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Effect of Flow on Plasma Instabilities

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Sub: CERTIFICATE

I certify that the thesis entitled "Effect of Flow on plasma instabilities", is the bona fide record of research work done by Deep Sarmah under my supervision.

To the best of my knowledge no part of this work has been submitted to any other university or institute for any degree.

Sudip Sen 12/12/02

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Abstract

The first chapter is an introduction to this thesis. We have discussed about the high growth of energy demand with the robust increase in world human population and industrialization. We have further discussed that the energy demand may be substantially met if the fusion energy (controlled) is used, prospected due to its cost effectiveness and for some other important factors also. The ultimate aim of the fusion research is to provide an even condition for a burning plasma and for this, certain well known parameter values have to be obtained: the product of density and energy confinement time has to be $3 \times 10^{14} cm^{-3}.s$ and the average plasma temperature has to be 10KeV. The most promising fusion device is a toroidal confinement system 'Tokamak' in which toroidal and poloidal magnetic field is issued for plasma confinement. In the tokamaks the target values of these parameters have not been obtained simultaneously. In experiments, where the temperature was adequate the confinement product inadequate and vice versa. The use of auxiliary heating to increase the plasma temperature in the tokamaks severely degraded the confinement, with no net result of effective approach to break even conditions. In 1982, ASDEX tokamak in Germany, discovered H-mode, where the deterioration of confinement quality at high temperature is avoided. After this discovery, as a result of continuous quest for high confinement, several other high confinement modes viz. VH-mode, ERS-mode, NCS-mode and RI-mode have been discovered. Still the challenge remains to enhance tokamak operation for the development and understanding the basic physics involved in the process that leads to the transition to these improved confinement modes. We have discussed some popular and arguably successful theories which tells that some mechanism(s) creates a (sheared) flow that gives rise to a (sheared) radial electric field ($\mathbf{E}$), (thereby $\mathbf{E} \times \mathbf{B}$ shear) stabilizing various instabilities and as a result fluctuations are suppressed and confinement is improved. A sheared flow is responsible for the transition to *all* improved modes in tokamaks in one way or other. Next we have presented the summary and conclusion of some of the fusion research related problems.

In the second chapter, the effect of a radially varying parallel equilibrium flow on the stability of the Rayleigh-Taylor (RT) mode is studied analytically in the presence of a

sheared magnetic field. It is shown that the parallel flow curvature can completely stabilize the RT mode. The flow curvature also has a robust effect on the radial structure of the mode. Possible implications of these theoretical findings to recent experiments are also discussed.

In the third chapter, the linear and quasilinear behaviour of the ITG drived perturbation with a parallel velocity shear is studied in a sheared slab geometry. Full analytic studies show that when the magnetic shear has the same sign as the second derivative of the parallel velocity with respect to the radial coordinate, the linear mode may become unstable and turbulent momentum transport increases. On the other hand, when the magnetic shear has opposite sign to the second derivative of the parallel velocity, the linear mode is completely stabilized and turbulent momentum transport reduces.

In the fourth chapter the well known instability associated with the ITG modes is shown to be stabilized in the presence of a radially varying parallel flow profile. This is contrary to the usual belief that the parallel flow shear is destabilizing for the ITG-like microinstabilities. The scale length of the flow profile required for this stabilization is rather modest and is usually observed in the toroidal flow profile measured during various improvement modes in the tokamak core. This shows that purely parallel flow can be used to create transport barriers to reduce the loss of particle and energy from the plasma. The improved mode formed by the parallel flow will, unlike the reverse shear mode, be non-transient in nature

List of publications

1. Role of parallel flow on the mitigation of Rayleigh-Taylor instability

   D. Sarmah, S. Sen and R. A. Cairns, Physics of Plasma, **8**, 986,(2001)

2. Effect of flow profile on low frequency drift-type waves in a dusty plasma

   M. Chakraborty, D. Sarmah, S. Sen and B.K. Saikia, Phys. Plasma, **8**, 3150 (2001)

3. On Possible Role of Parallel Flow Profile in the Transition of Enhanced Reversed Shear Modes

   D. Sarmah, S. Sen, A. Fukuyama, D.R. McCarthy and R.G. Storer, Journel of Plasma and Fusion Research, **4**, 258 (2001)

4. Effect of RF Waves on Negative Energy Perturbations in a Straight Stellarator

   S. Sen, A. Fukuyama and D. Sarmah, in Proc. of ICPP Conf. (Quebec, Canada), **II**, 688 (2000)

5. Nonlocal theory of drift type waves in a collissionless dusty plasma.

   D. Sarmah, M. Chakraborty and S. Sen, Phys. Plasmas, **9**, 1829 (2002).

6. Comments on Effect of Flow Profile on Low frequency Drift-type Waves in a Dusty Plasma

   M. Chakrraborty, D. Sarmah and S. Sen, Phys. Plasmas, **9**, 1483 (2002).

7. Mitigation of Low-frequency Oscillations by Parallel Inhomogeneous Flow

   D. Sarmah, S. Sen and A. Fukuyama Plasma Phys. Controlled Fusion, **44**, 351 (2002)

8. On Formation of Transport Barrier by RF Waves

   D. Sarmah and S. Sen (to be submitted to Phys. Rev. Lett.)

9. Effect of Parallel Flow Profile on the Toroidal ITG mode

   D. Sarmah, S. Sen and R. L. Dewar (to be submitted to Phys. Rev. Lett.)

Chapter 1

1.1 Introduction

With the growth of industrialization energy today is in prime demand like never before. In keeping parity with the expected 6 billion world population mark by 2050 and the continued high growth the world energy demand may be as high as 3 times by 2050 as compared to the 1990 level. The demand for energy is mostly met by coal, oil, natural gas and in lesser extent by the nuclear (fissile) fuels and by the renewable energy sources. But the supplies of oil and gas is becoming increasingly limited and costly in the present days and on the top of that, the environmental impact of the increased use of fossil fuel is of high concern. The alternative eco-friendly renewable sources of energy meet only 2 percent of the world energy need. The present day nuclear power plants have reduced the emission of green house gases by almost 15 percent. However, the known Uranium resources will satisfy our energy requirements only for another 50 to 80 years at the current rate of consumption without the help of a breeder reactor. But, because of the accidents in Chernobyl (USSR) and Three Mile Island (USA), the public awareness regarding the safety aspects have made nuclear fission reactors politically counter-productive. In contrast to the conventional fission power plants fusion may provide inherently safe, cost effective and abundant source of alternative energy generator.

The natural abundance of deuterium in hydrogen is one part in 6700. The mass of water in oceans is $1\ 4 \times 10^{21}$kg and mass of deuterium is therefore 4×10^{16}kg. The cost of deuterium is of the order of \$1 per gram and one gram allows the production of 300 GJ (electricity). The cost share of the deuterium fuel is therefore \$0.003 per GJ (electricity). This is negligible in comparison to today's consumer cost of \$30 per GJ.

The situation for Tritium is a little complex. Due to its short half-life it is virtually non-existent in nature. Tritium, however, may be bred from lithium using neutron fusion induced fission reactions.

$$Li^6 + n \rightarrow T + He^3 + 4.8 \tag{1.1}$$

$$Li^7 + n \rightarrow T + He^4 + n - 2.5MeV. \tag{1.2}$$

The natural abundance are 7.4% Li^6 and 92.6% Li^7. The cost of lithium is $ 40 per kg and so, the contribution of the lithium to the fuel cost is less than $ 0.001 per GJ (electricity).

When a nucleus of deuterium is fused with a nucleus of tritium, an α-particle is produced and a neutron is released. The nuclear rearrangement results in a reduction of total mass and consequently, in the Einsteinian way, energy is released in the form of kinetic energy as a reaction product.

$$_1D^2 +_1 T^3 \rightarrow_2 He^4(3.5MeV) +_0 n^1(14.1MeV) \tag{1.3}$$

In order that the nuclei of deuterium and tritium can fuse it is necessary to overcome the mutual repulsion due to their positive charges and as a result, the cross-section for fusion is small at low energies. However cross-section increases with energy, reaching a maximum at 100 keV, and a positive energy balance is possible if the fuel particle can be made to react before they lose their energy. To achieve this the particle must retain their energy and remain in the reacting region for a sufficient time. More precisely the product of this time and the density of the reacting particles must be sufficiently large.

In toroidal plasma confinement system tokamak, the plasma is confined by a magnetic field. The principal magnetic field is a toroidal field. However, this field alone does not allow confinement of the plasma. In order to have equilibrium in which the plasma pressure is balanced by the magnetic forces it is necessary to have also a poloidal magnetic field which in a tokamak is produced by the toroidal current in the plasma.

Impurities in plasma give rise to radiation losses and dilute the fuel. The restriction of their entry in to the plasma therefore plays a fundamental role in the successful operation of tokamaks. This requires a separation of the plasma from the vacuum vessel and this is done either by using a material limiter or by using magnetic field to produce a magnetic divertor.

The plasma pressure is the product of the particle density and temperature. The fact that the reactivity of the plasma increases with both of these quantities implies that in a reactor the pressure must be sufficiently high. The pressure which can be confined is determined by stability considerations and increases with the strength of the magnetic field. However, the magnitude of the toroidal field is limited by technological factor and cost.

The early tokamaks had energy confinement times of several milliseconds and ion temperatures of a few hundred of eV. The obvious need for the 1970s were to find out whether these conditions could be improved. But it soon became apparent that the energy confinement was anomalous. Large tokamaks were built to improve the confinement and the confinement time approaching 100ms had been obtained during 1980s.

The ultimate aim of the fusion plasma research is to provide the conditions for a burning plasma. The parameter values which have to achieve are well known: the product of density and energy confinement time has to be 3×10^{14} cm^{-3}-s and the average plasma temperature has to be 10KeV. The target values of these parameters have been obtained separately; however, in experiments where the temperature was adequate the confinement product inadequate, and vice versa. Simultaneous achievement of the two values was found to be difficult. The use of auxiliary heating to increase the plasma temperature in a tokamak gave rise to severe degradation of the confinement, with the net result of no effective approach to break-even conditions. The operational regime of tokamak plasma characterized by a confinement time that decreases strongly with increasing auxiliary heating power in termed the L- (low) mode. Recently, a new confinement regime has been discovered in the ASDEX tokamak in Germany [1] where the deterioration of the confinement quality at high heating power is avoided. This regime is called the H- (high) mode because of its high confinement characteristics. There is evidence that the L-mode characteristics will not lead to sufficient confinement for successful plasma burning. As a result there has been a continuing quest for other improved modes in tokamak in this decade, leading to the discovery of the VH- (very high) modes in the DIII-D [2], ERS- (enhanced reverse shear) modes in the TFTR [3], NCS- (negative central shear) modes in the DIII-D [4], and the RI- (radiation improved) modes in the TEXTOR-94 [5] and the IH- (improved high) modes in the DIII-D [6].

An important challenge for enhanced tokamak operation is the development and understanding of the basic physics involved in the process that leads to the transition to these improved confinement modes. A popular and arguably successful picture of these improved modes is that some mechanism(s) creates a (sheared) flow which gives rise to a (sheared) radial electric field (**E**), thereby (by the **E**$\times$ **B** shear) stabilizes various instabilities and as a result fluctuations are suppressed and confinement is improved. It is usually believed while a sheared poloidal flow is responsible for the H-mode transition, a sheared toroidal flow is the cause for the VH-mode transition. A hollow q profile, on the other hand, is necessary for the ERS or NCS modes. However, it is now widely accepted that the negative magnetic shear is not the only factor needed for the transport reduction in the ERS or NCS modes. Some of the clearest evidence comes from the comparison of the RS (reverse shear) and ERS (enhanced reverse shear) transition data in TFTR [7]. It shows that the RS phase is not necessarily an ERS phase and some other factor is needed to explain the transition. Theoretical study also indicates that the reversal of magnetic shear alone might have a little effect on the ITG-type microinstabilities [8]. Most recently, the $\mathbf{E} \times \mathbf{B}$ shear stabilization mechanism has been proposed to explain the core transport barriers formed in plasmas with negative or reverse magnetic shear regimes [9]. It is believed that the change in the radial electric field in the core is produced by a number of ways, for example, by toroidal flow ($v_{\phi i}$) [10] and/or by pressure gradient (∇p_i) [7] and more recently by poloidal flow ($v_{\theta i}$) [11]. Similarly recent RI mode experiments in the TEXTOR indicate that the shear in the toroidal flow also plays a crucial role in the transition to the RI mode [12]. The $\mathbf{E} \times \mathbf{B}$ velocity shear is also believed to be playing the key role in the IH-mode transition in the DIII-D [6]. However, while a sheared flow is responsible for the transition to *all* the improved modes in tokamaks in one way or other, there are many unsolved problems in the fusion research related to the flow shear stabilization of various plasma instabilities.

In this thesis, (i) we study the effect of a radially varying parallel equilibrium flow on the stability of the Rayleigh-Taylor (RT) mode analytically in the presence of a sheared magnetic field. It is shown that the parallel flow curvature can completely stabilize the RT mode. The flow curvature also has a robust effect on the radial structure of the mode. Possible implications of these theoretical findings to recent experiments are also discussed [13]. (ii) We then study the linear and quasilinear behaviour of the ITG driven perturbation with a parallel velocity shear in a sheared slab geometry. Full analytic studies show that when the magnetic shear has the same sign as the second derivative of the parallel velocity with respect to the radial coordinate, the linear mode may become unstable and turbulent momentum transport increases. On the other hand, when the magnetic shear has opposite sign to the second derivative of the parallel velocity, the linear mode is completely stabilized and turbulent momentum transport reduces [14-15]. (iii) The well known instability associated with the ITG modes is shown to be stabilized in the presence of a radially varying parallel flow profile. This is contrary to the usual belief that the parallel flow shear is destabilizing for the ITG-like microinstabilities. The scale length of the flow profile required for this stabilization is rather modest and is usually observed in the toroidal flow profile measured during various improvement modes in the tokamak core. This shows that purely parallel flow can be used to create transport barriers to reduce the loss of particle and energy from the plasma. The improved mode formed by the parallel flow will, unlike the reverse shear mode, be non-transient in nature [16].

In summary, we study the stability of RT, ITG and drift type instabilities in the presence of a radially varying equilibrium flow and report some interesting novel results [13-16].

Chapter 2

Effect of flow on Rayleigh-Taylor instability

2.1 Introduction

Heavy fluid supported by a lighter fluid in the presence of the gravitational field gives rise to Raleigh-Taylor (RT) instability [19]. This instability commonly occurs both in collisionless and collisional domain in the magnetized plasmas, where the role of the light fluid is played by the magnetic field. The collisionless interchange type instability (ballooning mode) can exist in the earth's plasma sphere as well as in the laboratory plasma. These collisionless mode arises due to an unfavorable curvature in the magnetic field (simulating an effective gravity) in the presence of a pressure gradient. It is believed that the RT instability can play a major role in the onset of equatorial spread F [18]. This instability is also known to be a major problem for a wide range of applications, from pulsed power technology to inertial confinement fusion. Conventional Z-pinch implosions are inherently unstable due to the RT instability which appears during run-in phase due to a steep density gradient across an accelerating interface between plasma and the magnetic field. A number of ways, like using of gas puff loads with thick shells, uniform fills [19-20], puff on puff [21-23], shear stabilization [24-25] and tailored density profiles [27-27], have been proposed to control or mitigate some of the most dangerous modes associated with RT type instabilities in the Z-pinch. Some of these methods have also been successfully implemented on several facilities in the past but only with a limited success.

Recently it has been proposed that the flow curvature (the second radial derivative) in the *perpendicular* flow might suppress microinstabilities and fluctuations [28-29]. It is shown that the flow curvature in the perpendicular flow indeed has robust effects both on the linear [30-31] and nonlinear mode stability [32] of drift-type waves. Studies on the resistive pressure gradient driven fluctuations [33] also show a similar result. Numerical simulation results of the ion temperature gradient (ITG) modes also seem to have reached a similar conclusion [34]. However, no such investigation on the effect of the flow shear/curvature in the *parallel* flow has ever been carried out for the RT type instability in the presence of magnetic shear. This is specially very important because in the recent staged Z-pinch experiments at the University of California at Riverside, an axial magnetic field is applied to shear stabilize the target plasma during implosions. The outer plasma shell is made of high-Z gas that is highly radiative and keeps the plasma cold during implosion. This allows both components of magnetic field diffuse easily through this plasma column, B_z diffuses outward and B_θ diffuses inward during the implosion phase. This creates a sheared profile of magnetic field that maintains the stability of the outer shell during the implosion [35]. This is also very important because the recent gas puff experiments on the 7-MA Saturn generator have identified a curved surface at the plasma-vacuum interface and observed at the same time an improvement in the implosion qüality [36]. In a subsequent simulation work [37], Douglas et. al. have shown a curved surface introduces an axial flow (note because $B_z > B_\theta$, the parallel flow is almost equal to the axial flow) which might result in the reduction in the RT growth. However, the exact cause of the reduction in the RT growth is due to the result of the velocity shear/curvature or because of the material advection along the interface is not known.

We develop a nonlocal theory of the RT mode in the presence of a radially varying parallel flow. Our full analytic analysis shows that the parallel flow curvature can stabilize the RT mode. The flow curvature also has a robust effect on the radial structure of the mode. The parallel flow shear, on the other hand, plays an insignificant role in this matter. These theoretical findings, apart from showing the parallel flow curvature as a major candidate to stabilize the RT mode, might also have an important role in determining the stability of staged Z-pinch and also in elucidating the recent gas puff experiments on the 7-MA Saturn generator which have identified a curved surface while observing an improvement in the implosion quality.

