using Kronecker's limit formula, with which Ramanujan was not familiar. Berndt and Chan [20] and Berndt, Chan, and Zhang [25] completed the incomplete list of Ramanujan by using some modular equations recorded by Ramanujan [48] in his notebooks. Most importantly, Berndt, Chan, and Zhang [25] derived general formulas for evaluating $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ in terms of Weber-Ramanujan class invariants. The lost notebook [49] also contains many formulas for $R(q)$ and theta-function identities giving more formulas for the explicit evaluation of $R(q)$. Kang [37]-[38] proved many of the claims made by Ramanujan. It appears that though Ramanujan's formulas are interesting, they generally are not very much amenable in the calculation of elegant values of $R(q)$. Here we find some of the evaluations of $R(q)$ and $S(q)$, by using the values of the quotients of theta-functions found in Chapter 5 and some other theta-function identities. Our evaluations are much easier than those of the previous authors.

In Chapter 7, we deal with Ramanujan's cubic continued fraction $G(q)$, defined by

$$G(q) := \frac{q^{1/3}}{1+}\,\frac{q+q^2}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^3+q^6}{1+}\cdots, \qquad |q|<1. \tag{1.2.5}$$

Ramanujan first introduced this continued fraction in his second letter to Hardy [22]. He also recorded this continued fraction on page 366 of his lost notebook [49], and claimed that there are many results of $G(q)$ which are analogous to $R(q)$. Motivated by Ramanuajan's claims, Chan [32] proved many new identities which probabiy were the identities vaguely referred to by Ramanujan. He established some reciprocity theorems for $G(q)$, found relations between $G(q)$ and the three continued fractions $G(-q)$, $G(q^2)$ and $G(q^3)$, and obtained some explicit evaluations of $G(q)$. For example, he proved the following relation between $G(q)$ and $G(q^3)$

$$G^3(q) = G(q^3)\frac{1-G(q^3)+G^2(q^3)}{1+2G(q^3)+4G^2(q^3)}. \tag{1.2.6}$$

But his proof of (1.2.6) is not satisfactory. In particular, the last deduction [32, (2.18), p. 347] is not an obvious one. In Section 7.2, we find an easy proof of (1.2.6).

By deriving some theta-function identities in Section 7.3 and Section 7.4, we give general formulas for the explicit evaluations of $G(-e^{-3\pi\sqrt{n}})$ and $G(e^{-3\pi\sqrt{n}})$ in Section 7.5. General formulas for the explicit evaluations of $G(-e^{-\pi\sqrt{n}})$ and $G(e^{\pi\sqrt{n}})$, were established by Berndt, Chan and Zhang [24].

In Section 7.6, we find three new beautiful eta-function identities [Theorems 7.6.1-7.6.3], and use them to derive two beautiful identities [Theorems 7.7.1-7.7.2] giving relations between $G(q)$ and the two continued fractions $G(q^5)$ and $G(q^7)$. For example, in Theorem 7.7.1, we prove that, if $v = G(q)$ and $w = G(q^5)$, then

$$v^6 - vw + 5vw(v^3 + w^3)(1 - 2vw) + w^6 = v^2w^2(16v^3w^3 - 20v^2w^2 + 20vw - 5).$$

Chapter 2

Schläfli-type "Mixed" Modular Equations

2.1 Introduction

The theory of modular equations began with the transformations of Gauss and Landen which give modular equations of degree 2 [13, pp. 30-31]. In 1825 Legendre explicitly found a modular equation of degree 3. In the next 100 years, many modular equations were discovered by E. Fielder, R. Fricke, A. G. Greenhill, C. Guetzlaff, M. Hanna, C. G. J. Jacobi, F. Klein, R. Russell, L. Schläfli, H. Weber, and others. Hanna's paper [34] contains a lot of references in the literature. However, Ramanujan

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Note: *The main results of this chapter have appeared in our papers [4] and [9]. A part of this chapter was presented at the 15th Annual Conference of the Ramanujan Mathematical Society, held in the Ramanujan Institute for Advanced Study in Mathematics, University of Madras, for which the author was awarded "Prof. M. Vengkataraman memorial best paper presentation award."*

recorded more modular equations than those of his predecessors combined. Chapters 19-21 of his second notebook [48] are almost completely devoted to modular equations. Many others can be found in the unorganized pages of his first and second notebooks [48]. He also recorded some modular equations in his lost notebook ([19], [49]) and his letters to Hardy [22]. For the introductory part of his modular equations one may also see [12], [13], [15] ,[16], or [41]. His work on modular equations is based on his theory of theta-functions. His general theta-function $f(a,b)$ is given by

$$f(a,b)=\sum_{k=-\infty}^{\infty} a^{\frac{k(k+1)}{2}} b^{\frac{k(k-1)}{2}}, \tag{2.1.1}$$

where $|ab|<1$. If we set $a=q^{2iz}$, $b=q^{-2iz}$, and $q=e^{\pi i\tau}$, where z is complex and $\mathrm{Im}(\tau)>0$, then $f(a,b)=\vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta-functions in its standard notations [62, p. 464].

Now, we recall the definition of a modular equation from Berndt's book [15].

The complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k):=\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2 sin^2\phi}}=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^2}{(n!)^2}k^{2n}=\frac{\pi}{2}{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2), \tag{2.1.2}$$

where $0<k<1$. The series representation in (2.1.2) is found by expanding the integrand in a binomial series and integrating termwise, and ${}_2F_1$ is the ordinary or Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n, \quad 0\le|z|<1,$$

where a, b, and c are complex numbers such that c is not a nonpositive integer. The number k is called the modulus of K, and $k':=\sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L, and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{2.1.3}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is implied by (2.1.3). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α. The multiplier m is defined by

$$m = \frac{K}{L}. \tag{2.1.4}$$

Similarly one can define Ramanujan's "mixed" modular equation or modular equation of composite degrees. Again, we recall from Chapter 20 of B. C. Berndt's book [15, p. 325]. Let K, K', L_1, L_1', L_2, L_2', L_3, and L_3' denote complete elliptic integrals of the first kind corresponding, in pairs,to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, and $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2, and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1\frac{K'}{K} = \frac{L_1'}{L_1}, \quad n_2\frac{K'}{K} = \frac{L_2'}{L_2}, \quad \text{and} \quad n_3\frac{K'}{K} = \frac{L_3'}{L_3} \tag{2.1.5}$$

hold. Then a "mixed" modular equation is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, and $\sqrt{\delta}$ that is induced by (2.1.5). In such an instance, we say that β, γ, and δ are of degrees n_1, n_2, and n_3, respectively, over α. Denoting $z_r = \phi^2(q^r)$, where $q = \exp(-\pi K'/K)$, $\phi(q) = f(q,q)$, $|q| < 1$; the multipliers m, and m' associated with α, β, and γ, δ, respectively are defined by

$$m = \frac{z_1}{z_{n_1}}, \quad m' = \frac{z_{n_2}}{z_{n_3}}. \tag{2.1.6}$$

Ramanujan probably used a lot of methods [12] in deriving his modular equations. Berndt ([14], [15], [17], [18], [19]) discussed all the modular equations recorded by Ramanujan in his notebooks

[48] and lost notebooks [49]. (One may also see [3]) But it is worthwhile to note that many of the Ramanujan's modular equations remained to be elucidated by the methods known to Ramanujan. Two new methods are employed to prove those results. One is parametrization of certain quantities and the other is the theory of modular forms. R.J. Evans in [33] used the theory of modular forms to verifys theta-function identities in a very remarkable way. Berndt has frequently used Evans' ideas also. But the main disadvantage of these methods is that one has to know the modular equation in advance. These methods do not give much insights to Ramanujan's discoveries. So deductions and proofs based on probable methods of Ramanujan [12] are preferred.

If the modular quantities $(2^4/\alpha(1-\alpha))^{1/24}$ and $(2^4/\beta(1-\beta))^{1/24}$ are connected in an equation, then such a modular equation is called a Schläfli-type modular equation (see, [41, p. 404]). In 1870, L. Schläfli studied such type of equations for prime degrees. Ramanujan not only rediscovered all the equations found by Schläfli but also discovered such type of equations associated with "mixed" modular equations. In fact, on pages 86 and 88 of his first notebook [48], Ramanujan recorded 12 Schläfli-type "mixed" modular equations. 11 of these were not recorded in his second notebook [48]. One of these 11 equations follows from a modular equation recorded by Ramanujan in Chapter 20 of his second notebook. This was first observed by K. G. Ramanathan [41, pp. 419-420]. Berndt [18] proved the other 10 equations by invoking the theory of modular forms. In this chapter, we prove 8 of these equations. Two are proved analytically by deriving some theta-function identities using Schröter's formulae. For the other equations we give elementary proofs by employing Ramanujan's modular equations of prime degrees, other "mixed" modular equations and Weber-type equations ([18], [41]). In the process, we also found two new Schläfli-type "mixed" modular equations [(2.3.19) and (2.3.60)].

Now we state the theorems which will be proved in this chapter.

We set

$$P := (256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{\frac{1}{48}}, \tag{2.1.7}$$

$$Q := \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{\frac{1}{48}} \tag{2.1.8}$$

$$R := \left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta}\right)^{\frac{1}{48}}, \tag{2.1.9}$$

and,

$$T := \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{\frac{1}{48}}. \tag{2.1.10}$$

Theorem 2.1.1 *([48, Vol. I, p. 86]; [18, p.380]). If α, β, γ, and δ have degrees 1, 3, 5, and 15, respectively, then*

$$T^4 + \frac{1}{T^4} - 2(P^2 + \frac{1}{P^2}) + 3 = 0. \tag{2.1.11}$$

Theorem 2.1.2 *([48, Vol. I, p. 86]; [18, p. 380]). If α, β, γ, and δ have degrees 1, 3, 11, and 33, respectively, then*

$$T^4 + \frac{1}{T^4} + 3(T^2 + \frac{1}{T^2}) - 2(P^2 + \frac{1}{P^2}) = 0. \tag{2.1.12}$$

Theorem 2.1.3 *([48, Vol. I, p. 86]; [18, p. 380]). If α, β, γ, and δ have degrees 1, 3, 5, and 15, respectively, then*

$$R^6 + \frac{1}{R^6} - 4\left(P^4 + \frac{1}{P^4}\right) + 10\left(P^2 + \frac{1}{P^2} - 1\right) = 0. \tag{2.1.13}$$

Theorem 2.1.4 *([48, Vol. I, p. 86]; [18, p. 380]). If α, β, γ, and δ have degrees 1, 3, 7, and 21, respectively, then*

$$R^8 + \frac{1}{R^8} + 7\left(R^6 + \frac{1}{R^6}\right) + 14\left(R^4 + \frac{1}{R^4}\right) + 21\left(R^2 + \frac{1}{R^2}\right) - 8\left(P^6 + \frac{1}{P^6}\right) + 42 = 0. \tag{2.1.14}$$

Theorem 2.1.5 *([48, Vol. I, p. 86]; [18, p. 380]). If α, β, γ, and δ have degrees 1, 3, 7, and 21, respectively, then*

$$Q^{16}+\frac{1}{Q^{16}}-5\left(Q^{12}+\frac{1}{Q^{12}}\right)+5\left(Q^{8}+\frac{1}{Q^{8}}\right)+6\left(Q^{4}+\frac{1}{Q^{4}}\right)-8(P^6+\frac{1}{P^6})+6=0. \quad (2.1.15)$$

Theorem 2.1.6 *([48, Vol. I, p. 86]; [18, p. 381]). If α, β, γ, and δ have degrees 1, 5, 7, and 35, respectively, then*

$$T^6+\frac{1}{T^6}+5\sqrt{2}\left(T^3+\frac{1}{T^3}\right)\left(P+\frac{1}{P}\right)-4\left(P^4+\frac{1}{P^4}\right)+10=0. \quad (2.1.16)$$

Theorem 2.1.7 *([48, Vol. I, p. 86]; [18, p. 381]). If α, β, γ, and δ have degrees 1, 5, 7, and 35, respectively, then*

$$R^4+\frac{1}{R^4}-\left(Q^6+\frac{1}{Q^6}\right)+5\left(Q^4+\frac{1}{Q^4}\right)-10\left(Q^2+\frac{1}{Q^2}\right)+15=0. \quad (2.1.17)$$

Theorem 2.1.8 *([48, Vol. I, p. 86]; [18, p. 381]). If α, β, γ, and δ have degrees 1, 5, 11, and 55, respectively, then*

$$T^6+\frac{1}{T^6}-5\left(T^4+\frac{1}{T^4}\right)+10\left(T^2+\frac{1}{T^2}\right)\left(P^2+\frac{1}{P^2}-1\right)-4\left(P^4+\frac{1}{P^4}\right)+10\left(P^2+\frac{1}{P^2}\right)-25=0. \quad (2.1.18)$$

The first two theorems will be proved in section 2 by deriving some theta-function identities by employing Schröter's formulae. The other modular equations will be proved by using Ramanujan's Schläfli-type modular equations of prime degrees, "mixed" modular equations, and Weber-type equations.

We shall make use of several results from Berndt's book [15] in our proofs. We record some of these results below for further reference. Page numbers refer to the location of the results in [15].

Entry 18. (p. 34) We have

$$(i) \qquad f(a,b) = f(b,a), \tag{2.1.19}$$

$$(ii) \qquad f(1,a) = 2f(a,a^3), \tag{2.1.20}$$

$$(iii) \qquad f(-1,a) = 0, \tag{2.1.21}$$

and, if n is an integer,

$$(iv) \qquad f(a,b) = a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n, b(ab)^{-n}). \tag{2.1.22}$$

Entry 22. (pp. 36-37) If $|q| < 1$, then

$$(i) \qquad \phi(q) := f(q,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2}, \tag{2.1.23}$$

$$(ii) \qquad \psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2}, \tag{2.1.24}$$

$$(iii) \quad f(-q) := f(-q,-q^2) = \sum_{k=0}^{\infty}(-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)/2} = (q;q)_\infty, \tag{2.1.25}$$

$$(iv) \qquad \chi(q) := (-q;q^2)_\infty, \tag{2.1.26}$$

where $(a;q)_\infty := \Pi_{k=0}^{\infty}(1-aq^k)$.

The third equality in (iii) is a statement of Euler's famous pentagonal number theorem. For an elementary proof and further references see G. E. Andrews' paper [2].

If we put $q = \exp(-\pi K'/K)$, $z = z_1$, and $x = \alpha$ in **Entries 10, 11, and 12** of chapter 17 (pp. 122-124) then we have the following results :

$$10(i) \qquad \phi(q) = \sqrt{z_1}, \tag{2.1.27}$$

10(ii) $$\phi(-q) = \sqrt{z_1}(1-\alpha)^{\frac{1}{4}}, \tag{2.1.28}$$

10(iv) $$\phi(q^2) = \sqrt{z_1}\{\frac{1}{2}(1+\sqrt{1-\alpha})\}^{\frac{1}{2}}, \tag{2.1.29}$$

10(vi) $$\phi(q^{\frac{1}{2}}) = \sqrt{z_1}(1+\sqrt{\alpha})^{\frac{1}{2}}, \tag{2.1.30}$$

10(vii) $$\phi(-q^{\frac{1}{2}}) = \sqrt{z_1}(1-\sqrt{\alpha})^{\frac{1}{2}}, \tag{2.1.31}$$

11(i) $$\psi(q) = \sqrt{\frac{z_1}{2}}\alpha^{\frac{1}{8}}q^{-\frac{1}{8}}, \tag{2.1.32}$$

11(ii) $$\psi(-q) = \sqrt{\frac{z_1}{2}}(\alpha(1-\alpha))^{\frac{1}{8}}q^{-\frac{1}{8}}, \tag{2.1.33}$$

11(iii) $$\psi(q^2) = \frac{1}{2}\sqrt{z_1}\alpha^{\frac{1}{4}}q^{-\frac{1}{4}}, \tag{2.1.34}$$

11(iv) $$\psi(q^4) = \frac{1}{2}\sqrt{\frac{1}{2}z_1}\{(1-\sqrt{1-\alpha})\}^{\frac{1}{2}}q^{-\frac{1}{2}}, \tag{2.1.35}$$

12(iii) $$f(-q^2) = \sqrt{z_1}2^{-\frac{1}{3}}\{\alpha(1-\alpha)\}^{\frac{1}{12}}q^{-\frac{1}{12}}, \tag{2.1.36}$$

12(v) $$\chi(q) = 2^{\frac{1}{6}}\{\alpha(1-\alpha)\}^{-\frac{1}{24}}q^{\frac{1}{24}}. \tag{2.1.37}$$

It is to be noted that, if we replace q by q^r, then z_1 and α will be replaced by z_r and corresponding square of the modulus, respectively.

If μ and ν are integers such that $\mu > \nu \geq 0$, then from Schröter's formulae (36.1), (36.2), (36.6), and (36.9) (pp. 67-69), we note that

$$\frac{1}{2}\{f(Aq^{\mu+\nu},q^{\mu+\nu}/A)f(Bq^{\mu-\nu},q^{\mu-\nu}/B)+f(-Aq^{\mu+\nu},-q^{\mu+\nu}/A)f(-Bq^{\mu-\nu},-q^{\mu-\nu}/B)\}$$

$$=\sum_{m=0}^{\mu-1}\left(\frac{A}{B}\right)^m q^{2\mu m^2} f\left(\frac{A^{\mu-\nu}}{B^{\mu+\nu}}q^{(2\mu+4m)(\mu^2-\nu^2)},\frac{B^{\mu+\nu}}{A^{\mu-\nu}}q^{(2\mu-4m)(\mu^2-\nu^2)},\right)$$

$$\times f\left(ABq^{2\mu+4\nu m},\frac{q^{2\mu-4\nu m}}{AB}\right), \tag{2.1.38}$$

$$\frac{1}{2}\{f(Aq^{\mu+\nu},q^{\mu+\nu}/A)f(Bq^{\mu-\nu},q^{\mu-\nu}/B)-f(-Aq^{\mu+\nu},-q^{\mu+\nu}/A)f(-Bq^{\mu-\nu},-q^{\mu-\nu}/B)\}$$

$$=A\sum_{m=0}^{\mu-1}(AB)^m q^{(2m+1)(\mu+\nu)+2\mu m^2} f\left(A^{\mu-\nu}B^{\mu+\nu}q^{(2\mu+4m+2)(\mu^2-\nu^2)},\frac{q^{(2\mu-4m-2)(\mu^2-\nu^2)}}{A^{\mu-\nu}B^{\mu+\nu}}\right)$$

$$\times f\left(\frac{A}{B}q^{4\mu+2\nu+4\nu m},\frac{B}{A}q^{-2\nu-4\nu m}\right), \tag{2.1.39}$$

$$\tfrac{1}{2}\{\phi(q^{\mu+\nu})\phi(q^{\mu-\nu})+\phi(-q^{\mu+\nu})\phi(-q^{\mu-\nu})\}+2q^{\mu/2}\psi(q^{2\mu+2\nu})\psi(q^{2\mu-2\nu})$$

$$=\sum_{m=0}^{\mu-1}q^{2\mu m^2}f(q^{(2\mu+4m)(\mu^2-\nu^2)},q^{(2\mu-4m)(\mu^2-\nu^2)})f(q^{2\nu m+\mu/2},q^{-2\nu m+\mu/2}), \tag{2.1.40}$$

and,

$$\psi(q^{\mu+\nu})\psi(q^{\mu-\nu})=q^{\mu^3/4-\mu/4}\psi(q^{2\mu(\mu^2-\nu^2)})f(q^{\mu+\mu\nu},q^{\mu-\mu\nu})$$

$$+\sum_{m=0}^{(\mu-3)/2}q^{\mu m(m+1)}f(q^{(\mu+2m+1)(\mu^2-\nu^2)},q^{(\mu-2m-1)(\mu^2-\nu^2)})f(q^{\mu+\nu+2\nu m},q^{\mu-\nu-2\nu m}), \tag{2.1.41}$$

where in (2.1.41) μ is odd.

From Entry 24(iii) (p. 39), Example (v) (p. 51), and (7.5) (p. 365) we note that

$$\chi(q) = \frac{f(-q^2)}{\psi(-q)}, \tag{2.1.42}$$

$$f(q,q^5) = \psi(-q^3)\chi(q), \tag{2.1.43}$$

and,

$$4qf(-q^{22})f(-q^2) = \phi(q^{11})\phi(q) - \phi(-q^{11})\phi(-q) - 4q^3\psi(q^{22})\psi(q^2). \tag{2.1.44}$$

2.2 Proofs of Theorems 2.1.1 and 2.1.2

Proof of Theorem 2.1.1. First of all we prove the following beautiful modular equation of Ramanujan.

Lemma 2.2.1 *([15, p. 280]). If β has degree 5 over α, then*

$$(\alpha\beta)^{\frac{1}{2}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{2}} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{6}} = 1. \tag{2.2.1}$$

According to Berndt [15, p. 282] direct proof of this modular equation, by methods known to Ramanujan, is not available. He verified this equation from the other equations of same degree, and conjectured that Ramanujan might have deduced in a same procedure. But we see that this equation follows from a very simple theta-function identity. Therefore Berndt may not be right in making his conjecture. In [51], Li-Chien Shen established this equation via classical ideas given in [62].

Proof of Lemma 2.2.1.

Applying (2.1.39) with $\mu = 3$, $\nu = 2$, $A = 1$, $B = -1$, we find that

$$\frac{1}{2}\{f(q^5, q^5)f(-q,-q) - f(-q^5,-q^5)f(q,q)\}$$

$$= \sum_{m=0}^{2}(-1)^m q^{5(2m+1)+6m^2} f(-q^{5(8+4m)}, -q^{5(4-4m)})f(-q^{16+8m}, -q^{-4-8m}). \tag{2.2.2}$$

Taking $p = q^5$, and utilizing (2.1.19), (2.1.21), (2.1.22), and (2.1.23) we deduce from (2.2.2) that

$$\frac{1}{2}\{\phi(p)\phi(-q) - \phi(-p)\phi(q)\} = -2qf(-p^4,-p^8)f(-q^4,-q^8). \tag{2.2.3}$$

Invoking (2.1.25), we deduce from above that

$$\phi(-p)\phi(q) - \phi(p)\phi(-q) = 4qf(-q^4)f(-p^4). \tag{2.2.4}$$

Replacing q by $q^{\frac{1}{2}}$ in (2.2.4), we find that

$$\phi(-p^{\frac{1}{2}})\phi(q^{\frac{1}{2}}) - \phi(p^{\frac{1}{2}})\phi(-q^{\frac{1}{2}}) = 4q^{\frac{1}{2}}f(-q^2)f(-p^2). \tag{2.2.5}$$

Transcribing (2.2.5) via (2.1.30), (2.1.31), and (2.1.36), we find that

$$\{(1+\sqrt{\alpha})(1-\sqrt{\beta})\}^{\frac{1}{2}} - \{(1-\sqrt{\alpha})(1+\sqrt{\beta})\}^{\frac{1}{2}} = 2^{\frac{4}{3}}\{(\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{12}}. \tag{2.2.6}$$

Squaring both sides of (2.2.6), and then simplifying, we arrive at (2.2.1), which completes the proof.

Proof of the main theorem.

Putting $\mu = 3$, $\nu = 2$, and replacing q by q^2 in (2.1.40), we find that

$$\frac{1}{2}\{\phi(q^{10})\phi(q^2) + \phi(-q^{10})\phi(-q^2)\} + 2q^3\psi(q^{20})\psi(q^4)$$

$$= \sum_{m=0}^{2} q^{12m^2} f(q^{5(12+8m)}, q^{5(12-8m)})f(q^{3+8m}, q^{3-8m})$$

$$= \phi(p^{12})\phi(q^3) + 2q^7 f(q,q^5)f(p^4,p^{20}), \tag{2.2.7}$$

where $p = q^5$ and we have also employed (2.1.19), (2.1.22), and (2.1.23).

Again, setting $\mu = 3$, $\nu = 2$, $A = B = 1$ in (2.1.39), we find that

$$\frac{1}{2}\{f(p,p)f(q,q) - f(-p,-p)f(-q,-q)\} = \sum_{m=0}^{2} q^{5(2m+1)+6m^2} f(p^{8+4m}, p^{4-4m}) f(q^{16+8m}, q^{-4-8m})$$

$$= 2qf(p^4,p^8)f(q^4,q^8) + q^9 f(1,q^{12})f(1,p^{12}), \tag{2.2.8}$$

where we have utilized (2.1.19) and (2.1.22).

In (2.2.8) we employ (2.1.20), (2.1.23), and (2.1.24), to deduce that

$$\frac{1}{2}\{\phi(p)\phi(q) - \phi(-p)\phi(-q)\} = 2qf(q^4,q^8)f(p^4,p^8) + 4q^9\psi(q^{12})\psi(p^{12}). \tag{2.2.9}$$

Multiplying both sides of (2.2.9) by 2, and then replacing q by q^2, we find that

$$\phi(p^2)\phi(q^2) - \phi(-p^2)\phi(-q^2) = 4q^2 f(q^8,q^{16})f(p^8,p^{16}) + 8q^{18}\psi(q^{24})\psi(p^{24}). \tag{2.2.10}$$

Putting $\mu = 3$, $\nu = 2$ in (2.1.41), and writing $p = q^5$, we obtain

$$\psi(p)\psi(q) = q^6\psi(p^6)f(q^9,q^{-3}) + f(p^4,p^2)f(q^5,q)$$

$$= q^3\psi(p^6)\phi(q^3) + f(q,q^5)f(p^2,p^4), \tag{2.2.11}$$

where we have again used (2.1.19), (2.1.22), and (2.1.23).

Raplacing q by q^4 in (2.2.11), we deduce that

$$\psi(p^4)\psi(q^4) = q^{12}\psi(p^{24})\phi(q^{12}) + f(q^4,q^{20})f(p^8,p^{16}). \tag{2.2.12}$$

Multiplying both sides of (2.2.12) by $4q^3$ and adding (2.2.10), we arrive at

$$\begin{aligned}
&\phi(p^2)\phi(q^2) - \phi(-p^2)\phi(-q^2) + 4q^3\psi(p^4)\psi(q^4) \\
&\qquad = 4q^2 f(q^8,q^{16})f(p^8,p^{16}) + 8q^{18}\psi(q^{24})\psi(p^{24}) \\
&\qquad +4q^{15}\psi(p^{24})\phi(q^{12}) + 4q^3 f(q^4,q^{20})f(p^8,p^{16}) \\
&\qquad = 4q^2 f(p^8,p^{16})[f(q^8,q^{16}) + qf(q^4,q^{20})] + 4q^{15}\psi(p^{24})[f(q^{12},q^{12}) \\
&\qquad +q^3 f(1,q^{24})] \\
&\qquad = 4q^2 f(p^8,p^{16})f(q,q^5) + 4q^{15}\psi(p^{24})\phi(q^3), \qquad (2.2.13)
\end{aligned}$$

where we have employed

$$f(a,b) = f(a^3b,ab^3) + af(\frac{b}{a},\frac{a}{b}a^4b^4), \qquad (2.2.14)$$

which can easily be deduced from Entries 30(ii) and 30(iii) [15, p. 46].