2.2 Stability Analysis

Following our earlier work [40], we carry out a nonlocal stability analysis and assume, for simplicity, a Cartesian coordinate system and a two-fluid model for the Rayleigh-Taylor modes in the presence of a radially varying parallel flow. We assume a low-β collisionless plasma (hence neglecting any electromagnetic fluctuations) with both equilibrium density variation and magnetic shear in x, i.e., $n_o(x) = n_{oo}exp(-x^2/2L_n^2)$, $\mathbf{B}(x) = B[\mathbf{e}_z + (x/L_s)\mathbf{e}_y]$ (it is important to note here that, in the staged Z-pinch equilibrium at the University of California at Riverside, the ratio $\frac{B_\theta}{B_z}$ is of the order of 0.6). Here, L_s is the magnetic shear scale length and x is the distance from the mode rational surface defined by $\mathbf{k}\ \mathbf{B} = 0$. We assume a uniform gravity force mg in the x direction for ions. We model the equilibrium parallel flow by a profile $V_{\|0}(x) = V_{\|00} + V_{\|00}x/L_{v1} + V_{\|00}x^2/2L_{v2}$, where the shear and the curvature contributions to the velocity profile are represented by the second and the third term in the Taylor series respectively; and $L_{v1} = (1/V_{\|oo}dV_{\|o}(x)/dx)^{-1}$; $L_{v2} = (1/V_{\|oo}d^2V_{\|o}(x)/dx^2)^{-1}$. Because of inhomogeneity in the x-direction, perturbations have the form $\phi(\mathrm{x}, t) = \phi(x)\exp\imath(k_y y + k_z z - \omega t)$. Ion inertia effects are retained to include the ion polarization drift, but electron inertia and ion and electron pressure are neglected for simplicity. We can write the linearized basic equation of continuity and momentum transfer for ions and electrons as follows:

$$\frac{\partial n_\alpha}{\partial t} + \nabla.(n_\alpha V_\alpha) = 0 \tag{2.1}$$

$$E = -\frac{1}{c}(V_e \times B) \tag{2.2}$$

$$m_\imath(\frac{\partial}{\partial t} + V_\imath.\nabla)V_\imath = eE + \frac{eV_\imath \times B}{c} + m_\imath g \tag{2.3}$$

Here

$$V_e = V_{\parallel 0}(x) + V_E$$

$$V_i = V_{\parallel 0}(x) + V_E + V_p + V_g$$

$$V_E = -c(\nabla_\perp \phi \times \mathbf{B})/B^2$$

$$V_p = (\frac{ic\omega}{B\omega_{ci}})\nabla_\perp \phi$$

$$V_g = -\frac{cm_i g}{eB}\mathbf{e}_y$$

$$\omega_{ci} = \frac{eB}{cm_i}$$

$$\nabla_\perp = ik_y \mathbf{e}_y + \mathbf{e}_x \frac{d}{dx}$$

Here α denotes species (e for electrons and i for ions), all other symbols are assumed to have the usual meaning unless otherwise stated explicitly. Parallel flow, $V_{\parallel 0}$, has therefore two effects. First, it introduces a Doppler shift, $k_\parallel V_{\parallel 0}(x)$, in all time derivatives and second, an extra term, $\mathbf{V}_E.\nabla V_{\parallel 0}(x)$, representing radial convection of ion momentum. It is the second term which makes the effect of parallel flow shear completely different from that of the perpendicular flow shear. We eliminate the Doppler shift by performing a Galilean transformation in the $\mathbf{e}_\parallel$ direction. It is important to mention here that the spatial variation in the Doppler shift in the mode frequency due to *parallel* flow is negligible for flute-type modes ($k_\parallel << k_\perp$). It is probably obvious as $d^n/dx^n(k_\parallel V_o(x)) << d^n/dx^n(k_\perp V_o(x))$ due to the fact that $k_\parallel << k_\perp$. So, one can eliminate the Doppler shift performing a Galilean transformation in the $\mathbf{e}_\parallel$ direction. This has also been noted elsewhere [39]. Now, using quasineutrality and the usual low frequency assumptions, we obtain radial eigenvalue equation:

$$\frac{d^2\phi}{dx^2} + p(x)\frac{d\phi}{dx} + q(x)\phi = 0 \tag{2.4}$$

where

$p(x) = \frac{n_0'}{n_0}$

$q(x) = -k_y^2 + \frac{k_y^2 g}{\omega^2}\frac{n_0'}{n_0} + \frac{k_y^2 k_{\parallel} g}{\omega^3}\frac{dV_{\parallel 00}}{dx}$

In deriving equation (2.4) we have retained terms up to the first order in $k_{\parallel}$ which is justified for the flute type mode ($k_{\parallel} \sim 0$). Now to remove the first derivative (with respect to x) term we make use of the transformation

$$\phi(x) = \psi(x)\exp(-\int^x \frac{p(\eta)}{2} d\eta)$$

when we get

$$\frac{d^2\psi}{dx^2} + Q(x)\psi = 0$$

where

$Q(x) = q(x) - \frac{p'(x)}{2} - \frac{p^2(x)}{4}$.

With the velocity and density profile described earlier, the radial eigenvalue equation now reduces to:

$$\frac{d^2\psi}{dx^2} + [T + Sx + Rx^2]\psi = 0 \qquad (2.5)$$

where

$T = \frac{1}{2L_n^2} - k_y^2$

$S = \frac{L_n k_y^3}{L_s L_{v1}} - \frac{k_y^2 g}{\omega^2 L_n^2}$

$R = \frac{L_n k_y^3}{L_s L_{v2}} - \frac{1}{4L_n^4}$

In deriving equation equation (2.5), we have assumed that $\omega \sim \sqrt{g/L_n} \sim V_{\|oo}$, which is usually true. Equation (2.5) is a simple Weber equation. Depending on the sign of R, we have two types of solutions. If $R < 0$, the solution satisfying the physical boundary condition, i.e., $\psi \rightarrow 0$ at $x = \pm\infty$ is given by

$$\psi(x) = \psi_o exp[-\sqrt{|R|}(x - x_o)^2]$$

where $x_o = S/2|R|$. So, in this case, the mode decays with x, i.e., it does not propagate and hence is intrinsically undamped. This solution therefore implies the existence of an unstable mode.

On the other hand, if $R > 0$, equation (2.5) has the solution

$$\psi(x) = \psi_o exp[-\imath\sqrt{|R|}(x + x_o)^2]$$

Thus, in this case we have a non-localized mode with outgoing energy flux at $x = \pm\infty$. Because of the convective wave energy leakage the perturbation will decay in time in the absence of any energy source feeding the wave. The wave is therefore damped. So, positive parallel flow curvature has a stabilizing role on the RT instability. Parallel flow shear, on the other hand, has an insignificant role in this matter. It can shift the potential, but does not alter the quadratic structure. It also shifts the center of the mode away from the $x = 0$ rational surface. The main stabilizing effect comes from the quadratic term which forms an anti-well pushing the wave function away from $x = 0$. These observations are the same as in the case of drift waves [39]. The corresponding dispersion relation is given by $T - \frac{S^2}{4R} = \imath\sqrt{R}$, which can be simplified for a weakly unstable situation ($\omega_i < \omega_r$)

and without the velocity shear contribution as:

$$\left(\frac{k_y^4 g^2}{L_n^4}\right) \frac{-(\omega_r - \omega_i)^2 + 4\omega_r^2\omega_i^2 + 4i\omega_r\omega_i|(\omega_r^2 - \omega_i^2)|}{|[4\omega_r\omega_i(\omega_r^2 - \omega_i^2)]^2 - [(\omega_r - \omega_i)^2 - 4\omega_r^2\omega_i^2]^2|} = 4R(T - i\sqrt{R})$$

Here ω_r and ω_i are the real and the imaginary parts of the eigenfrequency, respectively. It is easy to see that without the velocity field the linear growth rate of the RT instability assumes the familiar form $\sqrt{g/L_n}$. From the dispersion relation it is clear that the parallel flow curvature can stabilize the RT instability if $\frac{L_n k_y^3}{L_s L_{v2}} > \frac{1}{4L_n^4}$. If we assume $L_{v2} \sim L_n$, $k_y\rho_i \sim 0.1$ (where ρ_i is the ion-gyroradius) and $\frac{L_n}{\rho_i} \sim 100$, then the condition of stability can be further simplified to $\frac{L_s}{L_n} < 400$. We emphasize here that, these assumptions, although are usually true, are made only to facilitate comparison with the experimental data and no generality whatsoever is lost thereby. We have therefore obtained a condition of stability for the RT mode in the presence of axial flow curvature, which is likely to be satisfied in any staged Z-pinch type device having a very weak magnetic shear.

2.3 Conclusion

In summary, we have studied the effect of a radially varying parallel equilibrium flow on the stability of the Rayleigh-Taylor mode analytically. It is shown that the parallel flow curvature can completely stabilize the RT mode. The flow curvature also has a robust effect on the radial structure of the mode. Our work therefore shows that by suitably tailoring the parallel flow profile it is possible to completely stabilize the RT instability in a magnetized plasma. A weak magnetic shear is needed for the process, and hence a staged Z-pinch will be an ideal device for realizing this. This result might also have an important part to play in explaining the recent gas puff experiments on the 7-MA Saturn generator which have identified a curved surface while observing an improvement in the implosion quality [36]. This is because the subsequent simulation work of Douglas et al. [37] has shown that a curved surface introduces an axial flow and our work here shows that the flow curvature in such an axial flow can indeed stabilize the RT growth.

However, a definitive comparison is only possible when one gets additional data from this experiment.

Before we conclude, it is interesting to compare our results with a recent work of Shumlak and Roderick [40]. The authors in reference [40] find that the shear in the axial flow can mitigate the RT instability, whereas the result in this work, including both the flow shear and curvature in the parallel flow and also the magnetic shear, indicates that it is the flow *curvature* which plays the leading role in suppressing the RT instability. Furthermore, the result in reference [40] indicates that for the mitigation of RT instability, a *sufficiently strong* sheared flow is necessary, in this work we, however, find that the value of the flow curvature necessary for the mitigation of the RT instability is rather modest in the presence of even a very weak magnetic shear. So, possibly the most realistic way of stabilizing the RT instability is to use an axial flow with a properly tailored flow profile in conjunction with a sheared magnetic field. This possibility is already being explored on the Z-Accelerator at Sandia National Laboratories [41], where the magnetic shear comes from the axial magnetic field produced by using a twisted tungsten wire array and the radially varying axial velocity, presumably, from the hourglass shape of the array associated with its twisting. The initial result indeed shows a stabilizing effect.

Chapter 3

Effect of flow on slab ITG instability

3.1 Introduction

Arguably the most remarkable story of fusion research over the past decade is the discovery of the enhanced reverse shear modes (ERS modes) in Tokamak Fusion Test Reactor (TFTR) [3] and the negative central magnetic shear modes (NCS modes) in DIII-D [4]. It is not often that a system self-organizes to a higher energy state with such a large reduction of turbulence and transport when an additional source of free energy is applied to it [9]. It is usually believed that the ERS or NCS configurations can provide the characteristics desirable for a fusion reactor [42].

The understanding of the formation of transport barriers in the ERS or NCS plasma configurations is therefore fundamental to the development of techniques to control such barriers for tailoring profiles and for improving operating regimes further. This is especially significant because it is now widely accepted that the negative magnetic shear is not the only factor needed for the transport reduction in the ERS or NCS modes. Some of the clearest evidence comes from the comparison of the RS (reverse shear) and ERS (enhanced reverse shear) transition data in TFTR [7]. It shows that the RS phase is not necessarily an ERS phase and some other factor is needed to explain the transition. Theoretical study also indicates that the reversal of magnetic shear alone might have a little effect on the ITG-type microinstabilities [8].

Most recently, the $\mathbf{E} \times \mathbf{B}$ shear stabilization mechanism has been proposed to explain the core transport barriers formed in plasmas with negative or reverse magnetic shear regimes [9]. It is believed that the change in the radial electric field in the core is produced by a number of ways, for example, by toroidal flow ($v_{\phi i}$) [10] and/or by pressure gradient

(∇p_i) [7] and more recently by poloidal flow ($v_{\theta i}$) [11]. However, while this $\mathbf{E} \times \mathbf{B}$ shear stabilization mechanism *alone* can satisfactorily explain the confinement improvement in the edge, it may not be an obvious explanation for the core confinement improvement in the ERS and NCS plasma. For example, the formation of the ERS mode in TFTR has been reported [7] at values of $\gamma_{E\times B}$ ($\mathbf{E} \times \mathbf{B}$ shearing rate), as much as a factor of 3 below $\gamma_{\max}$ (the maximum linear growth rate), while for the suppression of turbulence-induced transport the condition $\gamma_{E\times B} \geq \gamma_{\max}$ needs to be satisfied. It is therefore natural that an explanation of these experimental results should be sought in the effects such optimized magnetic configurations have on microinstabilities and on the consequent transport.

In this work, we identify another effect which might be playing a key role in the reverse shear transition. We show here when the magnetic shear has the same sign as the second derivative of the parallel flow with respect to the radial coordinate, the ion temperature gradient (ITG) mode may become unstable. On the other hand, when the magnetic shear has opposite sign to the second derivative of the parallel velocity, the ITG mode is completely stabilized. This result, therefore, shows that it is the relative sign of the second radial derivative of the equilibrium parallel flow with respect to the magnetic shear which may be the key factor for the enhanced reverse shear transition.

3.2 Stability Analysis

We choose a simple model of the ITG-driven modes [44]. We adopt a two-fluid theory in a sheared slab geometry, $\mathbf{B} = B_o[\mathbf{z} + (x/L_s)\mathbf{y}]$, where L_s is the scale length of magnetic shear. The x, y and z directions in the sheared slab geometry are defined as the radial, poloidal and toroidal directions in the tokamak configuration. We assume a background plasma with all inhomogeneities only in the radial direction, where perturbations have the form $\phi(\mathbf{x}, t) = \phi(x)\exp[i(k_y y + k_z z - \omega t)]$. We ignore finite gyroradius effects by limiting consideration to the wavelength domain $k_\perp \rho_i \ll 1$, where ρ_i is the ion gyroradius. We then write down the equations of continuity for ions:

$$\frac{\partial \tilde{n}_i}{\partial t} + \nabla.(n_0 + \tilde{n}_i)(v_0 + \tilde{v}_\perp + \tilde{v}_\parallel) = 0 \tag{3.1}$$

where,

$\hat{b} = \vec{B}/|B|,\ \ P_i = P_{i0}(x) + \tilde{p}_i,\ \ n_i = n_0 + \tilde{n}_i$

$v_0 = v_{\parallel 0}(x)\hat{e}_\parallel$

$\tilde{v}_\perp = v_E + v_{Di} + v_P$

$v_E = \frac{c}{B}\hat{b} \times \nabla_\perp \phi$

$v_{Di} = \frac{c}{eBn_i}\hat{b} \times \nabla_\perp P_i,$

$v_P = -\frac{c^2 m_i}{eB^2}\left(\frac{\partial}{\partial t} + (v_0 + v_E + v_{Di}).\nabla\right)\nabla_\perp \phi$

Assuming electron dynamics to be adiabatic equation (3.1) can be reduced to

$$\left(\frac{\partial}{\partial t} + v_0.\nabla\right)(1-\nabla_\perp^2)\tilde{\phi} + v_D\left[1 + \left(\frac{1+\eta_i}{\tau}\right)\nabla_\perp^2\right]\nabla_y\tilde{\phi} - \hat{b}\times\nabla\tilde{\phi}\ \nabla(\nabla_\perp^2\tilde{\phi}) + \nabla_\parallel\tilde{v}_\parallel = 0 \tag{3.2}$$

Here, in deriving equation (3.2) we have re-scaled the quantities as

$\tilde{\phi} \equiv e\phi/T_e,\ \tilde{v}_\parallel \equiv \tilde{v}_{\parallel i}/c_s,\ \tilde{p} \equiv [\tilde{p}_i/P_{i0}](T_i/T_e),\ \tau \equiv \frac{T_e}{T_i},\ \mathcal{Y} \equiv \frac{\Gamma}{\tau},\ \mu \equiv \frac{\mu_\parallel \omega_{ci}}{c_s^2}$. Here, Γ is the ration of specific heats, and $\mu_\parallel$ is the parallel viscosity (due to either Landau damping or collisional viscosity) required for saturation of the turbulence.

Similarly, the parallel momentum equation for ions can be written as:

$$\frac{\partial \tilde{v}_{\parallel i}}{\partial t} + (v_E + v_{\parallel 0}(x)).\nabla(v_{\parallel 0}(x) + \tilde{v}_{\parallel i}) = -\frac{e}{m_i}\nabla_{\parallel}\phi - \frac{1}{m_i n_i}\nabla_{\parallel}P_i + \mu_{\parallel}\nabla_{\parallel}^2(v_0 + \tilde{v}_{\parallel i}) \quad (3.3)$$

and the re-scaled equation as:

$$\left(\frac{\partial}{\partial t} + v_0.\nabla\right)\tilde{v}_{\parallel} - \frac{v_0}{L_v}\nabla_y\tilde{\phi} + \hat{b}\times\nabla\tilde{\phi}.\nabla\tilde{v}_{\parallel} = -\nabla_{\parallel}\tilde{\phi} - \nabla_{\parallel}\tilde{p} + \mu\nabla_{\parallel}^2\tilde{v}_{\parallel} \quad (3.4)$$

Finally, the equation of adiabatic pressure evolution is written as:

$$\frac{\partial \tilde{p}_i}{\partial t} + v_E\ \nabla P_{i0} + v_E.\nabla\tilde{p}_i + v_0\ \nabla\tilde{p}_i + \Gamma P_{i0}\nabla_{\parallel}\tilde{v}_{\parallel i} = 0 \quad (3.5)$$

and the re-scaled version as:

$$\left(\frac{\partial}{\partial t} + v_0.\nabla\right)\tilde{p} + T_e v_D\left(\frac{1+\eta_i}{\tau}\right)\nabla_y\tilde{\phi} + \hat{b}\times\nabla_{\perp}\tilde{\phi}.\nabla\tilde{p} + \mathcal{Y}\nabla_{\parallel}\tilde{v}_{\parallel} = 0 \quad (3.6)$$

Here, all symbols are assumed to have the usual meaning unless otherwise stated explicitly. We make no attempt to speculate about the source of the parallel flow although a strongly peaked ion velocity parallel to the magnetic field is observed to coexist in tokamaks in the region where the plasma confinement is improved [45]. Parallel flow introduces a Doppler shift, $k_{\parallel}v_{\parallel o}(x)$, in all time derivatives and second, an extra term, $\mathbf{v}_E.\nabla v_{\parallel o}(x)$, representing radial convection of ion momentum. The second term makes the effect of parallel flow shear completely different from that of the perpendicular flow shear [43].

To model the equilibrium parallel velocity we assume a simple general case of variation with the radial distance x.

$v_0(x) = v_{00} + \frac{v_{00}}{L_{v1}} x + \frac{v_{00}}{2L_{v2}} x^2$

Linearizing equations (3.2,3.4,3.6), and neglecting $\mathcal{Y}$ (which gives corrections of order $(k_{\|}/k_{\perp})^4$), we write down the eigenvalue equation as

$$\frac{\partial^2 \tilde{\phi}}{\partial x^2} + \left[\Lambda - Qx + Px^2\right] = 0 \quad (3.7)$$

where,

$\Lambda = -k_y^2 + \frac{1-\Omega}{\Omega+K}$, $P = -\frac{J_2 S}{\Omega(\Omega+K)} + \frac{S^2}{\Omega^2}$, $Q = \frac{J_1}{\Omega(\Omega+K)} S$, $\Omega = \frac{\tilde{\omega}}{k_y v_D}$, $K = \frac{1+\eta_i}{\tau}$, $S = \frac{L_n}{L_s}$, $J_2 = \left(\frac{v_{00} L_n}{L_{v2}}\right)$, $J_1 = \left(\frac{v_{00} L_n}{L_{v1}}\right)$.

Equation (3.7) is a simple Weber equation. Depending on the sign of P, we have two types of solution. If $P < 0$, i.e.,

$$\frac{J_2 S}{\Omega(\Omega + K)} > \frac{S^2}{\Omega^2}$$

the solution which satisfies the physical boundary condition, i.e., $\phi \longrightarrow 0$ at $x = \pm\infty$ is given by

$$\phi(x) = \phi_o \exp[-1/2\sqrt{\mid P \mid}(x - x_o)^2] \quad (3.8)$$

where $x_o = |Q|/2|P|$. The wave therefore does not propagate and is intrinsically undamped.

On the other hand, if $P > 0$, Equation (3.7) has the solution

$$\phi(x) = \phi_o \exp[-i/2\sqrt{\mid P \mid}(x + x_o)^2] \tag{3.9}$$

So, in this case we have now a non-localized wave. The wave is therefore damped as in this case because of the convective wave energy leakage the perturbation will decay in time in the absence of any energy source feeding the wave.

From the above discussion it is clear that it is the parallel flow curvature which actually plays the key role in the stability of the mode. When the magnetic shear has the same sign as the parallel flow curvature, i.e., for positive magnetic shear ($L_s > 0$), parallel flow curvature acts to destabilize the mode. On the other hand, for the negative magnetic shear configuration ($L_s < 0$), i.e., when the magnetic shear has the opposite sign to the second derivative of the parallel flow with respect to the radial coordinate x, the parallel flow curvature acts to stabilize the mode. Flow curvature now forms an additional antiwell which pushes the wave function away from the mode rational surface, thereby enhancing stabilization.

The overall stability of the mode may also be obtained from the dispersion relation

$$\Lambda = \frac{Q^2}{4P} + i\sqrt{|P|}$$

which can be written more explicitly as:

$$(1+k_y^2)\Omega^2 + (K\kappa_y^2 - 1)\Omega = -\frac{J_1^2\Omega}{4(\Omega + K)\left(1 - \frac{J_2\Omega}{S(\Omega+K)}\right)} - iS(\Omega+K)\sqrt{\left|1 - \frac{J_2\Omega}{S(\Omega + K)}\right|} \tag{3.10}$$

We will now solve the eigenvalue equation by using a numerical code developed by Bai et al. [46]. For numerical solution we keep the contribution of $\mathcal{Y}$ when the eigenvalue equation reduces to

$$\frac{\partial^2 \tilde{\phi}}{\partial x^2} + \left[\Lambda - Qx + P_m x^2\right] = 0 \tag{3.11}$$

where,

$P_m = -\frac{J_2 S}{\Omega(\Omega+K)} + \frac{S^2}{\Omega^2(1-\frac{\Gamma}{\tau}\frac{s^2 x^2}{\Omega^2})}$,

It is clear that because of the x dependence of P_m, eq. (3.11) is not solvable analytically.

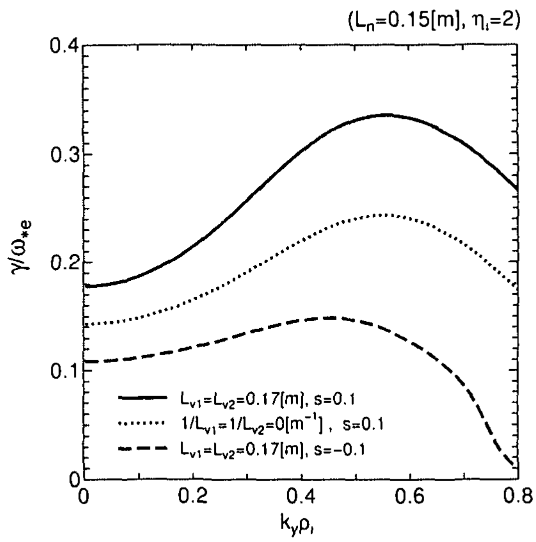

Figure 1: *Normalized growth rate of ITG mode with $k_y\rho_i$ for negative and positive magnetic shear.*

Fig. 1 shows that for the case of positive magnetic shear the ITG mode is more destabilized than the case when there is no parallel flow. On the other hand, for the negative magnetic shear case the growth rate of the ITG mode decreases in the presence of parallel flow curvature and the mode can be fully stabilized for $k_y\rho_i > 0.8$. Fig. 2 shows the plot of real frequencies in these three cases. Fig. 3 shows the variation of the ITG growth rate with the scale length of parallel flow curvature for the positive and negative magnetic shear cases. It reconfirms while for the positive shear case the growth rate of the ITG can

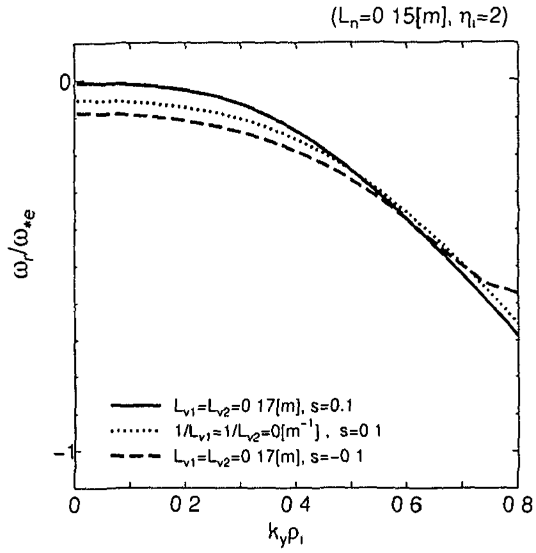

Figure 2

Figure 2: *Real part of the frequency of ITG mode with $k_y\rho_i$ for negative and positive magnetic shear.*

be very high, in the case of negative shear the ITG mode can be completely stabilized by decreasing the scale length of the parallel flow curvature.

3.3 Conclusion

In summary, we present here a model for transition to the enhanced reverse shear (ERS) or negative central shear (NCS) mode triggered in tokamaks. Our studies show that when the magnetic shear has the same sign as the second derivative of the parallel velocity with respect to the radial coordinate, the ITG mode may become more unstable. On the other hand, when the magnetic shear has the opposite sign to the second derivative of the parallel velocity, the ITG mode may be completely stabilized. This result is similar to what we have found earlier for the PVS instabilities [61,47]. So, the similar result with the ITG mode, considered nowadays as the dominant source of anomalous energy losses in the low confinement (L) mode, therefore, shows on a firmer footing that it is the relative sign of the second radial derivative of the equilibrium parallel flow with respect to the magnetic shear which may be the key factor for the enhanced reverse shear transition.

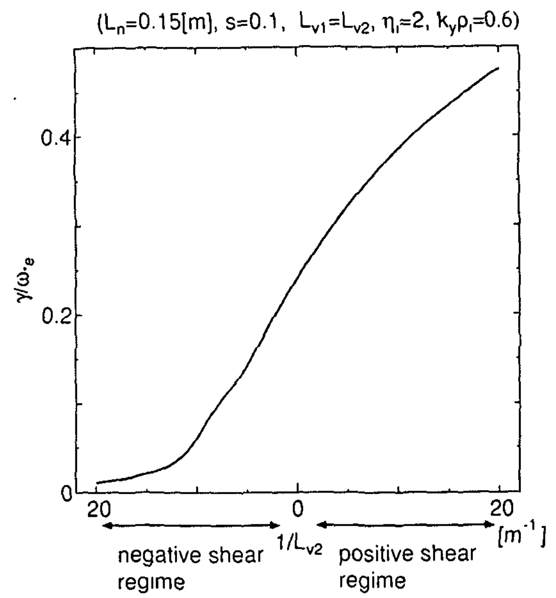

Figure 3: *Normalized growth rate of ITG mode with L_{v2} for negative and positive magnetic shear.*

Now, the real benefit of the outcome of this work is that it puts forward a novel idea of transport barrier formation. This is that, contrary to the usual notion that a parallel flow shear is always destabilizing, the destabilizing influence of the shear in the parallel flow can be changed altogether if one takes the effect of the flow curvature into account. The transverse curvature in the parallel flow can overcome the destabilizing influence of the shear and can render the low frequency modes stable and can thereby reduce the radial transport. This new scenario that parallel flow can be a viable candidate for the stabilization of instabilities is very promising (note the usual picture is that it is the perpendicular $\mathbf{E} \times \mathbf{B}$ flow which does the stabilization). This is because, in a tokamak, the parallel velocity is very nearly equal to the toroidal velocity whereas the perpendicular velocity to the poloidal velocity. Now, the poloidal rotation in tokamak suffers from several disadvantages over toroidal rotation, most notably that poloidal flow is efficiently damped by magnetic pumping. Indeed, experimentally measured damping time of poloidal flows is of the order of the ion-ion collision time or less. and hence is much shorter than the damping time of toroidal flows. As a result, poloidal rotation dies away immediately after the beams [in the neutral beam injection (NBI) heating] are turned off leaving the plasma rotation in the toroidal direction. Toroidal rotation, on the other hand, is dissipated only through the diffusive transport of momentum which is expected to reduce to low, neoclassical levels. *Stabilization by parallel flow, therefore, seems to offer much more attractive prospect for high performance tokamak operation.*

On experimental front, recent results from the JET have shown that the reduction of small-scale turbulence in optimized magnetic shear regimes is directly related to the existence of a strongly sheared toroidal velocity in the area of the internal transport barrier [45]. Furthermore, the clear evidence for the theory developed here comes from the STOR-M tokamak at Canada where *no* change in the radial electric field is observed during the transition to the improved modes indicating that it is the parallel component of the flow which might be playing the key role in the transition [48].

Chapter 4

Effect of Parallel Flow Profile on the Toroidal ITG mode

4.1 Introduction

In order that the tokamak becomes a leading contender for a fusion reactor it should develop a magnetic configuration that has good confinement and stability and a large fraction of bootstrap current. Understanding and control of turbulent transport and of its underlying driving agents is therefore a prerequisite in this process. Recent discoveries of various enhanced performance operational regimes like the H-modes [1], the VH-modes [2], the enhanced reverse shear modes (ERS-modes) [3] or negative central magnetic shear (NCS-modes) [4] and the radiative improved modes (RI-modes) [5] has opened up a new window for improved tokamak operation.

An important challenge for enhanced tokamak operation is the development and understanding of the basic physics involved in the process that leads to the transition to the improved confinement modes. While a sheared poloidal (toroidal) flow is found to be responsible for the H- (VH-) modes, a hollow q profile (hence normally a hollow current profile) is necessary for the ERS or NCS modes. Most tokamaks however operate with inductive current drive which in general produces a peaked current density profile at the magnetic axis because of the strong dependence of the plasma conductivity on the electron temperature. Only by noninductive current drive or transient techniques can a hollow current density profile be generated.

All these improved modes in the tokamak *core* seem to have a common feature that the formation of the transport barrier is usually accompanied by a jump in the toroidal velocity in the region where the transport barrier is formed. Although in the beginning of the ERS plasma in the TFTR usually a balanced injection was used resulting in almost no toroidal flow, in more recent shots using different applied torques from the neutral beam injection (NBI) it has been confirmed that the toroidal velocity also plays an important role in the TFTR ERS mode [49-50]. Similarly recent RI-modes experiments in the TEXTOR-94 indicate that the toroidal flow also plays a crucial role in the transition to the RI-mode [51]. It is usually believed that the (sheared) toroidal velocity gives rise to a (sheared) radial electric field (E) and thereby (by the $\mathbf{E} \times \mathbf{B}$ shear) suppressing fluctuations and improving the core confinement. However, while this $\mathbf{E} \times \mathbf{B}$ shear stabilisation mechanism *alone* can satisfactorily explain the confinement improvement in the edge, it may not be an obvious explanation for the core confinement improvement. This is because from the radial force balance equation

$$v_\perp \approx E/B_\phi = \frac{v_\phi B_\theta}{B_\phi} - v_\theta + \frac{\partial P_i/\partial r}{nZeB_\phi}$$

it is obvious that with $B_\phi \gg B_\theta$, v_ϕ can contribute only weakly to $v_\perp$. As a result to produce the same change in E, v_ϕ should change by $\frac{B_\phi}{B_\theta}(\gg 1)$ times that of v_ϕ. Experiments however indicate otherwise. It is also supported by the fact that the formation of the ERS mode in TFTR can occur at values of $\gamma_{E\times B}$ ($\mathbf{E} \times \mathbf{B}$ shearing rate) as much as a factor of 3 below γ_{MAX} (the maximum linear growth rate) while for the suppression of turbulence-induced transport the condition $\gamma_{E\times B} \geq \gamma_{MAX}$ need to be satisfied [52].

We suggest in this letter that the transport barrier may be created by purely parallel flow profile. We demonstrate that the parallel flow curvature stabilizes toroidal ITG modes, which is thought to be the likely mechanism for anomalous transport in the plasma [53, 54]. We thus demonstrate two aspects, first in the core confinement improvement, the important role of the toroidal rotation may not be only through the $\mathbf{E} \times \mathbf{B}$ shear (which is usually weak) but by the strong parallel component. Second, as the parallel flow in the tokamak is almost equal to the toroidal flow and as the toroidal flow can be generated by the NBI, there is no time limit for how long the parallel flow can be maintained. So, the transport barrier created by the toroidal flow will have a distinctive advantage over that created by the poloidal flow as the toroidal flow unlike the poloidal counterpart is not damped by the magnetic pumping.

4.2 Stability Analysis

We use usual (r, θ, ϕ) coordinates, corresponding to the minor radial, poloidal and toroidal directions, respectively, and consider the long-wavelength ($\kappa^2 a_i^2 \ll 1$) ITG modes for a large aspect-ratio circular tokamak. The purturbed potential can then be expressed as:

$$\tilde{\varphi}(r,\theta,\phi,t) = \varphi(r,\theta)\exp\{i(n\phi - m\theta - \omega t)\}$$

where r is the radial distance from the mode rational surface, i.e. $m = nq(r_0)$, and $s = rq'/q$ at $r = r_0$. Here, for simplicity, we will assume ions to be cold and will ignore the electron temperature gradient. Using fluid descriptions, the eigenvalue equation in the presence of a velocity field can be derived (ref. Appendix III, eq. (III.9)) in a straightforward way.

$$a_i^2\frac{\partial^2\tilde{\varphi}}{\partial r^2} - b\tilde{\varphi} + \frac{(\omega_* - \omega + i\omega\delta)}{(\omega+\omega_*(\frac{1+\eta_i}{\tau}))}\tilde{\varphi} - \left(\frac{\omega_*\epsilon_c}{\omega\kappa a_i}\right)^2\left(\frac{\partial}{\partial\theta} + i\kappa s r\right)^2\tilde{\varphi}$$

$$-2\epsilon_n\frac{\omega_*}{\omega}\left(\cos\theta + \frac{i\sin\theta}{\kappa}\frac{\partial}{\partial r}\right)\tilde{\varphi} + \frac{\kappa a_i c_s k_\parallel}{(\omega+\omega_*\left(\frac{1+\eta_i}{\tau}\right))\omega}\frac{\partial V_{\parallel 0}(r)}{\partial r}\tilde{\varphi} = 0 \qquad (4.1)$$

where

$\kappa = nq/r = m/r$

$s = (r/q)(dq/dr)$

$$a_i^2 = c^2 m_i T_e / e^2 B^2$$

$$c_s^2 = T_e / m_i$$

$$b = \kappa^2 a_i^2$$

$$\epsilon_n = q\epsilon_c = r_n / R$$

$$\tau = T_e / T_i$$

$$\omega_* = (\kappa c T_e / e B r_n)$$

$$\nabla_\parallel \tilde{\varphi} = i k_\parallel \tilde{\varphi}$$

$$\nabla_\parallel = \frac{1}{qR}\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\phi}\right).$$

Now to model the equilibrium parallel velocity we follow the toroidal velocity profile usually observed during the VH-mode, the NCS-mode, and the RI-mode experiments.

$$v_{\parallel o}(r) = v_{\parallel 00} - \left(\frac{r}{L_{v1}} + \frac{r^2}{L_{v2}^2}\right) v_{\parallel 0},$$

where

$$\frac{dv_{\parallel 0}}{dr} = -\frac{v_{\parallel 00}}{L_{v1}}, \; \frac{1}{2}\frac{d^2 v_{\parallel 0}}{dr^2} = -\frac{v_{\parallel 00}}{L_{v2}^2}.$$

We put

$$x = \kappa r s \Rightarrow r = \frac{x}{\kappa s} \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{\kappa s}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} \Rightarrow \frac{1}{\kappa s}\frac{\partial}{\partial r} \Rightarrow \frac{\partial}{\partial r} = \kappa s \frac{\partial}{\partial x}$$

$$\frac{\partial^2}{\partial x^2} = \frac{1}{(\kappa s)^2}\frac{\partial^2}{\partial r^2}$$

$$\frac{\partial^2}{\partial r^2} = (\kappa s)^2\frac{\partial^2}{\partial x^2}$$

Therefore we get

$$\left[\frac{\partial^2}{\partial x^2} - \sigma^2\left(\frac{\partial}{\partial\theta} + ix\right)^2 - \epsilon\left(\cos\theta + is\sin\theta\frac{\partial}{\partial x}\right) + \lambda\right]\tilde{\varphi} + \left[\frac{\kappa a_\imath c_s x}{bs^2 qR\omega(\omega + \omega^*\left(\frac{1+\eta_i}{\tau}\right))}\left(-\frac{v_{\|00}}{L_{v1}} - \frac{v_{\|00x}}{\kappa s L_{v2}^2}\right)\right]\tilde{\varphi} = 0 \tag{4.2}$$

where

$$\sigma = \frac{\epsilon_c}{bs}$$

$$\epsilon = \frac{2\epsilon_n}{bs^2}$$

and $v_{\|0}$ is the equilibrium parallel flow. The first term in Eq. (4.1) arises from the finite Larmour radius effect and the third from the ion sound. The parameter $i\delta$ represents the destabilizing effect of electron Landau resonance and the trapped electrons. The fourth is the effect of toroidal coupling. Parallel flow has two effects. First, it introduces a Doppler shift, $k_{\|}v_{\|0}$ (in deriving equation (4.1) we have neglected Doppler shifts due to equilibrium velocity), in all time derivatives, and second, an extra term $v_E\nabla v_{\|0}(x)$ representing the radial convection of the ion momentum. It is the second term which makes the effect of parallel flow shear completely different from that of perpendicular flow shear.

To reduce the two-dimensional (2-D) eigenmode problem to one-dimensional (1-D), we will apply the ballooning transformation. However, the validity of the conventional ballooning formalism in the presence of sheared flow is, in general, severely restricted. This is due to the fact that in an equilibrium with differential rotation, the Doppler-shifted frequencies varies from one flux surface to another and hence the conventional ballooning formalism, for which to lowest order (the local approximation) the perturbation is a superposition of *identical* single helicity modes, no longer applies. In other words, the eikonal solutions do not have pure exponential time dependence and are not eigenmodes in the system. However, we notice that the spatial variation in the Doppler shift, $k_{\|}v_{\|0}$, in the mode frequency due to *parallel* flow is negligible for flute-type modes ($k_{\|} \ll k_{\perp}$). It is probably obvious as $d^n/dx^n(k_{\|}v_{\|0}(x)) \ll d^n/dx^n(k_{\perp}v_0(x))$ due to the fact that $k_{\|} \ll k_{\perp}$ and that $v_{\|0} \sim v_{\perp 0}$ in the experiments. So, one can eliminate the Doppler shift by performing a Galilean transformation in the $\hat{e}_{\|}$ direction. This has been noted else where [17]. It can be noted that the effect of the *parallel* shear/curvature enters in the problem through an extra term $v_E \nabla v_{\|0}(x)$ representing radial convection of ion momentum [17]. So, once radial variation in the Doppler shift is neglected, the restriction on the applicability of the ballooning formalism no longer applies.