From (2.2.7) and (2.2.13), we find that

$$\begin{aligned}
&2[\phi(p^2)\phi(q^2) + 4q^3\psi(p^4)\psi(q^4)] \\
&\qquad = 2\phi(q^3)[f(p^{12},p^{12}) + p^3 f(1,p^{24})] + 4q^2 f(q,q^5)[f(p^8,p^{16}) + pf(p^4,p^{20})]. \qquad (2.2.15)
\end{aligned}$$

Employing (2.2.14), we deduce from (2.2.15) that

$$\phi(p^2)\phi(q^2) + 4q^3\psi(p^4)\psi(q^4) = \phi(q^3)\phi(p^3) + 2q^2 f(q,q^5)f(p,p^5). \qquad (2.2.16)$$

Now invoking (2.1.43) in (2.2.16), we find that

$$\phi(p^2)\phi(q^2) + 4q^3\psi(p^4)\psi(q^4) = \phi(q^3)\phi(p^3) + 2q^2\psi(-q^3)\psi(-p^3)\chi(q)\chi(p). \qquad (2.2.17)$$

Transcribing (2.2.17) via (2.1.27), (2.1.29), (2.1.30), (2.1.35), and (2.1.37), we arrive at

$$\frac{\sqrt{z_1 z_5}}{2}[(1+\sqrt{1-\alpha})(1+\sqrt{1-\gamma})]^{\frac{1}{2}} + \frac{\sqrt{z_1 z_5}}{2}[(1-\sqrt{1-\alpha})(1-\sqrt{1-\gamma})]^{\frac{1}{2}}$$
$$= \sqrt{z_3 z_{15}} + \sqrt{z_3 z_{15}}\, 2^{\frac{1}{3}} \frac{(\beta\delta(1-\beta)(1-\delta))^{\frac{1}{8}}}{(\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{24}}}. \tag{2.2.18}$$

If m and m' are the multipliers associated with z_1, z_3, and z_5, z_{15}, respectively then from (2.2.18), we find that

$$\sqrt{mm'}[\{(1+\sqrt{1-\alpha})(1+\sqrt{1-\gamma})\}^{\frac{1}{2}} + \{(1-\sqrt{1-\alpha})(1-\sqrt{1-\gamma})\}^{\frac{1}{2}}]$$
$$= 2 + 2^{\frac{4}{3}}\left[\frac{\{\beta\delta(1-\beta)(1-\delta)\}^3}{\alpha\gamma(1-\alpha)(1-\gamma)}\right]^{\frac{1}{24}}. \tag{2.2.19}$$

Squaring both sides of (2.2.19), we arrive at

$$mm'[(1+\sqrt{1-\alpha})(1+\sqrt{1-\gamma}) + (1-\sqrt{1-\alpha})(1-\sqrt{1-\gamma}) + 2\sqrt{\alpha\gamma}]$$
$$= 4 + 2^{\frac{8}{3}}\left[\frac{\{\beta\delta(1-\beta)(1-\delta)\}^3}{\alpha\gamma(1-\alpha)(1-\gamma)}\right]^{\frac{1}{12}} + 2^{\frac{10}{3}}\left[\frac{\{\beta\delta(1-\beta)(1-\delta)\}^3}{\alpha\gamma(1-\alpha)(1-\gamma)}\right]^{\frac{1}{24}}. \tag{2.2.20}$$

Simplification gives

$$mm'[1 + \sqrt{\alpha\gamma} + \sqrt{(1-\alpha)(1-\gamma)}]$$
$$= 2 + 2^{\frac{5}{3}}\left[\frac{\{\beta\delta(1-\beta)(1-\delta)\}^3}{\alpha\gamma(1-\alpha)(1-\gamma)}\right]^{\frac{1}{12}} + 2^{\frac{7}{3}}\left[\frac{\{\beta\delta(1-\beta)(1-\delta)\}^3}{\alpha\gamma(1-\alpha)(1-\gamma)}\right]^{\frac{1}{24}}. \tag{2.2.21}$$

Employing Lemma 2.2.1 in (2.2.21), we find that

$$mm' = \frac{1 + 2^{\frac{2}{3}}\left[\frac{\{\beta\delta(1-\beta)(1-\delta)\}^3}{\alpha\gamma(1-\alpha)(1-\gamma)}\right]^{\frac{1}{12}} + 2^{\frac{4}{3}}\left[\frac{\{\beta\delta(1-\beta)(1-\delta)\}^3}{\alpha\gamma(1-\alpha)(1-\gamma)}\right]^{\frac{1}{24}}}{1 - 2^{\frac{2}{3}}[\alpha\gamma(1-\alpha)(1-\gamma)]^{\frac{1}{6}}}. \tag{2.2.22}$$

This is a mixed modular equation of degrees 3, 5, 15. Another mixed modular equation of same degrees, but of reciprocating nature is given by

$$\frac{9}{mm'} = \frac{1 + 2^{\frac{2}{3}}\left[\frac{\{\alpha\gamma(1-\alpha)(1-\gamma)\}^3}{\beta\delta(1-\beta)(1-\delta)}\right]^{\frac{1}{12}} + 2^{\frac{4}{3}}\left[\frac{\{\alpha\gamma(1-\alpha)(1-\gamma)\}^3}{\beta\delta(1-\beta)(1-\delta)}\right]^{\frac{1}{24}}}{1 - 2^{\frac{2}{3}}[\beta\delta(1-\beta)(1-\delta)]^{\frac{1}{6}}}. \tag{2.2.23}$$

In (2.2.22) and (2.2.23) we substitute P and T from (2.1.7) and (2.1.10), respectively, to deduce

$$mm' = \frac{1 + P^4T^8 + 2P^2T^4}{1 - \frac{P^4}{T^4}} \tag{2.2.24}$$

and,

$$\frac{9}{mm'} = \frac{1 + \frac{P^4}{T^8} + 2\frac{P^2}{T^4}}{1 - P^4T^4} \tag{2.2.25}$$

respectively.

Multiplying (2.2.24) and (2.2.25), we find that

$$9 = \frac{(1 + P^4T^8 + 2P^2T^4)(1 + \frac{P^4}{T^8} + 2\frac{P^2}{T^4})}{(1 - \frac{P^4}{Q^4})(1 - P^4Q^4)}. \tag{2.2.26}$$

We rewrite (2.2.26) in the form

$$T^8 + \frac{1}{T^8} + 4 + (T^4 + \frac{1}{T^4})(2P^2 + \frac{2}{P^2} + 9) - 8(P^4 + \frac{1}{P^4}) = 0. \tag{2.2.27}$$

Factorizing (2.2.27), we find that

$$\left(T^4 + \frac{1}{T^4} - 2(P^2 + \frac{1}{P^2}) + 3\right)\left(T^4 + \frac{1}{T^4} + 4(P^2 + \frac{1}{P^2}) + 6\right) = 0. \tag{2.2.28}$$

Therefore we have

$$T^4 + \frac{1}{T^4} - 2(P^2 + \frac{1}{P^2}) + 3 = 0, \tag{2.2.29}$$

since the other factor can not be zero.

Thus we arrive at (2.1.11), which completes the proof of Theorem 2.1.1.

Proof of Theorem 2.1.2.

Setting $\mu = 6$, $\nu = 5$ in (2.1.40), and taking $p = q^{11}$, we find that

$$\frac{1}{2}\{\phi(p)\phi(q) + \phi(-p)\phi(-q)\} + 2q^3\psi(p^2)\psi(q^2)$$
$$= \sum_{m=0}^{5} q^{12m^2} f(p^{12+4m}, p^{12-4m}) f(q^{3+10m}, q^{3-10m}). \tag{2.2.30}$$

Replacing m by $m+3$ in the last three summands of the right hand side of (2.2.30), we find, after applying (2.1.22), that

$$q^{12(m+3)^2} f(p^{12+4(m+3)}, p^{12-4(m+3)}) f(q^{3+10(m+3)}, q^{3-10(m+3)})$$
$$= q^{12(m+3)^2} f(p^{24+4m}, p^{-4m}) f(q^{33+10m}, q^{-27-10m})$$
$$= q^{12m^2} p^{2m+3} f(p^{24+4m}, p^{-4m}) f(q^{3+10m}, q^{3-10m}). \tag{2.2.31}$$

Thus the right hand side of (2.2.30) may be rewritten in the form

$$\sum_{m=0}^{2} q^{12m^2} \{f(p^{12+4m}, p^{12-4m}) + p^{2m+3} f(p^{24+4m}, p^{-4m})\} f(q^{3+10m}, q^{3-10m}). \tag{2.2.32}$$

Taking $a = p^{2m+3}$ and $b = p^{3-2m}$ in (2.2.14), we deduce that

$$f(p^{3+2m}, p^{3-2m}) = f(p^{12+4m}, p^{12-4m}) + p^{2m+3} f(p^{24+4m}, p^{-4m}). \tag{2.2.33}$$

Employing (2.2.33) in (2.2.32), we find, after applying (2.1.19), that

$$\sum_{m=0}^{2} q^{12m^2} f(p^{3-2m}, p^{3+2m}) f(q^{3-10m}, q^{3+10m}). \tag{2.2.34}$$

Thus from (2.2.30), we find that

$$\phi(p)\phi(q) + \phi(-p)\phi(-q) + 4q^3\psi(p^2)\psi(q^2) = 2\sum_{m=0}^{2} q^{12m^2} f(p^{3-2m}, p^{3+2m}) f(q^{3-10m}, q^{3+10m}). \tag{2.2.35}$$

Employing (2.1.19) and (2.1.22), we can write (2.2.35) in the form

$$\phi(p)\phi(q)+\phi(-p)\phi(-q)+4q^3\psi(p^2)\psi(q^2)$$

$$=2f(p^3,p^3)f(q^3,q^3)+4q^4f(p,p^5)f(q,q^5). \tag{2.2.36}$$

By (2.1.23) and (2.1.43), (2.2.41) may be written as

$$\phi(p)\phi(q)+\phi(-p)\phi(-q)+4q^3\psi(p^2)\psi(q^2)$$

$$=2\phi(p^3)\phi(q^3)+4q^4\chi(q)\psi(-q^3)\chi(p)\psi(-p^3). \tag{2.2.37}$$

Employing (2.1.42) in (2.2.37), we find that

$$\phi(p)\phi(q)+\phi(-p)\phi(-q)+4q^3\psi(p^2)\psi(q^2)$$

$$=2\phi(p^3)\phi(q^3)+4q^4\psi(-q^3)\psi(-p^3)\frac{f(-q^2)f(-p^2)}{\psi(-q)\psi(-p)}. \tag{2.2.38}$$

Now, invoking (2.1.44) in (2.2.38), we arrive at

$$\phi(p)\phi(q)+\phi(-p)\phi(-q)+4q^3\psi(p^2)\psi(q^2)$$

$$=2\phi(p^3)\phi(q^3)+q^3\frac{\psi(-q^3)\psi(-p^3)}{\psi(-q)\psi(-p)}[\phi(p)\phi(q)-\phi(-p)\phi(-q)-4q^3\psi(p^2)\psi(q^2)]. \tag{2.2.39}$$

This can also be written as,

$$\left[1-q^3\frac{\psi(-q^3)\psi(-p^3)}{\psi(-q)\psi(-p)}\right]\phi(p)\phi(q)+\left[1+q^3\frac{\psi(-q^3)\psi(-p^3)}{\psi(-q)\psi(-p)}\right]$$

$$\times\left(\phi(-p)\phi(-q)+4q^3\psi(p^2)\psi(q^2)\right)=2\phi(q^3)\phi(p^3). \tag{2.2.40}$$

Transcribing (2.2.40) via (2.1.27), (2.1.28), (2.1.33), and (2.1.34), we find that

$$\left[1-\sqrt{\frac{z_3z_{33}}{z_1z_{11}}}\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{\frac{1}{8}}\right]\sqrt{z_1z_{11}}+\left[1+\sqrt{\frac{z_3z_{33}}{z_1z_{11}}}\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{\frac{1}{8}}\right]$$
$$\times[\sqrt{z_1z_{11}}((1-\alpha)(1-\gamma))^{\frac{1}{4}}+\sqrt{z_1z_{11}}(\alpha\gamma)^{\frac{1}{4}}]=2\sqrt{z_3z_{33}}. \qquad (2.2.41)$$

If m and m' are the multipliers associated with α, β, and γ, δ respectively then (2.2.41) may be written as,

$$\left[1-\frac{1}{\sqrt{mm'}}\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{\frac{1}{8}}\right]+\left[1+\frac{1}{\sqrt{mm'}}\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{\frac{1}{8}}\right]$$
$$\times\left((\alpha\gamma)^{\frac{1}{4}}+((1-\alpha)(1-\gamma))^{\frac{1}{4}}\right)=\frac{2}{\sqrt{mm'}}. \qquad (2.2.42)$$

Now, by Entry 7(i) [15, p. 363], we have

$$(\alpha\gamma)^{\frac{1}{4}}+((1-\alpha)(1-\gamma))^{\frac{1}{4}}=1-2\{16\alpha\gamma(1-\alpha)(1-\gamma)\}^{\frac{1}{12}}. \qquad (2.2.43)$$

Employing (2.2.43) in (2.2.42), we arrive at

$$\sqrt{mm'}=\frac{(\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{24}}+2^{\frac{1}{3}}(\beta\delta(1-\beta)(1-\delta))^{\frac{1}{8}}}{(\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{24}}-2^{\frac{1}{3}}(\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{8}}}. \qquad (2.2.44)$$

This is a mixed modular equation of degrees 3, 11, 33. Another mixed modular equation of same degrees, but of reciprocating nature is given by,

$$\frac{3}{\sqrt{mm'}}=\frac{(\beta\delta(1-\beta)(1-\delta))^{\frac{1}{24}}+2^{\frac{1}{3}}(\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{8}}}{(\beta\delta(1-\beta)(1-\delta))^{\frac{1}{24}}-2^{\frac{1}{3}}(\beta\delta(1-\beta)(1-\delta))^{\frac{1}{8}}}. \qquad (2.2.45)$$

Multiplying (2.2.44), and (2.2.45), we find that

$$3=\frac{(\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{24}}+2^{\frac{1}{3}}(\beta\delta(1-\beta)(1-\delta))^{\frac{1}{8}}}{(\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{24}}-2^{\frac{1}{3}}(\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{8}}}$$

$$\times\frac{(\beta\delta(1-\beta)(1-\delta))^{\frac{1}{24}}+2^{\frac{1}{3}}(\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{8}}}{(\beta\delta(1-\beta)(1-\delta))^{\frac{1}{24}}-2^{\frac{1}{3}}(\beta\delta(1-\beta)(1-\delta))^{\frac{1}{8}}}. \qquad (2.2.46)$$

Putting the expressions for P and T from (2.1.7) and (2.1.10) in (2.2.46), we find that

$$3=\frac{[2^{-\frac{1}{6}}PT^{-1}+2^{-\frac{1}{6}}P^3T^3]}{[2^{-\frac{1}{6}}PT^{-1}-2^{-\frac{1}{6}}P^3T^{-3}]}\frac{[2^{-\frac{1}{6}}PT+2^{-\frac{1}{6}}P^3T^{-3}]}{[2^{-\frac{1}{6}}PT-2^{-\frac{1}{6}}P^3T^3]}. \qquad (2.2.47)$$

Simplifying (2.2.47) we readily arrive at (2.1.12), which completes the proof of Theorem 2.1.2.

2.3 Proofs of Theorems 2.1.3-2.1.8

First Proof of Theorem 2.1.3.: For simplicity, we set

$$A:=(\alpha\beta)^{\frac{1}{8}},\ B:=((1-\alpha)(1-\beta))^{\frac{1}{8}}, C:=(\gamma\delta)^{\frac{1}{8}},\ \text{and}\ D:=((1-\gamma)(1-\delta))^{\frac{1}{8}} \qquad (2.3.1)$$

so that

$$P^6=2ABCD \text{ and } R^6=CD/AB. \qquad (2.3.2)$$

Now from Entry 11(xiv) ([15, p. 385]; [48, VOL. II, P. 247]), we obtain

$$AC+BD=1-P^2 \qquad (2.3.3)$$

This modular equation was also derived by Weber [61, p. 415].

Now from Entry 5(ii) [15, p. 230] we recall the following modular equation of degree 3.

$$(\alpha\beta)^{\frac{1}{4}}+((1-\alpha)(1-\beta))^{\frac{1}{4}}=1. \qquad (2.3.4)$$

Therefore, we have

$$(\alpha\beta(1-\alpha)(1-\beta))^{\frac{1}{4}} = (\alpha\beta)^{\frac{1}{4}}\left(1-(\alpha\beta)^{\frac{1}{4}}\right). \tag{2.3.5}$$

Employing (2.3.1) and (2.3.2), we rewrite (2.3.5) as

$$\frac{P^6}{2R^6} = \frac{1}{4} - \left((\alpha\beta)^{\frac{1}{4}} - \frac{1}{2}\right)^2 \tag{2.3.6}$$

Hence,

$$(\alpha\beta)^{\frac{1}{4}} = \frac{1}{2} \pm \sqrt{k_1}, \tag{2.3.7}$$

where

$$k_1 = \frac{1}{4} - \frac{P^6}{2R^6}.$$

Using (2.3.7) in (2.3.4), we find that

$$((1-\alpha)(1-\beta))^{\frac{1}{4}} = \frac{1}{2} \mp \sqrt{k_1}. \tag{2.3.8}$$

In a similar way, we obtain

$$(\gamma\delta)^{\frac{1}{4}} = \frac{1}{2} \pm \sqrt{k_2}, \tag{2.3.9}$$

and

$$((1-\gamma)(1-\delta))^{\frac{1}{4}} = \frac{1}{2} \mp \sqrt{k_2}, \tag{2.3.10}$$

where

$$k_2 = \frac{1}{4} - \frac{P^6R^6}{2}.$$

From (2.3.7), (2.3.8), (2.3.9), and (2.3.10), we find that

$$(AC)^2 + (BD)^2 = \frac{1}{2} \pm 2\sqrt{k_1k_2}. \tag{2.3.11}$$

From (2.3.3) and (2.3.11), we deduce that

$$\frac{1}{2} \pm 2\sqrt{k_1k_2} + P^6 = (1 - P^2)^2 \tag{2.3.12}$$

Isolating the term involving $\sqrt{k_1k_2}$ on one side of the equation, squaring both sides, and then substituting for k_1 and k_2, we easily arrive at (2.1.13). Thus we complete the proof.

Second Proof of Theorem 2.1.3.: From Entry 13(xiv) [15, p. 282], we note that

$$\left(\frac{Q}{R}\right)^3 + \left(\frac{R}{Q}\right)^3 = 2\left[\frac{1}{(16\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{12}}} - (16\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{12}}\right] \tag{2.3.13}$$

Also

$$(QR)^3 + \left(\frac{1}{QR}\right)^3 = 2\left[\frac{1}{(16\beta\delta(1-\beta)(1-\delta))^{\frac{1}{12}}} - (16\beta\delta(1-\beta)(1-\delta))^{\frac{1}{12}}\right] \tag{2.3.14}$$

Multiplying (2.3.13) and (2.3.14), and then using (2.1.7)-(2.1.10), we find that

$$Q^6 + \frac{1}{Q^6} + \left(R^6 + \frac{1}{R^6}\right) = 4\left(P^4 + \frac{1}{P^4} - T^4 - \frac{1}{T^4}\right) \tag{2.3.15}$$

By Entry 11(xv) [15, p. 385], we find that

$$Q^3 + \frac{1}{Q^3} = \sqrt{2}(P + \frac{1}{P}) \tag{2.3.16}$$

From Theorem 2.1.1., we note that

$$T^4 + \frac{1}{T^4} = 2(P^2 + \frac{1}{P^2}) - 3. \tag{2.3.17}$$

Using (2.3.16) and (2.3.17) in (2.3.15), we find that

$$2\left(P + \frac{1}{P}\right)^2 - 2 + \left(R^6 + \frac{1}{R^6}\right) = 4\left[P^4 + \frac{1}{P^4} - 2\left(P^2 + \frac{1}{P^2}\right) + 3\right]. \tag{2.3.18}$$

Simplifying (2.3.18), we easily arrive at (2.1.13), which completes the proof.

Proof of Theorem 2.1.4: First of all, we prove the following new Schläfli-type "mixed" modular equation of degrees 1, 3, 7, and 21.

Lemma 2.3.1 *If α, β, γ, and δ have degrees 1, 3, 7, and 21, respectively, then*

$$R^2+\frac{1}{R^2}=Q^4+\frac{1}{Q^4}-3, \tag{2.3.19}$$

where Q and R are given by (2.1.8) and (2.1.9), respectively.

Proof of Lemma 2.3.1: From Entry 19(i) of Chapter 20 [15, p. 426], we note that

$$\phi(q)\phi(q^{63})-\phi(q^7)\phi(q^9)=2qf(q^3)f(q^{21}). \tag{2.3.20}$$

By Corollary (i) of Entry 31 in Chapter 16 [15, p. 49]

$$\phi(q^9)=\phi(q)-2qf(q^3,q^{15}). \tag{2.3.21}$$

Employing (2.1.43) in (2.3.31), we deduce that

$$\phi(q^9)=\phi(q)-2q\chi(q^3)\psi(-q^9). \tag{2.3.22}$$

Now by Entry 2(ii) [15, p. 349], we note that

$$\psi(q^9)=\frac{1}{3q}\left\{\psi(q)-\frac{\phi(-q)}{\chi(-q^3)}\right\}. \tag{2.3.23}$$

Replacing q by $-q$ in (2.3.33), we find that

$$\psi(-q^9)=\frac{1}{3q}\left\{\frac{\phi(q)}{\chi(q^3)}-\psi(-q)\right\}. \tag{2.3.24}$$

Employing (2.3.34) in (2.3.32), we obtain

$$\phi(q^9)=\phi(q)-\frac{2}{3}\chi(q^3)\left\{\frac{\phi(q)}{\chi(q^3)}-\psi(-q)\right\}. \tag{2.3.25}$$

This may be rewritten in the form

$$\phi(q^9)=\frac{1}{3}\phi(q)+\frac{2}{3}\chi(q^3)\psi(-q). \tag{2.3.26}$$

Replacing q by q^7 in (2.3.36), we find that

$$\phi(q^{63}) = \frac{1}{3}\phi(q^7) + \frac{2}{3}\chi(q^{21})\psi(-q^7). \tag{2.3.27}$$

Using (2.3.36) and (2.3.37) in (2.3.30), we obtain

$$\phi(q)\chi(q^{21})\psi(-q^7) - \phi(q^7)\chi(q^3)\psi(-q) = 3qf(q^3)f(q^{21}). \tag{2.3.28}$$

Transcribing (2.3.38) by employing (2.1.27), (2.1.33), (2.1.37) and Entry 12(i) of Chapter 17 [15, pp. 122-124], and then simplifying, we deduce that

$$\frac{(\gamma(1-\gamma))^{\frac{1}{8}}}{(\delta(1-\delta))^{\frac{1}{24}}} - \frac{(\alpha(1-\alpha))^{\frac{1}{8}}}{(\beta(1-\beta))^{\frac{1}{24}}} = \frac{3}{\sqrt{mm'}}(\beta\delta(1-\beta)(1-\delta))^{\frac{1}{24}}, \tag{2.3.29}$$

where m and m' are the multipliers associated with α and β, and γ and δ, respectively.

Reciprocal of this mixed modular equation of degrees 1, 3, 7, and 21 is given by

$$\frac{(\beta(1-\beta))^{\frac{1}{8}}}{(\alpha(1-\alpha))^{\frac{1}{24}}} - \frac{(\delta(1-\delta))^{\frac{1}{8}}}{(\gamma(1-\gamma))^{\frac{1}{24}}} = \sqrt{mm'}(\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{24}}. \tag{2.3.30}$$

Multiplying (2.3.39) and (2.3.40), we find that

$$\frac{(\beta\gamma(1-\beta)(1-\gamma))^{\frac{1}{8}}}{(\alpha\delta(1-\alpha)(1-\delta))^{\frac{1}{24}}} - \frac{(\alpha\delta(1-\alpha)(1-\delta))^{\frac{1}{8}}}{(\beta\gamma(1-\beta)(1-\gamma))^{\frac{1}{24}}} - (\gamma\delta(1-\gamma)(1-\delta))^{\frac{1}{12}}$$

$$-(\alpha\beta(1-\alpha)(1-\beta))^{\frac{1}{12}} = 3(\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{\frac{1}{24}}. \tag{2.3.31}$$

Dividing both sides of (2.3.41) by $(\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{\frac{1}{24}}$, and then using (2.1.8) and (2.1.9), we find that

$$Q^4 + \frac{1}{Q^4} - R^2 - \frac{1}{R^2} = 3, \tag{2.3.32}$$

which is equivalent to (2.3.29). This completes the proof of the Lemma.

Proof of the main theorem: We note that

$$\frac{Q}{R} = \left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{\frac{1}{24}} \text{ and } QR = \left(\frac{\delta(1-\delta)}{\beta(1-\beta)}\right)^{\frac{1}{24}} \tag{2.3.33}$$

From Entry 19(ix) of Chapter 19 [15, p. 315], we find that

$$\left(\frac{Q}{R}\right)^4 + \left(\frac{R}{Q}\right)^4 + 7 = 2\sqrt{2}\left[(16\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{8}} + \frac{1}{(16\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{8}}}\right] \tag{2.3.34}$$

and

$$(QR)^4 + \left(\frac{1}{QR}\right)^4 + 7 = 2\sqrt{2}\left[(16\beta\delta(1-\beta)(1-\delta))^{\frac{1}{8}} + \frac{1}{(16\beta\delta(1-\beta)(1-\delta))^{\frac{1}{8}}}\right]. \tag{2.3.35}$$

Multiplying (2.3.34) and (2.3.35), and then using (2.1.7)-(2.1.10), we find that

$$Q^8 + \frac{1}{Q^8} + \left(R^8 + \frac{1}{R^8}\right) + 7\left(Q^4 + \frac{1}{Q^4}\right)\left(R^4 + \frac{1}{R^4}\right) + 49 = 8\left(P^6 + \frac{1}{P^6} + T^6 + \frac{1}{T^6}\right). \tag{2.3.36}$$

Employing Lemma 2.3.1, (2.3.36) can be written as

$$R^8 + \frac{1}{R^8} + \left(R^2 + \frac{1}{R^2} + 3\right)^2 + 7\left(R^4 + \frac{1}{R^4}\right)\left(R^2 + \frac{1}{R^2} + 3\right) + 47$$
$$= 8\left(P^6 + \frac{1}{P^6} + T^6 + \frac{1}{T^6}\right). \tag{2.3.37}$$

Again from Entry 5(xii) of Chapter 19 [15, p. 231], we find that

$$\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{\frac{1}{4}} + \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{\frac{1}{4}} = 2\sqrt{2}\left[\frac{1}{(16\alpha\beta(1-\alpha)(1-\beta))^{\frac{1}{8}}} - (16\alpha\beta(1-\alpha)(1-\beta))^{\frac{1}{8}}\right] \tag{2.3.38}$$

and

$$\left(\frac{\delta(1-\delta)}{\gamma(1-\gamma)}\right)^{\frac{1}{4}} + \left(\frac{\gamma(1-\gamma)}{\delta(1-\delta)}\right)^{\frac{1}{4}} = 2\sqrt{2}\left[\frac{1}{(16\gamma\delta(1-\gamma)(1-\delta))^{\frac{1}{8}}} - (16\gamma\delta(1-\gamma)(1-\delta))^{\frac{1}{8}}\right]. \tag{2.3.39}$$

Multiplying (2.3.38) and (2.3.39), and then using (2.1.7)-(2.1.10), we find that

$$T^{12} + \frac{1}{T^{12}} + Q^{12} + \frac{1}{Q^{12}} = 8\left(P^6 + \frac{1}{P^6} - R^6 - \frac{1}{R^6}\right) \tag{2.3.40}$$

Using Lemma 2.3.1. we deduce from (2.3.40) that

$$T^{12}+\frac{1}{T^{12}}=8\left(P^6+\frac{1}{P^6}-R^6-\frac{1}{R^6}\right)-\left(R^2+\frac{1}{R^2}+3\right)^3+3\left(R^2+\frac{1}{R^2}+3\right). \qquad (2.3.41)$$

Eliminating T from (2.3.37) and (2.3.41), we find that

$$64\left(8\left(P^6+\frac{1}{P^6}\right)-8\left(R^6+\frac{1}{R^6}\right)-\left(R^2+\frac{1}{R^2}+3\right)^3+3\left(R^2+\frac{1}{R^2}+3\right)+2\right)$$
$$=\left(R^8+\frac{1}{R^8}+\left(R^2+\frac{1}{R^2}+3\right)^2+7\left(R^4+\frac{1}{R^4}\right)\left(R^2+\frac{1}{R^2}+3\right)+47-8\left(P^6+\frac{1}{P^6}\right)\right)^2. \qquad (2.3.42)$$

Transferring to one side and then factoring by using **Mathematica**, we find that

$$\left(R^8+\frac{1}{R^8}+7\left(R^6+\frac{1}{R^6}\right)+14\left(R^4+\frac{1}{R^4}\right)+21\left(R^2+\frac{1}{R^2}\right)-8\left(P^6+\frac{1}{P^6}\right)+42\right)$$
$$\left(R^8+\frac{1}{R^8}+7\left(R^6+\frac{1}{R^6}\right)+30\left(R^4+\frac{1}{R^4}\right)+5\left(R^2+\frac{1}{R^2}\right)-8\left(P^6+\frac{1}{P^6}\right)+138\right)=0.$$

It can be shown (by numerically checking, or by using power series method) that the second factor is not identically 0. Thus, we arrive at (2.1.14), which comletes the proof of the theorem.

Proof of Theorem 2.1.5.: From Lemma 2.3.1 and Theorem 2.1 4 we easily deduce (2.1.15), which completes the proof

Proof of Theorem 2.1.6.: We set

$$A:=(\alpha\gamma)^{\frac{1}{8}},\quad B:=((1-\alpha)(1-\gamma))^{\frac{1}{8}},\quad C:=(\beta\delta)^{\frac{1}{8}},\ \text{and}\ D:=((1-\beta)(1-\delta))^{\frac{1}{8}} \qquad (2.3.43).$$

so that

$$P^6=2ABCD \text{ and } T^6=CD/AB \qquad (2.3.44)$$

Employing (2.3.43) in the Weber-Ramanujan "mixed" modular equation of degrees 1, 5, 7, and 35 ([48, Vol. I, p. 309]; [18, p. 392]; [41, p. 416], we note that

$$(1+AC+BD)^2-4(AC+BD+ABCD)-(1+AC+BD)(2ABCD)^{1/3}-2(2ABCD)^{2/3}=0. \qquad (2.3.45)$$

Using (2.3.44) we may rewrite (2.3.45) as

$$(AC+BD)^2-(2+P^2)(AC+BD)+1-P^2-2P^4-2P^6=0. \tag{2.3.46}$$

Solving for $AC+BD$, we find that

$$AC+BD=\frac{2+P^2\pm P\sqrt{8+9P^2+8P^4}}{2}. \tag{2.3.47}$$

Now by Entry 19(i) [15, p. 314]

$$(\alpha\gamma)^{\frac{1}{8}}+((1-\alpha)(1-\gamma))^{\frac{1}{8}}=1. \tag{2.3.48}$$

Therefore

$$(\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{8}}=(\alpha\gamma)^{\frac{1}{8}}\left(1-(\alpha\gamma)^{\frac{1}{8}}\right). \tag{2.3.49}$$

Employing (2.3.43) and (2.3.44), we rewrite (2.3.49) as

$$\frac{P^3}{\sqrt{2}T^6}=\frac{1}{4}-\left((\alpha\gamma)^{\frac{1}{8}}-\frac{1}{2}\right)^2 \tag{2.3.50}$$

Hence,

$$(\alpha\gamma)^{\frac{1}{8}}=\frac{1}{2}\pm\sqrt{k_1}, \tag{2.3.51}$$

where

$$k_1=\frac{1}{4}-\frac{P^3}{\sqrt{2}T^3}.$$

Using (2.3.51) in (2.3.48), we find that

$$((1-\alpha)(1-\gamma))^{\frac{1}{8}}=\frac{1}{2}\mp\sqrt{k_1}. \tag{2.3.52}$$

Proceeding in a similar way, we obtain

$$(\beta\delta)^{\frac{1}{8}}=\frac{1}{2}\pm\sqrt{k_2}, \tag{2.3.53}$$

and

$$((1-\beta)(1-\delta))^{\frac{1}{8}} = \frac{1}{2} \mp \sqrt{k_2}, \tag{2.3.54}$$

where

$$k_2 = \frac{1}{4} - \frac{P^3T^3}{\sqrt{2}}.$$

From (2.3.51), (2.3.52), (2.3.53), and (2.3.54), we deduce that

$$AC + BD = \frac{1}{2} \pm 2\sqrt{k_1 k_2}. \tag{2.3.55}$$

From (2.3.47) and (2.3.55), we find that

$$\frac{1}{2} \pm 2\sqrt{k_1 k_2} = \frac{2 + P^2 \pm P\sqrt{8 + 9P^2 + 8P^4}}{2}. \tag{2.3.56}$$

Thus,

$$4\sqrt{k_1 k_2} = 1 + P^2 \pm P\sqrt{8 + 9P^2 + 8P^4}. \tag{2.3.57}$$

Squaring both sides of (2.3.57), and then simplifying by employing the expressions for k_1 and k_2, we arrive at

$$10P^2 + 10P^4 + 2\sqrt{2}P^3(T^3 + \frac{1}{T^3}) = \pm 2P\sqrt{8 + 9P^2 + 8P^4}. \tag{2.3.58}$$

Squaring both sides of (2.3.58) we easily deduce (2.1.16), which completes the proof.

Remark: Applying the same procedure to Weber's modular equation [41, p. 416]

$$\left[(1 - AC - BD)^2 - 2P^6\right]^2 = P^6(1 + AC + BD), \tag{2.3.59}$$

we easily derive the new Schläfli-type "mixed" modular equation

$$T^{12}+\frac{1}{T^{12}}-18\left(T^6+\frac{1}{T^6}\right)+18\sqrt{2}\left(T^3+\frac{1}{T^3}\right)\left(P^3+\frac{1}{P^3}\right)-8\left(P^6+\frac{1}{P^6}\right)-54=0, \quad (2.3.60)$$

where, now, in the expressions for P and T, α, β, γ, and δ are of degrees 1, 3, 7, and 21, respectively.

Proof of Theorem 2.1.7.: We note that

$$\frac{Q}{R}=\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{\frac{1}{24}} \text{ and } QR=\left(\frac{\delta(1-\delta)}{\beta(1-\beta)}\right)^{\frac{1}{24}} \quad (2.3.61)$$

From Entry 19(ix) of Chapter 19 [15, p. 315], we find that

$$\left(\frac{Q}{R}\right)^4+\left(\frac{R}{Q}\right)^4+7=2\sqrt{2}\left[(16\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{8}}+\frac{1}{(16\alpha\gamma(1-\alpha)(1-\gamma))^{\frac{1}{8}}}\right]. \quad (2.3.62)$$

Also

$$(QR)^4+\left(\frac{1}{QR}\right)^4+7=2\sqrt{2}\left[(16\beta\delta(1-\beta)(1-\delta))^{\frac{1}{8}}+\frac{1}{(16\beta\delta(1-\beta)(1-\delta))^{\frac{1}{8}}}\right]. \quad (2.3.63)$$

Multiplying (2.3.62) and (2.3.63), and then using (2.1.7)-(2.1.10), we find that

$$Q^8+\frac{1}{Q^8}+\left(R^8+\frac{1}{R^8}\right)+7\left(Q^4+\frac{1}{Q^4}\right)\left(R^4+\frac{1}{R^4}\right)+49=8\left(P^6+\frac{1}{P^6}+T^6+\frac{1}{T^6}\right). \quad (2.3.64)$$

Now from the Ramanujan's Schläfli-type "mixed" modular equation in Notebook-I [48, p. 86], which was proved by Ramanathan ([41, p. 420]; [18, p. 379-380]), we note that

$$2\left(P^2+\frac{1}{P^2}\right)=u^2-u-2, \quad (2.3.65)$$

where

$$u=Q^2+\frac{1}{Q^2}. \quad (2.3.66)$$

Therefore, we have

$$8\left(P^6+\frac{1}{P^6}\right)=(u^2-u-2)^3-12(u^2-u-2). \quad (2.3.67)$$

Again from Entry 13(xiv) of Chapter 19 [15, p. 282], we find that

$$\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{\frac{1}{8}}+\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{\frac{1}{24}}=2\left[\frac{1}{(16\alpha\beta(1-\alpha)(1-\beta))^{\frac{1}{12}}}-(16\alpha\beta(1-\alpha)(1-\beta))^{\frac{1}{12}}\right] \quad (2.3.68)$$

and,

$$\left(\frac{\delta(1-\delta)}{\gamma(1-\gamma)}\right)^{\frac{1}{8}}+\left(\frac{\gamma(1-\gamma)}{\delta(1-\delta)}\right)^{\frac{1}{24}}=2\left[\frac{1}{(16\gamma\delta(1-\gamma)(1-\delta))^{\frac{1}{12}}}-(16\gamma\delta(1-\gamma)(1-\delta))^{\frac{1}{12}}\right] \quad (2.3.69)$$

Multiplying (2.3.68) and (2.3.69), and then using (2.1.7)-(2.1.10), we find that

$$T^6+\frac{1}{T^6}+Q^6+\frac{1}{Q^6}=4\left(P^4+\frac{1}{P^4}-R^4-\frac{1}{R^4}\right). \quad (2.3.70)$$

Thus, we have

$$T^6+\frac{1}{T^6}=u^4-3u^3-3u^2+7u-4-4v, \quad (2.3.71)$$

where

$$v=R^4+\frac{1}{R^4}. \quad (2.3.72)$$

Employing (2.3.67) and (2.3.71) in (2.3.64), and then simplifying, we find that

$$v^2+(7u^2+18)v-u^6+3u^5-4u^4+13u^3+26u^2-56u+65=0. \quad (2.3.73)$$

Factoring (2.3.73), we obtain

$$(v-u^3+5u^2-7u+5)(v+u^3+2u^2+7u+13)=0. \quad (2.3.74)$$

Thus, we deduce that

$$v-u^3+5u^2-7u+5=0, \quad (2.3.75)$$

since the other factor never vanishes. Putting the expressions for u and v from (2.3.66) and (2.3.72), respectively, we easily deduce (2.1.17), which completes the proof.

Proof of Theorem 2.1.8.: For simplicity, we set

$$A := (\alpha\gamma)^{\frac{1}{8}},\ \ B := ((1-\alpha)(1-\gamma))^{\frac{1}{8}},\ C := (\beta\delta)^{\frac{1}{8}},\ \text{and}\ D := ((1-\beta)(1-\delta))^{\frac{1}{8}} \tag{2.3.76}$$

so that

$$P^6 = 2ABCD \text{ and } T^6 = CD/AB \tag{2.3.77}$$

Now from the Weber-Ramanujan "mixed" modular equation of degrees 1, 5, 11, and 55 ([48, Vol. I, p. 309]; [18, p. 391-392]; [41, p. 415-416]), we note that

$$U^3 - W(4U^2 + V) - UW^2 + 4W^3 = 0, \tag{2.3.78}$$

where

$$U = 1 - AC - BD,$$

$$V = 4(AC + BD - ABCD),$$

$$\text{and}\ \ W = (2ABCD)^{1/3} = P^2.$$

Setting x:=AC+BD, and then using (2.3.76), we may rewrite (2.3.77) as

$$1 + 3x^2 - 4P^2 - 4P^2x^2 + 2P^8 - P^4 + 4P^6 = (3 + x^2 - 4P^2 - P^4)x. \tag{2.3.79}$$

From Entry 7(i) of [15, p. 363], we note that

$$A^2 + B^2 = 1 - 2\left(\frac{P}{T}\right)^2 \tag{2.3.80}$$

Thus

$$(AB)^2 = A^2\left(1 - \frac{2P^2}{T^2} - A^2\right). \tag{2.3.81}$$

After some simplification, we arrive at

$$A^2 = a \pm \left(a^2 - \frac{P^6}{2T^6}\right)^{1/2}, \tag{2.3.82}$$

and

$$B^2 = a \mp \left(a^2 - \frac{P^6}{2T^6}\right)^{1/2}, \tag{2.3.83}$$

where

$$a = \frac{1}{2} - \frac{P^2}{T^2}. \tag{2.3.84}$$

Similarly, we find that

$$C^2 = b \pm \left(b^2 - \frac{P^6T^6}{2}\right)^{1/2}, \tag{2.3.85}$$

and

$$D^2 = b \mp \left(b^2 - \frac{P^6T^6}{2}\right)^{1/2}, \tag{2.3.86}$$

where

$$b = \frac{1}{2} - P^2T^2. \tag{2.3.87}$$

Therefore, we obtain

$$x^2 = 2ab + 2\sqrt{k_1k_2} + P^6, \tag{2.3.88}$$

where

$$k_1 = a^2 - \frac{P^6}{2T^6}, \tag{2.3.89}$$

and

$$k_2 = b^2 - \frac{P^6T^6}{2}. \tag{2.3.90}$$

Employing the value of x^2 from (2.3.88) in (2.3.79), and then simplifying by using **Mathematica**, we complete the theorem.

Chapter 3

Weber-Ramanujan's Class Invariants

3.1 Introduction

We set

$$(a;q)_\infty = \Pi_{n=0}^{\infty}(1-aq^n), \qquad |q|<1,$$

and, recall from (2.1.26) that

$$\chi(q) = (-q;q^2)_\infty. \tag{3.1.1}$$

If $q = \exp(-\pi\sqrt{n})$, where n is any positive rational number, then Weber-Ramanujan's class invariants G_n and g_n are defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q), \qquad g_n := 2^{-1/4}q^{-1/24}\chi(-q). \tag{3.1.2}$$

Note: *This chapter is identical to our paper [6], which has been accepted for publication in the Journal of Indian Mathematical Society.*

In his book, H. Weber [61] calculated 105 class invariants, or monic, irreducible polynomials satisfied by them. He was motivated to calculate class invariants so that he could construct Hilbert class fields. At the scattered places in his first notebook [48], Ramanujan recorded 107 class invariants, or monic, irreducible polynomials satisfied by them. On pages 294-299 in his second notebook [48], he recorded a table of 77 class invarints, three of which are not found in the first notebook. By the time Ramanujan wrote his paper [47], he came to know about Weber's work, and therefore his table of 46 class invariants in [47] does not contain any that are found in Weber's book [61]. Except for g_{325} and G_{363}, all of the remaining values are found in his notebooks [48]. G.N. Watson [54]-[60] established 28 of these 46 class invariants. Ten of the class invariants had been proved by using Ramanujan's modular equations and the rest had been proved by using his unrigorous "empirical process" . So, after Watson's work, 18 invariants of Ramanujan from his paper [47] and notebooks [48] remained to be verified. These 18 class invarints are: G_{65}, G_{69}, G_{77}, G_{117}, G_{141}, G_{145}, G_{153}, G_{205}, G_{213}, G_{217}, G_{265}, G_{301} , G_{441}, G_{445}, G_{505}, G_{553}, g_{90}, and g_{198}. These invariants are proved by B.C. Berndt, H.H. Chan, L.C. Zhang [24], [26]. In [24], five of the invariants, viz., G_{117}, G_{153}, G_{441}, g_{90}, and g_{198}, are proved by employing two new theorems that relate G_{9n} with G_n, and g_{9n} with g_n, respectively. In [26], they used modular equations to prove six of the remaining thirteen invariants. To prove the other seven invariants via modular equations, one needs modular equations of degrees 31, 41, 43, 53, 79, 89, and, 101. But, only for degree 31 Ramanujan recorded modular equations, for he recorded no modular equations for the other degrees. They could not utilize those modular equations of degree 31 to effect a proof for G_{217}. They [26] proved all the remaining invariants, including G_{217}, by using Kronecker's limit formula, an idea completely unknown to Ramanujan, and Watson's "empirical process." For a detail discussion on their evaluations see Berndt's book [18].

All this is now rigorous by papers of Zhang & Chan

In Section 3.3, we shall establish the class invariant G_{217} by using Ramanujan's modular equations of degrees 7 and 31. In Section 3.4, we employ, for the first time, some of the Schläfli-type "mixed" modular equations discussed in Chapter 2, along with some other Schläfli-type modular equations of prime degrees to evaluate Ramanujan's class invariants G_{15}, G_{21}, G_{33}, G_{39}, G_{55}, and G_{65}. It is worthwhile to note that our evaluation of G_{65} is much more easier than that of Berndt, Chan, and Zhang [18], [26]. Most important feature of our method is that we can also simultaneously get the values of $G_{5/3}$, $G_{7/3}$, $G_{11/3}$, $G_{13/3}$, $G_{11/5}$, and $G_{13/5}$. Previously, these values were found by verifications. We also note that, these class invariants can be utilised to find some of the explicit values of the famous Rogers-Ramanujan continued fraction, $R(q)$, defined by

$$R(q) := \frac{q^{1/5}}{1} \, \frac{q}{+1+} \, \frac{q^2}{1} \, \frac{q^3}{+1+}, \qquad |q| < 1, \tag{3.1.3}$$

some of the values of Ramanujan's product of theta-functions $a_{m,n}$ (m, n are positive integers), recorded on pages 338-339 of his first notebook [48], and defined by

$$a_{m,n} := ne^{-(\pi/4)(n-1)}\sqrt{m/n}\,\frac{\psi(e^{-\pi\sqrt{mn}})\phi(-e^{-2\pi\sqrt{mn}})}{\psi(e^{-\pi\sqrt{m/n}})\phi(-e^{-2\pi\sqrt{m/n}})}$$

and, the values of the quotient of eta-functions, λ_n, recorded by Ramanujan on page 212 of his lost notebook [49], and defined by

$$\lambda_n := \frac{e^{\pi}/2\sqrt{n/3}}{3\sqrt{3}}\{(1+e^{-\pi\sqrt{n/3}})(1-e^{-2\pi\sqrt{n/3}})(1-e^{-4\pi\sqrt{n/3}})..\}^6.$$

For details of the above evaluations see [25], [27], and [30].

We complete this introduction by noting that, since from (2.1.37),

$$\chi(q) = 2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24},$$

it follows from (3.1.2) that

$$G_n = \{4\alpha(1-\alpha)/q\}^{-1/24} \quad \text{and} \qquad G_{r^2n} = \{4\beta(1-\beta)/q\}^{-1/24}, \tag{3.1.4}$$

where β has degree r over α and $q = \exp(-\pi\sqrt{n})$.

3.2 Preliminary Lemmas

In this section we state some lemmas which will be used in our evaluation.

Lemma 3.2.1 *([18, p. 247]; [26])If β has degree r over α, then β has degree p over $1-\alpha$, where p and r are two coprime positive integers.*

In the next three lemmas we state three Schläfli-type modular equations of Ramanujan [15, pp. 231, 282, 315] for prime degrees.

Lemma 3.2.2 *Let*

$$P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \quad \textit{and} \quad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}$$

Then

$$Q + \frac{1}{Q} + 2\sqrt{2}\left(P - \frac{1}{P}\right) = 0,$$

where β has degee 3 over α.

Lemma 3.2.3 *Let*

$$P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \quad \textit{and} \quad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}.$$

Then

$$Q + \frac{1}{Q} + 2\left(P - \frac{1}{P}\right) = 0,$$

where β has degee 5 over α.

Lemma 3.2.4 *Let*

$$P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \quad \textit{and} \quad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}.$$

Then

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left(P + \frac{1}{P}\right),$$

where β has degee 7 over α.

In the following three lemmas, we state three of Ramanujan's Schläfli-type modular equations for composite degrees.

Lemma 3.2.5 *([48, Vol. I, p. 86], [15, p. 324]) If α, β, γ, and δ have degrees 1, 3, 5, and 15, respectively, then*

$$Q^3 + \frac{1}{Q^3} = \sqrt{2}\left(P + \frac{1}{P}\right). \tag{3.2.1}$$

Lemma 3.2.6 *([48, Vol. I, p. 88], [18, p. 381]) If α, β, γ, and δ have degrees 1, 3, 13, and 39, respectively, then*

$$Q^4 + \frac{1}{Q^4} - 3\left(Q^2 + \frac{1}{Q^2}\right) - \left(R^2 + \frac{1}{R^2}\right) + 3 = 0. \tag{3.2.2}$$

Lemma 3.2.7 *([48, Vol. I, p. 88], [18, p. 381]) If α, β, γ, and δ have degrees 1, 5, 13, and 65, respectively, then*

$$Q^6 + \frac{1}{Q^6} - 5\left(Q + \frac{1}{Q}\right)^2\left(R + \frac{1}{R}\right)^2 - \left(R^4 + \frac{1}{R^4}\right) = 0. \tag{3.2.3}$$

The next lemma due to Landau [40, p. 53] will be very useful in simplifying some of our redicals.

much older

Lemma 3.2.8 *If $a^2 - qb^2 = d^2$, a perfect square, then*

$$\sqrt{a + b\sqrt{q}} = \sqrt{\frac{a+d}{2}} + (sgnb)\sqrt{\frac{a-d}{2}}. \tag{3.2.4}$$

Our last lemma is originaly due to Bruce Reznick. For a proof via Chebyshev polynomials one may see [28, p. 150].

Lemma 3.2.9 *If $a, b \geq 1/2$, then*

$$\{(8a^2-1) + \sqrt{(8a^2-1)^2 - 1}\}^{1/4} = \sqrt{a + \frac{1}{2}} + \sqrt{a - \frac{1}{2}} \tag{3.2.5}$$

and

$$\{(32b^3 - 6b) + \sqrt{(32b^3 - 6b)^2 - 1}\}^{1/6} = \sqrt{b + \frac{1}{2}} + \sqrt{b - \frac{1}{2}}. \tag{3.2.6}$$

Method due to Berndt, Chan, Zhang

3.3 Class invariant G_{217}

Theorem 3.3.1

$$G_{217} = \left(\sqrt{\frac{11+4\sqrt{7}}{2}} + \sqrt{\frac{9+4\sqrt{7}}{2}}\right)^{1/2} \left(\sqrt{\frac{12+5\sqrt{7}}{4}} + \sqrt{\frac{16+5\sqrt{7}}{4}}\right)^{1/2}.$$

Proof: From Entries 19(i) and 19(iii) of Berndt's book [15, p. 314], we note that

$$\left(\frac{(1-\beta)^7}{(1-\alpha)}\right)^{1/8} - \left(\frac{\beta^7}{\alpha}\right)^{1/8} = m\left(1 - (\alpha\beta(1-\alpha)(1-\beta)^{1/8}\right), \tag{3.3.1}$$

and

$$\left(\frac{\alpha^7}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^7}{(1-\beta)}\right)^{1/8} = \frac{7}{m}\left(1 - (\alpha\beta(1-\alpha)(1-\beta)^{1/8}\right), \tag{3.3.2}$$

where β has degree 7 over α, and m is the multiplier connecting α and β.

Multiplying (3.3.1) and (3.3.2), we find that

$$\alpha(1-\beta) + \beta(1-\alpha) = A\left[7(1-A)^2 + (\alpha\beta)^{3/4} + ((1-\alpha)(1-\beta))^{3/4}\right], \tag{3.3.3}$$

where $A = (\alpha\beta(1-\alpha)(1-\beta))^{1/8}$.

Now, by the first equality of Entry 19(i) of Berndt's book [15, p. 314], we obtain

$$(\alpha\beta)^{3/4} + ((1-\alpha)(1-\beta))^{3/4} = 1 - 6A + 9A^2 - 2A^3. \tag{3.3.4}$$

From (3.3.3) and (3.3.4), we deduce that

$$\alpha(1-\beta) + \beta(1-\alpha) = 2A\left(4 - 10A + 8A^2 - A^3\right). \tag{3.3.5}$$

Now, suppose, $G_{31/7} = (4\alpha(1-\alpha))^{-1/24}$. If β has degree 7 over α, then, by (3.1.4), we find that $G_{217} = (4\beta(1-\beta))^{-1/24}$.

Thus,

$$\frac{1}{A} = \sqrt{2}(G_{217}G_{31/7})^3. \tag{3.3.6}$$

We now recall the following two modular equations of degree 31 from Entries 22(ii) and (iii) of Berndt's book [15, p. 439].

$$1 + (\alpha\beta)^{1/4} + ((1-\alpha)(1-\beta))^{1/4} - 2((\alpha\beta)^{1/8} + ((1-\alpha)(1-\beta))^{1/8} + A)$$
$$= 2A^{1/2}(1 + (\alpha\beta)^{1/8} + ((1-\alpha)(1-\beta))^{1/8})^{1/2}, \tag{3.3.7}$$

and

$$1 + (\alpha\beta)^{1/4} + ((1-\alpha)(1-\beta))^{1/4} - \left(\frac{1}{2}\{1 + (\alpha\beta)^{1/2} + ((1-\alpha)(1-\beta))^{1/2}\}\right)^{1/2}$$
$$= (\alpha\beta)^{1/8} + ((1-\alpha)(1-\beta))^{1/8} + A, \tag{3.3.8}$$

where β has degree 31 over α.

Replacing α by $1-\alpha$ in (3.3.7) and (3.3.8), and employing Lemma 3.2.1, we obtain

$$1 + \{(1-\alpha)\beta\}^{1/4} + \{\alpha(1-\beta)\}^{1/4} - 2[\{(1-\alpha)\beta\}^{1/8} + \{\alpha(1-\beta)\}^{1/8} + A]$$
$$= 2A^{1/2}[1 + \{(1-\alpha)\beta\}^{1/8} + \{\alpha(1-\beta)\}^{1/8}]^{1/2}, \tag{3.3.9}$$

and

$$1 + \{(1-\alpha)\beta\}^{1/4} + \{\alpha(1-\beta)\}^{1/4} - \left[\frac{1}{2}\{1 + \sqrt{(1-\alpha)\beta} + \sqrt{\alpha(1-\beta)}\}\right]^{1/2}$$
$$= \{(1-\alpha)\beta\}^{1/8} + \{\alpha(1-\beta)\}^{1/8} + A, \tag{3.3.10}$$

where, now, β has degree 7 over α.

From (3.3.5), (3.3.9) and (3.3.10), we arrive at (simplification is done by using **Mathematica**)

$$1-160A-66848A^2-4978240A^3+88485264A^4-657312128A^5+2752494208A^6$$

$$-7235315456A^7+12522323040A^8-14470630912A^9+11009976832A^{10}-5258497024A^{11}$$

$$+1415764224A^{12}-159303680A^{13}-4278272A^{14}-20480A^{15}+256A^{16}=0. \qquad (3.3.11)$$

Factoring the left side of (3.3.11) by using **Mathematica**, we find that

$$(1-376A+1048A^2-752A^3+4A^4)(1-8A+24A^2-16A^3+4A^4)(1+224A$$

$$+15088A^2-80192A^3+166728A^4-160384A^5+60352A^6+1792A^7+16A^8)=0. \qquad (3.3.12)$$

Thus,

$$1-376A+1048A^2-752A^3+4A^4=0. \qquad (3.3.13)$$

Since from the other two factors we will not get positive real values of A.