To determine the radial mode structure, the solution of the fully 2-D eigenmode problem must be obtained within the framework of ballooning formalism, this means solving the problem to a higher order. The problem then separates into two distinctive radial length scales. To leading order, the problem reduces to the usual 1-D eigenmode equation (with radial variable appearing only as parameter), which determines the mode structure along the magnetic field lines. The next order equation then determines the radial mode structure. In the usual theory of high n ballooning mode, one maps the poloidal angle θ on to an extended coordinate χ with $-\infty < \chi < \infty$ and writes the perturbation in the form

$$\tilde{\varphi}(\theta, x) = \sum_m e^{-\imath m\theta} \int_{-\infty}^{\infty} e^{\imath m\chi} \hat{\phi}(\chi, x) d\chi,$$

where

$$\hat{\phi} = A(x) F(\chi, x) \exp[-ix(\chi + \chi_0)].$$

Here χ_0 is an arbitrary phase of the eikonal. A(x) is assumed to vary on some scale intermediate between the equilibrium scale length and the perpendicular wavelength. Now to leading order (in $n^{-1/2}$ expansion), the ballooning equation becomes

$$\left(\sigma^2 \frac{d^2}{d\chi^2} + (\chi + \chi_0)^2 + \epsilon[\cos\chi + s(\chi + \chi_0)\sin\chi] + p_1 x + p_2 x^2 - \lambda\right) F(\chi, x) = 0 \tag{4.3}$$

To explore its implication for radial mode structure and stability of toroidal ITG mode one needs the higher order ballooning theory. In the higher order theory[18] χ_0 is obtained from the equation $(\partial\lambda/\partial\chi_0)(x, \chi_0) = 0$ and A(x) satisfies

$$\frac{\partial^2\lambda}{\partial\chi_0^2} \cdot \frac{d^2 A}{dx^2} + [2(\lambda - \lambda_0) - 2p_1 x - 2p_2 x^2]A = 0, \tag{4.4}$$

where

$$\lambda = \frac{1}{bs^2}\left[\frac{\omega_* - \omega - \imath\omega\delta}{\omega - \omega_*\left(\frac{1+\eta_i}{\tau}\right)} - b\right],$$

$$\lambda_0 = \imath\frac{\epsilon_c}{bs},$$

$$p_1 = \frac{\kappa a_i c_s v_{\parallel 00}}{bs^2 L_{v1} q R\omega\left(\omega + \omega_*\left(\frac{1+\eta_i}{\tau}\right)\right)},$$

$$p_2 = \frac{a_i c_s v_{\parallel 00}}{bs^3 L_{v2}^2 q R\omega\left(\omega + \omega_*\left(\frac{1+\eta_i}{\tau}\right)\right)},$$

Equation (4.2) is a simple Weber equation. When p_2 is positive and $\partial^2\lambda/\partial\chi_0^2 > 0$ ($\partial^2\lambda/\partial\chi_0^2 > 0$ is necessary in order that the mode be most unstable), $A(x)$ is localized Gaussian function. However, an important change is introduced by the velocity term for the negative magnetic shear. $A(x)$ is then given by

$$A(x) = \exp\left[-\imath\frac{1}{2}\left(|p_2|/|\frac{\partial^2\lambda}{\partial\chi_0^2}|\right)^{1/2}(x + x_0)^2\right], \tag{4.5}$$

where $x_0 = p_1/|p_2|$. So, the mode envelop is now radially outgoing, which is reminiscent of the equivalent slab problem. Velocity curvature in the toroidal problem like magnetic shear in the corresponding slab problem creates an *antiwell* in the radial direction. The wave energy is therefore convected outward. The eigenvalue is given by

$$\lambda = \lambda_0 + \frac{p_1^2}{2|p_2|} - \imath\frac{1}{2}\left(|p_2||\frac{\partial^2\lambda}{\partial\chi_0^2}|\right)^{1/2}. \quad (4.6)$$

This also shows damping contribution in the global eigenvalue. Thus, both the radially outgoing nature and the damping contribution in the global eigenvalue unambiguously show that parallel flow profile might *stabilize* toroidal ITG waves which otherwise escape magnetic shear damping!

4.3 Conclusions

In conclusion, we have shown in this article that the transport barriers may be created by purely parallel flow profile. We have demonstrated that the curvature in the parallel flow stabilizes toroidal ITG modes, which have been identified as the likely mechanism for anomalous transport in the plasma. We thus demonstrated two aspects. First, in regards to core confinement improvement, the important role of the toroidal rotation may not be only through the $E \times B$ shear (which may be weak), but also through the strong parallel component. Second, as the parallel flow in the tokamak is almost equal to the toroidal flow and as the toroidal flow can be generated by the NBI, there is no limit for how long the parallel flow can be maintained. So, the transport barrier created by the toroidal flow will have a distinctive advantage over that created by the poloidal flow as the toroidal flow unlike the poloidal counterpart is not damped by the magnetic pumping.

Chapter 5

Summary and Conclusion

The first chapter is an introduction to this thesis. We have discussed about the high growth of energy demand with the robust increase in world human population and industrialization. We have further discussed that the energy demand may be substantially met if the fusion energy (controlled) is used, prospected due to its cost effectiveness and for some other important factors also. The ultimate aim of the fusion research is to provide an even condition for a burning plasma and for this, certain well known parameter values have to be obtained: the product of density and energy confinement time has to be $3 \times 10^{14} cm^{-3}.s$ and the average plasma temperature has to be 10KeV. The most promising fusion device is a toroidal confinement system 'Tokamak' in which toroidal and poloidal magnetic field is issued for plasma confinement. In the tokamaks the target values of these parameters have not been obtained simultaneously. In experiments, where the temperature was adequate the confinement product inadequate and vice versa. The use of auxiliary heating to increase the plasma temperature in the tokamaks severely degraded the confinement, with no net result of effective approach to break even conditions. In 1982, ASDEX tokamak in Germany, discovered H-mode, where the deterioration of confinement quality at high temperature is avoided. After this discovery, as a result of continuous quest for high confinement, several other high confinement modes viz. VH-mode, ERS-mode, NCS-mode and RI-mode have been discovered. Still the challenge remains to enhance tokamak operation for the development and understanding the basic physics involved in the process that leads to the transition to these improved confinement modes. We have discussed some popular and arguably successful theories which tells that some mechanism(s) creates a (sheared) flow that gives rise to a (sheared) radial electric field (E), (thereby $\mathbf{E} \times \mathbf{B}$ shear) stabilizing various instabilities and as a result fluctuations are suppressed and confinement is improved. A sheared flow is responsible for the transition to *all* improved modes in tokamaks in one way or other. Next we have presented the summary and conclusion of some of the fusion research related problems.

In the second chapter we have studied the effect of a radially varying parallel equilibrium flow on the stability of the Rayleigh-Taylor mode analytically. It is shown that the parallel flow curvature can completely stabilize the RT mode. The flow curvature also has a robust effect on the radial structure of the mode. Our work therefore shows that by suitably tailoring the parallel flow profile it is possible to completely stabilize the RT instability in a magnetized plasma. A weak magnetic shear is needed for the process, and hence a staged Z-pinch will be an ideal device for realizing this. This result might also have an important part to play in explaining the recent gas puff experiments on the 7-MA Saturn generator which have identified a curved surface while observing an improvement in the implosion quality [18]. This is because the subsequent simulation work of Douglas et al. [19] has shown that a curved surface introduces an axial flow and our work here shows that the flow curvature in such an axial flow can indeed stabilize the RT growth. However, a definitive comparison is only possible when one gets additional data from this experiment.

It is interesting to compare our results with a recent work of Shumlak and Roderick. The authors in reference find that the shear in the axial flow can mitigate the RT instability, whereas the result in this work, including both the flow shear and curvature in the parallel flow and also the magnetic shear, indicates that it is the flow *curvature* which plays the leading role in suppressing the RT instability. Furthermore, the result in reference indicates that for the mitigation of RT instability, a *sufficiently strong* sheared flow is necessary, in this work we, however, find that the value of the flow curvature necessary for the mitigation of the RT instability is rather modest in the presence of even a very weak magnetic shear. So, possibly the most realistic way of stabilizing the RT instability is to use an axial flow with a properly tailored flow profile in conjunction with a sheared magnetic field. This possibility is already being explored on the Z-Accelerator at Sandia National Laboratories, where the magnetic shear comes from the axial magnetic field produced by using a twisted tungsten wire array and the radially varying axial velocity, presumably, from the hourglass shape of the array associated with its twisting. The initial result indeed shows a stabilizing effect.

In the third chapter we present a model for transition to the enhanced reverse shear (ERS) or negative central shear (NCS) mode triggered in tokamaks. Our studies show that when the magnetic shear has the same sign as the second derivative of the parallel velocity with respect to the radial coordinate, the ITG mode may become more unstable. On the other hand, when the magnetic shear has the opposite sign to the second derivative of the parallel velocity, the ITG mode may be completely stabilized. This result is similar to what we have found earlier for the PVS instabilities [47,55]. So, the similar result with the ITG mode, considered nowadays as the dominant source of anomalous energy losses in the low confinement (L) mode, therefore, shows on a firmer footing that it is the relative sign of the second radial derivative of the equilibrium parallel flow with respect to the magnetic shear which may be the key factor for the enhanced reverse shear transition.

Now, the real benefit of the outcome of this work is that it puts forward a novel idea of transport barrier formation. This is that, contrary to the usual notion that a parallel flow shear is always destabilizing, the destabilizing influence of the shear in the parallel flow can be changed altogether if one takes the effect of the flow curvature into account. The transverse curvature in the parallel flow can overcome the destabilizing influence of the shear and can render the low frequency modes stable and can thereby reduce the radial transport. This new scenario that parallel flow can be a viable candidate for the stabilization of instabilities is very promising (note the usual picture is that it is the perpendicular $\mathbf{E} \times \mathbf{B}$ flow which does the stabilization). This is because, in a tokamak, the parallel velocity is very nearly equal to the toroidal velocity whereas the perpendicular velocity to the poloidal velocity. Now, the poloidal rotation in tokamak suffers from several disadvantages over toroidal rotation, most notably that poloidal flow is efficiently damped by magnetic pumping. Indeed, experimentally measured damping time of poloidal flows is of the order of the ion-ion collision time or less. and hence is much shorter than the damping time of toroidal flows. As a result, poloidal rotation dies away immediately after the beams [in the neutral beam injection (NBI) heating] are turned off leaving the plasma rotation in the toroidal direction. Toroidal rotation, on the other hand, is dissipated only through the diffusive transport of momentum which is expected to reduce to low, neoclassical levels. *Stabilization by parallel flow, therefore, seems to offer much more attractive prospect for high performance tokamak operation.*

On experimental front, recent results from the JET have shown that the reduction of small-scale turbulence in optimized magnetic shear regimes is directly related to the existence of a strongly sheared toroidal velocity in the area of the internal transport barrier [45]. Furthermore, the clear evidence for the theory developed here comes from the STOR-M tokamak at Canada where *no* change in the radial electric field is observed during the transition to the improved modes indicating that it is the parallel component of the flow which might be playing the key role in the transition [48].

In the fourth chapter we have shown that the transport barriers may be created by purely parallel flow profile. We have demonstrated that the curvature in the parallel flow stabilizes toroidal ITG modes, which have been identified as the likely mechanism for anomalous transport in the plasma. We thus demonstrated two aspects. First, in regards to core confinement improvement, the important role of the toroidal rotation may not be only through the $E \times B$ shear (which may be weak), but also through the strong parallel component. Second, as the parallel flow in the tokamak is almost equal to the toroidal flow and as the toroidal flow can be generated by the NBI, there is no limit for how long the parallel flow can be maintained. So, the transport barrier created by the toroidal flow will have a distinctive advantage over that created by the poloidal flow as the toroidal flow unlike the poloidal counterpart is not damped by the magnetic pumping.

Appendix I

Calculations regarding Chapter 2

Let us define:

$\mathbf{B}(x) = B_0[\hat{e}_z + (x/L_s)\hat{e}_y]$

$\phi(\mathbf{x}, t) = \phi(x)\exp i(k_y y + k_z z - \omega t)$

$V_{\|0}(x) = v_{\|0}\hat{e}_{\|}.$

$\tilde{v}_e = V_E$

$\tilde{v}_i = V_E + V_p + V_g$

$V_e = V_{\|0}(x) + \tilde{v}_e$

$V_i = V_{\|0}(x) + \tilde{v}_i$

$V_E = -c(\nabla_\perp\phi \times \mathbf{B})/B_0^2 = -\frac{c}{B_0}\left(ik_y\hat{e}_x\phi - \hat{e}_y\frac{d\phi}{dx} + \hat{e}_z\frac{x}{L_s}\frac{d\phi}{dx}\right)$

$V_p = (\frac{ic\omega}{B_0\omega_{ci}})\nabla_\perp\phi = (\frac{ic\omega}{B_0\omega_{ci}})(ik_y\hat{e}_y\phi + \hat{e}_x\frac{d\phi}{dx})$

$V_g = -\frac{cm_i g}{eB_0}\hat{e}_y$

$\omega_{ci} = \frac{eB}{cm_i}$

$\nabla_\perp = ik_y\hat{e}_y + \hat{e}_x\frac{d}{dx}$

here $\mathbf{B}(x)$ is the shear magnetic field; L_s is the magnetic shear scale length, x is the distance from mode rational surface which is defined as $\mathbf{k}.\mathbf{B} = 0$, $\phi(\mathrm{x}, t)$ is the form of perturbed potential, $v_{\|0}$ is the equilibrium parallel flow varying in x-direction, g is the acceleration due to gravity, e is the charge of an electron, m_i is the ion mass, $\hat{e}_y$ and $\hat{e}_z$ are unit vectors in y and z direction respectively, c is the speed of light, k_y and k_z are the wave numbers in y and z direction respectively and all the other symbols have their usual meaning unless stated otherwise.

Continuity equation for ions

$$\frac{\partial \tilde{n}_i}{\partial t} + \nabla.[(n_0 + \tilde{n}_i)(v_{\|0} + \tilde{v}_i + V_g)] = 0 \qquad \text{(I.1)}$$

$$\Rightarrow \frac{\partial \tilde{n}_i}{\partial t} + \nabla_\perp.[(n_0 + \tilde{n}_i)(v_{\|0} + \tilde{v}_i + V_g)] + \nabla_\|[(n_0 + \tilde{n}_i)(v_{\|0} + \tilde{v}_i + V_g)] = 0$$

here $\tilde{n}_i$, n_0; $\tilde{v}_{\perp i}$ and $\tilde{v}_{\|i}$ is respectively the perturbed ion density, equilibrium ion density; perturbed perpendicular and parallel ion velocities.

$$\Rightarrow \frac{\partial \tilde{n}_i}{\partial t} + \nabla_\perp.[n_0 \tilde{v}_{\perp i}] + \nabla_\perp.[\tilde{n}_i V_g] + \nabla_\|.[n_0 \tilde{v}_{\|i}] + \nabla_\|[\tilde{n}_i v_{\|0}] = 0 \qquad \text{(I.2)}$$

We will evaluate separately each terms of the equation (I.2) as below:

Let us take the second term of equation (I.2),

$\nabla_\perp.[n_0 \tilde{v}_{\perp i}]$.

$$= \nabla_\perp \left[n_0 \left(-\frac{c}{B_0} i k_y \phi + \frac{i c \omega}{B_0 \omega_{ci}} \frac{d\phi}{dx}\right) \hat{e}_x + n_0 \left(\frac{c}{B_0}\frac{d\phi}{dx} + \frac{i c\omega}{B_0 \omega_{ci}} i k_y \phi\right) \hat{e}_y\right]$$

$$= i k_y n_0 \left(\frac{c}{B_0}\frac{d\phi}{dx} + \frac{i c \omega}{B_0 \omega_{ci}} i k_y \phi\right) + \left(-\frac{c}{B_0} i k_y\right)\left[n_0 \frac{d\phi}{dx} + n_0' \phi\right] + \frac{i c \omega}{B_0 \omega_{ci}}\left[n_0 \frac{d^2\phi}{dx^2} + n_0' \frac{d\phi}{dx}\right]$$

$$= -\frac{i k_y^2 c \omega n_0}{B_0 \omega_{ci}} \phi - \frac{i k_y c}{B_0} n_0' \phi + \frac{i c\omega}{B_0 \omega_{ci}} n_0 \frac{d^2\phi}{dx^2} + \frac{i c \omega}{B_0 \omega_{ci}} n_0' \frac{d\phi}{dx}$$

The third, fourth and fifth terms of equation (I.2)

$$\nabla_{\perp}[\tilde{n}_i V_g] + \nabla_{\parallel}[\tilde{n}_i v_{\parallel 0}] + \nabla_{\parallel}[n_0 \tilde{v}_{\parallel i}]$$

$$= -ik_y \frac{cm_i g}{B_0 e}\tilde{n}_i + ik_{\parallel} v_{\parallel 0}\tilde{n}_i + ik_{\parallel} n_0 \tilde{v}_{\parallel i}$$

Let us take momentum balance equation for ions

$$m_i n_i \left[\frac{\partial \tilde{v}_{\parallel i}}{\partial t} + (v_E.\nabla v_{\parallel 0})\right] = -en_i \nabla_{\parallel}\phi \qquad \text{(I.3)}$$

$$\Rightarrow \tilde{v}_{\parallel i} = \frac{e}{im_i\omega} ik_{\parallel}\phi - \frac{ik_y c}{i\omega B_0}\frac{dv_{\parallel 0}(x)}{dx}\phi$$

$$= \frac{ek_{\parallel}}{m_i\omega}\phi - \frac{k_y c}{\omega B_0}\frac{dv_{\parallel 0}(x)}{dx}\phi$$

Therefore we get,

$$\nabla_{\parallel}[n_0 \tilde{v}_{\parallel i}] = ik_{\parallel} n_0 \left[\frac{ek_{\parallel}}{m_i\omega}\phi - \frac{k_y c}{\omega B_0}\frac{dv_{\parallel 0}}{dx}\phi\right]$$

Now let us rearrange the evaluated quantities of equation (I.2)

$$i\omega\tilde{n}_i + \left[-\frac{ik_y^2 c\omega n_0}{B_0\omega_{ci}}\phi - \frac{ik_y c}{B_0}n_0'\phi + \frac{ic\omega}{B_0\omega_{ci}}n_0\frac{d^2\phi}{dx^2} + \frac{ic\omega}{B_0\omega_{ci}}n_0'\frac{d\phi}{dx}\right]$$

$$-ik_y\frac{cm_i}{B_0 e}g\tilde{n}_i + ik_{\parallel}v_{\parallel 0}\tilde{n}_i + \frac{iek_{\parallel}^2}{m_i\omega}n_0\phi - \frac{ik_{\parallel}k_y c n_0}{\omega B_0}\frac{dv_{\parallel 0}}{dx}\phi = 0 \qquad \text{(I.4)}$$

Let us take the momentum balance equation for electrons

$$m_e n_e \left[\frac{\partial \tilde{v}_{\parallel e}}{\partial t} + v_E.\nabla v_{\parallel 0}\right] = -en_e\nabla_{\parallel}\phi \qquad \text{(I.5)}$$

$$\Rightarrow m_e\left[-i\omega\tilde{v}_{\parallel e} + \left(-\frac{c}{B_0}ik_y\frac{dv_{\parallel 0}}{dx}\phi\right)\right] = -eik_{\parallel}\phi$$

$$\Rightarrow \tilde{v}_{\parallel e} = \frac{-iek_{\parallel}}{-im_e\omega}\phi + \frac{ik_y c}{-i\omega B_0}\frac{dv_{\parallel 0}(x)}{dx}\phi$$

$$= \frac{ek_{\parallel}}{m_e\omega}\phi - \frac{k_y c}{\omega B_0}\frac{dv_{\parallel 0}(x)}{dx}\phi$$

Again, the continuity equation for electrons is

$$\frac{\partial \tilde{n}_e}{\partial t} + \nabla.[(n_0 + \tilde{n}_e)(v_{\parallel 0} + \tilde{v}_e)] = 0 \tag{I.6}$$

$$\Rightarrow \frac{\partial \tilde{n}_e}{\partial t} + \nabla_{\perp}.[n_0\tilde{v}_{\perp e}] + \nabla_{\parallel}.[n_0\tilde{v}_{\parallel e}] + \nabla_{\parallel}[\tilde{n}_e v_{\parallel 0}] = 0$$

$$\Rightarrow -i\omega\tilde{n}_e + \nabla_{\perp}.[n_0\tilde{v}_{\perp e}] + ik_{\parallel}n_0\left[\frac{ek_{\parallel}}{\omega m_e}\phi - \frac{ck_y}{B_0\omega}\frac{dv_{\parallel 0}}{dx}\phi\right] + ik_{\parallel}v_{\parallel 0}\tilde{n}_e = 0 \tag{I.7}$$