We can rewrite (3.3.13) as

$$A^2\left(4A^2+\frac{1}{A^2}-376\left(2A+\frac{1}{A}\right)+1048\right)=0. \qquad (3.3.14)$$

Since $A^2\neq 0$, we find that

$$\left(2A+\frac{1}{A}\right)^2-376\left(2A+\frac{1}{A}\right)+1044=0. \qquad (3.3.15)$$

Solving (3.3.15), we find that

$$2A+\frac{1}{A}=188+70\sqrt{7}. \qquad (3.3.16)$$

Hence,

$$\frac{1}{A}-2A=2\sqrt{17409+6580\sqrt{7}}. \qquad (3.3.17)$$

From (3.3.16) and (3.3.17), we obtain

$$\frac{1}{A} = 94 + 35\sqrt{7} + \sqrt{17409 + 6580\sqrt{7}}. \tag{3.3.18}$$

Now, by Lemma 2.2.4, we note that

$$Q + \frac{1}{Q} = 2\left(2A + \frac{1}{A}\right) - 7, \tag{3.3.19}$$

where $Q = (G_{217}/G_{31/7})^4$. From (3.3.16) and (3.3.19), we find that

$$Q + \frac{1}{Q} = 369 + 140\sqrt{7}. \tag{3.3.20}$$

Solving (3.3.20) for Q, we obtain

$$Q = \frac{1}{2}(369 + 140\sqrt{7} + \sqrt{273357 + 103320\sqrt{7}}). \tag{3.3.21}$$

From (3.3.6), (3.3.18) and (3.3.21), we deduce that

$$G_{217} = \left(\frac{94 + 35\sqrt{7}}{\sqrt{2}} + \sqrt{\frac{17409 + 6580\sqrt{7}}{2}}\right)^{1/6} \left(\frac{369 + 140\sqrt{7}}{2} + \sqrt{\frac{273357 + 103320\sqrt{7}}{4}}\right)^{1/8}. \tag{3.3.22}$$

Now, substituting $a = (14 + 5\sqrt{7})/4$ in Lemma 2.2.9, we find that

$$\left(\frac{369 + 140\sqrt{7}}{2} + \sqrt{\frac{273357 + 103320\sqrt{7}}{4}}\right)^{1/4} = \sqrt{\frac{12 + 5\sqrt{7}}{4}} + \sqrt{\frac{16 + 5\sqrt{7}}{4}}.$$

Hence, it remains to show that

$$\frac{94 + 35\sqrt{7}}{\sqrt{2}} + \sqrt{\frac{17409 + 6580\sqrt{7}}{2}} = \left(\sqrt{\frac{11 + 4\sqrt{7}}{2}} + \sqrt{\frac{9 + 4\sqrt{7}}{2}}\right)^3,$$

which is a routine work. This completes the theorem.

3.4 Class invariants from "mixed" modular equations

Theorem 3.4.1

$$G_{15} = 2^{1/4}\left(\frac{1+\sqrt{5}}{2}\right)^{1/3} \quad \textit{and} \quad G_{5/3} = 2^{1/4}\left(\frac{\sqrt{5}-1}{2}\right)^{1/3}.$$

Proof: If

$$G_n = (4\alpha(1-\alpha))^{-1/24}$$

and β, γ, and δ have degrees 3, 5, and 15, respectively, over α, then by (3.1.4), we obtain

$$G_{9n} = (4\beta(1-\beta))^{-1/24}, \quad G_{25n} = (4\gamma(1-\gamma))^{-1/24} \quad \text{and} \quad G_{225n} = (4\delta(1-\delta))^{-1/24}. \tag{3.4.1}$$

Employing Lemma 3.2.3, we find that

$$\left(\frac{G_n}{G_{25n}}\right)^3 + \left(\frac{G_{25n}}{G_n}\right)^3 = 2\left[(G_nG_{25n})^2 - \frac{1}{(G_nG_{25n})^2}\right]. \tag{3.4.2}$$

Putting $n = 1/15$ in (3.4.2), we obtain

$$\left(\frac{G_{15}}{G_{5/3}}\right)^3 + \left(\frac{G_{5/3}}{G_{15}}\right)^3 = 2\left[\left(G_{15}G_{5/3}\right)^2 - \frac{1}{\left(G_{15}G_{5/3}\right)^2}\right], \tag{3.4.3}$$

where we have used the fact that, $G_n = G_{1/n}$.

Now, by Lemma 3.2.5, we obtain that

$$\left(\frac{G_nG_{225n}}{G_{9n}G_{25n}}\right)^{3/2} + \left(\frac{G_{9n}G_{25n}}{G_nG_{225n}}\right)^{3/2} = \sqrt{2}\left[(G_nG_{9n}G_{25n}G_{225n})^{1/2} + \frac{1}{(G_nG_{9n}G_{25n}G_{225n})^{1/2}}\right]. \tag{3.4.4}$$

Putting $n = 1/15$ in (3.4.4), we find that

$$\left(\frac{G_{15}}{G_{5/3}}\right)^3 + \left(\frac{G_{5/3}}{G_{15}}\right)^3 = \sqrt{2}\left[\left(G_{15}G_{5/3}\right) - \frac{1}{\left(G_{15}G_{5/3}\right)}\right]. \tag{3.4.5}$$

Setting $x := G_{15}G_{5/3}$ in (3.4.3) and (3.4.5), we deduce that

$$2(x^2 - \frac{1}{x^2}) = \sqrt{2}(x + \frac{1}{x}). \tag{3.4.6}$$

As $x + \frac{1}{x} > 0$, from (3.4.6), we conclude that

$$\sqrt{2}(x - \frac{1}{x}) = 1. \tag{3.4.7}$$

Solving (3.4.7) for x, we find that

$$x = G_{15}G_{5/3} = \sqrt{2}. \tag{3.4.8}$$

Using this value of x in (3.4.5), we deduce that

$$y^3 + \frac{1}{y^3} = 3, \tag{3.4.9}$$

where, $y = G_{15}/G_{5/3}$.

Solving (3.4.9) for y^3, we find that

$$y^3 = \left(\frac{G_{15}}{G_{5/3}}\right)^3 = \frac{3+\sqrt{5}}{2}. \tag{3.4.10}$$

From (3.4.8) and (3.4.10), we obtain

$$G_{15}^6 = y^3x^3 = 2^{3/2}\frac{3+\sqrt{5}}{2} = \sqrt{2}(3+\sqrt{5}), \tag{3.4.11}$$

and

$$G_{5/3}^6 = y^{-3}x^3 = 2^{3/2}\frac{2}{3+\sqrt{5}} = \sqrt{2}(3-\sqrt{5}), \tag{3.4.12}$$

Now, from Lemma 3.2.8, we see that

$$\sqrt{3 \pm \sqrt{5}} = \sqrt{5/2} \pm \sqrt{1/2}.$$

Thus, from (3.4.11) and (3.4.12), we can arrive at the required values of G_{15} and $G_{5/3}$.

Theorem 3.4.2

$$G_{21} = \left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)^{1/4}\left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^{1/6} \quad \textit{and} \quad G_{7/3} = \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{1/4}\left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^{1/6}.$$

Proof: As in the proof of Theorem 3.4.1, if

$$G_n = (4\alpha(1-\alpha))^{-1/24}$$

and β, γ, and δ have degrees 3, 7, and 21, respectively, over α, then by (3.1.4),

$$G_{9n} = (4\beta(1-\beta))^{-1/24}, \quad G_{49n} = (4\gamma(1-\gamma))^{-1/24} \quad \text{and} \quad G_{441n} = (4\delta(1-\delta))^{-1/24}. \tag{3.4.13}$$

Therefore, by Lemma 3.2.2, we find that

$$\left(\frac{G_n}{G_{9n}}\right)^6 + \left(\frac{G_{9n}}{G_n}\right)^6 = 2\sqrt{2}\left[(G_nG_{9n})^3 - \frac{1}{(G_nG_{9n})^3}\right]. \tag{3.4.14}$$

Putting $n = 1/21$ in (3.4.14), we deduce that

$$\left(\frac{G_{21}}{G_{7/3}}\right)^6 + \left(\frac{G_{7/3}}{G_{21}}\right)^6 = 2\sqrt{2}\left[\left(G_{21}G_{7/3}\right)^3 - \frac{1}{\left(G_{21}G_{7/3}\right)^3}\right], \tag{3.4.15}$$

where we have again used the fact that, $G_n = G_{1/n}$.

Now, by Theorem 2.1.4, we deduce that

$$R^4 + \frac{1}{R^4} + 7\left(R^3 + \frac{1}{R^3}\right) + 14\left(R^2 + \frac{1}{R^2}\right) + 21\left(R + \frac{1}{R}\right) - 8\left(P^6 + \frac{1}{P^6}\right) + 42 = 0, \tag{3.4.16}$$

where, now,

$$R = \left(\frac{G_nG_{9n}}{G_{49n}G_{441n}}\right) \quad \text{and} \quad P^2 = 1/\left(G_nG_{9n}G_{49n}G_{441n}\right).$$

Putting $n = 1/21$ in (3.4.16), we find that

$$\left(G_{21}G_{7/3}\right)^6 + \frac{1}{\left(G_{21}G_{7/3}\right)^6} = 16 \tag{3.4.17}$$

Solving (3.4.17) for $\left(G_{21}G_{7/3}\right)^6$, we obtain

$$\left(G_{21}G_{7/3}\right)^6 = 8 + 3\sqrt{7} = \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^2. \tag{3.4.18}$$

Employing (3.4.18) in (3.4.15), we find that

$$\left(\frac{G_{21}}{G_{7/3}}\right)^6 + \left(\frac{G_{7/3}}{G_{21}}\right)^6 = 2\sqrt{2}\left(\frac{3+\sqrt{7}}{\sqrt{2}} - \frac{\sqrt{2}}{3+\sqrt{7}}\right) = 4\sqrt{7}. \tag{3.4.19}$$

Solving (3.4.19) for $\left(G_{21}/G_{7/3}\right)^6$, we obtain

$$\left(\frac{G_{21}}{G_{7/3}}\right)^6 = 3\sqrt{3} + 2\sqrt{7} = \left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)^3. \tag{3.4.20}$$

From (3.4.18) and (3.4.20), we obtain

$$G_{21}^{12} = \left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)^3 \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^2, \tag{3.4.21}$$

and

$$G_{7/3}^{12} = \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3 \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^2. \tag{3.4.22}$$

From (3.4.21) and (3.4.22), we get the required values of G_{21} and $G_{7/3}$ as given in the theorem.

Theorem 3.4.3

$$G_{33} = \left(\frac{3+\sqrt{11}}{\sqrt{2}}\right)^{1/6} \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/2} \quad \textit{and} \quad G_{11/3} = \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^{1/6} \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/2}.$$

Proof: In this case also, if

$$G_n = (4\alpha(1-\alpha))^{-1/24}$$

and β, γ, and δ have degrees 3, 11, and 33, respectively, over α, then by (3.1.4),

$$G_{9n} = (4\beta(1-\beta))^{-1/24}, \quad G_{121n} = (4\gamma(1-\gamma))^{-1/24} \quad \text{and} \quad G_{1089n} = (4\delta(1-\delta))^{-1/24}. \tag{3.4.23}$$

By putting $n = 1/33$ in (3.4.14), we deduce that

$$\left(\frac{G_{33}}{G_{11/3}}\right)^6 + \left(\frac{G_{11/3}}{G_{33}}\right)^6 = 2\sqrt{2}\left[\left(G_{33}G_{11/3}\right)^3 - \frac{1}{\left(G_{33}G_{11/3}\right)^3}\right]. \tag{3.4.24}$$

Now, by Theorem 2.1.2, we find that

$$T^2 + \frac{1}{T^2} + 3\left(T + \frac{1}{T}\right) - 2\left(P^2 + \frac{1}{P^2}\right) = 0, \tag{3.4.25}$$

where, now,

$$T = \frac{G_n G_{121n}}{G_{9n}G_{1089n}} \qquad \text{and} \qquad P^2 = 1/\left(G_n G_{9n} G_{121n} G_{1089n}\right).$$

Putting $n = 1/33$ in (3.4.25), we find that

$$\left(G_{33}G_{11/3}\right)^2 + \frac{1}{\left(G_{33}G_{11/3}\right)^2} = 4. \tag{3.4.26}$$

Solving (3.4.26) for $\left(G_{33}G_{11/3}\right)^2$, we obtain

$$\left(G_{33}G_{11/3}\right)^2 = 2 + \sqrt{3} = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^2. \tag{3.4.27}$$

Employing (3.4.27) in (3.4.24), we find that

$$\left(\frac{G_{33}}{G_{11/3}}\right)^6 + \left(\frac{G_{11/3}}{G_{33}}\right)^6 = 20. \tag{3.4.28}$$

Solving (3.4.28) for $\left(G_{33}/G_{11/3}\right)^6$, we obtain

$$\left(\frac{G_{33}}{G_{11/3}}\right)^6 = 10 + 3\sqrt{11} = \left(\frac{3+\sqrt{11}}{\sqrt{2}}\right)^2. \qquad (3.4.29)$$

From (3.4.27) and (3.4.29), we obtain

$$G_{33}^{12} = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^6 \left(\frac{3+\sqrt{11}}{\sqrt{2}}\right)^2, \qquad (3.4.30)$$

and

$$G_{11/3}^{12} = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^6 \left(\frac{\sqrt{2}}{3+\sqrt{11}}\right)^2. \qquad (3.4.31)$$

From (3.4.30) and (3.4.31), we can easily find the values of G_{11} and $G_{11/3}$.

Theorem 3.4.4

$$G_{39} = 2^{1/4}\left(\frac{\sqrt{13}+3}{2}\right)^{1/6}\left(\sqrt{\frac{5+\sqrt{13}}{8}} + \sqrt{\frac{\sqrt{13}-3}{8}}\right)$$

and

$$G_{13/3} = 2^{1/4}\left(\frac{\sqrt{13}+3}{2}\right)^{1/6}\left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{\sqrt{13}-3}{8}}\right).$$

Proof: As above, if

$$G_n = (4\alpha(1-\alpha))^{-1/24}$$

and β, γ, and δ have degrees 3, 13, and 39, respectively, over α, then

$$G_{9n} = (4\beta(1-\beta))^{-1/24}, \quad G_{169n} = (4\gamma(1-\gamma))^{-1/24} \quad \text{and} \quad G_{1521n} = (4\delta(1-\delta))^{-1/24}. \qquad (3.4.32)$$

Putting $n = 1/39$ in (3.4.14), we Find that

$$\left(\frac{G_{39}}{G_{13/3}}\right)^6 + \left(\frac{G_{13/3}}{G_{39}}\right)^6 = 2\sqrt{2}\left[\left(G_{39}G_{13/3}\right)^3 - \frac{1}{\left(G_{39}G_{13/3}\right)^3}\right]. \qquad (3.4.33)$$

Now, by Lemma 3.2.6, we obtain

$$Q^2 + \frac{1}{Q^2} - 3\left(Q + \frac{1}{Q}\right) - \left(R + \frac{1}{R}\right) = 0, \tag{3.4.34}$$

where

$$Q = \frac{G_{9n}G_{169n}}{G_nG_{1521n}} \qquad \text{and} \qquad T = \frac{G_nG_{9n}}{G_{169n}G_{1521n}}.$$

Putting $n = 1/39$ in (3.4.34), we find that

$$\left(\frac{G_{39}}{G_{13/3}}\right)^4 + \left(\frac{G_{13/3}}{G_{39}}\right)^4 - 3\left(\frac{G_{39}}{G_{13/3}}\right)^2 + \left(\frac{G_{13/3}}{G_{39}}\right)^2 + 1 = 0. \tag{3.4.35}$$

Therefore, we obtain

$$\left(\frac{G_{39}}{G_{13/3}}\right)^2 + \left(\frac{G_{13/3}}{G_{39}}\right)^2 = \frac{3+\sqrt{13}}{2}. \tag{3.4.36}$$

Solving (3.4.36) for $\left(G_{39}/G_{13/3}\right)^2$, we find that

$$\left(\frac{G_{39}}{G_{13/3}}\right)^2 = \frac{1}{2}\left[\frac{3+\sqrt{13}}{2} + \sqrt{\frac{3+3\sqrt{13}}{2}}\right]. \tag{3.4.37}$$

Employing (3.4.37) in (3.4.33), we find that

$$\left(G_{39}G_{13/3}\right)^3 - \frac{1}{\left(G_{39}G_{13/3}\right)^3} = \frac{27+7\sqrt{13}}{4\sqrt{2}}. \tag{3.4.38}$$

Solving (3.4.38) for $\left(G_{39}G_{13/3}\right)^3$, we find that

$$\left(G_{39}G_{13/3}\right)^3 = \frac{1}{2}\left[\frac{27+7\sqrt{13}}{4\sqrt{2}} + \sqrt{\frac{747+189\sqrt{13}}{16}}\right]. \tag{3.4.39}$$

Since, $747^2 - 13.189^2 = 306^2$, by Lemma 3.2.8, we note that

$$\sqrt{747+189\sqrt{13}} = \sqrt{\frac{747+306}{2}} + \sqrt{\frac{747-306}{2}} = \frac{21+9\sqrt{13}}{\sqrt{2}}.$$

Thus, from (3.4.39), we find that

$$\left(G_{39}G_{13/3}\right)^3 = \sqrt{2}(3+\sqrt{13}). \tag{3.4.40}$$

From (3.4.37) and (3.4.40), we obtain

$$G_{39}^{12} = (3+\sqrt{13})^2 \left[\frac{3+\sqrt{13}}{2} + \sqrt{\frac{3+3\sqrt{13}}{2}}\right]^3. \tag{3.4.41}$$

Therefore,

$$G_{39} = 2^{1/4}(3+\sqrt{13})^{1/6} \left[\frac{3+\sqrt{13}}{4} + \frac{1}{2}\sqrt{\frac{3+3\sqrt{13}}{2}}\right]^{1/4}. \tag{3.4.42}$$

Now, putting $a = (1+\sqrt{13})/8$ in Lemma 3.2.9, we obtain

$$\left(\frac{3+\sqrt{13}}{4} + \sqrt{\frac{3+3\sqrt{13}}{8}}\right)^{1/4} = \sqrt{\frac{5+\sqrt{13}}{8}} + \sqrt{\frac{\sqrt{13}-3}{8}}.$$

Thus, we arrive at the required value of G_{39}. Similarly, we can get the value of $G_{13/3}$.

Theorem 3.4.5

$$G_{55} = 2^{1/4}(2+\sqrt{5})^{1/4}\left(\sqrt{\frac{7+\sqrt{5}}{8}} + \sqrt{\frac{\sqrt{5}-1}{8}}\right)$$

and

$$G_{11/5} = 2^{1/4}(2+\sqrt{5})^{1/4}\left(\sqrt{\frac{7+\sqrt{5}}{8}} - \sqrt{\frac{\sqrt{5}-1}{8}}\right).$$

Proof: As in the previous proofs, if

$$G_n = (4\alpha(1-\alpha))^{-1/24}$$

and β, γ, and δ have degrees 5, 11, and 55, respectively, over α, then by (3.1.4)

$$G_{25n} = (4\beta(1-\beta))^{-1/24}, \quad G_{121n} = (4\gamma(1-\gamma))^{-1/24} \quad \text{and} \quad G_{3025n} = (4\delta(1-\delta))^{-1/24}. \tag{3.4.43}$$

By putting $n = 1/55$ in (3.4.2), we deduce that

$$\left(\frac{G_{55}}{G_{11/5}}\right)^3 + \left(\frac{G_{11/5}}{G_{55}}\right)^3 = 2\left[\left(G_{55}G_{11/5}\right)^2 - \frac{1}{\left(G_{55}G_{11/5}\right)^2}\right]. \tag{3.4.44}$$

By Theorem 3.1.8, we deduce that

$$T^3 + \frac{1}{T^3} - 5\left(T^2 + \frac{1}{T^2}\right) + 10\left(T + \frac{1}{T}\right)(P^2 + \frac{1}{P^2} - 1)$$
$$-4\left(P^4 + \frac{1}{P^4}\right) + 10\left(P^2 + \frac{1}{P^2}\right) - 25 = 0, \tag{3.4.45}$$

where

$$T = \frac{G_nG_{121n}}{G_{25n}G_{3025n}} \qquad \text{and} \qquad P^2 = 1/\left(G_nG_{25n}G_{121n}G_{3025n}\right).$$

Putting $n = 1/55$ in (3.4.45), we find that

$$4\left(P^4 + \frac{1}{P^4}\right) - 30\left(P^2 + \frac{1}{P^2}\right) + 53 = 0, \tag{3.4.46}$$

where, now, $P^2 = 1/\left(G_{55}G_{11/5}\right)^2$.

From (3.4.46), we deduce that

$$P^2 + \frac{1}{P^2} = \frac{15 + 3\sqrt{5}}{4}. \tag{3.4.47}$$

Solving for $1/P^2$, we find that

$$\frac{1}{P^2} = \frac{15 + 3\sqrt{5}}{4} + \frac{1}{4}\sqrt{\frac{103 + 45\sqrt{5}}{2}}. \tag{3.4.48}$$

Since, $103^2 - 5.45^2 = 22^2$, we see from Lemma 3.2.8 that

$$\sqrt{103 + 45\sqrt{5}} = \sqrt{\frac{125}{2}} + \sqrt{\frac{81}{2}} = \frac{9 + 5\sqrt{5}}{\sqrt{2}}.$$

Thus

$$\frac{1}{P^2} = 3 + \sqrt{5}. \tag{3.4.49}$$

Employing (3.4.49) in (3.4.44), we find that

$$\left(\frac{G_{55}}{G_{11/5}}\right)^3 + \left(\frac{G_{11/5}}{G_{55}}\right)^3 = \frac{9+5\sqrt{5}}{2}. \tag{3.4.50}$$

Solving (3.4.50) for $\left(G_{55}/G_{11/5}\right)^3$, we obtain

$$\left(\frac{G_{55}}{G_{11/5}}\right)^3 = \frac{1}{2}\left(\frac{9+5\sqrt{5}}{2} + \sqrt{\frac{95+45\sqrt{5}}{2}}\right). \tag{3.4.51}$$

From (3.4.49) and (3.4.51), we deduce that

$$G_{55}^{12} = (3+\sqrt{5})^3 \left(\frac{9+5\sqrt{5}}{4} + \sqrt{\frac{95+45\sqrt{5}}{8}}\right)^2. \tag{3.4.52}$$

Thus,

$$G_{55} = 2^{1/4}(2+\sqrt{5})^{1/6} \left(\frac{9+5\sqrt{5}}{4} + \sqrt{\frac{95+45\sqrt{5}}{8}}\right)^{1/6}. \tag{3.4.53}$$

Now, substituting $b = (3+\sqrt{5})/8$ in Lemma 3.2.9, we see that

$$\left(\frac{9+5\sqrt{5}}{4} + \sqrt{\frac{95+45\sqrt{5}}{8}}\right)^{1/6} = \sqrt{\frac{7+\sqrt{5}}{8}} + \sqrt{\frac{\sqrt{5}-1}{8}}. \tag{3.4.54}$$

This completes the evaluation of G_{55}. The value of $G_{11/5}$ can be deduced similarly by using (3.4.49) and (3.4.51).

Theorem 3.4.6

$$G_{65} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4} \left(\frac{\sqrt{5}+1}{2}\right)^{1/4} \left(\sqrt{\frac{9+\sqrt{65}}{8}} + \sqrt{\frac{1+\sqrt{65}}{8}}\right)^{1/2}$$

and

$$G_{13/5} = \left(\frac{\sqrt{13}-3}{2}\right)^{1/4} \left(\frac{\sqrt{5}-1}{2}\right)^{1/4} \left(\sqrt{\frac{9+\sqrt{65}}{8}} + \sqrt{\frac{1+\sqrt{65}}{8}}\right)^{1/2}.$$

Proof: Here also, if

$$G_n = (4\alpha(1-\alpha))^{-1/24}$$

and β, γ, and δ have degrees 5, 13, and 65, respectively, over α, then as in the previous proofs, by (3.1.4)

$$G_{25n} = (4\beta(1-\beta))^{-1/24}, \quad G_{169n} = (4\gamma(1-\gamma))^{-1/24} \quad \text{and} \quad G_{4225n} = (4\delta(1-\delta))^{-1/24}. \tag{3.4.55}$$

By putting $n = 1/65$ in (3.4.2), we deduce that

$$\left(\frac{G_{65}}{G_{13/5}}\right)^3 + \left(\frac{G_{13/5}}{G_{65}}\right)^3 = 2\left[\left(G_{65}G_{13/5}\right)^2 - \frac{1}{\left(G_{65}G_{13/5}\right)^2}\right]. \tag{3.4.56}$$

By Lemma 3.2.7, we deduce that

$$Q^3 + \frac{1}{Q^3} - 5\left(Q + \frac{1}{Q} + 2\right)\left(R + \frac{1}{R} + 2\right) - \left(R^2 + \frac{1}{R^2}\right), \tag{3.4.57}$$

where, now,

$$Q = \frac{G_{25n}G_{169n}}{G_nG_{4225n}} \qquad \text{and} \qquad R = \frac{G_nG_{25n}}{G_{169n}G_{4225n}}.$$

Putting $n = 1/65$ in (3.4.57), we find that

$$\left(Q^3 + \frac{1}{Q^3}\right) - 20\left(Q + \frac{1}{Q}\right) - 42 = 0, \tag{3.4.58}$$

where, now, $Q = \left(G_{65}/G_{13/5}\right)^2$.

From (3.4.58), we obtain

$$Q + \frac{1}{Q} = \frac{3+\sqrt{65}}{2}. \tag{3.4.59}$$

Solving for $G_{65}/G_{13/5}$, we find that

$$\frac{G_{65}}{G_{13/5}} = \sqrt{\frac{7+\sqrt{65}}{8}} + \sqrt{\frac{\sqrt{65}-1}{8}}. \tag{3.4.60}$$

Invoking (3.4.60) in (3.4.56), we find that

$$2\left[\left(G_{65}G_{13/5}\right)^2-\frac{1}{\left(G_{65}G_{13/5}\right)^2}\right]=\sqrt{74+10\sqrt{65}}. \tag{3.4.61}$$

Solving (3.4.61) for $\left(G_{65}G_{13/5}\right)^2$, we find that

$$\left(G_{65}G_{13/5}\right)^2=\frac{1}{4}\left(\sqrt{74+10\sqrt{65}}+\sqrt{90+10\sqrt{65}}\right). \tag{3.4.62}$$

Thus, we deduce that

$$G_{65}G_{13/5}=\frac{1}{2}\left(\sqrt{74+10\sqrt{65}}+\sqrt{90+10\sqrt{65}}\right)^{1/2}. \tag{3.4.63}$$

Since, $90^2-65.10^2=40^2$, from Lemma 3.2.8, we see that

$$\sqrt{90+10\sqrt{65}}=\sqrt{\frac{130}{2}}+\sqrt{\frac{50}{2}}=5+\sqrt{65}.$$

Hence,

$$G_{65}G_{13/5}=\sqrt{\frac{9+\sqrt{65}}{8}}+\sqrt{\frac{1+\sqrt{65}}{8}}. \tag{3.4.64}$$

From (3.4.60) and (3.4.64), we deduce that

$$G_{65}^2=\left(\sqrt{\frac{7+\sqrt{65}}{8}}+\sqrt{\frac{\sqrt{65}-1}{8}}\right)\left(\sqrt{\frac{9+\sqrt{65}}{8}}+\sqrt{\frac{1+\sqrt{65}}{8}}\right). \tag{3.4.65}$$

and

$$G_{13/5}^2=\left(\sqrt{\frac{7+\sqrt{65}}{8}}-\sqrt{\frac{\sqrt{65}-1}{8}}\right)\left(\sqrt{\frac{9+\sqrt{65}}{8}}+\sqrt{\frac{1+\sqrt{65}}{8}}\right). \tag{3.4.66}$$

Now, simple calculation shows that

$$\left(\sqrt{\frac{7+\sqrt{65}}{8}}\pm\sqrt{\frac{\sqrt{65}-1}{8}}\right)^2=\left(\frac{\sqrt{13}\pm 3}{2}\right)\left(\frac{\sqrt{5}\pm 1}{2}\right). \tag{3.4.67}$$

Using (3.4.67) in (3.4.65) and (3.4.66), we easily arrive at the required class invariants.

As mentioned in the Introduction, we have seen that our evaluation of G_{65} is much easier than that of Berndt, Chan and Zhang [18], [25].

Chapter 4

Eta-Function Identities

4.1 Introduction

The classical Dedekind eta-function $\eta(z)$ is defined by

$$\eta(z) = e^{\pi iz/12}\Pi_{n=1}^{\infty}(1 - e^{2\pi inz}), \quad \text{Im}z > 0. \tag{4.1.1}$$

Following Ramanujan's notations, we set $q = \exp(2\pi iz)$ and

$$f(-q) = q^{-1/24}\eta(z). \tag{4.1.2}$$

In the unorganized portions of his second notebook [48], Ramanujan recorded without proofs 25 beautiful identities involving quotients of only eta-functions and no other theta-functions.

Note: *This chapter is identical to our paper [5], which has been accepted for publication in the Indian Journal of Mathematics.*

B. C. Berndt and L.-C. Zhang proved some of Ramanujan's eta-functions identities in [23]. Proofs of all 25 identities recorded by Ramanujan are given in Chapter 25 of Berndt's book [17]. 19 identities were proved by employing modular equations and parameterization and 6 were proved by invoking to the theory of modular forms. But in many of their proofs via parameterization, and in all the proofs via modular forms the identities must be known in advance. So those proofs may be merely called verifications. In this chapter, we prove five of these identities by using Ramanujan's other eta-functions identities and one of our newly derived identities. The remarkable feature of our method is that new identities can also be derived by this method. We note that in Section 7.6 of our last chapter, we find three new theta-function identities in course of deducing some modular equations for Ramanujan's cubic continued fraction.

4.2 Ramanujan's Identities

We will prove the following eta-functions identities of Ramanujan:

Theorem 4.2.1 *([48, p. 314]; [17, p. 186]) If*

$$u = \frac{f(-q^3)f(-q^6)}{q^{\frac{3}{4}}f(-q^9)f(-q^{18})} \quad and \quad v = \frac{f(-q)f(-q^2)}{qf(-q^9)f(-q^{18})},$$

then

$$u^4 = v^3 + 3v^2 + 9v. \tag{4.2.1}$$

Theorem 4.2.2 *([48, p. 330]; [17, p. 218]) If*

$$P = \frac{f(-q^6)f(-q^5)}{q^{\frac{1}{4}}f(-q^2)f(-q^{15})} \quad and \quad Q = \frac{f(-q^3)f(-q^{10})}{q^{\frac{3}{4}}f(-q)f(-q^{30})},$$

then

$$PQ + 1 + \frac{1}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2. \tag{4.2.2}$$

Theorem 4.2.3 *([48, p. 313]; [17, p. 230]) If*

$$P = \frac{f(-q)f(-q^2)}{q^{\frac{1}{2}}f(-q^5)f(-q^{10})} \quad \text{and} \quad Q = \frac{f(-q^3)f(-q^6)}{q^{\frac{3}{2}}f(-q^{15})f(-q^{30})},$$

then

$$PQ + \frac{25}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 - 3\left(\frac{Q}{P} + \frac{P}{Q} + 2\right). \tag{4.2.3}$$

Theorem 4.2.4 *([48, p. 327]; [17, p. 233]) If*

$$P = \frac{\psi(q)}{q^{\frac{1}{2}}\psi(q^5)} \quad \text{and} \quad Q = \frac{\psi(q^3)}{q^{\frac{3}{2}}\psi(q^{15})},$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 + 3\frac{Q}{P} + 3\frac{P}{Q} - \left(\frac{P}{Q}\right)^2. \tag{4.2.4}$$

Theorem 4.2.5 *([48, p. 327]; [17, p. 235]) If*

$$P = \frac{\phi(q)}{\phi(q^5)} \quad \text{and} \quad Q = \frac{\phi(q^3)}{\phi(q^{15})},$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 + 3\frac{Q}{P} + 3\frac{P}{Q} - \left(\frac{P}{Q}\right)^2. \tag{4.2.5}$$

Ramnujan incorrectly recorded the entry in Theorem 4.2.5 as

$$5PQ + \frac{1}{PQ} = \left(\frac{P}{Q}\right)^2 + 3\frac{Q}{P} + 3\frac{P}{Q} - \left(\frac{Q}{P}\right)^2.$$

In Theorem 4.2.1 we have slightly changed the notations used by Ramanujan.

The existing proofs of these theorems in the literature [17, Chapter 25] involves heavy amount of algebraic manipulation and could not have been accomplished without prior knowledge of the identities. Here we prove these theorems by using other eta-function identities whose proofs via modular equations can be found directly.

4.3 Proof of Theorem 4.2.1.

We set

$$P := \frac{f(-q)}{q^{\frac{1}{12}} f(-q^3)}, \qquad Q := \frac{f(-q)}{q^{\frac{1}{3}} f(-q^9)},$$
$$R := \frac{f(-q^2)}{q^{\frac{1}{6}} f(-q^6)} \quad \text{and,} \quad S := \frac{f(-q^2)}{q^{\frac{2}{3}} f(-q^{18})}. \tag{4.3.1}$$

We note that

$$PR = \frac{v}{u} \quad \text{and} \quad QS = v. \tag{4.3.2}$$

From Entry 56 [17, p. 210], we find that

$$Q^3 + S^3 = v^2 + 3v. \tag{4.3.3}$$

Employing (4.3.1) in Entry 1(iv) [15, p. 346], we find that

$$\left(1 + \frac{9}{Q^3}\right)^3 = 1 + \frac{27}{P^{12}}. \tag{4.3.4}$$

Replacing q by q^2 in (4.3.4), and then employing (4.3.1), we obtain

$$\left(1 + \frac{9}{S^3}\right)^3 = 1 + \frac{27}{R^{12}}. \tag{4.3.5}$$

We may rewrite (4.3.4) and (4.3.5) in simplified form as

$$\frac{Q^9}{P^{12}} = Q^6 + 9Q^3 + 27 \tag{4.3.6}$$

and

$$\frac{S^9}{R^{12}} = S^6 + 9S^3 + 27, \tag{4.3.7}$$

respectively.

Multiplying (4.3.6) and (4.3.7), we obtain

$$(QS)^6 + 9(QS)^3(Q^3 + S^3 + 9) + 27(Q^6 + S^6) + 243(Q^3 + S^3) + 729 = \frac{(QS)^9}{(PR)^{12}}. \tag{4.3.8}$$

Invoking (4.3.2) and (4.3.3) in (4.3.8), and then simplifying, we arrive at

$$\frac{u^{12}}{v^3} - (v^2 + 3v + 9)^3 = 0. \tag{4.3.9}$$

Factorizing (4.3.9), we find that

$$\left(\frac{u^4}{v} - (v^2 + 3v + 9)\right)\left(\frac{u^8}{v^2} + (v^2 + 3v + 9)^2 + \frac{u^4}{v}(v^2 + 3v + 9)\right) = 0. \tag{4.3.10}$$

From (4.3.10) we readily arrive at (4.2.1), since the second factor on the left hand side can not be 0. Thus we complete the proof.

4.4 Proof of Theorem 4.2.2.

We set

$$L_1 := \frac{f(-q)}{q^{\frac{1}{12}} f(-q^3)}, \qquad L_2 := \frac{f(-q^5)}{q^{\frac{5}{12}} f(-q^{15})},$$
$$M_1 := \frac{f(-q^2)}{q^{\frac{1}{6}} f(-q^6)} \qquad \text{and,} \quad M_2 := \frac{f(-q^{10})}{q^{\frac{5}{6}} f(-q^{30})}, \tag{4.4.1}$$

so that

$$P = \frac{L_2}{M_1} \qquad \text{and} \qquad Q = \frac{M_2}{L_1}. \tag{4.4.2}$$

Employing (4.4.1) in Entry 51 [17, p. 204], we obtain

$$(L_1M_1)^2 + \frac{9}{(L_1M_1)^2} = \left(\frac{M_1}{L_1}\right)^6 + \left(\frac{M_1}{L_1}\right)^6. \tag{4.4.3}$$

Replacing q by q^5 in the same Entry, and then employing (4.4.1) again, we find that

$$(L_2M_2)^2 + \frac{9}{(L_2M_2)^2} = \left(\frac{M_2}{L_2}\right)^6 + \left(\frac{M_2}{L_2}\right)^6. \tag{4.4.4}$$

Multiplying (4.4.3) and (4.4.4), and then applying (4.4.2), we obtain

$$(L_1L_2M_1M_2)^2 + \frac{81}{(L_1L_2M_1M_2)^2} - \left\{\left(\frac{M_1L_2}{M_2L_1}\right)^6 + \left(\frac{M_2L_1}{M_1L_2}\right)^6\right\}$$
$$= \left(\frac{Q}{P}\right)^6 + \left(\frac{P}{Q}\right)^6 - 9\left\{(PQ)^2 + \frac{1}{(PQ)^2}\right\}. \tag{4.4.5}$$

Now employing (4.4.1) in Entry 59 [17, p. 214], we find that

$$\frac{L_2M_2}{L_1M_1} + \frac{L_1M_1}{L_2M_2} = \left(\frac{M_2L_1}{M_1L_2}\right)^3 + \left(\frac{M_1L_2}{M_2L_1}\right)^3 + 4. \tag{4.4.6}$$

Using (4.4.2) in (4.4.6), we obtain

$$\left(\frac{M_2L_1}{M_1L_2}\right)^3 + \left(\frac{M_1L_2}{M_2L_1}\right)^3 = PQ + \frac{1}{PQ} - 4. \tag{4.4.7}$$

Squaring both sides of (4.4.7), we arrive at

$$\left(\frac{M_2L_1}{M_1L_2}\right)^6 + \left(\frac{M_1L_2}{M_2L_1}\right)^6 = \left(PQ + \frac{1}{PQ} - 4\right)^2 - 2. \tag{4.4.8}$$

From (4.4.5) and (4.4.8), we obtain

$$(L_1L_2M_1M_2)^2 + \frac{81}{(L_1L_2M_1M_2)^2} = \left(PQ + \frac{1}{PQ} - 4\right)^2 - 2 + \left(\frac{Q}{P}\right)^6 + \left(\frac{P}{Q}\right)^6$$
$$-9\left\{(PQ)^2 + \frac{1}{(PQ)^2}\right\}. \tag{4.4.9}$$

Employing (4.4.1) in Entry 60 [17, p. 215], and then using (4.4.2), we find that

$$L_1L_2M_1M_2 + \frac{9}{L_1L_2M_1M_2} = \left(\frac{Q}{P}\right)^3 - 4\left(\frac{Q}{P}\right) - 4\left(\frac{P}{Q}\right) + \left(\frac{P}{Q}\right)^3 \tag{4.4.10}$$

Squaring both sides of (4.4.10), and then invoking (4.4.9), we obtain

$$\left(PQ + \frac{1}{PQ} - 4\right)^2 - 2 + \left(\frac{Q}{P}\right)^6 + \left(\frac{P}{Q}\right)^6 - 9\left\{(PQ)^2 + \frac{1}{(PQ)^2}\right\}$$
$$= \left\{\left(\frac{Q}{P}\right)^3 - 4\left(\frac{Q}{P}\right) - 4\left(\frac{P}{Q}\right) + \left(\frac{P}{Q}\right)^3\right\}^2 - 18. \tag{4.4.11}$$

Simplifying (4.4.11), we arrive at

$$\left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 - \left(\frac{Q}{P}\right)^4 - \left(\frac{P}{Q}\right)^4 + (PQ)^2 + \frac{1}{(PQ)^2} + PQ + \frac{1}{PQ}. \tag{4.4.12}$$

Factorizing (4.4.12), we find that

$$\left(\left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 + PQ + \frac{1}{PQ}\right)\left(PQ + \frac{1}{PQ} - \left(\frac{Q}{P}\right)^2 - \left(\frac{P}{Q}\right)^2 + 1\right) = 0. \tag{4.4.13}$$

Thus, we arrive at

$$PQ + \frac{1}{PQ} - \left(\frac{Q}{P}\right)^2 - \left(\frac{P}{Q}\right)^2 + 1 = 0. \tag{4.4.14}$$

since the other factor can not be 0. This completes the proof.

4.5 Proof of Theorem 4.2.3.

First of all, we prove the following new eta-function identity.

Lemma 4.5.1 *If*

$$P_1 = \frac{f(-q)f(-q^3)}{q^{\frac{2}{3}}f(-q^5)f(-q^{15})} \quad \text{and} \quad Q_1 = \frac{f(-q^2)f(-q^6)}{q^{\frac{4}{3}}f(-q^{10})f(-q^{30})},$$

then

$$\left(\frac{P_1}{Q_1}\right)^3+\left(\frac{Q_1}{P_1}\right)^3=10+P_1Q_1+\frac{25}{P_1Q_1}-2\left(41+4P_1Q_1+\frac{100}{P_1Q_1}\right)^{1/2}. \tag{4.5.1}$$

Proof of Lemma 4.5.1.: Let

$$R_1=\frac{f(q)f(q^3)}{q^{\frac{2}{3}}f(q^5)f(q^{15})}.$$

By Entries 12(i) and (iii) in Chapter 17 of [15, p. 124], we find that

$$R_1=\sqrt{\frac{z_1z_3}{z_5z_{15}}}\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{\frac{1}{24}}. \tag{4.5.2}$$

and

$$Q_1=\sqrt{\frac{z_1z_3}{z_5z_{15}}}\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{\frac{1}{12}}, \tag{4.5.3}$$

where β, γ, and δ have degrees 3, 5, 15, respectively, over α. From (4.5.2) and (4.5.3) we readily see that

$$\frac{Q_1}{R_1}=\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{\frac{1}{24}}, \tag{4.5.4}$$

and

$$\frac{R_1^2}{Q_1}=\sqrt{\frac{z_1z_3}{z_5z_{15}}}. \tag{4.5.5}$$

Now by Entries 11(xii), (xiii), respectively, in Chapter 20 of [15, p. 384], we note that

$$\left(\frac{\gamma\delta}{\alpha\beta}\right)^{\frac{1}{4}}+\left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{\frac{1}{4}}+\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{\frac{1}{4}}-2\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{\frac{1}{8}}$$
$$\times\left\{1+\left(\frac{\gamma\delta}{\alpha\beta}\right)^{\frac{1}{8}}+\left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{\frac{1}{8}}\right\}=\frac{z_1z_3}{z_5z_{15}} \tag{4.5.6}$$

and

$$\left(\frac{\alpha\beta}{\gamma\delta}\right)^{\frac{1}{4}}+\left(\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}\right)^{\frac{1}{4}}+\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{\frac{1}{4}}-2\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{\frac{1}{8}}$$

$$\times\left\{1+\left(\frac{\alpha\beta}{\gamma\delta}\right)^{\frac{1}{8}}+\left(\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}\right)^{\frac{1}{8}}\right\}=25\frac{z_1z_3}{z_5z_{15}}. \tag{4.5.7}$$

For simplicity, we set

$$x:=\left(\frac{\gamma\delta}{\alpha\beta}\right)^{\frac{1}{8}}+\left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{\frac{1}{8}} \quad\text{and}\quad y:=\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{\frac{1}{8}},$$

so that

$$\frac{R_1^3}{Q_1^3}=y. \tag{4.5.8}$$

Then from (4.5.6), we find that

$$x=y\pm\left(4y+\frac{z_1z_3}{z_5z_{15}}\right)^{\frac{1}{2}}. \tag{4.5.9}$$

Also, from (4.5.7), we find the reciprocal equation of (4.5.9) as

$$\frac{x}{y}=\frac{1}{y}\pm\left(\frac{4}{y}+25\frac{z_5z_{15}}{z_1z_3}\right)^{\frac{1}{2}}. \tag{4.5.10}$$

Combining (4.5.9) and (4.5.10), we obtain

$$y\pm\left(4y+\frac{z_1z_3}{z_5z_{15}}\right)^{\frac{1}{2}}=1\pm y\left(\frac{4}{y}+25\frac{z_5z_{15}}{z_1z_3}\right)^{\frac{1}{2}}. \tag{4.5.11}$$

Employing (4.5.4), (4.5.5), and (4.5.8) in (4.5.11), we find that

$$\frac{R_1^3}{Q_1^3}\pm\left(4\frac{R_1^3}{Q_1^3}+\frac{R_1^4}{Q_1^2}\right)^{\frac{1}{2}}=1\pm\frac{R_1^3}{Q_1^3}\left(4\frac{Q_1^3}{R_1^3}+25\frac{Q_1^2}{R_1^4}\right)^{\frac{1}{2}} \tag{4.5.12}$$

We rewrite (4.5.12) as

$$R_1^3-Q_1^3=\pm R_1^3\left(4\frac{Q_1^3}{R_1^3}+25\frac{Q_1^2}{R_1^4}\right)^{\frac{1}{2}}\mp Q_1^3\left(4\frac{R_1^3}{Q_1^3}+\frac{R_1^4}{Q_1^2}\right)^{\frac{1}{2}}. \tag{4.5.13}$$

Squaring both sides of (4.5.13), and then simplifying, we arrive at

$$R_1^6+Q_1^6=10R_1^3Q_1^3+25R_1^2Q_1^2+R_1^4Q_1^4-2R_1^3Q_1^3\left(41+4R_1Q_1+\frac{100}{R_1Q_1}\right)^{\frac{1}{2}}. \tag{4.5.14}$$

Dividing both sides of (4.5.14) by $R_1^3Q_1^3$, we find that

$$\frac{R_1^3}{Q_1^3}+\frac{Q_1^3}{R_1^3}=10+R_1Q_1+\frac{25}{R_1Q_1}-2\left(41+4R_1Q_1+\frac{100}{R_1Q_1}\right)^{\frac{1}{2}}. \tag{4.5.15}$$

If we replace q by $-q$ then R_1Q_1 transforms to P_1Q_1 and R_1^3/Q_1^3 transforms to P_1^3/Q_1^3. Thus (4.5.15) is transformed to (4.5.1), which completes the proof of the lemma.

Proof of the main theorem: We set

$$L_1:=\frac{f(-q)}{q^{\frac{1}{6}}f(-q^5)}, \qquad L_2:=\frac{f(-q^3)}{q^{\frac{1}{2}}f(-q^{15})},$$

$$M_1:=\frac{f(-q^2)}{q^{\frac{1}{3}}f(-q^{10})} \quad \text{and,} \quad M_2:=\frac{f(-q^6)}{qf(-q^{30})}, \tag{4.5.16}$$

so that

$$P=L_1M_1 \qquad \text{and} \qquad Q=L_2M_2. \tag{4.5.17}$$

Employing (4.5.16) in Entry 53 [17, p. 206], we find that

$$L_1M_1+\frac{5}{L_1M_1}=\left(\frac{L_1}{M_1}\right)^3+\left(\frac{M_1}{L_1}\right)^3. \tag{4.5.18}$$

Replacing q by q^3 in the same entry, and then employing (4.5.16), we find that

$$L_2M_2+\frac{5}{L_2M_2}=\left(\frac{L_2}{M_2}\right)^3+\left(\frac{M_2}{L_2}\right)^3. \tag{4.5.19}$$

Multiplying (4.5.18) and (4.5.19), and then applying (4.5.17), we obtain

$$\left(\frac{L_1L_2}{M_1M_2}\right)^3+\left(\frac{M_2L_1}{M_1L_2}\right)^3+\left(\frac{L_2M_1}{M_2L_1}\right)^3+\left(\frac{M_1M_2}{L_1L_2}\right)^3$$
$$=PQ+\frac{25}{PQ}+5\left(\frac{Q}{P}+\frac{P}{Q}\right). \tag{4.5.20}$$

Employing (4.5.16) and (4.5.17) in Entry 59 [17, p. 214], we find that

$$\left(\frac{M_2L_1}{M_1L_2}\right)^3+\left(\frac{L_2M_1}{M_2L_1}\right)^3=\frac{Q}{P}+\frac{P}{Q}-4. \tag{4.5.21}$$

From (4.5.20) and (4.5.21), we find that

$$\left(\frac{L_1L_2}{M_1M_2}\right)^3+\left(\frac{M_1M_2}{L_1L_2}\right)^3=PQ+\frac{25}{PQ}+4\left(\frac{Q}{P}+\frac{P}{Q}\right)+4. \tag{4.5.22}$$

Now invoking (4.5.16) and (4.5.17) in the above lemma, we find that

$$\left(\frac{L_1L_2}{M_1M_2}\right)^3+\left(\frac{M_1M_2}{L_1L_2}\right)^3=10+PQ+\frac{25}{PQ}-2\left(41+4PQ+\frac{100}{PQ}\right)^{1/2}. \tag{4.5.23}$$

From (4.5.22) and (4.5.23) , we obtain

$$\left(41+4PQ+\frac{100}{PQ}\right)^{1/2}=3-2\left(\frac{P}{Q}+\frac{Q}{P}\right). \tag{4.5.24}$$

Squaring both sides of (4.5.24), and then simplifying, we readily deduce (4.2.3), completing the proof of the theorem.

4.6 Proof of Theorem 4.2.4

We note from Entry 24(iii) [15, p. 39] that

$$\psi(q)=\frac{f^2(-q^2)}{f(-q)}. \tag{4.6.1}$$

Therefore P and Q can be reformulated as

$$P=\frac{f(-q^5)f^2(-q^2)}{q^{\frac{1}{2}}f(-q)f^2(-q^{10})} \quad \textit{and} \quad Q=\frac{f(-q^{15})f^2(-q^6)}{q^{\frac{3}{2}}f(-q^3)f^2(-q^{30})}.$$

Now we set

$$L_1 := \frac{f(-q)}{q^{\frac{1}{6}} f(-q^5)}, \qquad L_2 := \frac{f(-q^3)}{q^{\frac{1}{2}} f(-q^{15})},$$

$$M_1 := \frac{f(-q^2)}{q^{\frac{1}{3}} f(-q^{10})} \qquad \text{and,} \quad M_2 := \frac{f(-q^6)}{q f(-q^{30})}, \tag{4.6.2}$$

so that

$$P = \frac{M_1{}^2}{L_1} \qquad \text{and} \qquad Q = \frac{M_2{}^2}{L_2}. \tag{4.6.3}$$

Employing (4.6.2) in Entry 53 [17, p. 206], we obtain

$$L_1 M_1 + \frac{5}{L_1 M_1} = \left(\frac{L_1}{M_1}\right)^3 + \left(\frac{M_1}{L_1}\right)^3 \tag{4.6.4}$$

Replacing q by q^3 in the same entry, and then employing (4.6.2), we find that

$$L_2 M_2 + \frac{5}{L_2 M_2} = \left(\frac{L_2}{M_2}\right)^3 + \left(\frac{M_2}{L_2}\right)^3 \tag{4.6.5}$$

Using (4.6.3) we may rewrite (4.6.4) in the form

$$\frac{M_1{}^3}{P} + \frac{5P}{M_1{}^3} = \frac{M_1{}^3}{P^3} + \frac{P^3}{M_1{}^3}. \tag{4.6.6}$$

Thus we arrive at

$$M_1{}^6 = \frac{P^4(P^2-5)}{P^2-1}. \tag{4.6.7}$$

Similarly from (4.6.3) and (4.6.5), we deduce that

$$M_2{}^6 = \frac{Q^4(Q^2-5)}{Q^2-1}. \tag{4.6 8}$$

Employing (4.6.2) in (59.10) of [17, p. 215], we find that

$$\left(\frac{L_2}{L_1}\right)^3 + \left(\frac{M_2}{M_1}\right)^3 = \left(\frac{L_2 M_2}{L_1 M_1}\right)^2 - \left(\frac{L_2 M_2}{L_1 M_1}\right) \tag{4.6.9}$$

Invoking (4.6.3) in (4.6.9), and then simplifying we deduce that

$$\left(\frac{M_2}{M_1}\right)^3 = \frac{1+P/Q}{\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^3}. \tag{4.6.10}$$

From (4.6.7), (4.6.8) and (4.6.10), we find that

$$\frac{Q^4(Q^2-5)(P^2-1)}{P^4(P^2-5)(Q^2-1)} = \left[\frac{1+P/Q}{\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^3}\right]^2. \tag{4.6.11}$$

This can be readily seen to be equivalent to (4.2.4). Thus we complete the proof.

4.7 Proof of Theorem 4.2.5.

We note from Entry 24(iii) [15, p. 39] that

$$\phi(q) = \frac{f^2(q)}{f(-q^2)}. \tag{4.7.1}$$

Therefore P and Q can be reformulated as

$$P = \frac{f^2(q)f(-q^{10})}{f(-q^2)f^2(q^5)} \quad and \quad Q = \frac{f^2(q^3)f(-q^{30})}{f(-q^6)f^2(q^{15})},$$

Let

$$R = \frac{f^2(-q)f(-q^{10})}{f(-q^2)f^2(q^{-5})} \quad and \quad S = \frac{f^2(q^{-3})f(-q^{30})}{f(-q^6)f^2(q^{-15})},$$

Then

$$R = \frac{{L_1}^2}{M_1} \qquad \text{and} \qquad S = \frac{{L_2}^2}{M_2}, \tag{4.7.2}$$

where L_1, L_2, M_1, and M_2 are same as in the proof of Theorem 4.2.4. We may write (4.6.4) in the form

$$\frac{{L_1}^3}{R} + \frac{5R}{{L_1}^3} = \frac{{L_1}^3}{R^3} + \frac{R^3}{{L_1}^3}. \tag{4.7.3}$$

Thus we arrive at

$$L_1{}^6 = \frac{R^4(R^2-5)}{R^2-1}. \tag{4.7.4}$$

Similarly we can deduce that

$$L_2{}^6 = \frac{S^4(S^2-5)}{S^2-1}. \tag{4.7.5}$$

Invoking (4.7.2) in (4.6.9), and then simplifying, we obtain

$$\left(\frac{L_2}{L_1}\right)^3 = \frac{1+R/S}{\left(\frac{R}{S}\right)^2 - \left(\frac{R}{S}\right)^3}. \tag{4.7.6}$$

From (4.7.4), (4.7.5), and (4.7.6), we arrive at

$$\frac{S^4(S^2-5)(R^2-1)}{R^4(R^2-5)(S^2-1)} = \left[\frac{1+R/S}{\left(\frac{R}{S}\right)^2 - \left(\frac{R}{S}\right)^3}\right]^2. \tag{4.7.7}$$

This can be readily seen to be equivalent to

$$RS + \frac{5}{RS} = \left(\frac{S}{R}\right)^2 + 3\frac{S}{R} + 3\frac{R}{S} - \left(\frac{R}{S}\right)^2. \tag{4.7.8}$$

Replacing q by $-q$, we see that RS transforms to PQ and R/S transforms to P/Q. Thus, we obtain

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 + 3\frac{Q}{P} + 3\frac{P}{Q} - \left(\frac{P}{Q}\right)^2 \tag{4.7.9}$$

This completes the proof.