The second term of equation (I.7) is

$$\nabla_{\perp}[n_0\tilde{v}_{\perp e}]$$

$$= \left[ik_y\hat{e}_y + \hat{e}_x\frac{d}{dx}\right].\left[n_0\left(-\frac{c}{B_0}\right)\left\{ik_y\hat{e}_x\phi - \frac{d\phi}{dx}\hat{e}_y\right\}\right]$$

$$= ik_y\frac{c}{B_0}n_0\frac{d\phi}{dx} - \frac{c}{B_0}ik_y\left[n_0\frac{d\phi}{dx} + n_0'\phi\right]$$

Therefore equation (I.7) becomes

$$-i\omega\tilde{n}_e - ik_y\frac{c}{B_0}n_0'\phi + \frac{ik_{\parallel}^2 en_0}{\omega m_e}\phi - \frac{ik_{\parallel}k_y cn_0}{B_0\omega}\frac{dv_{\parallel 0}}{dx}\phi + ik_{\parallel}v_{\parallel 0}\tilde{n}_e = 0$$

$$\Rightarrow (-\omega + k_{\parallel}v_{\parallel 0})\tilde{n}_e = \left[\frac{k_y c}{B_0}n_0' + \frac{k_{\parallel}k_y cn_0}{B_0\omega}\frac{dv_{\parallel 0}}{dx} - \frac{k_{\parallel}^2 en_0}{\omega m_e}\right]\phi$$

Taking Galilean transformation in $\hat{e}_{\parallel}$ direction we eliminate $k_{\parallel}v_{\parallel 0}$

$$\Rightarrow \tilde{n}_e = \frac{1}{-\omega}\left[\frac{k_y c}{B_0}n_0' + \frac{k_{\parallel}k_y cn_0}{B_0\omega}\frac{dv_{\parallel 0}}{dx} - \frac{k_{\parallel}^2 en_0}{\omega m_e}\right]\phi \tag{I.8}$$

Rearranging the equation (I.4) we get,

$$\frac{c\omega n_0}{B_0\omega_{ci}}\frac{d^2\phi}{dx^2} + \frac{c\omega n_0'}{B_0\omega_{ci}}\frac{d\phi}{dx} + \left[-\frac{k_y^2 c\omega n_0}{B_0\omega_{ci}} - \frac{k_y cn_0'}{B_0} + \frac{k_{\parallel}^2 en_0}{m_i\omega} - \frac{k_{\parallel}k_y cn_0}{\omega B_0}\frac{dv_{\parallel 0}}{dx}\right]\phi + \left[-\omega - \frac{k_y cm_i g}{eB_0}\right]\tilde{n}_i = 0$$

Now we apply quasineutrality condition (i.e $\tilde{n}_e \sim \tilde{n}_i$) in (I.4) i.e we replace $\tilde{n}_i$ in equation (I.4) by $\tilde{n}_e$ from equation (I.8). Thus we get

$$\Rightarrow \frac{d^2\phi}{dx^2} + \frac{n_0'}{n_0}\frac{d\phi}{dx}$$

$$+ \left[\frac{B_0\omega_{ci}}{c\omega n_0}\left(-\frac{k_y^2 c\omega n_0}{B_0\omega_{ci}}\right) - \frac{B_0\omega_{ci}k_y c n_0'}{c\omega n_0 B_0} + \frac{B_0\omega_{ci}k_\parallel^2 e n_0}{c\omega n_0 m_i \omega} - \frac{B_0\omega_{ci}k_\parallel k_y c n_0}{c\omega n_0 \omega B_0}\frac{dv_{\parallel 0}}{dx}\right]\phi$$

$$+ \left[\frac{B_0\omega_{ci}}{c\omega n_0}(-\omega) - \frac{B_0\omega_{ci}k_y c m_i g}{c\omega n_0 e B_0}\right]\left[-\frac{k_y c n_0'}{\omega B_0} - \frac{k_\parallel k_y c n_0}{\omega B_0 \omega}\frac{dv_{\parallel 0}}{dx} + \frac{k_\parallel^2 e n_0}{\omega^2 m_e}\right]\phi = 0$$

$$\Rightarrow \frac{d^2\phi}{dx^2} + \frac{n_0'}{n_0}\frac{d\phi}{dx} + \left[-k_y^2 - \frac{k_y\omega_{ci}}{\omega}\frac{n_0'}{n_0} + \frac{k_\parallel^2\omega_{ci}^2}{\omega^2} - \frac{k_\parallel k_y\omega_{ci}}{\omega^2}\frac{dv_{\parallel 0}}{dx}\right]\phi$$

$$+ \left[-\frac{B_0\omega_{ci}}{cn_0}\left(-\frac{k_y c n_0'}{\omega B_0}\right) - \frac{B_0\omega_{ci}}{cn_0}\left(-\frac{k_\parallel k_y c n_0}{\omega^2 B_0}\frac{dv_{\parallel 0}}{dx}\right) - \frac{B_0\omega_{ci}}{cn_0}\frac{k_\parallel^2 e n_0}{\omega^2 m_e}\right]\phi$$

$$+ \left[-\frac{k_y m_i\omega_{ci}g}{e\omega n_0}\left(-\frac{k_y c n_0'}{\omega B_0}\right) - \frac{\omega_{ci}k_y m_i g}{\omega n_0 e}\left(-\frac{k_\parallel k_y c n_0}{\omega B_0 \omega}\frac{dv_{\parallel 0}}{dx}\right) - \frac{\omega_{ci}k_y m_i g k_\parallel^2 e n_0}{\omega n_0 e \omega^2 m_e}\right]\phi = 0$$

$$\Rightarrow \frac{d^2\phi}{dx^2} + \frac{n_0'}{n_0}\frac{d\phi}{dx} + \left[-k_y^2 - \frac{k_y\omega_{ci}}{\omega}\frac{n_0'}{n_0} + \frac{k_\parallel^2\omega_{ci}^2}{\omega^2} - \frac{k_\parallel k_y\omega_{ci}}{\omega^2}\frac{dv_{\parallel 0}}{dx} + \frac{k_y\omega_{ci}}{\omega}\frac{n_0'}{n_0}\right]\phi$$

$$+ \left[\frac{k_\parallel k_y\omega_{ci}}{\omega^2}\frac{dv_{\parallel 0}}{dx} - \frac{k_\parallel^2\omega_{ci}eB_0}{\omega^2 c m_e} + \frac{k_y^2\omega_{ci}c m_i g n_0'}{\omega^2 e B_0 n_0} + \frac{k_y^2 k_\parallel\omega_{ci}c m_i g}{\omega^3 e B_0}\frac{dv_{\parallel 0}}{dx} - \frac{k_y k_\parallel^2\omega_{ci}m_i g}{\omega^3 m_e}\right]\phi = 0$$

$$\Rightarrow \frac{d^2\phi}{dx^2} + \frac{n_0'}{n_0}\frac{d\phi}{dx} + \left[-k_y^2 + \frac{k_\parallel^2\omega_{ci}^2}{\omega^2} - \frac{k_\parallel^2\omega_{ci}eB_0}{\omega^2 c m_e} + \frac{k_y^2 g}{\omega^2}\frac{n_0'}{n_0} + \frac{k_y^2 k_\parallel g}{\omega^3}\frac{dv_{\parallel 0}}{dx} - \frac{k_y k_\parallel^2\omega_{ci}m_i g}{\omega^3 m_e}\right]\phi = 0 \quad (I.9)$$

Neglecting the terms involving $k_\parallel^2$ ($k_\parallel \sim 0$ is justified for flute type mode) we get,

$$\frac{d^2\phi}{dx^2} + p(x)\frac{d\phi}{dx} + q(x)\phi = 0 \quad (I.10)$$

where

$$p(x) = \frac{n_0'}{n_0}$$

$q(x) = -k_y^2 + \frac{k_y^2 g}{\omega^2}\frac{n_{0'}}{n_0} + \frac{k_y^2 k_\| g}{\omega^3}\frac{dv_{\|0}}{dx}.$

This is the equation (2.4) in Chapter 2.

Using transformation

$\phi(x) = \psi(x)\exp(-\int^x \frac{p(\eta)}{2} d\eta)$ in equation (I.10), we get

$$\frac{d^2\psi}{dx^2} + Q(x)\psi = 0 \tag{I.11}$$

where

$Q(x) = q(x) - \frac{p'(x)}{2} - \frac{p^2(x)}{4}.$

We use the velocity and density profile as described below:

$n_o(x) = n_{oo} exp(-x^2/2L_n^2),$

$v_{\|0} = v_{00} + v_{00}x/L_{v1} + v_{00}x^2/2L_{v2}$

Here

$L_{v1} = (1/v_{00} dv_0(x)/dx)^{-1},$

$L_{v2} = (1/v_{00} d^2 v_0(x)/dx^2)^{-1}$

and L_n is the density gradient scale length.

By putting $k_\| = \frac{k_y x}{L_s}$ we get

$Q(x) = -k_y^2 + \frac{1}{2L_n^2} - \frac{k_y^2 g x}{\omega^2 L_n^2} + \frac{k_y^3 g v_{00} x}{\omega^3 L_s L_{v1}} + \frac{k_y^3 g v_{00} x^2}{\omega^3 L_s L_{v2}} + \frac{x^2}{4L_n^4}$

Assuming $v_{00} \sim \omega$ and noting $\omega^2 L_n \sim g$ we get

$$Q(x) = -k_y^2 + \frac{1}{2L_n^2} - \frac{k_y^2 g x}{\omega^2 L_n^2} + \frac{k_y^3 L_n x}{L_s L_{v1}} + \frac{k_y^3 L_n x^2}{L_s L_{v2}} + \frac{x^2}{4L_n^4}$$

With this equation (I.11) becomes

$$\frac{d^2\psi}{dx^2} + [T + Sx + Rx^2]\psi = 0 \qquad \text{(I.12)}$$

where

$$T = \frac{1}{2L_n^2} - k_y^2$$

$$S = \frac{L_n k_y^3}{L_s L_{v1}} - \frac{k_y^2 g}{\omega^2 L_n^2}$$

$$R = \frac{L_n k_y^3}{L_s L_{v2}} - \frac{1}{4L_n^4}.$$

This is the equation (2.5) in Chapter 2.

Now we get two types of solution for equation (I.12)

1. If $R < 0$, the solution satisfying the physical boundary condition, i.e., $\psi \rightarrow 0$ at $x = \pm\infty$ is given by

$$\psi(x) = \psi_o exp[-\sqrt{|R|}(x - x_o)^2]$$

where $x_o = S/2|R|$.

2. If $R > 0$, equation (I.12) has the solution

$$\psi(x) = \psi_o exp[-i\sqrt{|R|}(x + x_o)^2]$$

The corresponding dispersion relation is given by $T - \frac{S^2}{4R} = i\sqrt{R}$

The dispersion relation can be derived as follows:

We start from eigen value equation (I.12), which is

$\frac{\partial^2\psi}{\partial x^2} + [T + Sx + Rx^2]\psi = 0$

Let us consider $z = x + x_0$.

This will transform the equation (I.12) to

$\frac{\partial^2\psi}{\partial z^2} + [T + Rz^2 - (2Rx_0 - S)z + (Rx_0 - S)x_0]\psi = 0$

Now we put

$x_0 = \frac{S}{2R}$

This will give

$\frac{\partial^2\psi}{\partial z^2} + [T + Rz^2 - \frac{S^2}{4R}]\psi = 0$

Now we reconvert the independent variable z to x

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} + [T + R(x + x_0)^2 - \frac{S^2}{4R}]\psi = 0 \quad \text{(I.13)}$$

Again let us take the solution for $R > 0$

$\psi(x) = \psi_0 \left[-\frac{i\sqrt{R}}{2}(x + x_0)^2 \right]$

Twice differentiating with respect to x we get

$$\frac{\partial^2 \psi}{\partial x^2} + i\sqrt{R}\psi(x) + R(x + x_0)^2 \psi(x) = 0 \qquad \text{(I.14)}$$

Equation (I.13) and (I.14) will exactly match when

$i\sqrt{R} = T - \frac{S^2}{4R}$

This is our dispersion relation.

From the dispersion relation

$\frac{S^2}{4R} = T - i\sqrt{R}$

$\Rightarrow S^2 = 4R(T - i\sqrt{R})$

Without the velocity shear contribution we get (hence $S = -\frac{k_y^2 g}{\omega^2 L_n^2}$)

$$\left(\frac{k_y^4 g^2}{L_n^4} \right) \frac{-(\omega_r - \omega_i)^2 + 4\omega_r^2 \omega_i^2 + 4i\omega_r \omega_i |(\omega_r^2 - \omega_i^2)|}{|[4\omega_r \omega_i (\omega_r^2 - \omega_i^2)]^2 - [(\omega_r - \omega_i)^2 - 4\omega_r^2 \omega_i^2]^2|} = 4R(T - i\sqrt{R})$$

Here ω_r and ω_i are the real and the imaginary parts of the eigenfrequency, respectively.

Appendix II

Calculations regarding Chapter 3

Let us start with the continuity equation for ions

$$\frac{\partial n_i}{\partial t} + \nabla.(n_i v) = 0 \qquad \text{(II.1)}$$

$$\Rightarrow \frac{\partial \tilde{n}_i}{\partial t} + \nabla.(n_0 + \tilde{n}_i)(v_0 + \tilde{v}_\perp + \tilde{v}_\parallel) = 0,\ v_0 = v_{\parallel 0}(x)\hat{e}_\parallel$$

$$\Rightarrow \frac{\partial \tilde{n}_i}{\partial t} + \nabla_\perp.(n_0\tilde{v}_\perp) + \nabla_\parallel.(\tilde{n}_i v_0) + \nabla_\parallel(n_0\tilde{v}_\parallel) = 0$$

$$\Rightarrow \frac{\partial \tilde{n}_i}{\partial t} + (v_0\ \nabla_\parallel)\tilde{n}_i + n_0(\nabla_\perp.\tilde{v}_\perp) + (\tilde{v}_\perp\ \nabla_\perp)n_0 + n_0\nabla_\parallel\tilde{v}_\parallel = 0 \qquad \text{(II.2)}$$

$$\tilde{v}_\perp = v_E + v_{Di} + v_P$$

$$v_E = \frac{c}{B}\hat{b} \times \nabla_\perp\phi$$

$$v_{Di} = \frac{c}{eBn_i}\hat{b} \times \nabla_\perp P_i,\ \hat{b} = \vec{B}/|B|,\ P_i = P_{i0}(x) + \tilde{p}_i$$

$$v_P = -\frac{c^2 m_i}{eB^2}\left(\frac{\partial}{\partial t} + (v_0 + v_E + v_{Di})\ \nabla\right)\nabla_\perp\phi$$

$$= -\frac{c^2 m_i}{eB^2}\nabla_\perp\frac{\partial\tilde{\phi}}{\partial t} - \frac{c^2 m_i}{eB^2}(v_0.\nabla_\parallel)\nabla_\perp\phi - \frac{c^2 m_i}{eB^2}\left(\frac{c}{B}\hat{b} \times \nabla_\perp\phi.\nabla_\perp\right)\nabla_\perp\phi$$
$$- \frac{c^2 m_i}{eB^2}\left(\frac{c}{eBn_i}\hat{b} \times \nabla_\perp P_i.\nabla_\perp\right)\nabla_\perp\phi$$

here n_0, $\tilde{n}_i$, v_0, $\tilde{v}_\perp$, $\tilde{v}_\parallel$, P_{i0} and $\tilde{p}_i$ are equilibrium density, perturbed density, equilibrium parallel velocity, perturbed perpendicular velocity, perturbed parallel velocity, radial equilibrium ion pressure and perturbed ion pressure respectively. The all other symbols have their usual meaning.

We consider perturbation as

$$\phi(\mathbf{x}, t) = \phi(x)\exp[\imath(k_y y + k_z z - \omega t)].$$

Let us evaluate the terms from equation (II.2) starting with

$$(\tilde{v}_\perp . \nabla_\perp) n_0$$

For the v_E component of $\tilde{v}_\perp$,

$$(\tfrac{c}{B}\hat{b} \times \nabla_\perp \phi . \nabla_\perp) n_0$$

$$= \left(\tfrac{c}{B} \left(-\hat{z} \tfrac{x}{L_s} \tfrac{\partial \phi}{\partial x} + \hat{y} \tfrac{\partial \phi}{\partial x} - \hat{x} \tfrac{\partial \phi}{\partial y} \right) . \left(\hat{x} \tfrac{\partial}{\partial x} + \hat{y} \tfrac{\partial}{\partial y} \right) \right) n_0$$

$$= \tfrac{c}{B} \left(-\tfrac{\partial \phi}{\partial y} \right) \tfrac{\partial n_0}{\partial x}$$

$$= -\tfrac{c}{B} \tfrac{\partial n_o}{\partial x} \nabla_y \phi \qquad \text{(II.3)}$$

For the $v_{D\imath}$ component of $\tilde{v}_\perp$,

$$\nabla_\perp . \left(n_\imath \tfrac{c}{eBn_\imath} \hat{b} \times \nabla_\perp P_\imath \right)$$

$$= \left(\hat{x} \tfrac{\partial}{\partial x} + \hat{y} \tfrac{\partial}{\partial y} \right) . \left(n_\imath \tfrac{c}{eBn_\imath} \hat{b} \times \nabla_\perp P_\imath \right)$$

$$= \tfrac{c}{eB} \tfrac{\partial}{\partial y} \left(\tfrac{\partial P_{\imath 0}(x)}{\partial x} \right) - \tfrac{c}{eB} \tfrac{\partial^2 \tilde{p}_\imath}{\partial x \partial y} + \tfrac{c}{eB} \tfrac{\partial^2 \tilde{p}_\imath}{\partial x \partial y}$$

$$= 0 \qquad \text{(II.4)}$$

For the v_P component of $\tilde{v}_\perp$,

First two terms of v_P

$$=\left(\left(-\frac{c^2m_i}{eB^2}\left(\frac{\partial}{\partial t}+v_0.\nabla_{\|}\right)\nabla_{\perp}\phi\right).\nabla_{\perp}\right)n_0$$

$$=\left(-\frac{c^2m_i}{eB^2}\left(\frac{\partial}{\partial t}+v_0.\nabla_{\|}\right)\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}\right)\phi\right).\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}\right)n_0$$

$$=-\frac{c^2m_i}{eB^2}\left(\frac{\partial}{\partial t}+v_0.\nabla_{\|}\right)\frac{\partial\phi}{\partial x}\frac{\partial n_0}{\partial x} \tag{II.5}$$

For third term of v_P

$$=\left(\left(-\frac{c^2m_i}{eB^2}\left(\left(\frac{c}{B}\hat{b}\times\nabla_{\perp}\phi\right).\nabla_{\perp}\right)\nabla_{\perp}\phi\right).\nabla_{\perp}\right)n_0$$

$$=\left(-\frac{c^3m_i}{eB^3}\left(\left(\left(-\hat{z}\frac{x}{L_s}\frac{\partial\phi}{\partial x}+\hat{y}\frac{\partial\phi}{\partial x}-\hat{x}\frac{\partial\phi}{\partial y}\right).\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}\right)\right)\left(\hat{x}\frac{\partial\phi}{\partial x}+\hat{y}\frac{\partial\phi}{\partial y}\right)\right).\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}\right)\right)n_0$$

$$=\left(-\frac{c^3m_i}{eB^3}\left(-\frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial\phi}{\partial x}\frac{\partial n_0}{\partial x}\right)\right)$$

$$=-\frac{c^3m_i}{eB^3}\left(-\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x^2}\frac{\partial n_0}{\partial x}-\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}\frac{\partial^2 n_0}{\partial x^2}+\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial n_0}{\partial x}\right) \tag{II.6}$$

These are all higher order terms and have subsequently been neglected.

$$\frac{\partial P_{i0}(x)}{\partial x}=n_0\frac{\partial T_i}{\partial x}+T_i\frac{n_0}{\partial x}$$

$$=T_i\frac{\partial n_0}{\partial x}\left(1+\frac{\frac{1}{T_i}\frac{\partial T_i}{\partial x}}{\frac{1}{n_0}\frac{\partial n_0}{\partial r}}\right) \tag{II.7}$$

For fourth term of v_P (starting with $\nabla_{\perp}.(n_iv_P)$)

$$\nabla_{\perp}.n_i\left(-\frac{c^2m_i}{eB^2}\left(\frac{c}{eBn_i}\hat{b}\times P_i.\nabla_{\perp}\right)\nabla_{\perp}\phi\right)$$

$$=\nabla_{\perp}.\left(-\frac{c^3m_i}{eB^3}(\hat{b}\times\nabla_{\perp}P_i.\nabla_{\perp})\nabla_{\perp}\phi\right)$$

$\tilde{p}_i$ yields higher order terms, so they are neglected.

$$=\nabla_{\perp}.\left(-\frac{c^3m_i}{e^2B^3}\left(\left(\hat{y}\frac{x}{L_s}+\hat{z}\right)\times\left(\hat{x}\frac{\partial P_{i0}(x)}{\partial x}\right).\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}\right)\right)\left(\hat{x}\frac{\partial\phi}{\partial x}+\hat{y}\frac{\partial\phi}{\partial y}\right)\right)$$

$$=-\frac{c^3 m_i}{e^2 B^3}\left(\frac{\partial P_{i0}(x)}{\partial x}\frac{\partial^3\phi}{\partial x^2\partial y}+\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 P_{i0}(x)}{\partial x^2}+\frac{\partial P_{i0}(x)}{\partial x}\frac{\partial^3\phi}{\partial y^3}\right)$$

The second term is a higher order term $\left(\sim \frac{1}{L_n^2}\right)$ so we neglect it.

$$=-\frac{c^3 m_i}{e^2 B^3}\left\{\frac{\partial P_{i0}(x)}{\partial x}(\nabla_x^2\nabla_y\phi+\nabla_y^2\nabla_y\phi)\right\}$$

$$=-\frac{c^3 m_i}{e^2 B^3}\frac{\partial P_{i0}(x)}{\partial x}\nabla_\perp^2\nabla_y\phi$$

$$=-\frac{c^3 m_i T_i}{e^2 B^3}\frac{\partial n_0}{\partial x}(1+\eta_i)\nabla_\perp^2\nabla_y\phi \qquad \text{(II.8) (using (II.7))}$$

To evaluate $n_0(\nabla_\perp.\tilde{v}_\perp)$ of equation (II.2)

For the v_E component of $\tilde{v}_\perp$

$$=n_0\left(\nabla_\perp\cdot\left(\frac{c}{B}\hat{b}\times\nabla_\perp\phi\right)\right)$$

$$=n_0\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}\right)\cdot\left(\frac{c}{B}\left(-\hat{z}\frac{x}{L_s}\frac{\partial\phi}{\partial x}+\hat{y}\frac{\partial\phi}{\partial x}-\hat{x}\frac{\partial\phi}{\partial y}\right)\right)$$

$$=0 \qquad \text{(II.9)}$$

For v_{Di}, it is already worked out.

For the v_P component of $\tilde{v}_\perp$

First part

$$n_0\nabla_\perp\cdot\left(-\frac{c^2 m_i}{eB^2}\left(\frac{\partial}{\partial t}+v_0.\nabla_\parallel\right)\nabla_\perp\phi\right)$$

$$=n_0\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}\right)\cdot\left(-\frac{c^2 m_i}{eB^2}\left(\frac{\partial}{\partial t}\left(\hat{x}\frac{\partial\phi}{\partial x}+\hat{y}\frac{\partial\phi}{\partial y}\right)+v_0\nabla_\parallel\left(\hat{x}\frac{\partial\phi}{\partial x}+\hat{y}\frac{\partial\phi}{\partial y}\right)\right)\right)$$

$$=n_0\left(-\frac{c^2m_i}{eB^2}\left(\left(\frac{\partial^3\phi}{\partial x^2\partial t}+\frac{\partial^3\phi}{\partial y^2\partial t}\right)+\frac{\partial}{\partial x}\left(v_0\frac{\partial\nabla_\parallel\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(v_0\frac{\partial\nabla_\parallel\phi}{\partial y}\right)\right)\right)$$

$$=-\frac{n_0c^2m_i}{eB^2}\left(\frac{\partial}{\partial t}+v_0.\nabla_\parallel\right)\nabla_\perp^2\phi-n_0\frac{c^2m_i}{eB^2}\frac{\partial v_0}{\partial x}\frac{\partial\nabla_\parallel\phi}{\partial x} \tag{II.10}$$

The last term is neglected as it is a small quantity ($k_\parallel$ made the term smaller).

Now,

$$n_0\left(\nabla_\perp.-\frac{c^2m_i}{eB^2}\left(v_E.\nabla_\perp\right)\nabla_\perp\phi\right)$$

$$=-\frac{n_0c^3m_i}{eB^3}\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}\right).\left(\frac{\partial\phi}{\partial x}\nabla_\perp\frac{\partial\phi}{\partial y}-\frac{\partial\phi}{\partial y}\nabla_\perp\frac{\partial\phi}{\partial x}\right)$$

$$=-\frac{n_0c^3m_i}{eB^3}\left(\frac{\partial\phi}{\partial x}\nabla_\perp^2\frac{\partial\phi}{\partial y}+\nabla_\perp\frac{\partial\phi}{\partial y}.\nabla_\perp\frac{\partial\phi}{\partial x}-\frac{\partial\phi}{\partial y}\nabla_\perp^2\frac{\partial\phi}{\partial x}-\nabla_\perp\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\right)$$

$$=-\frac{n_0c^3m_i}{eB^3}\left(\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}\right)\nabla_\perp^2\phi$$

$$=-\frac{n_0c^3m_i}{eB^3}\left(\left(\hat{b}\times\nabla_\perp\phi\right).\nabla_\perp\right)\nabla_\perp^2\phi \tag{II.11}$$

We rewrite the equation (II.1) (considering the relevant terms from equation II.2, II.3, II.4,II.5, II.6, II.7,II.8, II.9, II.10 and II.11)

$$\frac{\partial\tilde{n}_i}{\partial t}+(v_0.\nabla_\parallel)\tilde{n}_i-\frac{n_0c^2m_i}{eB^2}\left(\frac{\partial}{\partial t}+v_0.\nabla_\parallel\right)\nabla_\perp^2\phi-\frac{c}{B}\frac{\partial n_e}{\partial x}\nabla_y\phi-\frac{c^3m_iT_i}{e^2B^3}\frac{\partial n_0}{\partial x}\left(1+\eta_i\right)\nabla_\perp^2\nabla_y\phi$$
$$-\frac{n_0c^3m_i}{eB^3}\left(\left(\hat{b}\times\nabla_\perp\phi\right).\nabla_\perp\right)\nabla_\perp^2\phi+n_0\nabla_\parallel\tilde{v}_\parallel=0$$

Put $\tilde{n}_i=n_0\tilde{\phi}$ and $\phi=\frac{T_e}{e}\tilde{\phi}$

$$\Rightarrow\frac{\partial\tilde{\phi}}{\partial t}+(v_0.\nabla_\parallel)\tilde{\phi}-\frac{T_ec^2m_i}{e^2B^2}\left(\frac{\partial}{\partial t}+v_0.\nabla_\parallel\right)\nabla_\perp^2\tilde{\phi}-\frac{cT_e}{eBn_0}\frac{\partial n_e}{\partial x}\nabla_y\tilde{\phi}-\frac{T_ec^3m_iT_i}{e^3B^3n_0}\frac{\partial n_0}{\partial x}\left(1+\eta_i\right)$$
$$\nabla_\perp^2\nabla_y\tilde{\phi}+\nabla_\parallel\tilde{v}_\parallel-\frac{n_0c^3T_e^2m_i}{e^3B^3}\left(\left(\hat{b}\times\nabla_\perp\phi\right).\nabla_\perp\right)\nabla_\perp^2\phi=0.$$

$$\Rightarrow\left(\frac{\partial}{\partial t}+v_0.\nabla\right)(1-\nabla_\perp^2)\tilde{\phi}+v_D\left[1+\left(\frac{1+\eta_i}{\tau}\right)\nabla_\perp^2\right]\nabla_y\tilde{\phi}$$
$$-\hat{b}\times\nabla\tilde{\phi}.\nabla(\nabla_\perp^2\tilde{\phi})+\nabla_\parallel\tilde{v}_\parallel=0 \tag{II.12}$$

This is our equation (3.2) of Chapter 3.

To simplify the quantities we re-scale time and distance to units of ω_{ci}^{-1} and $\rho_s(= c_s/\omega_{ci})$ and the other quantities as stated below:

$\tilde{\phi} \equiv e\phi/T_e, \tilde{v}_{\parallel} \equiv \tilde{v}_{\parallel i}/c_s, \tilde{p} \equiv [\tilde{p}_i/ < P_{i0} >](T_i/T_e), P_i =< P_{i0}(x) > +\tilde{p}_i, n_i = n_0 + \tilde{n}_i, v_D \equiv -\frac{d(\ln n_0)}{dx}, \tau \equiv \frac{T_e}{T_i}, \mathcal{Y} \equiv \frac{\Gamma}{\tau}, \mu \equiv \frac{\mu_{\parallel}\omega_{ci}}{c_s^2}, \eta_i = \frac{d \ln T_i}{d \ln n_0}, v_0 = \frac{<v_\varphi>}{c_s}, c_s^2 = \frac{T_e}{m_i}, \omega_{ci} = \frac{eB_0}{cm_i}$.

Let us take momentum balance equation

$$m_i n_i \left(\frac{\partial(v_{\parallel 0}(x) + \tilde{v}_{\parallel i})}{\partial t} + (v_E + v_{\parallel 0}(x)).\nabla(v_{\parallel 0}(x) + \tilde{v}_{\parallel i}) \right)$$
$$= -en_i \nabla_{\parallel}\phi - \nabla_{\parallel}(P_{i0} + \tilde{p}_i) + m_i n_i \mu_{\parallel} \nabla_{\parallel}^2 v_{\parallel i} \qquad \text{(II.13)}$$

$$\Rightarrow \frac{\partial \tilde{v}_{\parallel i}}{\partial t} + (v_E + v_{\parallel 0}(x)).\nabla(v_{\parallel 0}(x) + \tilde{v}_{\parallel i}) = -\frac{e}{m_i}\nabla_{\parallel}\phi - \frac{1}{m_i n_i}\nabla_{\parallel}P_i + \mu_{\parallel}\nabla_{\parallel}^2(v_0 + \tilde{v}_{\parallel i})$$

$$\Rightarrow \frac{\partial \tilde{v}_{\parallel i}}{\partial t} + v_E.\nabla v_{\parallel 0}(x) + v_E.\nabla \tilde{v}_{\parallel i} + v_{\parallel 0}(x).\nabla \tilde{v}_{\parallel i} = -\frac{e}{m_i}\nabla_{\parallel}\phi - \frac{1}{m_i n_i}\nabla_{\parallel}\tilde{p}_i + \mu_{\parallel}\nabla_{\parallel}^2 \tilde{v}_{\parallel i}$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + v_{\parallel 0}(x).\nabla\right)\tilde{v}_{\parallel i} + v_E.\nabla v_{\parallel 0}(x) + v_E.\nabla \tilde{v}_{\parallel i} = -\frac{e}{m_i}\nabla_{\parallel}\phi - \frac{1}{m_i n_i}\nabla_{\parallel}\tilde{p}_i + \mu_{\parallel}\nabla_{\parallel}^2 \tilde{v}_{\parallel i}$$

Let us evaluate each terms separately

$$v_E.\nabla v_{\parallel 0}(x)$$

$$= \left(\frac{c}{B}\hat{b} \times \nabla_{\perp}\phi\right).\nabla v_{\parallel 0}(x)$$

$$= \left(\frac{c}{B}\left(\hat{z} + \hat{y}\frac{x}{L_s}\right) \times \nabla_{\perp}\phi\right).\nabla v_{\parallel 0}(x)$$

$$= \left(\frac{cT_e}{eB}\left(\hat{z} + \hat{y}\frac{x}{L_s}\right) \times \nabla_{\perp}\tilde{\phi}\right).\nabla v_{\parallel 0}(x)$$

$$= \frac{cT_e}{eB}\left(\hat{y}\partial_x\tilde{\phi} - \hat{x}\partial_y\tilde{\phi} - \hat{z}\frac{x}{L_s}\partial_x\tilde{\phi}\right).\nabla v_{\parallel 0}(x)$$

$=-\frac{cT_e}{eB}\partial_y\tilde{\phi}\partial_x v_{\|0}$

$=-\frac{cT_e}{eB}\frac{v_0}{L_v}\nabla_y\tilde{\phi},\ \frac{1}{L_v} = \frac{1}{v_0}\frac{\partial v_{\|0}(x)}{\partial x}$

$$=-\frac{c_s^2}{\omega_{ci}}\frac{v_{\|0}(x)}{L_v}\nabla_y\tilde{\phi},\ \frac{c_s^2}{\omega_{ci}} = \frac{cT_e}{eB} \quad \text{(II.14)}$$

In all the cases $\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}$

$v_E.\nabla\tilde{v}_{\|i}$

$=\left(\frac{c}{B}\hat{b} \times \nabla_\perp\phi\right).\nabla c_s\tilde{v}_\|$

$=\frac{c_s cT_e}{eB}\hat{b} \times \nabla\tilde{\phi}.\nabla\tilde{v}_\|$

$=\frac{c_s^3}{\omega_{ci}}\hat{b} \times \nabla\tilde{\phi}.\nabla\tilde{v}_\|$

$-\frac{e}{m_i}\nabla_\|\phi$

$=-\frac{eT_e}{m_i e}\nabla_\|\tilde{\phi}$

$=-c_s^2\nabla_\|\tilde{\phi}$

$=-\frac{1}{m_i n_i}\nabla_\|\tilde{p}_i$

$=-\frac{P_{i0}T_e}{m_i n_i T_i}\nabla_\|\tilde{p}$

$=-c_s^2\nabla_\|\tilde{p},\ \frac{P_{i0}(x)}{n_i T_i} = 1$

$\mu_\|\nabla_\|^2\tilde{v}_{\|i}$

$=\mu_\| c_s\nabla_\|^2\tilde{v}_\|$

We rewrite equation (II.13),

$$\Rightarrow \frac{\omega_{ci}}{c_s^2}\left(\frac{\partial}{\partial t} + v_0.\nabla\right)\tilde{v}_{\parallel} - \frac{v_0}{L_v}\nabla_y\tilde{\phi} + \hat{b}\times\nabla\tilde{\phi}.\nabla\tilde{v}_{\parallel} = -\nabla_{\parallel}\tilde{\phi} - \nabla_{\parallel}\tilde{p} + \frac{\mu_{\parallel}\omega_{ci}}{c_s^2}\nabla_{\parallel}^2\tilde{v}_{\parallel}$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + v_0.\nabla\right)\tilde{v}_{\parallel} - \frac{v_0}{L_v}\nabla_y\tilde{\phi} + \hat{b}\times\nabla\tilde{\phi}.\nabla\tilde{v}_{\parallel} = -\nabla_{\parallel}\tilde{\phi} - \nabla_{\parallel}\tilde{p} + \mu\nabla_{\parallel}^2\tilde{v}_{\parallel} \qquad \text{(II.15)}$$

This is our equation (3.4) of Chapter 3.

Here $\mu = \frac{\mu_{\parallel}\omega_{ci}}{c_s^2}$.

Let us take the pressure balance equation

$$\frac{(\partial(P_{i0}+\tilde{p}_i)}{\partial t} + (v_E + v_0).\nabla(P_{i0} + \tilde{p}_i) + \Gamma(P_{i0} + \tilde{p}_i)\nabla_{\parallel}(v_0 + \tilde{v}_{\parallel}) = 0 \qquad \text{(II.16)}$$

$$\Rightarrow \frac{\partial\tilde{p}_i}{\partial t} + v_E.\nabla P_{i0} + v_E.\nabla\tilde{p}_i + v_0.\nabla\tilde{p}_i + \Gamma P_{i0}\nabla_{\parallel}\tilde{v}_{\parallel i} = 0$$

$$\Rightarrow n_0 T_e\left(\frac{\partial}{\partial t} + v_0.\nabla\right)\tilde{p} + v_E.\nabla P_{i0} + v_E.\nabla\tilde{p}_i + \Gamma P_{i0}\nabla_{\parallel}\tilde{v}_{\parallel i} = 0.$$

Here,

$$\tilde{p}_i = T_e n_0 \tilde{p}$$

$$v_0.\nabla P_{i0} = 0$$

$$\Rightarrow \left(n_0 T_e\frac{\partial}{\partial t} + v_0.\nabla\right)\tilde{p} + v_E.\nabla P_{i0} + v_E.\nabla\tilde{p}_i + \Gamma P_{i0}\nabla_{\parallel}\tilde{v}_{\parallel i} = 0$$

$$v_E.\nabla P_{i0}$$

$$= \left(\frac{c}{B}\hat{b}\times\nabla_{\perp}\phi\right).\nabla(n_0 T_i)$$

$=\frac{cT_e}{eB}(-\hat{x}\nabla_y\tilde{\phi}).\nabla(n_0\frac{\partial T_i}{\partial x}+T_i\frac{\partial n_0}{\partial x})$

$=-\frac{cT_e}{eB}\nabla_y\tilde{\phi}T_i n_0\frac{1}{n_0}\frac{\partial n_0}{\partial x}\left(1+\frac{d\ln T_i}{d\ln n_0}\right)$

$=T_e n_0 v_D\left(\frac{1+\eta_i}{\tau}\right)\nabla_y\tilde{\phi}$

$v_E.\nabla\tilde{p}_i$

$=\left(\frac{c}{B}\hat{b}\times\nabla_\perp\phi\right)\nabla\tilde{p}_i$

$=\frac{cT_e}{eB}\hat{b}\times\nabla_\perp\tilde{\phi}.\nabla\tilde{p}_i$

$=T_e n_0\hat{b}\times\nabla_\perp\tilde{\phi}.\nabla\tilde{p},\ \frac{cT_e}{eB}=\frac{\rho_i^2}{\omega_{ci}}=$

$=\Gamma P_{i0}\nabla_\|\tilde{v}_{\|i}$

$=\Gamma n_0 T_i\nabla_\|\tilde{v}_{\|i}$

$=n_0 T_e c_s\frac{\Gamma}{\tau}\nabla_\|\tilde{v}_\|$

$=n_0 T_e c_s\mathcal{Y}\nabla_\|\tilde{v}_\|$

Considering the relevant terms, we rewrite equation (II.16),

$$n_0 T_e\left(\frac{\partial}{\partial t}+v_0.\nabla\right)\tilde{p}+T_e n_0 v_D\left(\frac{1+\eta_i}{\tau}\right)\nabla_y\tilde{\phi}+n_0 T_e\hat{b}\times\nabla_\perp\tilde{\phi}.\nabla\tilde{p}+n_0 T_e c_s\mathcal{Y}\nabla_\|\tilde{v}_\|=0$$

$$\Rightarrow\left(\frac{\partial}{\partial t}+v_0.\nabla\right)\tilde{p}+T_e v_D\left(\frac{1+\eta_i}{\tau}\right)\nabla_y\tilde{\phi}+\hat{b}\times\nabla_\perp\tilde{\phi}.\nabla\tilde{p}+\mathcal{Y}\nabla_\|\tilde{v}_\|=0 \qquad \text{(II.17)}$$

This is our equation (3.6) of Chapter 3.