It is worthwhile to note that in [32], H. H. Chan used this identity to find the value of $G(-e^{-\sqrt{5}\pi})$, where $G(q)$ is the Ramanujan's cubic continued fraction defined in (1.2.5).

Chapter 5

Explicit Evaluations of Theta-Functions

5.1 Introduction

We recall the following special type of Ramanujan's theta-functions from Chapter 2.

$$\phi(q) := f(q,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2}, \tag{5.1.1}$$

where $|q| < 1$.

If $K(k)$ and ${}_2F_1$ denote the complete elliptic integral of the first kind and ordinary or Gaussian hypergeometric function as defined in (2.1.3) and (2.1.4), respectively, then one of the most fundamental results in the theory of elliptic functions [15, p. 102] is

$$\phi^2(q) = \frac{2}{\pi}K(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \quad q = \exp(-\pi K'/K),$$

Note: *The results of this chapter are identical to our paper [10].*

$K' = K(k')$, and $k' = \sqrt{1-k^2}$ is the complementary modulus. So, an evaluation of any one of the functions ϕ, ${}_2F_1$, or, K yields an evaluation of the other two functions. But such evaluations may not be very explicit. For example, if $K(k)$ is known for a certain value of k, it may be difficult or impossible to determine explicitly K' and q. Conversely, it is possible to evaluate $\phi(q)$ for certain q, but it may be impossible to determine k. Several values of ${}_2F_1$ and $K(k)$ have been determined by I. J. Zucker [65] and G. S. Joyce and Zucker [36]. Ramanujan recorded many values of $\phi(q)$ in his notebooks [48], some of which are new and some of which are classical. In his second notebook [48], Ramanujan recorded the values of $\phi(e^{-\pi})$, $\phi(e^{-\sqrt{2}\pi})$, $\phi(e^{-2\pi})$, and $\phi(e^{-5\pi})$. The first three values are classical and can be found in Whittaker and Watson's text [62, p. 525], while the value of $\phi(e^{-5\pi})$ is new. Using theta transformation formula Berndt [15, p. 210] and Joyce and Zucker [36] obtained the evaluation independently. Ramanujan also recorded many values of ϕ, as well as values of ψ and f in his first notebook [48]. All the elementary values of ϕ, ψ and f are easy consequences of the "catalogue "of evaluations given by Ramanujan in chapter 17 of his second notebook [15, pp. 122-124]. At scattered places in his first notebook [48], Ramanujan also recorded the nonelementary values of $\phi(e^{-n\pi})$ for n =3, 7, 9, and 45. The evaluation for $n = 3$ can be found in Zucker's paper [65]. For proofs of all the non-elementary values one may see Berndt [18, Chapter 35], and Berndt and Chan [21]. In [21], Berndt and Chan also found three new explicit evaluations of $\phi(e^{-n\pi})$ for $n = 13$, 27 and 63. It is worthwhile to note that Ramanujan recorded most of his values for $\phi(e^{-n\pi})$ in terms of $\phi(e^{-\pi})$. But, since the value

$$\phi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma\left(\frac{3}{4}\right)},$$

is well known [62, p. 525], so $\phi(e^{-n\pi})$ is determined explicitly. In this chapter, we give some more transparent proofs for the evaluation of the non-elementary values of $\phi(e^{-n\pi})$ claimed by Ramanujan by employing some of his modular equations and class invariants. We also evaluate some new explicit non-elementary values of $\phi(e^{-n\pi})$.

In Section 5.2, we find a new proof, different from that of B. C. Berndt [15, p. 352], for one of the modular equations recorded by Ramanujan in his second notebook, and combine this with his class invariants to arrive at Borweins' [31, p. 145] formula for $\phi(e^{-9n\pi})/\phi(e^{-n\pi})$ and deduce a number of evaluations.

In Section 5.3, we find a general formula for explicit evaluation of $\phi(e^{-5n\pi})/\phi(e^{-n\pi})$ and again deduce some evaluations.

In Section 5.4, we prove the evaluation for $\phi^2(e^{-7\pi})$. Berndt [18, p. 336] and Berndt and Chan [21, p. 289] have also proved this evaluation. But, in [18] **Mathematica** is used to simplify the very complicated nested radicals, and in [21], the simplification of the radicals are very cumbersome and, they agreed that Ramanujan might have found a more transparent proof. We believe that, our evaluation is close to Ramanujan's proof.

But Chan's proof is easier in [18]

In Sections 5.5, 5.6, and, 5.7 we find three new evaluations $\phi(e^{-15\pi})$, $\phi(e^{-21\pi})$, and $\phi(e^{-35\pi})$, respectively from "mixed " modular equations. Three of the modular equations are found by Ramanujan and another one by us in Chapter 2.

5.2 Evaluations of $\phi(e^{-9n\pi})/\phi(e^{-n\pi})$

J. M. and P. B. Borwein [31, p. 145] first observed that class invariants could be used to calculate certain values of $\phi(e^{-n\pi})$. In fact, they observed the following theorem.

Theorem 5.2.1 *For any positive rational number n^2,*

$$3\frac{\phi(e^{-9n\pi})}{\phi(e^{-n\pi})} = 1 + \sqrt{2}\frac{G_{9n^2}}{G_{n^2}^3}. \tag{5.2.1}$$

Berndt [18] and Berndt and Chan [21] also provide a proof for this theorem. Here, we give another proof using Ramanujan's modular equations. First of all, we recall the following beautiful modular equation of Ramanujan [15, Entry 3(i), p. 352].

Lemma 5.2.2 *Let β and γ have degrees 3 and 9, respectively, over α. Let m denote the multiplier connecting α and β, and let m' be the multiplier relating β and γ. Then*

$$\frac{3}{\sqrt{mm'}} = 1 + 4^{1/3}\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/24} \tag{5.2.2}$$

We provide here a proof somewhat different from that in [15].

Proof of Lemma 5.2.2: We recall from (2.3.31) that

$$\phi(q) = \phi(q^9) + 2qf(q^3, q^{15}) \tag{5.2.3}$$

Since by (2.1.43)

$$f(q, q^5) = \psi(-q^3)\chi(q), \tag{5.2.4}$$

so, from (5.2.3), we find that

$$\phi(q) = \phi(q^9) + 2q\psi(-q^9)\chi(q^3). \tag{5.2.5}$$

Transcribing (5.2.5) via Entries 10(i), 11(ii) and 12(v) [15; pp. 122-124], we arrive at

$$\sqrt{mm'} = 1 + 4^{1/3}\left(\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}\right)^{1/24} \tag{5.2.6}$$

Reciprocating (in the sense of Entry 24(v) [15, p. 216]) the above modular equation, we can easily arrive at (5.2.2).

Proof of Theorem 5.2.1. If we set $q = e^{-n\pi}$, it can be easily seen from (3.1.4) and (5.2.2) that

$$\frac{3}{\sqrt{mm'}} = 1 + \sqrt{2}\frac{G_{9n^2}}{G_{n^2}^3}, \tag{5.2.7}$$

where , now, $m = \phi^2(e^{-n\pi})/\phi^2(e^{-3n\pi})$ and $m' = \phi^2(e^{-3n\pi})/\phi^2(e^{-9n\pi})$. Thus, the proof is complete.

A number of evaluations claimed by Ramanujan follows as corollaries from the above theorem.

Corollary 5.2.3 *([48, Vol.-I, p. 287]).*

$$\frac{\phi(e^{-9\pi})}{\phi(e^{-\pi})} = \frac{1+\sqrt[3]{2(\sqrt{3}+1)}}{3}. \tag{5.2.8}$$

Proof: We put $n = 1$ in (5.2.1) to arrive at

$$3\frac{\phi(e^{-9\pi})}{\phi(e^{-\pi})} = 1 + \sqrt{2}\frac{G_9}{G_1^3}. \tag{5.2.9}$$

Since from Berndt's book [18, p. 189]

$$G_1 = 1 \qquad \text{and} \quad G_9 = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/3},$$

we easily find the required evaluation.

Corollary 5.2.4 *([48, volume II, Chap. 18]; [15, p. 210])*

$$(\sqrt{5}+\sqrt{3})\phi(e^{-\pi\sqrt{5}/3}) = (3+\sqrt{3})\phi(e^{-3\pi\sqrt{5}}). \tag{5.2.10}$$

Proof. Putting $n = \sqrt{5}/3$ in (5.2.1), we find that

$$3\frac{\phi(e^{-3\pi\sqrt{5}})}{\phi(e^{-\pi\sqrt{5}/3})} = 1 + \sqrt{2}\frac{G_5}{G_{5/9}^3}. \tag{5.2.11}$$

Again from Berndt's book [18, p. 189 and p. 345], we note that

$$G_5 = \left(\frac{1+\sqrt{5}}{2}\right)^{1/4}, \quad \text{and} \quad G_{5/9} = G_{9/5} = (\sqrt{5}+2)^{1/4}\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^{1/3}$$

Employing these values in (5.2.11), and then simplifying, we find that

$$3\frac{\phi(e^{-3\pi\sqrt{5}})}{\phi(e^{-\pi\sqrt{5}/3})} = \frac{3-\sqrt{3}}{\sqrt{5}-\sqrt{3}}. \tag{5.2.12}$$

Thus,

$$\frac{\phi(e^{-3\pi\sqrt{5}})}{\phi(e^{-\pi\sqrt{5}/3})} = \frac{3-\sqrt{3}}{3(\sqrt{5}-\sqrt{3})} = \frac{\sqrt{5}+\sqrt{3}}{3+\sqrt{3}}. \tag{5.2.13}$$

Similarly, some other explicit values of $\phi(e^{-9n\pi})/\phi(e^{-n\pi})$, for positive rational n^2, can also be evaluated if the corresponding class invariants are known. For example, Berndt and Chan [21] have found the evaluations corresponding to $n = 5$ claimed by Ramanujan and two new evaluations corresponding to $n = 3$ and $n = 7$.

5.3 Evaluations of $\phi(e^{-5n\pi})/\phi(e^{-n\pi})$

Theorem 5.3.1 *For any positive rational number n^2,*

$$5\frac{\phi^2(e^{-5n\pi})}{\phi^2(e^{-n\pi})} = 1 + 2\frac{G_{25n^2}}{G_{n^2}^5}. \tag{5.3.1}$$

Proof: If β has degree 5 over α and m is the multiplier for degree 5, then from Chapter 19 of Ramanujan's second notebook [15, Entry 13(iv), p. 281]

$$\frac{5}{m} = 5\frac{\phi^2(q^5)}{\phi^2(q)} = 1 + 2^{\frac{4}{3}}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{\frac{1}{24}}. \tag{5.3.2}$$

Putting $q = e^{-\pi n}$ in (5.3.2) and then employing (3.1.4), we easily arrive at (5.3.1).

As in the Theorem 5.2.1, a number of evaluations follow from Theorem 5.3.1 if the corresponding class invariants are known. We give a couple of examples below.

Corollary 5.3.2 *[18, p. 327]*

$$\frac{\phi(e^{-5\pi})}{\phi(e^{-\pi})} = \frac{1}{\sqrt{5\sqrt{5}-10}}. \tag{5.3.3}$$

Proof: Putting $n = 1$ in (5.3.1), we obtain

$$5\frac{\phi^2(e^{-5\pi})}{\phi^2(e^{-\pi})} = 1 + 2\frac{G_{25}}{G_1^5}. \tag{5.3.4}$$

Since from Berndt's book [18, p. 190]

$$G_1 = 1 \qquad \text{and} \qquad G_{25} = \frac{1+\sqrt{5}}{2},$$

we see that

$$5\frac{\phi^2(e^{-5\pi})}{\phi^2(e^{-\pi})} = 2 + \sqrt{5} = \frac{1}{\sqrt{5}-2}. \tag{5.3.5}$$

Now it is easy to arrive at (5.3.3).

The following evaluation is new.

Corollary 5.3.3

$$\frac{\phi^2(e^{-15\pi})}{\phi^2(e^{-3\pi})} = \frac{1}{5}\left[1 + \sqrt{2}\left(\frac{1+\sqrt{5}}{1+\sqrt{3}}\right)\left(\frac{\sqrt{5}+\sqrt{3}}{2\sqrt{2}} + \frac{(15)^{1/4}}{2}\right)\right]. \tag{5.3.6}$$

Proof: We put $n = 3$ in (5.3.1), to obtain

$$5\frac{\phi^2(e^{-15\pi})}{\phi^2(e^{-3\pi})} = 1 + 2\frac{G_{225}}{G_9^5}. \tag{5.3.7}$$

Using the values of G_{225} and G_9 from Berndt's book [18], one can get the required assertion. As the value of $\phi(e^{-3\pi})$ is already known [18, p. 327], $\phi(e^{-15\pi})$ can be found explicitly, provided the appropriate root is extracted. In section 5.5 we find a simple evaluation for $\phi(e^{-15\pi})$.

5.4 Evaluation of $\phi^2(e^{-7\pi})$.

Theorem 5.4.1

$$\frac{\phi^2(e^{-7\pi})}{\phi^2(e^{-\pi})} = \frac{(28)^{1/8}}{14}\left(\sqrt{13+\sqrt{7}} + \sqrt{7+3\sqrt{7}}\right). \tag{5.4.1}$$

Berndt [18] and Berndt and Chan [21] proved this evaluation claimed by Ramanujan. But, simplifications of radicals are too cumbersome. They agreed that Ramanujan might have had a more transparent proof. We think the following proof is very close to that of Ramanujan.

Proof: From Entry 19(i), (ii), and (viii) of Chapter 19 of Ramanujan's second notebook [15, pp. 314-315], we note the following modular equations.

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1, \tag{5.4.2}$$

$$\frac{7}{m} = \frac{1 - 4\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}}{(\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}}, \tag{5.4.3}$$

and,

$$m - \frac{7}{m} = 2((\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8})(2 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}), \tag{5.4.4}$$

where β has degree 7 over α and m is the multiplier connecting α and β.

From the above equations it can be easily deduced that

$$\frac{49}{m^2} = 1 + 4(\alpha\beta(1-\alpha)(1-\beta))^{1/8} + 24\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24} - 16(\alpha(1-\alpha))^{5/12}(\beta(1-\beta))^{1/12}. \tag{5.4.5}$$

Now, in (5.4.5) we put $\alpha = 1/2$, so that $q = e^{-\pi}$, and then we find by invoking (3.1.4) that

$$49\frac{\phi^4(e^{-7\pi})}{\phi^4(e^{-7\pi})} = 1 + 2\sqrt{2}G_{49}^{-3} + 6\sqrt{2}G_{49} - 8G_{49}^{-2}. \tag{5.4.6}$$

From Berndt's Book [18],

$$G_{49} = \frac{7^{1/4} + \sqrt{4+\sqrt{7}}}{2}. \tag{5.4.7}$$

Simple calculations give

$$G_{49}^{-1} = \frac{\sqrt{4+\sqrt{7}} - 7^{1/4}}{2}, \tag{5.4.8}$$

$$8G_{49}^{-2} = 4\left(2 + \sqrt{7} - 7^{1/4}\sqrt{4+\sqrt{7}}\right), \tag{5.4.9}$$

and

$$2\sqrt{2}G_{49}^{-3} = 2(4+\sqrt{7}) - \sqrt{2}\sqrt{7}.7^{1/4} - 3\sqrt{2}.7^{1/4}. \tag{5.4.10}$$

not simpler than Chan's proof

Employing (5.4.7), (5.4.9), and (5.4.10) in (5.4.6), and then simplifying using (see Lemma 3.2.8)

$$\sqrt{4+\sqrt{7}} = \sqrt{7/2} + \sqrt{1/2},$$

we find that

$$49\frac{\phi^4(e^{-7\pi})}{\phi^4(e^{-7\pi})} = 7 + 5\sqrt{2}.7^{1/4} + 4\sqrt{7} + \sqrt{2}.7^{3/4}. \tag{5.4.11}$$

Taking square root of both sides, we obtain

$$7\frac{\phi^2(e^{-7\pi})}{\phi^2(e^{-7\pi})} = \frac{(28)^{1/8}}{2}\left(20 + 8\sqrt{2}.7^{1/4} + 4\sqrt{7} + 2\sqrt{2}.7^{3/4}\right)^{1/2}. \tag{5.4.12}$$

Since

$$2\sqrt{2}7^{3/4} + 8\sqrt{2}7^{1/4} = 2\sqrt{(13+\sqrt{7})(7+3\sqrt{7})}$$

and

$$20 + 4\sqrt{7} = (13+\sqrt{7}) + (7+3\sqrt{7}),$$

it is easy to arrive at (5.4.1) from (5.4.12).

5.5 A simple evaluation of $\phi(e^{-15\pi})$

Theorem 5.5.1

$$\frac{\phi(e^{-15\pi})}{\phi(e^{-\pi})} = \frac{1}{\sqrt[4]{2\sqrt{3}-3}\ \sqrt{5-\sqrt{5}}} + \frac{\sqrt{3}-1}{\sqrt{2}\ \sqrt{5\sqrt{5}-5}\ \sqrt[4]{6\sqrt{3}-9}}. \tag{5.5.1}$$

Proof: From Entries 11(i)and 11(iv) of Chapter 20 of Ramanujan's second notebook [15, p. 383], we note the following modular equations.

$$(\beta\gamma)^{1/8} + \{(1-\beta)(1-\gamma)\}^{1/8} = \sqrt{\frac{m}{m'}}, \tag{5.5.2}$$

and

$$1+(\beta\gamma)^{1/8}+\{(1-\beta)(1-\gamma)\}^{1/8}=4^{1/3}\left(\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24} \tag{5.5.3}$$

where β, γ, and δ have degrees 3, 5, and 15, respectively, over α.

From (5.5.2) and (5.5.3), we find that

$$1+\sqrt{\frac{m}{m'}}=4^{1/3}\left(\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24} \tag{5.5.4}$$

Setting $\alpha=1/2$ in (5.5.4), and then using (3.1.4), we deduce that

$$1+\frac{\phi(e^{-\pi})\phi(e^{-15\pi})}{\phi(e^{-3\pi})\phi(e^{-5\pi})}=\sqrt{2}\frac{G_{225}}{G_9^2G_{25}^2}. \tag{5.5.5}$$

Again from Berndt's book [18, pp. 189-190 and 195], we note that

$$G_9=\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/3} \qquad G_{25}=\frac{1+\sqrt{5}}{2},$$

and

$$G_{225}=\left(\frac{1+\sqrt{5}}{2}\right)(2+\sqrt{3})^{1/3}\left(\frac{\sqrt{4+\sqrt{15}}+(15)^{1/4}}{2}\right).$$

Employing these values in (5.5.5), and then simplifying, we find that

$$\frac{\phi(e^{-\pi})\phi(e^{-15\pi})}{\phi(e^{-3\pi})\phi(e^{-5\pi})}=\frac{\sqrt{2}(15)^{1/4}}{1+\sqrt{5}}+\frac{\sqrt{3}-1}{1+\sqrt{5}}, \tag{5.5.6}$$

where in the simplification, we also used (see Lemma 3.2.8)

$$\sqrt{4+\sqrt{15}}=\sqrt{5/2}+\sqrt{3/2}.$$

Using (5.3.3) and the value ([18, p. 327]; [21, p. 280])

$$\phi(e^{-3\pi})/\phi(e^{-\pi})=1/\sqrt[4]{6\sqrt{3}-9},$$

in (5.5.6), we find that

$$\frac{\phi(e^{-15\pi})}{\phi(e^{-\pi})} = \frac{\sqrt{2}(15)^{1/4} + \sqrt{3} - 1}{(1 + \sqrt{5})(\sqrt[4]{6\sqrt{3} - 9})(\sqrt{5\sqrt{5} - 10})}. \tag{5.5.7}$$

Simplifying (5.5.7), we easily arrive at (5.5.1).

5.6 Evaluation of $\phi(e^{-21\pi})$

We shall use the following "mixed" modular equation of degrees 1, 3, 7, 21 found by us in Chapter 2.

Lemma 5.6.1 *Let β, γ, and δ have degrees 3, 7, and 21, respectivly, over α. Let m denote the multiplier connecting α and β, and let m' be the multiplier relating γ and δ. Then*

$$\frac{3}{\sqrt{mm'}}(\beta\delta(1-\beta)(1-\delta))^{1/24} = \left(\frac{\gamma^3(1-\gamma)^3}{\delta(1-\delta)}\right)^{1/24} - \left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/24} \tag{5.6.1}$$

In the above lemma, we now set $\alpha = 1/2$, so that $q = e^{-\pi}$ to obtain

$$\frac{3}{\sqrt{mm'}} = \frac{G_9 G_{441}^2}{G_{49}^3} - G_9^2 G_{441}. \tag{5.6.2}$$

Thus

$$\frac{\phi(e^{-3\pi})\phi(e^{-21\pi})}{\phi(e^{-\pi})\phi(e^{-7\pi})} = \frac{G_9 G_{441}}{3}\left(\frac{G_{441}}{G_{49}^3} - G_9\right). \tag{5.6.3}$$

Using the requisite class invariants from Berndt's book [18], and the known values of $\phi(e^{-\pi})$, $\phi(e^{-3\pi})$, and $\phi(e^{-7\pi})$, we can find an explicit value of $\phi(e^{-21\pi})$ from (5.6.3).

5.7 Evaluation of $\phi(e^{-35\pi})$

We shall use the following "mixed" modular equation of degrees 1, 5, 7, 35 recorded by Ramanujan and proved by Berndt [15, p. 423].

Lemma 5.7.1 *Let β, γ, and δ have degrees 5, 7, and 35, respectivly, over α. Let m denote the multiplier connecting α and β, and let m' be the multiplier relating γ and δ. Then*

$$\sqrt{\frac{m'}{m}} = \frac{\{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/24} - \{16\alpha\delta(1-\alpha)(1-\delta)\}^{1/8}}{\{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/24} + \{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/8}}. \tag{5.7.1}$$

In the above lemma, we set $\alpha = 1/2$, so that $q = e^{-\pi}$ to obtain

$$\sqrt{\frac{m'}{m}} = \frac{G_{25}^{-1}G_{49}^{-1} - G_{1225}^{-3}}{G_{25}^{-1}G_{49}^{-1} + G_{25}^{-3}G_{49}^{-3}}. \tag{5.7.2}$$

Thus,

$$\frac{\phi(e^{-5\pi})\phi(e^{-7\pi})}{\phi(e^{-\pi})\phi(e^{-35\pi})} = \frac{G_{25}^{-1}G_{49}^{-1} - G_{1225}^{-3}}{G_{25}^{-1}G_{49}^{-1} + G_{25}^{-3}G_{49}^{-3}}. \tag{5.7.3}$$

Using the requisite Ramanujan's class invariants from Berndt's book [18], and the known values of $\phi(e^{-\pi})$, $\phi(e^{-5\pi})$, and $\phi(e^{-7\pi})$, we can find an explicit value of $\phi(e^{-35\pi})$ from (5.7.3).

Note that, evaluation for $\phi^2(e^{-35\pi})$ can be found by using (5.3.1). But to get the value of $\phi(e^{-35\pi})$ our evaluation is less tedious.

Concluding remarks: 1. The following theta-transformation formula for ϕ [15, p. 43] can be used to find the explicit evaluations of $\phi(e^{-\pi/n})$, when the corresponding values of $\phi(e^{-\pi n})$ are known.

If $a, b > 0$ with $ab = \pi$, then $\sqrt{a}\phi(e^{-a^2}) = \sqrt{b}\phi(e^{-b^2})$.

As for example, if $a^2 = \pi/\sqrt{r}$, then $\phi(e^{-\pi/\sqrt{r}}) = r^{1/4}\phi(e^{-\pi\sqrt{r}})$; and hence, by putting $r = 25$, we obtain $\phi(e^{-\pi/5}) = \sqrt{5}\phi(e^{-5\pi})$.

2. It is to be noted that the values of $\phi(e^{-n\pi})/\phi(e^{-\pi})$ are algebraic. In fact, Berndt, Chan, and Zhang [27, p. 610] proved the following general theorem.

Let m and n be positive integers. Then $\phi(e^{-mn\pi})/\phi(e^{-n\pi})$ is algebraic. Furthermore, if m is odd, then $\sqrt{2m}\phi(e^{-mn\pi})/\phi(e^{-n\pi})$ is an algebraic integer dividing $2\sqrt{m}$, while if m is even, then $2\sqrt{m}\phi(e^{-mn\pi})/\phi(e^{-n\pi})$ is an algebraic integer dividing $4\sqrt{m}$.

3. In Chapter 6 and Chapter 7 we give many interesting general formulas for the explicit evaluations of theta-functions in contexts with Rogers-Ramanujan continued fraction and Ramanujan's cubic continued fraction.

Chapter 6

Evaluation of Rogers-Ramanujan Continued Fraction

6.1 Introduction

We recall that, for $|q| < 1$, the famous Rogers-Ramanujan continued fraction $R(q)$ is defined by

$$R(q) := \frac{q^{1/5}}{1} \; \frac{q}{+1+} \; \frac{q^2}{1} \; \frac{q^3}{+1+\cdots}. \qquad (6.1.1)$$

We also set $S(q) = -R(-q)$. In the literature, considerable attention has been given in finding the explicit values of $R(q)$ when $q = e^{-\pi\sqrt{n}}$, for several positive rational values of n. In fact, In his first and second letters to Hardy [22], Ramanujan communicated several explicit values of $R(q)$ and $S(q)$. Watson [52], [53] proved Ramanujan's claims in those letters. Moreover,

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Note: *Some parts of this chapter consist of our paper [7].*

in both letters, Ramanujan asserted that "$R(e^{-\pi\sqrt{n}})$ can be exactly found if n be any positive rational quantity". In both his first [48] and lost notebooks [49], Ramanujan recorded several other evaluations. In particular, on page 210 of his lost notebook [49], Ramanujan provided a list of evaluations and intended evaluations. In [42]-[46], K.G. Ramanathan made the first attempt to find a uniform method to evaluate $R(q)$ by using Kronecker's limit formula, with which Ramanujan was not probably familiar. B.C. Berndt and H.H. Chan [bcbchancontinued] and Berndt, Chan and L.-C. Zhang [25] completed the incomplete list of Ramanujan by using some modular equations recorded by Ramanujan in his notebooks [48]. Most importantly, in [25], Berndt, Chan, and Zhang derived general formulas for evaluating $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ in terms of Weber-Ramanujan class invariants. The lost notebook [49] also contains many formulas for $R(q)$ and theta-function identities giving more formulas for the explicit evaluation of $R(q)$. S.-Y. Kang [37], [38] proved many of the claims made by Ramanujan. It appeared that though Ramanujan's formulas are interesting, they generally are not very much amenable in the calculation of elegant values of $R(q)$. We would like to refer the expository paper by Berndt, Chan, Huang , Kang, Sohn, and Son [29] to know about the knowledge available in the literature till the publication of that paper. In this chapter, we find some of the evaluations of $R(q)$ and $S(q)$. Our evaluations are more transparent than those of the previous authors.

First of all, in section 6.2, we establish some beautiful theta-function identities recorded by Ramanujan in the unorganized pages of both his first and second notebooks [48]. Berndt [17], [18] proved these identities via parameterization.

Secondly in section 6.3, we give some more theorems for the explicit evaluation of the quotients of theta-functions, as found in Section 4.2, by combining Weber-Ramanujan's class invariants with the identities proved in section 6.2 and some other theta-function identities. In our last section, we find some of the evaluations of $R(q)$ and $S(q)$, by using the values of the quotients of theta-functions found in section 6.3 and some other identities.