We'll now evaluate each term from equation (II.12)

$\left(\frac{\partial}{\partial t} + v_0.\nabla\right)(1 - \nabla_\perp^2)\tilde{\phi}$

$=\frac{\partial\tilde{\phi}}{\partial t} - \left(\frac{\partial}{\partial t}\nabla_\perp^2\right)\tilde{\phi} + (v_0.\nabla)\tilde{\phi} - (v_0.\nabla)\nabla_\perp^2\tilde{\phi}$

$=-i\omega\tilde{\phi} + i\omega\frac{\partial^2\tilde{\phi}}{\partial x^2} - i\omega k_y^2\tilde{\phi} + ik_\parallel v_0\tilde{\phi} - ik_\parallel v_0\frac{\partial^2\tilde{\phi}}{\partial x^2} + ik_y^2 k_\parallel v_0\tilde{\phi}$

$v_D\left[1 + \left(\frac{1+\eta_i}{\tau}\right)\nabla_\perp^2\right]\nabla_y\tilde{\phi}$

$=ik_y v_D\tilde{\phi} + v_D\left(\frac{1+\eta_i}{\tau}\right)ik_y\frac{\partial^2\tilde{\phi}}{\partial x^2} + v_D i^3 k_y^3\left(\frac{1+\eta_i}{\tau}\right)\tilde{\phi}$

$=ik_y v_D\tilde{\phi} + ik_y v_D\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial^2\tilde{\phi}}{\partial x^2} - ik_y^3 v_D\left(\frac{1+\eta_i}{\tau}\right)\tilde{\phi}$

$-\hat{b}\times\nabla\tilde{\phi}.\nabla(\nabla_\perp^2\tilde{\phi})$

This higher order term is neglected

$\nabla_\parallel\tilde{v}_\parallel$

$=ik_\parallel\tilde{v}_\parallel$

Now we evaluate each term of equation (II.15)

First term

$\left(\frac{\partial}{\partial t} + v_0.\nabla\right)\tilde{v}_\parallel$

$=-i\omega\tilde{v}_\parallel + ik_\parallel v_0\tilde{v}_\parallel$

$-\frac{v_0}{L_v}\nabla_y\tilde{\phi}$

$=-\frac{ik_y v_0}{L_v}\tilde{\phi}$

$\hat{b} \times \tilde{\phi}.\nabla \tilde{v}_{\|}$

This is a higher order quantity and is neglected.

$\nabla_{\|}\tilde{\phi}$

$=\imath k_{\|}\tilde{\phi}$

$\nabla_{\|}\tilde{p}$

$=\imath k_{\|}\tilde{p}$

$-\mu \nabla_{\|}^2 \tilde{v}_{\|}$

$=\mu k_{\|}^2$

This quantity is neglected as it involves the higher order term $k_{\|}^2$.

Now we evaluate each term of equation (II.17).

First term

$\left(\frac{\partial}{\partial t} + v_0.\nabla\right)\tilde{p}$

$=-\imath\omega\tilde{p} + \imath k_{\|} v_0 \tilde{p}$

$v_D\left(\frac{1+\eta_\imath}{\tau}\right)\nabla_y\tilde{\phi}$

$=\imath k_y v_D\left(\frac{1+\eta_\imath}{\tau}\right)\tilde{\phi}$

$\hat{b} \times \nabla\tilde{\phi}.\nabla\tilde{p}$

This is a higher order term and is neglected.

$\mathcal{Y}\nabla_{\|}\tilde{v}_{\|}$

$=\imath k_{\|}\mathcal{Y}\tilde{v}_{\|}$

This term is neglected, as it gives only the higher order correction.

We rewrite equation (II.17) with the evaluated relevant terms.

$$\Rightarrow -\imath\omega\tilde{p} + \imath k_{\|}v_0\tilde{p} + \imath k_y v_D \left(\frac{1+\eta_i}{\tau}\right)\tilde{\phi} = 0$$

$$\Rightarrow \tilde{p} = \frac{k_y v_D}{\omega - k_{\|}v_0}\left(\frac{1+\eta_i}{\tau}\right)\tilde{\phi} \qquad \text{(II.18)}$$

We write equation (II.15) with the relevant terms.

$$\Rightarrow -\imath\omega\tilde{v}_{\|} + \imath k_{\|}v_0\tilde{v}_{\|} - \imath k_y \frac{v_0}{L_v}\tilde{\phi} + \imath k_{\|}\tilde{\phi} + \imath k_{\|}\tilde{p} = 0$$

$$\Rightarrow (\omega - k_{\|}v_0)\tilde{v}_{\|} = k_{\|}\tilde{\phi} - \frac{k_y v_0}{L_v}\tilde{\phi} + k_{\|}\left[\frac{k_y v_D}{\omega - k_{\|}v_0}\left(\frac{1+\eta_i}{\tau}\right)\tilde{\phi}\right]$$

Here in the last term we have put $\tilde{p}$

$$\Rightarrow \tilde{v}_{\|} = \frac{k_{\|}\tilde{\phi}}{\omega - k_{\|}v_0} - \frac{k_y v_0\tilde{\phi}}{L_v(\omega - k_{\|}v_0)} + \frac{k_y k_{\|}v_D}{(\omega - k_{\|}v_0)(\omega - k_{\|}v_0)}\left(\frac{1+\eta_i}{\tau}\right)\tilde{\phi} \qquad \text{(II.19)}$$

Now we write equation (II.12) by putting the value of $\tilde{v}_{\|}$

$$\left\{(\omega - k_{\|}v_0) + k_y v_D\left(\frac{1+\eta_i}{\tau}\right)\right\}\frac{\partial^2\tilde{\phi}}{\partial x^2} - k_y^2(\omega - k_{\|}v_0)\tilde{\phi} - (\omega - k_{\|}v_0)\tilde{\phi} + k_y v_D\tilde{\phi} - k_y^3 v_D\left(\frac{1+\eta_i}{\tau}\right)\tilde{\phi} + \frac{k_{\|}^2\tilde{\phi}}{\omega - k_{\|}v_0} - \frac{k_y k_{\|}v_0\tilde{\phi}}{L_v(\omega - k_{\|}v_0)} + \frac{k_y k_{\|}^2 v_D}{(\omega - k_{\|}v_0)(\omega - k_{\|}v_0)}\left(\frac{1+\eta_i}{\tau}\right)\tilde{\phi} = 0$$

Substituting $\omega - k_{\|}v_0 = \tilde{\omega}$

$$\Rightarrow \left\{\tilde{\omega} + k_y v_D \left(\frac{1+\eta_i}{\tau}\right)\right\} \frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \tilde{\omega}\tilde{\phi} - \tilde{\omega}\tilde{\phi} + k_y v_D \tilde{\phi} - k_y^3 v_D \left(\frac{1+\eta_i}{\tau}\right) \tilde{\phi} + \frac{k_\parallel^2 \tilde{\phi}}{\omega - k_\parallel v_0} - \frac{k_y k_\parallel v_0 \tilde{\phi}}{L_v \tilde{\omega}} + \frac{k_y k_\parallel^2 v_D}{\tilde{\omega}\tilde{\omega}} \left(\frac{1+\eta_i}{\tau}\right) \tilde{\phi} = 0$$

$$\Rightarrow \left\{\tilde{\omega} + k_y v_D \left(\frac{1+\eta_i}{\tau}\right)\right\} \frac{\partial^2 \tilde{\phi}}{\partial x^2} - k_y^2 \left\{\tilde{\omega} + k_y v_D \left(\frac{1+\eta_i}{\tau}\right)\right\} \tilde{\phi} + k_y v_D \tilde{\phi} - \tilde{\omega}\tilde{\phi} - \frac{k_y^2 v_0 x}{L_v L_S \tilde{\omega}} + \frac{k_y^2 x^2}{L_s^2 \tilde{\omega}^2} \left\{\tilde{\omega} + k_y v_D \left(\frac{1+\eta_i}{\tau}\right)\right\} \tilde{\phi} = 0$$

$$\Rightarrow \frac{\partial^2 \tilde{\phi}}{\partial x^2} + \left[-k_y^2 + \frac{k_y v_D - \tilde{\omega}}{\tilde{\omega} + k_y v_D \left(\frac{1+\eta_i}{\tau}\right)} - \frac{v_0 k_y^2 x}{L_v L_s \tilde{\omega}\left(\tilde{\omega} + k_y v_D \left(\frac{1+\eta_i}{\tau}\right)\right)} + \frac{k_y^2 x^2}{L_s^2 \tilde{\omega}^2}\right] \tilde{\phi} = 0$$

$$\Rightarrow \frac{\partial^2 \tilde{\phi}}{\partial x^2} + \left[-k_y^2 + \frac{k_y v_D - \tilde{\omega}}{\tilde{\omega} + k_y v_D \left(\frac{1+\eta_i}{\tau}\right)} - \frac{v_0 k_y^2 L_n^2 v_D^2 x}{L_v L_s \tilde{\omega}\left\{\tilde{\omega} + k_y v_D \left(\frac{1+\eta_i}{\tau}\right)\right\}} + \frac{k_y^2 x^2}{L_s^2 \tilde{\omega}^2}\right] \tilde{\phi} = 0 \tag{II.20}$$

We rearrange (II.20)

$$\Rightarrow \frac{\partial^2 \tilde{\phi}}{\partial x^2} + \left[-k_y^2 + \frac{1 - \frac{\tilde{\omega}}{k_y v_D}}{\frac{\tilde{\omega}}{k_y v_D} + \frac{1+\eta_i}{\tau}} - \frac{\frac{v_0 L_n}{L_v}}{\frac{\tilde{\omega}}{k_y v_D}\left(\frac{\tilde{\omega}}{k_y v_D} + \frac{1+\eta_i}{\tau}\right)} \frac{L_n}{L_s} x + \frac{\frac{L_n^2}{L_S^2} x^2}{\frac{\tilde{\omega}^2}{k_y^2 v_D^2}}\right] \tilde{\phi} = 0 \tag{II.21}$$

We assume the velocity profile as described in the main text.

Let us define,

$$J_2 = \left(\frac{v_{00} L_n}{L_{v2}}\right)^2.$$

This term is due to flow curvature.

Let us substitute

$$\Omega = \frac{\tilde{\omega}}{k_y v_D}$$

$$K = \frac{1+\eta_i}{\tau}$$

$$S = \frac{L_n}{L_s}.$$

We get the eigenvalue equation (II.21) as

$$\Rightarrow \frac{\partial^2 \tilde{\phi}}{\partial x^2} + \left[-k_y^2 + \frac{1-\Omega}{\Omega+K} + \left(-\frac{J_2^{\frac{1}{2}} S}{\Omega(\Omega+K)} + \frac{S^2}{\Omega^2} \right) x^2 \right] \tilde{\phi} = 0.$$

We further substitute

$$\Lambda = -k_y^2 + \frac{1-\Omega}{\Omega+K}$$

$$P = -\frac{J_2^{\frac{1}{2}} S}{\Omega(\Omega+K)} + \frac{S^2}{\Omega^2}$$

$$= \frac{S^2}{\Omega^2} \left(1 - \frac{J_2^{\frac{1}{2}} \Omega}{S(\Omega+K)} \right)$$

$$\Rightarrow \frac{\partial^2 \tilde{\phi}}{\partial x^2} + [\Lambda + P x^2] = 0 \tag{II.22}$$

For $P > 0$ we get $\left(\frac{J_2^{\frac{1}{2}} S}{\Omega(\Omega+K)} < \frac{S^2}{\Omega^2} \right)$ stable mode. From this expression we see that the increase in the positive flow curvature renders the mode unstable.

The dispersion relation

$$\Lambda = i\sqrt{|P|}$$

$$\Rightarrow \Lambda - i\sqrt{|P|} = 0$$

$$\Rightarrow -k_y^2 + \frac{1-\Omega}{\Omega+K} - i\sqrt{\left| \frac{S^2}{\Omega^2} \left(1 - \frac{J_2^{\frac{1}{2}} \Omega}{S(\Omega+K)} \right) \right|} = 0$$

$$\Rightarrow \left(-k_y^2 + \frac{1-\Omega}{\Omega+K} \right) \Omega(\Omega + K) - \imath S(\Omega + K) \sqrt{\left| 1 - \frac{J_2^{\frac{1}{2}} \Omega}{S(\Omega+K)} \right|} = 0$$

$$\Rightarrow (1 + k_y^2)\Omega^2 + (K\kappa_y^2 - 1)\Omega = -iS(\Omega + K) \sqrt{\left| 1 - \frac{J_2^{\frac{1}{2}} \Omega}{S(\Omega+K)} \right|} \tag{II.23}$$

This dispersion relation (II.23) is same as the equation (3.10) when the shear flow term included.

Appendix III

Calculation regarding Chapter 4

Let us start with the pressure balance equation

$$\frac{\partial P_i}{\partial t} + (\vec{v}_E + v_0).\vec{\nabla} P_i + \Gamma \nabla_{\|} \tilde{v}_{\|i} = 0 \tag{III.1}$$

here $P_i(r), \tilde{p}_i, v_0, \tilde{v}_{\|i}$ and Γ respectively are equilibrium total radial ion pressure, perturbed ion pressure, equilibrium parallel velocity, perturbed ion parallel velocity and ratio of specific heats. The other quantities have their usual meaning. Here $\vec{v}_E$ is the perturbed perpendicular velocity ($\vec{v}_E = \frac{c}{B^2}\vec{B} \times \nabla_{\perp}\tilde{\varphi} = \frac{c}{B^2}(B_\theta\hat{\theta} + B_\phi\hat{\phi}) \times \nabla_{\perp}\tilde{\varphi} = \frac{c}{B}\hat{b} \times \nabla_{\perp}\tilde{\varphi}$, B is the magnetic field and $\hat{b} = \vec{B}/B$.

We have considered perturbation structure of various quantities in following form

$$\tilde{\varphi} = \varphi(r, \theta)\exp\{i(n\phi - m\theta - \omega t)\}$$

Let us evaluate each term from equation (III.1)

$$\frac{\partial P_i}{\partial t} = \frac{\partial (P_{i0}(r) + \tilde{p}_i)}{\partial t}$$

$$= \frac{\partial \tilde{p}_i}{\partial t}$$

$$= -i\omega \tilde{p}_i$$

$$v_0.\nabla P_i$$

$$= v_0.\nabla \tilde{p}_i$$

$=ik_{\|}v_0\tilde{p}_i$

This term gives rise to Doppler shift.

$\vec{v}_E.\vec{\nabla}(P_{i0}(r)+\tilde{p}_i)$

$=\vec{v}_E.\vec{\nabla}P_{i0}(r)$, $\tilde{p}_i$ term will give higher order quantity and is therefore neglected.

$$=\vec{v}_E.\hat{r}\frac{\partial}{\partial r}P_{i0}(r)$$

$$=\frac{c(B_\theta\hat{\theta}+B_\phi\hat{\phi})\times\vec{\nabla}_\perp\tilde{\varphi}}{B^2}.\hat{r}\frac{\partial P_{i0}(r)}{\partial r}$$

$$=\frac{c}{B^2}\left(-\hat{\phi}B_\theta\frac{\partial\tilde{\varphi}}{\partial r}+\hat{\theta}B_\phi\frac{\partial\tilde{\varphi}}{\partial r}-\hat{r}\frac{1}{r}B_\phi\frac{\partial\tilde{\varphi}}{\partial\theta}\right).\hat{r}\frac{\partial P_{i0}}{\partial r}$$

$$=\frac{c}{B^2}\left(-\frac{B_\phi}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\frac{\partial P_{i0}}{\partial r}\right)$$

$$=-\frac{c}{B^2}\frac{B_\phi}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\frac{\partial P_{i0}}{\partial r}$$

$$=-\frac{c}{B^2}\frac{B_\phi}{r}\frac{\partial(n_0(T_e+T_i))}{\partial r}\frac{\partial\tilde{\varphi}}{\partial\theta}$$

$$=-\frac{c}{B^2}\frac{B_\phi}{r}\left(n_0\frac{\partial T_i}{\partial r}+T_i\frac{\partial n_0}{\partial r}\right)\frac{\partial\tilde{\varphi}}{\partial\theta}$$

$$=-\frac{c}{B^2}\frac{B_\phi}{r}T_i\frac{\partial n_0}{\partial r}\left(1+\frac{n_0\frac{\partial T_i}{\partial r}}{T_i\frac{\partial n_0}{\partial r}}\right)\frac{\partial\tilde{\varphi}}{\partial\theta}$$

$$=-\frac{c}{Br}\frac{T_e}{\tau}\frac{\partial n_0}{\partial r}\left(1+\frac{dlnT_i}{dlnn_0}\right)\frac{\partial\tilde{\varphi}}{\partial\theta}$$

where $\tau=\frac{T_e}{T_i}$

$$=-\frac{c}{Br}\frac{T_e}{\tau}\frac{\partial n_0}{\partial r}(1+\eta_i)\frac{\partial\tilde{\varphi}}{\partial\theta}$$

$\Gamma P_{i0}\vec{\nabla}_{\|}\tilde{v}_{\|i}$

$= \Gamma P_{i0}\frac{1}{qR}\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\phi}\right)\tilde{v}_{\parallel i}$

$= \frac{\Gamma}{\tau}\frac{n_0 T_e}{qR}\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\phi}\right)\tilde{v}_{\parallel i}$

here $P_{i0} = \frac{n_0 T_e}{\tau}$

The quantity $\frac{\Gamma}{\tau}$ is neglected as it gives the correction of order $\left(\kappa_{\parallel}\right)^4$

Rearranging the evaluated terms of equation (III.1)

$-i\omega\tilde{p}_i - \frac{cT_e}{Br\tau}\frac{\partial n_0}{\partial r}\left(1+\eta_i\right)\frac{\partial\tilde{\varphi}}{\partial\theta} = 0$

$$\Rightarrow \tilde{p}_i = \frac{icT_e}{\omega rB}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial\tilde{\varphi}}{\partial\theta} \qquad \text{(III.2)}$$

Let us take momentum balance equation

$$m_i n_i\left(\frac{\partial\tilde{v}_{\parallel i}}{\partial t} + \left(\vec{v}_E + v_{\parallel 0}(r)\right).\vec{\nabla}\left(\tilde{v}_{\parallel i} + v_{\parallel 0}(r)\right)\right) = -en_i\nabla_{\parallel}\tilde{\varphi} - \vec{\nabla}_{\parallel}\tilde{p}_i + \mu_{\parallel}\vec{\nabla}_{\parallel}^2\tilde{v}_{\parallel i} \qquad \text{(III.3)}$$

Let us evaluate each term from equation (III.3)

$m_i n_0\frac{\partial\tilde{v}_{\parallel i}}{\partial t}$

$= -i\omega m_i n_0\tilde{v}_{\parallel i}$

$m_i n_i v_{\parallel 0}(r).\nabla\tilde{v}_{\parallel i}$

$= m_i n_i v_{\parallel 0}(r) i k_{\parallel}\tilde{v}_{\parallel i}$

This gives a frequency shift (Doppler shift) term.