6.2 Theta-function Identities

The following identity was recorded by Ramanujan on page 295 of his first notebook [48]. Berndt [18] proved this by papametrization. Here we give an alternate proof directly from theta-functions.

Theorem 6.2.1 *If $\phi(q)$, $\psi(q)$, and $\chi(q)$ are as defined in (2.1.23), (2.1.24), and (2.1.26), respectively, then*

$$\psi^2(-q) + 5q\psi^2(-q^5) = \frac{\phi^2(q)}{\chi(q)\chi(q^5)}. \tag{6.2.1}$$

Proof: From Entries 9(vii) and 10(v) of Berndt's book [15, p. 258 and 262, respectively], we find that

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi(-q^5)f(-q^5)}{\chi(-q)}, \tag{6.2.2}$$

where f is as defined in (2.1.25).

From Entry 24(iii) of the same book [15, p. 39], we note that

$$f(q) = \frac{\phi(q)}{\chi(q)}. \tag{6.2.3}$$

From (6.2.2) and (6.2.3), we deduce that

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q^5)}{\chi(-q)\chi(-q^5)}. \tag{6.2.4}$$

Again, we recall from Entry 9(iii) [15, p. 258] that

$$\phi^2(q) - \phi^2(q^5) = 4q\chi(q)f(-q^5)f(-q^{20}). \tag{6.2.5}$$

Replacing q by $-q$ in (6.2.5), we deduce that

$$\phi^2(-q^5) = \phi^2(-q) + 4q\chi(-q)f(q^5)f(-q^{20}). \tag{6.2.6}$$

Employing (6.2.6) in (6.2.4), we find that

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q)}{\chi(-q)\chi(-q^5)} + 4q\frac{f(q^5)f(-q^{20})}{\chi(-q^5)}. \tag{6.2.7}$$

Now, by Entry 24(iii) [15, p. 39], we find that

$$f(-q^4) = \psi(q^2)\chi(-q^2). \tag{6.2.8}$$

Using (6.2.8) in (6.2.7), we obtain

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q)}{\chi(-q)\chi(-q^5)} + 4q\frac{f(q^5)\psi(q^{10})\chi(-q^{10})}{\chi(-q^5)}. \tag{6.2.9}$$

From Entry 24(iv) [15, p. 39], we note that

$$\chi(q)\chi(-q) = \chi(-q^2). \tag{6.2.10}$$

Thus, from (6.2.9), we obtain

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q)}{\chi(-q)\chi(-q^5)} + 4qf(q^5)\psi(q^{10})\chi(-q^5). \tag{6.2.11}$$

From Entry 25(iv) [15, p. 40], we note that

$$\phi(q)\psi(q^2) = \psi^2(q). \tag{6.2.12}$$

Employing (6.2.3) and (6.2.12), with q replaced by q^5, we conclude from (6.2.11) that

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi^2(-q)}{\chi(-q)\chi(-q^5)} + 4q\psi^2(q^5). \tag{6.2.13}$$

Replacing q by $-q$ in (6.2.13), we complete the proof of the theorem.

The next theorem was recorded by Ramanujan on page 4 of his second notebook [48]. It is extremely useful in our calculations. Berndt [17, p. 202] proved this theorem via parameterization. Here we prove this from theta-function identities.

Theorem 6.2.2

$$\frac{\chi^5(q)}{\chi(q^5)} = 1 + 5q\frac{\psi^2(-q^5)}{\psi^2(-q)}. \tag{6.2.14}$$

Proof: From Theorem 6.2.1, we find that

$$1 + 5q\frac{\psi^2(-q^5)}{\psi^2(-q)} = \frac{\phi^2(q)}{\chi(q)\chi(q^5)\psi^2(-q)}. \tag{6.2.15}$$

Now, from Entry 24(iii) [15, p. 39], we note that

$$\chi(q) = \sqrt[3]{\frac{\phi(q)}{\psi(-q)}}. \tag{6.2.16}$$

Employing (6.2.16) in (6.2.15) we arrive at (6.2.14), which completes the proof.

6.3 Explicit evaluations of theta-functions

Theorem 6.3.1

$$(i) \quad e^{-\pi\sqrt{n}}\frac{\psi^2(-e^{-5\pi\sqrt{n}})}{\psi^2(-e^{-\pi\sqrt{n}})} = \frac{1}{5}\left(2\frac{G_n^5}{G_{25n}} - 1\right). \tag{6.3.1}$$

and

$$(ii) \quad e^{-\pi\sqrt{n}}\frac{\psi^2(e^{-5\pi\sqrt{n}})}{\psi(e^{-\pi\sqrt{n}})} = \frac{1}{5}\left(1 - 2\frac{g_n^5}{g_{25n}}\right). \tag{6.3.2}$$

Proof: From Theorem 6.2.2 and the definition of G_n from (3.1.2), we easily arrive at (6.3.1) by putting $q = \exp(-\pi\sqrt{n})$.

Replacing q by $-q$ in Theorem 6.2.2 and then using the definition of g_n from (6.3.1), we arrive at (6.3.2) by again putting $q = \exp(-\pi\sqrt{n})$.

If the class invariants are known, then we can explicitly find the value of the quotients of the right hand side expressions of the theorem. We give some examples below.

Corollary 6.3.2

$$e^{-\pi}\frac{\psi^2(-e^{-5\pi})}{\psi^2(-e^{-\pi})} = \frac{1}{5\sqrt{5}+1}. \tag{6.3.3}$$

Proof: Putting $n = 1$ in Theorem 6.3.1(i), we find that

$$e^{-\pi}\frac{\psi^2(-e^{-5\pi})}{\psi^2(-e^{-\pi})} = \frac{1}{5}\left(2\frac{G_1^5}{G_{25}} - 1\right) \tag{6.3.4}$$

From Berndt's book [18, p. 189],

$$G_1 = 1 \quad \text{and} \quad G_{25} = \frac{1+\sqrt{5}}{2}. \tag{6.3.5}$$

Employing (6.3.5) in (6.3.4), and then simplifying we complete the proof.

This was also evaluated by Kang [38] by using a different method.

Corollary 6.3.3

$$e^{-\pi/\sqrt{5}}\frac{\psi^2(-e^{-\sqrt{5}\pi})}{\psi^2(-e^{-\pi/\sqrt{5}})}=\frac{1}{\sqrt{5}}. \tag{6.3.6}$$

Proof: We put $n=1/5$ in Theorem 6.3.1(i), to obtain

$$e^{-\pi/\sqrt{5}}\frac{\psi^2(-e^{-\sqrt{5}\pi})}{\psi^2(-e^{-\pi/\sqrt{5}})}=\frac{1}{5}\left(2G_5^4-1\right). \tag{6.3.7}$$

Now, note from Berndt's book [18, p.189], that

$$G_5=\left(\frac{1+\sqrt{5}}{2}\right)^{1/4}. \tag{6.3.8}$$

We easily complete the proof by (6.3.7) and (6.3.8).

Corollary 6.3.4

$$e^{-\pi\sqrt{3/5}}\frac{\psi^2(-e^{-\pi\sqrt{15}})}{\psi^2(-e^{-\pi\sqrt{3/5}})}=\frac{3-\sqrt{5}}{5+\sqrt{5}}. \tag{6.3.9}$$

Proof: Putting $n=3/5$ in Theorem 6.3.1(i), we obtain

$$e^{-\pi\sqrt{3/5}}\frac{\psi^2(-e^{-\pi\sqrt{15}})}{\psi^2(-e^{-\pi\sqrt{3/5}})}=\frac{1}{5}\left(2\frac{G_{3/5}^5}{G_{15}}-1\right). \tag{6.3.10}$$

From Berndt's book [18, p. 341], we again note that

$$G_{15}=2^{-1/12}(1+\sqrt{5})^{1/3} \quad \text{and} \quad G_{3/5}=2^{-1/12}(\sqrt{5}-1)^{1/3}. \tag{6.3.11}$$

Employing (6.3.11) in (6.3.10), and then simplifying we arrive at (6.3.4).

Since from Chapter 4 (Theorem 4.2.1), for $q = e^{-\pi\sqrt{n}}$, n positive rational, the explicit formulas for $\phi^2(q^5)/\phi^2(q)$ is known, we now derive an identity by which the corresponding values of the quotient $\psi^2(-q^5)/\psi^2(-q)$ may be found.

Theorem 6.3.5

$$q\frac{\psi^2(-q^5)}{\psi^2(-q)} = \frac{1 - \phi^2(q^5)/\phi^2(q)}{(5\phi^2(q^5)/\phi^2(q)) - 1}. \tag{6.3.12}$$

Proof: We replace q by $-q$ in (6.2.4) and then divide the resulting identity by (6.2.1) to obtain

$$\frac{\phi^2(q^5)}{\phi^2(q)} = \frac{\psi^2(-q) + q\psi^2(-q^5)}{\psi^2(-q) + 5q\psi^2(-q^5)}. \tag{6.3.13}$$

This is indeed equivalent to (6.3.12).

6.4 Evaluation of $R(q)$ and $S(q)$

The following important formula about $R(q)$ was found by Watson in Ramanujan's notebooks and proved by him [52].

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{6.4.1}$$

Replacing q by $-q$ and q^2, respectively, we find that

$$\frac{1}{S^5(q)} + 11 - S^5(q) = \frac{f^6(q)}{qf^6(q^5)}, \tag{6.4.2}$$

and

$$\frac{1}{R^5(q^2)} - 11 - R^5(q^2) = \frac{f^6(-q^2)}{q^2f^6(-q^{10})}, \tag{6.4.3}$$

respectively.

From (6.4.2) and (6.4.3), we see that, to find the explicit values of $S(q)$ and $R(q^2)$, for $q = e^{-\pi\sqrt{n}}$, it is enough to find the expressions on the right sides, because solving the quadratic equations (6.4.2) and (6.4.3), one can easily obtain

$$S^5(q) = \sqrt{c_1^2+1} + c_1, \tag{6.4.4}$$

where

$$2c_1 = \frac{f^6(q)}{qf^6(q^5)} + 11,$$

and

$$R^5(q^2) = \sqrt{c_2^2+1} - c_2, \tag{6.4.5}$$

where

$$2c_2 = \frac{f^6(-q^2)}{q^2f^6(-q^{10})} + 11.$$

See also the papers by Berndt, Chan, and Zhang [25] and Kang [38]. In the sequel, we shall see that c_1 and c_2 can be obtained by combining some simple theta-function identities recorded by Ramanujan in Chapter 16 of his second notebook [48] with the explicit values of the quotients of theta-functions discussed in Section 6.3.

Theorem 6.4.1 *If $\phi(q)$, $\psi(q)$, and $f(q)$ are as defined in (2.1.23), (2.1.24), and (2.1.25), respectively, then*

$$(i) \quad F_1(q) := \frac{f^6(q)}{qf^6(q^5)} = \frac{\psi^2(-q)}{q\psi^2(-q^5)} \times \frac{\phi^4(q)}{\phi^4(q^5)}, \tag{6.4.6}$$

and

$$(ii) \quad F_2(q) := \frac{f^6(-q^2)}{q^2f^6(-q^{10})} = \frac{\phi^2(q)}{\phi^2(q^5)} \times \frac{\psi^4(q)}{q^2\psi^4(q^5)}. \tag{6.4.7}$$

Proof: From Entries 24(ii) and 24(iv) [15, p. 39], we note that

$$f^3(q) = \phi^2(q)\psi(-q), \tag{6.4.8}$$

and

$$f^3(-q^2) = \phi(q)\psi^2(-q). \tag{6.4.9}$$

From (6.4.8) and (6.4.9), it is easy to arrive at (6.4.6) and (6.4.7).

The values of $F_1(q)$ and $F_2(q)$ can be determined explicitly for $q = e^{-\pi\sqrt{n}}$ by employing (5.3.1) and (6.3.1). Thus, (6.4.4) and (6.4.5) gives explicit evaluations for $S(q)$ and $R(q^2)$. We give some examples below.

Corollary 6.4.2

$$(i) \quad S^5(e^{-\pi/\sqrt{5}}) = \sqrt{\left(\frac{5\sqrt{5}-11}{2}\right)^2+1} - \frac{5\sqrt{5}-11}{2}, \tag{6.4.10}$$

and

$$(ii) \quad R^5(e^{-2\pi/\sqrt{5}}) = \sqrt{\left(\frac{5\sqrt{5}+11}{2}\right)^2+1} - \frac{5\sqrt{5}+11}{2}. \tag{6.4.11}$$

Proof: As in Corollary 6.3.3, by putting $n = 1/\sqrt{5}$ in (5.3.1), it can be easily seen that

$$\frac{\phi^2(e^{-\sqrt{5}\pi})}{\phi^2(e^{-\pi/\sqrt{5}})} = \frac{1}{\sqrt{5}}. \tag{6.4.12}$$

Putting $q = e^{-\pi/\sqrt{5}}$ in (6.4.6) and (6.4.7), and then employing (6.4.12) and Corollary 6.3.3, we obtain

$$F_1(e^{-\pi/\sqrt{5}}) = 5\sqrt{5}, \tag{6.4.13}$$

and

$$F_2(e^{-\pi/\sqrt{5}}) = 5\sqrt{5}, \tag{6.4.14}$$

respectively. Using these results in (6.4.4) and (6.4.5) we complete the proof.

Remarks. Corollary 6.4.2 (i) was recorded by Ramanujan on page 210 of his lost notebook [48], and first proved by Ramanathan [43]. Berndt, Chan and Zhang [25] and Kang [38] also proved this.

Corollary 6.4.3

$$(i) \quad S^5(e^{-\pi\sqrt{3/5}}) = \frac{-3-5\sqrt{5}+\sqrt{30(5+\sqrt{5})}}{4}. \tag{6.4.15}$$

and

$$(ii) \quad R^5(e^{-2\pi\sqrt{3/5}}) = \frac{-147-55\sqrt{5}+\sqrt{36750+16170\sqrt{5}}}{4}. \tag{6.4.16}$$

Proof: As in Corollary 6.3.4, by putting $n = \sqrt{3/5}$ in (5.3.1), it can be easily seen that

$$\frac{\phi^2(e^{-\sqrt{15}\pi})}{\phi^2(e^{-\pi\sqrt{3/5}})} = \frac{2}{5-\sqrt{5}}. \tag{6.4.17}$$

Putting $q = e^{-\pi\sqrt{3/5}}$ in (6.4.6) and (6.4.7), and then employing (6.4.17) and Corollary 6.3.4, we obtain

$$F_1(e^{-\pi\sqrt{3/5}}) = \frac{5(5+\sqrt{5})}{2}, \tag{6.4.18}$$

and

$$F_2(e^{-\pi\sqrt{3/5}}) = \frac{5(25+11\sqrt{5})}{2}, \tag{6.4.19}$$

respectively. Invoking (6.4.18) and (6.4.19) in (6.4.4) and (6.4.5), respectively, we complete the proof.

Remark: Corollary 6.4.3(i) was incompletely recorded by Ramanujan on page 210 of his lost notebook [48], and was first proved by Ramanathan [44]. Other proofs may be found in [20], [25], and [38].

Proceeding in the same lines as in the above corollaries we can find the values of $S^5(e^{-\pi\sqrt{n}})$ and $R^5(e^{-2\pi\sqrt{n}})$, for $n = 1, 9/5, 11/5, 13/5, 3, 17/5, 21/5, 29/5, 41/5, 9, 53/5, 89/5$, and $101/5$, as the corresponding class invariants are known [18, Chapters 34-35] in these cases. Using the reciprocity theorems by Ramanathan [43], namely, if α and β are positive and $\alpha\beta = 1/5$, then

Both due to Ramanujan

$$\left\{\left(\frac{\sqrt{5}-1}{2}\right)^5 + S^5(e^{-\pi\alpha})\right\}\left\{\left(\frac{\sqrt{5}-1}{2}\right)^5 + S^5(e^{-\pi\beta})\right\} = 5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^5,$$

and

$$\left\{\left(\frac{\sqrt{5}+1}{2}\right)^5 + R^5(e^{-2\pi\alpha})\right\}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^5 + R^5(e^{-2\pi\beta})\right\} = 5\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^5,$$

one can easily obtain $S^5(e^{-\pi/\sqrt{25n}})$ and $R^5(e^{-2\pi/\sqrt{25n}})$, for the values of n stated above. We omit the details. It is worthwhile to mention that , Ramanathan [42]-[46], Berndt and Chan [20], Berndt, Chan, and Zhang [25], and Kang [38] also found some of these continued fractions. But in [42]-[46], Ramanathan used Kroneckers limit formula, in [25], one has to solve two quadratic equations, and in [20] and [38], the calculations are very lengthy. Surely our formulas are much more amenable than the previous authors.

We complete this chapter by noting that Jinhee Yi [63]-[64] has recently found many new eta-function identities and modular equations from which a number of explicit evaluations of $R(q)$ and $S(q)$ follow without depending upon class invariants.

Chapter 7

Ramanujan's Cubic Continued Fraction

7.1 Introduction

Let, for $|q| < 1$,

$$G(q) := \frac{q^{1/3}}{1} \, \frac{}{+} \, \frac{q+q^2}{1} \, \frac{}{+} \, \frac{q^2+q^4}{1} \, \frac{}{+} \, \frac{q^3+q^6}{1} \, \frac{}{+\dots}, \tag{7.1.1}$$

denotes Ramanujan's cubic continued fraction, first introduced by him in his second letter to Hardy [22]. Ramanujan also recorded this continued fraction on page 366 of his lost notebook [49], and claimed that there are many results of $G(q)$ which are analogous to Rogers-Ramanujan continued fraction $R(q)$. Motivated by Ramanuajan's claims, H.H. Chan [32] proved many new identities which probably were the identities vaguely reffered by Ramanujan. He established some reciprocity theorems for $G(q)$, found relations between $G(q)$ and the three continued fractions $G(-q)$, $G(q^2)$ and $G(q^3)$ and obtained some explicit evaluations of $G(q)$.

Note: *Some parts of this chapter consist of our papers [8] and [9].*

We note that his proof of the relation between $G(q)$ and $G(q^3)$ is not satisfactory. In particular, the last deduction [32, (2.18), p. 347] is not an obvious one. In Section 7.2 of this chapter, we find an easy proof of this relation.

In Section 7.3, we establish some theta-function identities recorded by Ramanujan in the unorganized pages of both his first and second notebooks [48]. Berndt [17] also proved these identities via parameterization.

In Section 7.4, we give some more theorems for the explicit evaluation of the quotients of theta-functions by using the identities found in the previous section.

In Section 7.5, we combine the theorems found in Section 7.4 with some other theta-function identities to deduce a number of explicit evaluations for $G(q)$. In fact, we have found general formulas for the explicit evaluations of $G(-e^{-3\pi\sqrt{n}})$ and $G(e^{3\pi\sqrt{n}})$. General formulas for the explicit evaluations of $G(-e^{-\pi\sqrt{n}})$ and $G(e^{\pi\sqrt{n}})$, were established by Berndt, Chan and Zhang [24].

In Section 7.6, we give three new eta-function identities, and use them in our final section to find two new identities giving relations between $G(q)$ and the two continued fractions $G(q^5)$ and $G(q^7)$.

7.2 A Relation Between $G(q)$ and $G(q^3)$

H.H. Chan [32] found the following beautiful relation connecting $G(q)$ and $G(q^3)$. As we already mentioned in the Introduction, his proof is not satisfactory. Here we give a simple proof of his theorem.

Theorem 7.2.1 *If $G(q)$ is as defined in (7.1.1), then*

$$G^3(q) = G(q^3)\frac{1 - G(q^3) + G^2(q^3)}{1 + 2G(q^3) + 4G^2(q^3)}. \tag{7.2.1}$$

Proof: From Entry 1(i) [15, p. 345], we note that

$$1 + \frac{1}{G(q)} = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}, \tag{7.2.2}$$

and

$$1 + \frac{1}{G^3(q)} = \frac{\psi^4(q)}{q\psi^4(q^3)}, \tag{7.2.3}$$

where $\psi(q)$ is as defined in (2.1.24).

Replacing q by q^3 in (7.2.2), we find that

$$1 + \frac{1}{G(q^3)} = \frac{\psi(q)}{q\psi(q^9)}. \tag{7.2.4}$$

Now, from Entry 1(ii) [15, p. 345], we note that

$$\left(1 + 3q\frac{\psi(-q^9)}{\psi(-q)}\right)^3 = 1 + 9q\frac{\psi^4(-q^3)}{\psi^4(-q)}. \tag{7.2.5}$$

Replacing q by $-q$ in (7.2.3) and (7.2.4), and then using the resultant identities in (7.2.5), we find that

$$\left(1 - \frac{3w}{1+w}\right)^3 = 1 - \frac{9u}{1+u}, \tag{7.2.6}$$

where $w = G(-q^3)$ and $u = G^3(-q)$.

Solving (7.2.6) for u, we find that

$$u = \frac{1 - \{(1-2w)/(1+w)\}^3}{8 + \{(1-2w)/(1+w)\}^3}. \tag{7.2.7}$$

Simplifying (7.2.7), we obtain

$$u = w\frac{1-w+w^2}{1+2w+4w^2}. \tag{7.2.8}$$

Replacing q by $-q$ in (7.2.8), we complete the proof.

7.3 A Theta-Function Identity

The following theorem was recorded by Ramanujan on page 4 of his second notebook [48]. It is extremely useful in our calculations. Berndt [17, p. 202] proved this theorem via parameterization. Here we prove this from theta-function identities.

Theorem 7.3.1 *If χ and ψ are as defined in (2.1.26) and (2.1.24), respectively, then*

$$\frac{\chi^3(q)}{\chi(q^3)} = 1 + 3q\frac{\psi(-q^9)}{\psi(-q)}, \tag{7.3.1}$$

Proof: From Corollary (ii) of Chapter 16 in Berndt's book [15, p. 49], we find that

$$\psi(q) - q\psi(q^9) = f(q^3, q^6). \tag{7.3.2}$$

Using Jacobi's triple product identity [15, Entry 19, p. 35], Berndt [15, p. 350] proved that

$$f(q, q^2) = \frac{\phi(-q^3)}{\chi(-q)}. \tag{7.3.3}$$

Replacing q by q^3 in (7.3.3), and then using the resultant identity in (7.3.2), we find that

$$\psi(q) - q\psi(q^9) = \frac{\phi(-q^9)}{\chi(-q^3)}. \tag{7.3.4}$$

Now, from Corollary (i) [15, p. 49] and (2.1.43), we find that

$$\phi(-q^9) = \phi(-q) + 2q\psi(q^9)\chi(-q^3). \tag{7.3.5}$$

Invoking (7.3.5) in (7.3.4), we deduce that

$$\psi(q) - 3q\psi(q^9) = \frac{\phi(-q)}{\chi(-q^3)}. \tag{7.3.6}$$

Thus,

$$1 - 3q\frac{\psi(q^9)}{\psi(q)} = \frac{\phi(-q)}{\chi(-q^3)\psi(q)}. \tag{7.3.7}$$

Now from Entry 24(iii) [15, p. 39], we note that

$$\chi(q) = \sqrt[3]{\frac{\phi(q)}{\psi(-q)}}. \tag{7.3.8}$$

Replacing q by $-q$ in (7.3.7) and then using (7.3.8), we complete the proof of the theorem.

7.4 Explicit evaluations of theta-functions

Theorem 7.4.1

$$(i) \quad e^{-\pi\sqrt{n}}\frac{\psi(-e^{-9\pi\sqrt{n}})}{\psi(-e^{-\pi\sqrt{n}})} = \frac{1}{3}\left(\sqrt{2}\frac{G_n^3}{G_{9n}} - 1\right) \tag{7.4.1}$$

and

$$(ii) \quad e^{-\pi\sqrt{n}}\frac{\psi(e^{-9\pi\sqrt{n}})}{\psi(e^{-\pi\sqrt{n}})} = \frac{1}{3}\left(1 - \sqrt{2}\frac{g_n^3}{g_{9n}}\right) \tag{7.4.2}$$

Proof: From Theorem 7.3.1 and the definition of G_n from (3.1.2), we easily arrive at (7.4.1). To prove (ii), we replace q by $-q$ in Theorem 7.3.1 and then use the definition of g_n from (3.1.2).

Since, G_{9n} and g_{9n} can be calculated from the respective values of G_n and g_n [24], from the above theorem, we see that the certain quotients of theta-functions on the right sides can be evaluated if the corresponding values of G_n and g_n are known. We give only a couple of examples below.

Corollary 7.4.2

$$e^{-\pi}\frac{\psi(-e^{-9\pi})}{\psi(-e^{-\pi})}=\frac{\sqrt[3]{2(\sqrt{3}-1}-1}{3}. \tag{7.4.3}$$

Proof: Putting $n=1$ in Theorem 7.4.1(i), we find that

$$e^{-\pi}\frac{\psi(-e^{-9\pi})}{\psi(-e^{-\pi})}=\frac{1}{3}\left(\sqrt{2}\frac{G_1^3}{G_9}-1\right). \tag{7.4.4}$$

From Berndt's book [16, p. 189],

$$G_1=1 \quad \text{and} \quad G_9=\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/3}. \tag{7.4.5}$$

Employing (7.4.5) in (7.4.4), and simplifying we complete the proof.

From Entry 11(ii) [15, p. 123], we find that

$$\psi(-e^{-\pi})=\phi(e^{-\pi})2^{-3/4}e^{\pi/8}. \tag{7.4.6}$$

Since

$$\phi(e^{-\pi})=\frac{\pi^{1/4}}{\Gamma\left(\frac{3}{4}\right)}$$

is classical [62], (7.4.3) and (7.4.6) provide an explicit evaluation for $\psi(-e^{-9\pi})$.

Corollary 7.4.3

$$e^{-\pi\sqrt{5}/3}\frac{\psi(-e^{-3\pi\sqrt{5}})}{\psi(-e^{-\pi\sqrt{5}/3})}=\frac{(3+\sqrt{5})(\sqrt{5}-\sqrt{3})-2}{6}. \tag{7.4.7}$$

Proof: Putting $n=5/9$ in Theorem 7.4.1(i), we obtain

$$e^{-\pi\sqrt{5}/3}\frac{\psi(-e^{-3\pi\sqrt{5}})}{\psi(-e^{-\pi\sqrt{5}/3})}=\frac{1}{3}\left(\sqrt{2}\frac{G_{5/9}^3}{G_5}-1\right). \tag{7.4.8}$$

Now, from Berndt's book [18, pp. 189 and 345], we note that

$$G_5 = \left(\frac{1+\sqrt{5}}{2}\right)^{1/4} \quad \text{and} \quad G_{5/9} = (\sqrt{5}+2)^{1/4}\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^{1/3} \tag{7.4.9}$$

Employing (7.4.9) in (7.4.8), and then simplifying we arrive at (7.4.7).

Since by Theorem 5.3.3 of Chapter 5, we know the explicit formula for $\phi(q^9)/\phi(q)$, for $q = e^{-\pi\sqrt{n}}$, n positive rational, we now derive an identity by which the corresponding values of the quotients $\psi(-q^9)/\psi(-q)$ may be found.

Theorem 7.4.4

$$q\frac{\psi(-q^9)}{\psi(-q)} = \frac{1-\phi(q^9)/\phi(q)}{(3\phi(q^9)/\phi(q))-1}. \tag{7.4.10}$$

Proof: Replacing q by $-q$ in (7.3.4) and (7.3.6) and then dividing the first resulting identity by the second, we find that

$$\frac{\phi(q^9)}{\phi(q)} = \frac{\psi(-q)+q\psi(-q^9)}{\psi(-q)+3q\psi(-q^9)}. \tag{7.4.11}$$

It is now easy to see that (7.4.10) and (7.4.11) are equivalent.

7.5 Explicit formulas for $G(-e^{-3\pi\sqrt{n}})$ and $G(e^{-3\pi\sqrt{n}})$

Berndt, Chan and Zhang [24] have found general formulas for $G(-e^{-\pi\sqrt{n}})$ and $G(e^{-\pi\sqrt{n}})$ by employing the formulas connecting G_n and G_{9n}, and g_n and g_{9n}, respectively. Using the formulas for the explicit evaluations of the quotients of theta-functions found in the previous section, we can find the general formulas for $G(-e^{-3\pi\sqrt{n}})$ and $G(e^{-3\pi\sqrt{n}})$

From Entry 1(i) [15, p. 345], we find that

$$G(-q^3) = \frac{-q\psi(-q^9)/\psi(-q)}{1+q\psi(-q^9)/\psi(-q)}. \tag{7.5.1}$$

These are subsumed under result of Berndt, Chan, Zhang

Replacing q by $-q$ in (7.5.1), we find that

$$G(q^3) = \frac{q\psi(q^9)/\psi(q)}{1 - q\psi(q^9)/\psi(q)}. \tag{7.5.2}$$

Taking $q = e^{-\pi\sqrt{n}}$ in (7.5.1) and (7.5.2), we find the following formulas for $G(-e^{-3\pi\sqrt{n}})$ and $G(e^{-3\pi\sqrt{n}})$.

Theorem 7.5.1

$$(i) \quad G(-e^{-3\pi\sqrt{n}}) = \frac{-e^{-\pi\sqrt{n}}\psi(-e^{-9\pi\sqrt{n}})/\psi(-e^{-\pi\sqrt{n}})}{1 + e^{-\pi\sqrt{n}}\psi(-e^{-9\pi\sqrt{n}})/\psi(-e^{-\pi\sqrt{n}})} \tag{7.5.3}$$

and

$$(ii) \quad G(e^{-3\pi\sqrt{n}}) = \frac{e^{-\pi\sqrt{n}}\psi(e^{-9\pi\sqrt{n}})/\psi(e^{-\pi\sqrt{n}})}{1 - e^{-\pi\sqrt{n}}\psi(e^{-9\pi\sqrt{n}})/\psi(e^{-\pi\sqrt{n}})}. \tag{7.5.4}$$

Combining with Theorem 7.4.1, a number of explicit evaluations follow. We give a couple of examples below.

Corollary 7.5.2

$$G(-e^{-3\pi}) = \frac{1 - \sqrt[3]{2(\sqrt{3}-1)}}{2 + \sqrt[3]{2(\sqrt{3}-1)}}. \tag{7.5.5}$$

Proof: Putting $n = 1$ in Theorem 7.5.1 (i), and then using Corollary 7.4.2, we arrive at (7.5.5).

Corollary 7.5.3

$$G(-e^{-\pi\sqrt{5}}) = \frac{(\sqrt{5}-\sqrt{3})(\sqrt{5}-3)}{4}. \tag{7.5.6}$$

Proof: In this case we put $n = 5/9$ in Theorem 7.5.1 (i), and then use Corollary 7.4.3, to obtain

$$G(-e^{-\pi\sqrt{5}}) = \frac{2 - (\sqrt{5}-\sqrt{3})(3+\sqrt{5})}{4 + (\sqrt{5}-\sqrt{3})(3+\sqrt{5})}. \tag{7.5.7}$$

Simplifying (7.5.7), we complete the proof.

Remark: For different proofs of Corollary 7.5.3, see [24] and [32].

7.6 Three eta-function Identities

In this section, we prove three eta-function identities which we will use in our next section.

Theorem 7.6.1 *If*

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \textit{and} \quad Q = \frac{\psi(q^5)}{q^{5/4}\psi(q^{15})},$$

then

$$(PQ)^2 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 + 5\left(\frac{Q}{P}\right)^2 + 5\left(\frac{P}{Q}\right)^2 + 5\left(\frac{Q}{P} - \frac{P}{Q}\right) - \left(\frac{P}{Q}\right)^3 \tag{7.6.1}$$

Proof. We note from Entry 24(iii) [15, p. 39] that

$$\psi(q) = \frac{f^2(-q^2)}{f(-q)}. \tag{7.6.2}$$

Therefore P and Q can be reformulated as

$$P = \frac{f(-q^3)f^2(-q^2)}{q^{1/4}f(-q)f^2(-q^6)} \quad \text{and} \quad Q = \frac{f(-q^{15})f^2(-q^{10})}{q^{5/4}f(-q^5)f^2(-q^{30})}.$$

Now we set

$$L_1 := \frac{f(-q)}{q^{1/12}f(-q^3)}, \qquad L_2 := \frac{f(-q^5)}{q^{5/12}f(-q^{15})},$$

$$M_1 := \frac{f(-q^2)}{q^{1/6}f(-q^6)} \qquad \text{and,} \quad M_2 := \frac{f(-q^{10})}{q^{5/6}f(-q^{30})}, \tag{7.6.3}$$

so that

$$P = \frac{{M_1}^2}{L_1} \qquad \text{and} \qquad Q = \frac{{M_2}^2}{L_2}. \tag{7.6.4}$$

Employing (7.6.3) in Entry 51 [17, p. 204], we obtain

$$(L_1M_1)^2 + \frac{9}{(L_1M_1)^2} = \left(\frac{L_1}{M_1}\right)^6 + \left(\frac{M_1}{L_1}\right)^6. \tag{7.6.5}$$

Replacing q by q^5 in the same entry, and then using (7.6.3), we find that

$$(L_2M_2)^2 + \frac{9}{(L_2M_2)^2} = \left(\frac{L_2}{M_2}\right)^6 + \left(\frac{M_2}{L_2}\right)^6. \tag{7.6.6}$$

Using (7.6.4) we may rewrite (7.6 .5) in the form

$$\frac{{M_1}^6}{P^2} + \frac{9P^2}{{M_1}^6} = \left(\frac{M_1}{P}\right)^6 + \left(\frac{P}{M_1}\right)^6. \tag{7.6.7}$$

Thus we arrive at

$${M_1}^{12} = \frac{P^8(P^4-9)}{P^4-1}. \tag{7.6.8}$$

Similarly from (7.6.4) and (7.6.6), we deduce that

$${M_2}^{12} = \frac{Q^8(Q^4-9)}{Q^4-1}. \tag{7.6.9}$$

Employing (7.6.3) in (59.10) [17, p. 215], we find that

$$\left(\frac{L_2}{L_1}\right)^3 + \left(\frac{M_2}{M_1}\right)^3 = \left(\frac{L_2M_2}{L_1M_1}\right)^2 - \left(\frac{L_2M_2}{L_1M_1}\right). \tag{7.6.10}$$

Invoking (7.6.4) in (7.6.10), and then simplifying, we deduce that

$$\left(\frac{M_2}{M_1}\right)^3 = \frac{1+P/Q}{\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^3}. \tag{7.6.11}$$

From (7.6.8), (7.6.9), and (7.6.11), we find that

$$\frac{Q^8(Q^4-9)(P^4-1)}{P^8(P^4-9)(Q^4-1)} = \left[\frac{1+P/Q}{\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^3}\right]^4. \tag{7.6.12}$$

Setting $x := P/Q$ and $y := PQ$, and then simplifying, we deduce that

$$\frac{(y^2 - 9x^2)(x^2y^2 - 1)}{(y^2 - x^2)(x^2y^2 - 9)} = \left(\frac{1+x}{1-x}\right)^4. \tag{7.6.13}$$

Further simplifications give

$$(1 + x^2)(9x^3 - y^2 - 5xy^2 - 5x^2y^2 + 5x^4y^2 - 5x^5y^2 + x^6y^2 + x^3y^4) = 0. \tag{7.6.14}$$

Since the first factor never vanishes, we deduce that

$$9x^3 - y^2 - 5xy^2 - 5x^2y^2 + 5x^4y^2 - 5x^5y^2 + x^6y^2 + x^3y^4 = 0. \tag{7.6.15}$$

Thus,

$$y^2 + \frac{9}{y^2} = \frac{1}{x^3} + \frac{5}{x^2} + 5x^2 + 5\left(\frac{1}{x} - x\right) - x^3, \tag{7.6.16}$$

which is readily seen to be equivalent to (7.6.1).

Remark Since by Entry 24(iii) [15, p. 39],

$$\phi(-q) = \frac{f^2(-q)}{f(-q^2)},$$

proceeding as above, we see that, if

$$P = \frac{\phi(-q)}{\phi(-q^3)} \quad \text{and} \quad Q = \frac{\phi(-q^5)}{\phi(-q^{15})},$$

then (7.6.1) holds. Replacing q by $-q$, we see that the same identity holds if

$$P = \frac{\phi(q)}{\phi(q^3)} \quad \text{and} \quad Q = \frac{\phi(q^5)}{\phi(q^{15})}.$$

Theorem 7.6.2 *If*

$$P = \frac{f(-q)f(-q^7)}{q^{2/3}f(-q^3)f(-q^{21})} \quad \textit{and} \quad Q = \frac{f(-q^2)f(-q^{14})}{q^{4/3}f(-q^6)f(-q^{42})},$$

then

$$\left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3 = 10 + PQ + \frac{9}{PQ} - 2\left(25 + 4PQ + \frac{36}{PQ}\right)^{1/2}. \tag{7.6.17}$$

Proof: Let

$$R = \frac{f(q)f(q^7)}{q^{2/3}f(q^3)f(q^{21})}.$$

By Entries 12(i) and (iii) in Chapter 17 of [15, p. 124] we find that

$$R = \sqrt{mm'}\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/24} \tag{7.6.18}$$

and

$$Q = \sqrt{mm'}\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/12}, \tag{7.6.19}$$

where β, γ, and δ have degrees 3, 7, 21, respectively, over α and m and m' are the multipliers connecting α, β and γ, δ, respectively.

From (7.6.18) and (7.6.19), we readily see that

$$\frac{Q}{R} = \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/24} \tag{7.6.20}$$

and

$$\frac{R^2}{Q} = \sqrt{mm'}. \tag{7.6.21}$$

Now by Entries 13(v) and 13(vi) in Chapter 20 of [15, p. 384], we note the "mixed" modular equations

$$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4}+\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}-2\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8}$$
$$\times\left\{1+\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8}\right\}=mm' \tag{7.6.22}$$

and

$$\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/4}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/4}+\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/4}-2\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/8}$$
$$\times\left\{1+\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/8}\right\}=\frac{9}{mm'}, \tag{7.6.23}$$

respectively.

For simplicity, we set

$$x:=\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8} \quad\text{and}\quad y:=\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8},$$

so that

$$\frac{R^3}{Q^3}=y. \tag{7.6.24}$$

Then from (7.6.22), we find that

$$x=y\pm(4y+mm')^{1/2}. \tag{7.6.25}$$

Also, from (7.6.23), we find the reciprocal equation of (7.6.25) as

$$\frac{x}{y}=\frac{1}{y}\pm\left(\frac{4}{y}+\frac{9}{mm'}\right)^{1/2}. \tag{7.6.26}$$

Combining (7.6.25) and (7.6.26), we obtain

$$y\pm(4y+mm')^{1/2}=1\pm y\left(\frac{4}{y}+\frac{9}{mm'}\right)^{1/2}. \tag{7.6.27}$$

Employing (7.6.20), (7.6.21), and (7.6.24) in (7.6.27), we find that

$$\frac{R^3}{Q^3} \pm \left(4\frac{R^3}{Q^3} + \frac{R^4}{Q^2}\right)^{1/2} = 1 \pm \frac{R^3}{Q^3}\left(4\frac{Q^3}{R^3} + 9\frac{Q^2}{R^4}\right)^{1/2}. \tag{7.6.28}$$

We rewrite (7.6.28) as

$$R^3 - Q^3 = \pm R^3\left(4\frac{Q^3}{R^3} + 9\frac{Q^2}{R^4}\right)^{1/2} \mp Q^3\left(4\frac{R^3}{Q^3} + \frac{R^4}{Q^2}\right)^{1/2}. \tag{7.6.29}$$

Squaring both sides of (7.6.29), and then simplifying, we arrive at

$$R^6 + Q^6 = 10R^3Q^3 + 9R^2Q^2 + R^4Q^4 - 2R^3Q^3\left(25 + 4RQ + \frac{36}{RQ}\right)^{1/2}. \tag{7.6.30}$$

Dividing both sides of (7.6.30) by R^3Q^3, we find that

$$\left(\frac{R}{Q}\right)^3 + \left(\frac{Q}{R}\right)^3 = 10 + RQ + \frac{9}{RQ} - 2\left(25 + 4RQ + \frac{36}{RQ}\right)^{1/2}. \tag{7.6.31}$$

If we replace q by $-q$ then RQ transforms to PQ and $(R/Q)^3$ transforms to $(P/Q)^3$. Thus (7.6.31) is transformed to (7.6.17), which completes the proof of the theorem.

Theorem 7.6.3 *If*

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \textit{and} \quad Q = \frac{\psi(q^7)}{q^{7/4}\psi(q^{21})},$$

then

$$k_1(PQ)^3 + k_2(PQ) = k_3(PQ)^2 + k_4\left(\frac{P}{Q}\right)^2 - k_5, \tag{7.6.32}$$

where

$$k_1 = \left(\frac{P}{Q}\right)^8 - 1, \quad k_2 = 14P^4\left(\left(\frac{P}{Q}\right)^4 - 1\right), \quad k_3 = P^4(7 - P^4),$$

$$k_4 = 7P^4(P^4 - 3), \quad \textit{and,} \quad k_5 = 27\left(\frac{P}{Q}\right)^4 - 7P^4\left(3 + 3\left(\frac{P}{Q}\right)^4 - P^4\right). \tag{7.6.33}$$

Proof. Proceeding as in Theorem 7.6.1, if we set

$$L_1 := \frac{f(-q)}{q^{1/12}f(-q^3)}, \qquad L_2 := \frac{f(-q^7)}{q^{7/12}f(-q^{21})},$$
$$M_1 := \frac{f(-q^2)}{q^{1/6}f(-q^6)} \quad \text{and,} \quad M_2 := \frac{f(-q^{14})}{q^{7/6}f(-q^{42})}, \tag{7.6.34}$$

so that

$$P = \frac{{M_1}^2}{L_1} \quad \text{and} \quad Q = \frac{{M_2}^2}{L_2}, \tag{7.6.35}$$

we find that

$${M_1}^{12} = \frac{P^8(P^4-9)}{P^4-1}, \tag{7.6.36}$$

and

$${M_2}^{12} = \frac{Q^8(Q^4-9)}{Q^4-1}. \tag{7.6.37}$$

Employing (7.6.34) and (7.6.35) in Theorem 7.6.2, we deduce that

$$\left(\frac{M_1M_2}{PQ}\right)^3 + \left(\frac{PQ}{M_1M_2}\right)^3 = 10 + \frac{(M_1M_2)^3}{PQ} + \frac{9PQ}{(M_1M_2)^3} - 2\left(25 + \frac{4(M_1M_2)^3}{PQ} + \frac{36PQ}{(M_1M_2)^3}\right)^{1/2}. \tag{7.6.38}$$

Simplifying (7.6.38), we find that

$$ax + \frac{b}{x} + 10 = 2\left(25 + \frac{4x}{PQ} + \frac{36PQ}{x}\right)^{1/2}, \tag{7.6.39}$$

where

$$x = (M_1M_2)^3, \quad a = \frac{1}{PQ} - \frac{1}{(PQ)^3}, \quad \text{and} \quad b = 9PQ - (PQ)^3. \tag{7.6.40}$$

Squaring both sides of (7.6.39), and then simplifying, we deduce that

$$a^2k + b^2 + 2abx^2 = x\left(c + dx^2\right), \tag{7.6.41}$$

where

$$k = x^2, \quad c = 144PQ - 20b \quad \text{and} \quad d = \frac{16}{PQ} - 20a. \tag{7.6.42}$$

Squaring both sides of (7.6.41), and then rearranging the terms, we arrive at

$$a^4k^2 + b^4 + 6a^2b^2k - 2cdk = x^2\left(c^2 + d^2k - 4a^3bk - 4ab^3\right). \tag{7.6.43}$$

Squaring both sides of (7.6.43), and then transferring to one side, we find that

$$\left(a^4k^2 + b^4 + 6a^2b^2k - 2cdk\right)^2 - k\left(c^2 + d^2k - 4a^3bk - 4ab^3\right)^2 = 0. \tag{7.6.44}$$

From (7.6.36), (7.6.37), (7.6.40), and (7.6.42), we note that

$$k = \frac{(PQ)^8(P^4-9)(Q^4-9)}{(P^4-1)(Q^4-1)}. \tag{7.6.45}$$

Substituting the expressions for a, b, c, d, and k from (7.6.40), (7.6.42), and (7.6.45) in (7.6.44), and then factoring by **Mathematica**, we deduce that

$$y^{10}(y^4-9)^4A(y,z)B(y,z) = 0, \tag{7.6.46}$$

where $y = PQ$, $z = P/Q$,

$$A(y,z) = -27z^4 + 21y^2z^2 - 21y^2z^4 + 21y^2z^6 - y^3 - 14y^3z^2 + 14y^3z^6 + y^3z^8 + 7y^4z^2 - 7y^4z^4 + 7y^4z^6 - y^6z^4,$$

and

$$B(y,z) = 27z^4 - 21y^2z^2 + 21y^2z^4 - 21y^2z^6 - y^3 - 14y^3z^2 + 14y^3z^6 + y^3z^8 - 7y^4z^2 + 7y^4z^4 - 7y^4z^6 + y^6z^4.$$

It can be seen that the first three factors in (7.6.46) are not identically zero. Thus, we deduce that

$$B(y,z) = 0. \tag{7.6.47}$$

It is now easy to see that (7.6.32) and (7.6.46) are equivalent.

Remark: Since by Entry 24(iii) [15, p. 39]

$$\phi(-q) = \frac{f^2(-q)}{f(-q^2)},$$

proceeding as above, it can be seen that, if

$$P = \frac{\phi(-q)}{\phi(-q^3)} \quad \text{and} \quad Q = \frac{\phi(-q^7)}{\phi(-q^{21})},$$

then (7.6.32) holds. Replacing q by $-q$, we also see that the same identity holds if

$$P = \frac{\phi(q)}{\phi(q^3)} \quad \text{and} \quad Q = \frac{\phi(q^7)}{\phi(q^{21})}.$$

7.7 Relations of $G(q)$ with $G(q^5)$ and $G(q^7)$

probably got idea from Jinhee Yi

In this section we find relations between $G(q)$ and the two continued fractions $G(q^5)$ and $G(q^7)$.

Theorem 7.7.1 *Let for* $|q| < 1$, $v = G(q)$ *and* $w = G(q^5)$. *Then*

$$v^6 - vw + 5vw(v^3 + w^3)(1 - 2vw) + w^6 = v^2w^2(16v^3w^3 - 20v^2w^2 + 20vw - 5). \tag{7.7.1}$$

Proof. From (7.2.3), we note that

$$P^4 = 1 + \frac{1}{v^3} \quad \text{and} \quad Q^4 = 1 + \frac{1}{w^3}, \tag{7.7.2}$$

where

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \text{and} \quad Q = \frac{\psi(q^5)}{q^{5/4}\psi(q^{15})}.$$

From the identity in Theorem 7.6.1, we see that

$$(PQ)^2 + \frac{9}{(PQ)^2} - 5\left(\frac{P}{Q}\right)^2 - 5\left(\frac{Q}{P}\right)^2 = \frac{P}{Q}\left(\left(\frac{Q}{P}\right)^4 + 5\left(\frac{Q}{P}\right)^2 - \left(\frac{P}{Q}\right)^2 - 5\right). \tag{7.7.3}$$

Squaring both sides of (7.7.3), and then simplifying, we deduce that

$$\begin{aligned}(PQ)^4 + \frac{81}{(PQ)^4} + 15\left(\frac{Q}{P}\right)^4 + 15\left(\frac{P}{Q}\right)^4 + 120 - 10Q^4 - 10P^4 - \frac{90}{P^4} - \frac{90}{Q^4}\\ = \left(\frac{P}{Q}\right)^2\left(\left(\frac{Q}{P}\right)^8 + \left(\frac{P}{Q}\right)^4 + 15\left(\frac{Q}{P}\right)^4 + 15\right).\end{aligned} \tag{7.7.4}$$

Squaring both sides of (7.7.4), and then using (7.7.2), we can deduce that

$$G(v,w)H(v,w) = 0, \tag{7.7.5}$$

where

$$G(v,w) = v^6 - vw + 5v^4w + 5v^2w^2 - 10v^5w^2 - 20v^3w^3 + 5vw^4 + 20v^4w^4 - 10v^2w^5 - 16v^5w^5 + w^6,$$

and

$$\begin{aligned}H(v,w) = {} & v^{12} + v^7w - 5v^{10}w + v^2w^2 - 10v^5w^2 + 20v^8w^2 + 10v^{11}w^2 + 5v^3w^3 - 35v^6w^3 + 10v^9w^3 + \\ & 5v^4w^4 - 5v^7w^4 + 80v^{10}w^4 - 10v^2w^5 + 110v^5w^5 + 10v^8w^5 + 16v^{11}w^5 - 35v^3w^6 + 386v^6w^6 + \\ & 280v^9w^6 + vw^7 - 5v^4w^7 + 440v^7w^7 + 320v^{10}w^7 + 20v^2w^8 + 10v^5w^8 + 80v^8w^8 + 10v^3w^9 + \\ & 280v^6w^9 + 320v^9w^9 - 5vw^{10} + 80v^4w^{10} + 320v^7w^{10} + 256v^{10}w^{10} + 1010v^2w^{11} + 16v^5w^{11} + w^{12}.\end{aligned}$$

From the definitions of v and w, we note that $v = O(q^{1/3})$ and $w = O(q^{5/3})$ as q tends to 0. So the first factor in (7.7.5) vanishes for q sufficiently small. Hence by the identity theorem, $G(v,w)$ vanishes for $|q| < 1$. Thus,

$$v^6 - vw + 5v^4w + 5v^2w^2 - 10v^5w^2 - 20v^3w^3 + 5vw^4 + 20v^4w^4 - 10v^2w^5 - 16v^5w^5 + w^6 = 0, \tag{7.7.6}$$

which is equivalent to (7.7.1). Thus we complete the proof.

Theorem 7.7.2 *Let for $|q| < 1$, $v = G(q)$ and $w = G(q^7)$. Then*

$$v^8 - vw - 56v^3w^3(v^2 + w^2) + 7vw(v^3 + w^3)(1 - 8v^3w^3) + 28v^2w^2(v^4 + w^4) = v^4w^4(21 - 64v^3w^3). \tag{7.7.7}$$

Proof. From (7.2.3), we find that

$$P^4 = 1 + \frac{1}{v^3} \quad \text{and} \quad Q^4 = 1 + \frac{1}{w^3}, \tag{7.7.8}$$

where

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \text{and} \quad Q = \frac{\psi(q^7)}{q^{7/4}\psi(q^{21})}.$$

Now, squaring both sides of the identity in Theorem 7.6.3, we find that

$$(k_2^2 + k_1^2(PQ)^4 + 2k_3k_5)(PQ)^2 + 2k_4k_5\left(\frac{P}{Q}\right)^2 = k_6, \tag{7.7.9}$$

where $k_1 - k_5$ are as given in Theorem 7.6.3, and

$$k_6 = k_3^2(PQ)^4 + k_4^2\left(\frac{P}{Q}\right)^4 + k_5^2 + 2k_3k_4P^4 - 2k_1k_2(PQ)^4.$$

Squaring both sides of (7.7.9), and then using (7.7.8), we deduce that

$$(1 + v^3)^3 A(v, w)B(v, w) = 0, \tag{7.7.10}$$

where,

$$A(v, w) = v^8 - vw + 7v^4w + 28v^6w^2 - 56v^5w^3 + 7vw^4 + 21v^4w^4 - 56v^7w^4 - 56v^3w^5 + 28v^2w^6 - 56v^4w^7 - 64v^7w^7 + w^7,$$

and

$$B(v,w) = v^{16} + v^9w - 7v^{12}w + v^2w^2 - 14v^5w^2 + 49v^8w^2 - 28v^{14}w^2 + 28v^7w^3 - 196v^{10}w^3 - 112v^{13}w^3 - 56v^6w^4 + 385v^9w^4 + 763v^{12}w^4 + 56v^{15}w^4 - 14v^2w^5 + 56v^5w^5 + 406v^8w^5 + 840v^{11}w^5 - 56v^4w^6 + 196v^7w^6 + 2604v^{10}w^6 + 1568v^{13}w^6 + 28v^3w^7 + 196v^6w^7 - 1960v^9w^7 - 3080v^{12}w^7 + 64v^{15}w^7 + 49v^2w^8 + 406v^5w^8 - 4920v^8w^8 - 3248v^{11}w^8 + 3136v^{14}w^8 + vw^9 + 385v^4w^9 - 1960v^7w^9 - 1568v^{10}w^9 + 1792v^{13}w^9 - 196v^3w^{10} + 2604v^6w^{10} - 1568v^9w^{10} - 3584v^{12}w^{10} + 840v^5w^{11} - 3248v^8w^{10} + 3584v^{10}w^{11} + 7168v^{14}w^{11} - 7vw^{12} + 763v^4w^{12} - 3080v^7w^{12} - 3584v^{10}w^{12} - 112v^3w^{13} + 1568v^6w^{13} + 1792v^9w^{13} - 28v^2w^{14} + 3136v^8w^{14} + 7168v^{11}w^{14} + 4096v^{14}w^{14} + 56v^4w^{15} + 64v^7w^{15} + w^{16}.$$

From the definitions of v and w, we see that $v = O(q^{1/3})$ and $w = O(q^{7/3})$ as q tends to 0. Hence the second factor of (7.7.10) vanishes for q sufficiently small. By the identity theorem that factor vanishes for $|q| < 1$. Thus we arrive at

$$v^8 - vw + 7v^4w + 28v^6w^2 - 56v^5w^3 + 7vw^4 + 21v^4w^4 - 56v^7w^4 - 56v^3w^5 + 28v^2w^6 - 56v^4w^7 - 64v^7w^7 + w^7 = 0, \tag{7.7.11}$$

which is equivalent to (7.7.7).

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Vita

Nayandeep Deka Baruah was born on February 1, 1972, iSibsagar district of Assam, India. He started his college education at Cotton College, Guwāti, in 1987 and received a degree of Bachelor of Science (B. Sc.) in Mathematics under Gaāti University in 1992. In 1993, he joined Indian Institute of Technology (IIT), Kanpur, foris masters degree and successfully completed with a degree of Master of Science (M. Sc.) egree in Mathematics in 1995. In 1995 he qualified National Eligibility Test (NET) for J|or Research Fellowship (JRF) and Lecturership conducted by UGC-CSIR. In the same yea|e also qualified Graduate Aptitude Test in Engineering (GATE) conducted by Department Education, MHRD, Govt. of India. From March, 1996 to January, 1997 he worked as a leirer of Mathematics at Assam University, Silchar. Since February 1997, he has been wo|ng as a lecturer in the Department of Mathematical Sciences at Tezpur University, AssamHe has published research papers in refereed national as well as international journals. Hi|reas of interest are Number Theory, Elliptic Functions (especially Theta-functions), q-SerieContinued Fractions, and other topics related to Ramanujan's Mathematics.