$m_i n_0 v_E.\nabla v_{\parallel 0}(r)$

$= -\frac{m_i n_0 c}{Br}\frac{\partial v_{\|0}(r)}{\partial r}\frac{\partial \tilde{\varphi}}{\partial \theta}$

$e n_0 \nabla_{\|}\tilde{\varphi}$

$= e n_0 \left(\frac{\hat{\theta}B_\theta + \hat{\phi}B_\phi}{B}\right).\vec{\nabla}\tilde{\varphi}$

We define $\nabla_{\|} = \frac{\hat{\theta}B_\theta + \hat{\phi}B_\phi}{B}.\vec{\nabla} = \frac{1}{qR}\left(\frac{\partial}{\partial \theta} + q\frac{\partial}{\partial \phi}\right) = ik_{\|}$

$= \frac{e n_0}{B}\left(\frac{B\theta}{r}\frac{\partial \tilde{\varphi}}{\partial \theta} + \frac{B_\phi}{R}\frac{\partial \tilde{\varphi}}{\partial \phi}\right)$

$= \frac{e n_0}{qR}\left(\frac{\partial \tilde{\varphi}}{\partial \theta} + q\frac{\partial \tilde{\varphi}}{\partial \phi}\right)$

$\nabla_{\|}\tilde{p}_i ; \nabla_{\|}P_{i0}(r) = 0$

$= \frac{B_\theta}{Br}\frac{\partial \tilde{p}_i}{\partial \theta} + \frac{B_\phi}{BR}\frac{\partial \tilde{p}_i}{\partial \phi}$

$= \frac{1}{qR}\left(\frac{\partial}{\partial \theta} + q\frac{\partial}{\partial \phi}\right)\tilde{p}_i$

$\mu_{\|}\nabla_{\|}^2 \tilde{v}_{\|i}$

$= \frac{\mu_{\|}}{q^2 R^2}\left(\frac{\partial}{\partial \theta} + q\frac{\partial}{\partial \phi}\right)^2 \tilde{v}_{\|i}$

Here $q = \frac{rB_\phi}{RB_\theta}$ and all the terms defined above have their usual meaning.

Rearranging the evaluated quantities of equation (III.3) we obtain

$$-i\omega m_i n_0 \tilde{v}_{\|i} - \frac{m_i n_0 c}{Br}\frac{\partial v_{\|0}(r)}{\partial r}\frac{\partial \tilde{\varphi}}{\partial \theta} = -\frac{e n_0}{qR}\left(\frac{\partial}{\partial \theta} + q\frac{\partial}{\partial \phi}\right)\tilde{\varphi} - \frac{1}{qR}\left(\frac{\partial}{\partial \theta} + q\frac{\partial}{\partial \theta}\right)\tilde{p}_i$$

$$+\frac{\mu_{\|}}{q^2 R^2}\left(\frac{\partial}{\partial \theta} + q\frac{\partial}{\partial \theta}\right)^2 \tilde{v}_{\|i}$$

The last quantity is a real quantity and the others are imaginary. Equating real and imaginary quantities and putting $\tilde{p}_i$ from equation (III.2)

$$\Rightarrow -\imath\omega m_i n_0 \tilde{v}_{\|i} - \frac{m_i n_0 c}{Br}\frac{\partial v_{\|0}(r)}{\partial r}\frac{\partial\tilde{\varphi}}{\partial\theta} = -\frac{en_0}{qR}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)\tilde{\varphi} - \frac{1}{qR}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\theta}\right)\frac{\imath cT_e}{\omega rB}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial\tilde{\varphi}}{\partial\theta}$$

$$\Rightarrow -\imath\omega m_i n_0 \tilde{v}_{\|i} = -\frac{en_0}{qR}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)\tilde{\varphi} - \frac{1}{qR}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)\frac{\imath cT_e}{\omega rB}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial\tilde{\varphi}}{\partial\theta} + \frac{m_i n_0 c}{Br}\frac{\partial v_{\|0}(r)}{\partial r}\frac{\partial\tilde{\varphi}}{\partial\theta}$$

$$\Rightarrow \tilde{v}_{\|i} = -\frac{en_0}{qR(-\imath\omega m_i n_0)}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)\tilde{\varphi} - \frac{\imath cT_e}{qR\omega r B(-\imath\omega m_i n_0)}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)\frac{\partial\tilde{\varphi}}{\partial\theta}$$

$$+\frac{\imath c}{\omega Br}\frac{\partial v_{\|0}(r)}{\partial r}\frac{\partial\tilde{\varphi}}{\partial\theta}$$

$$\Rightarrow \tilde{v}_{\|i} = -\frac{\imath e}{qR\omega m_i}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)\tilde{\varphi} - \frac{1}{qR}\frac{\imath cT_e m}{\omega^2 rBm_i n_0}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)\tilde{\varphi}$$

$$+\frac{cm}{\omega Br}\frac{\partial v_{\|0}(r)}{\partial r}\tilde{\varphi} \qquad \text{(III.4)}$$

Now let us consider a quantity (this will appear later while evaluating the continuity equation)

$$\nabla_{\|}(N\tilde{v}_{\|i})$$

$$= n_0 \imath k_{\|}\tilde{v}_{\|i}$$

$$= n_0 \imath k_{\|}\left\{-\frac{\imath e}{\omega qRm_i}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)\tilde{\varphi} - \frac{\imath cT_e m}{\omega^2 qRBn_0 m_i r}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\theta}\right)\frac{1}{B}\tilde{\varphi} + \frac{cm}{\omega Br}\frac{\partial v_{\|0}(r)}{\partial r}\tilde{\varphi}\right\}$$

$$= n_0\left\{-\frac{\imath e}{\omega qRm_i}\imath k_{\|}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)\tilde{\varphi} - \frac{\imath cT_e m}{\omega^2 qRBn_0 m_i r}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\imath k_{\|}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\theta}\right)\frac{1}{B}\tilde{\varphi}\right\}$$

$$+n_0\left\{\frac{cm}{\omega Br}\frac{\partial v_{\|0}(r)}{\partial r}\imath k_{\|}\tilde{\varphi}\right\}$$

Now let us replace $\imath k_{\|}$ by $\frac{1}{qR}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)$, that will give

$$=\frac{\imath n_0 e}{q^2 R^2 \omega m_\imath}\left[-\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)^2\tilde{\varphi}-\frac{mcT_e}{\imath eBn_0}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_\imath}{\tau}\right)\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)^2\tilde{\varphi}\right]$$

$$+n_0\frac{cm}{\omega Br}\frac{\partial v_{\|0}(r)}{\partial r}\frac{1}{qR}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)\tilde{\varphi}$$

$$=-\frac{\imath n_0 e}{q^2 R^2 \omega^2 m_\imath}\left\{\omega+\omega_*\left(\frac{1+\eta_\imath}{\tau}\right)\right\}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)^2\tilde{\varphi}+\frac{n_0 cm}{qR\omega Br}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)\frac{\partial v_{\|0}(r)}{\partial r}\tilde{\varphi} \qquad \text{(III.5)}$$

We now evaluate continuity equation for ions

$$\frac{\partial N}{\partial t}+\vec{\nabla}_\perp\left(N\tilde{v}_{\perp\imath}\right)+\nabla_\|(N\tilde{v}_{\|\imath})=0 \qquad \text{(III.6)}$$

We now evaluate each terms of equation (III.6)

$\frac{\partial N}{\partial t}$, $N=n_0(r)+\tilde{n}_\imath$

$= -\imath\omega\tilde{n}_\imath$

$=-\imath\omega n_0\frac{e}{T_e}(1-\imath\delta)\tilde{\varphi}$; using $\imath\delta$ model

$\nabla_\|(v_{\|0}\tilde{n}_\imath)$

$= v_{\|0}\imath k_\|\tilde{n}_\imath$

$= \imath k_\| v_{\|0}\tilde{n}_\imath$

This term will only shift (Doppler shift) the frequency ω.

$\vec{\nabla}_\perp.(N\tilde{v}_{\perp\imath})$

$= \vec{\nabla}_\perp\left(N\vec{v}_E\right)+\vec{\nabla}_\perp.(Nv_{D\imath})+\vec{\nabla}_\perp\left(Nv_P\right)$

where

$$\vec{v}_E = \frac{c}{B}\hat{b} \times \vec{\nabla}_\perp \tilde{\varphi}$$

$$\vec{v}_{Di} = \frac{c}{eBN}\hat{b} \times \vec{\nabla}_\perp P_i,\ P_i = P_{i0}(r) + \tilde{p}_i$$

$$\vec{v}_P = -\frac{c^2 m_i}{eB^2}\left(\frac{\partial}{\partial t} + (v_0 + \vec{v}_E + \vec{v}_{Di})\ \vec{\nabla}\right)\vec{\nabla}_\perp \tilde{\varphi}.$$

We define

$R = R_0 + r\cos\theta$; R and r major and minor radii respectively of a tokamak.

$B = B_0 R_0/R$, B_0 is the field at magnetic axis i.e. at the mid plane $R = R_0$

$$\vec{\nabla}_\perp\ (N\vec{v}_E)$$

$$= \vec{\nabla}_\perp\ n_0(r)\vec{v}_E$$

$$= \vec{\nabla}_\perp\ \left(\frac{n_0(r)c}{B}\hat{b} \times \vec{\nabla}_\perp \tilde{\varphi}\right)$$

$$= \nabla_\perp\ \left(\frac{n_0 c}{B}(\hat{\theta} + \hat{\phi}) \times \left(\hat{r}\frac{\partial\tilde{\varphi}}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\right)\right)$$

$$= \nabla_\perp\ \left(\frac{n_0 c}{B}\left(-\hat{\phi}\frac{\partial\tilde{\varphi}}{\partial r} + \hat{\theta}\frac{\partial\tilde{\varphi}}{\partial r} - \frac{\hat{r}}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\right)\right)$$

$$= \frac{n_0 c}{B}\nabla_\perp\ \left(-\hat{\phi}\frac{\partial\tilde{\varphi}}{\partial r} + \hat{\theta}\frac{\partial\tilde{\varphi}}{\partial r} - \frac{r}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\right) + \left(-\hat{\phi}\frac{\partial\tilde{\varphi}}{\partial r} + \hat{\theta}\frac{\partial\tilde{\varphi}}{\partial r} - \frac{\hat{r}}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\right)\ \nabla_\perp\left(\frac{n_0 c}{B}\right)$$

$$= \frac{n_0 c}{B}\frac{1}{rR}\left[\frac{\partial}{\partial r}\left\{rR\left(-\frac{1}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\right)\right\} + \frac{\partial}{\partial\theta}\left(R\frac{\partial\tilde{\varphi}}{\partial r}\right)\right] + \left(-\hat{\phi}\frac{\partial\tilde{\varphi}}{\partial r} + \hat{\theta}\frac{\partial\tilde{\varphi}}{\partial r} - \frac{\hat{r}}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\right)\ \left\{\hat{r}\frac{\partial}{\partial r}\left(\frac{n_0 c}{B}\right) + \frac{\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{n_0 c}{B}\right)\right\}$$

$$= \frac{n_0 c}{B}\frac{1}{rR}\left\{-\frac{\partial}{\partial r}\left(R\frac{\partial\tilde{\varphi}}{\partial\theta}\right) + \frac{\partial}{\partial\theta}\left(R\frac{\partial\tilde{\varphi}}{\partial r}\right)\right\} - \frac{1}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\frac{\partial}{\partial r}\left(\frac{n_0 c}{B}\right) + \frac{\partial\tilde{\varphi}}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{n_0 c}{B}\right)$$

$$= \frac{n_0 c}{B}\left[-\frac{1}{rR}R\frac{\partial^2\tilde{\varphi}}{\partial r\,\partial\theta} - \frac{1}{rR}\frac{\partial\tilde{\varphi}}{\partial\theta}\cos\theta + \frac{1}{rR}R\frac{\partial^2\tilde{\varphi}}{\partial r\,\partial\theta} + \frac{1}{rR}\frac{\partial\tilde{\varphi}}{\partial r}(-r\sin\theta)\right] - \frac{1}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\frac{c}{B}\frac{\partial n_0}{\partial r} - \frac{1}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\frac{n_0 c\cos\theta}{BR}$$
$$+ \frac{\partial\tilde{\varphi}}{\partial r}\frac{1}{r}n_0 c\frac{(-r\sin\theta)}{BR}$$

$$=-\frac{2n_0c}{BR}\left(\sin\theta\frac{\partial\tilde{\varphi}}{\partial r}+\frac{\cos\theta}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\right)-\frac{c}{rB}\frac{\partial n_0}{\partial r}\frac{\partial\tilde{\varphi}}{\partial\theta}$$

$$\vec{\nabla}_\perp.(N\vec{v}_{Di})$$

$$=\vec{\nabla}_\perp.\left(N\frac{c\hat{b}\times\vec{\nabla}_\perp P_i}{eBN}\right),\ P_i=P_{i0}+\tilde{p}_i$$

$$=\vec{\nabla}_\perp.\left(\frac{c\hat{b}\times\vec{\nabla}_\perp P_i}{eB}\right)$$

$$=\vec{\nabla}_\perp.\left(\frac{c\hat{b}\times\vec{\nabla}_\perp P_{i0}(r)}{eB}\right)+\vec{\nabla}_\perp.\left(\frac{c\hat{b}\times\vec{\nabla}_\perp\tilde{p}_i}{eB}\right)$$

$$\vec{\nabla}_\perp.\left(\frac{c}{eB}\hat{b}\times\vec{\nabla}_\perp\tilde{p}_i\right)$$

$$=\frac{c}{eB}\vec{\nabla}_\perp.(\hat{b}\times\vec{\nabla}_\perp p_i)+\hat{b}\times\vec{\nabla}_\perp p_i.\vec{\nabla}_\perp\left(\frac{c}{eB}\right)$$

$$=\frac{c}{eB}\nabla_\perp.\left(-\hat{\phi}\frac{\partial\tilde{p}_i}{\partial r}+\hat{\theta}\frac{\partial\tilde{p}_i}{\partial r}-\frac{\hat{r}}{r}\frac{\partial\tilde{p}_i}{\partial\theta}\right)+\left(-\hat{\phi}\frac{\partial\tilde{p}_i}{\partial r}+\hat{\theta}\frac{\partial\tilde{p}_i}{\partial r}-\frac{\hat{r}}{r}\frac{\partial\tilde{p}_i}{\partial\theta}\right).\nabla_\perp\left(\frac{c}{eB}\right)$$

$$=\frac{c}{eB}\frac{1}{rR}\left[\frac{\partial}{\partial r}\left\{rR\left(-\frac{1}{r}\frac{\partial\tilde{p}_i}{\partial\theta}\right)\right\}+\frac{\partial}{\partial\theta}\left(R\frac{\partial\tilde{p}_i}{\partial r}\right)\right]+\left(-\hat{\phi}\frac{\partial\tilde{p}_i}{\partial r}+\hat{\theta}\frac{\partial\tilde{p}_i}{\partial r}-\frac{\hat{r}}{r}\frac{\partial\tilde{p}_i}{\partial\theta}\right).\frac{c}{e}\left\{\hat{r}\frac{\partial}{\partial r}\left(\frac{1}{B}\right)+\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{B}\right)\right\}$$

$$=\frac{c}{eB}\frac{1}{rR}\left\{-\frac{\partial}{\partial r}\left(R\frac{\partial\tilde{p}_i}{\partial\theta}\right)+\frac{\partial}{\partial\theta}\left(R\frac{\partial\tilde{p}_i}{\partial r}\right)\right\}+\left(-\hat{\phi}\frac{\partial\tilde{p}_i}{\partial r}+\hat{\theta}\frac{\partial\tilde{p}_i}{\partial r}-\frac{\hat{r}}{r}\frac{\partial\tilde{p}_i}{\partial\theta}\right).\frac{c}{e}\left\{\frac{\hat{r}\cos\theta}{BR}+\frac{\hat{\theta}(-r\sin\theta)}{rBR}\right\}$$

$$=\frac{c}{eB}\frac{1}{rR}\left\{-R\frac{\partial^2\tilde{p}_i}{\partial r\partial\theta}-\frac{\partial\tilde{p}_i}{\partial\theta}\cos\theta+R\frac{\partial^2\tilde{p}_i}{\partial r\partial\theta}+\frac{\partial\tilde{p}_i}{\partial r}(-r\sin\theta)\right\}-\frac{\partial\tilde{p}_i}{\partial r}\frac{c}{e}\frac{\sin\theta}{BR}-\frac{1}{r}\frac{\partial\tilde{p}_i}{\partial\theta}\frac{c}{e}\frac{\cos\theta}{BR}$$

$$=-\frac{2c}{eBR}\left(\frac{\cos\theta}{r}\frac{\partial\tilde{p}_i}{\partial\theta}+\sin\theta\frac{\partial\tilde{p}_i}{\partial r}\right)$$

Putting $\tilde{p}_i(=\frac{icT_e}{\omega rB}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial\tilde{\phi}}{\partial\theta})$ from equation (III.2) we get

$$=-\frac{2c}{eBR}\left[\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left\{\frac{icT_e}{\omega rB}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial\tilde{\varphi}}{\partial\theta}\right\}+\sin\theta\frac{\partial}{\partial r}\left\{\frac{icT_e}{\omega rB}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial\tilde{\varphi}}{\partial\theta}\right\}\right]$$

$$=-\frac{2c}{eBR}\left[\frac{\cos\theta}{r}\frac{icT_e}{\omega rB}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}+\frac{\cos\theta}{r}\frac{icT_e}{\omega r}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial\tilde{\varphi}}{\partial\theta}\frac{(-r\sin\theta)}{BR}\right]$$
$$-\frac{2c}{eBR}\left[\sin\theta\frac{icT_e}{\omega rB}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial^2\tilde{\varphi}}{\partial r\partial\theta}+\sin\theta\frac{icT_e}{\omega r}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial\tilde{\varphi}}{\partial\theta}\frac{\cos\theta}{BR}-\sin\theta\frac{icT_e}{\omega B}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\frac{\partial\tilde{\varphi}}{\partial\theta}\frac{1}{r^2}\right]$$

$$= -\frac{2c}{eBR}\left\{\frac{\cos\theta}{r}\frac{\imath cT_e}{\omega rB}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_\imath}{\tau}\right)\frac{\partial^2\tilde{\phi}}{\partial\theta^2} + \sin\theta\frac{\imath cT_e}{\omega rB}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_\imath}{\tau}\right)\frac{\partial^2\tilde{\phi}}{\partial r\partial\theta}\right\} + \frac{2c}{eBR}\sin\theta\frac{\imath cT_e}{\omega B}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_\imath}{\tau}\right)\frac{\partial\tilde{\varphi}}{\partial\theta}\frac{1}{r^2}$$

$$= -\frac{\imath 2c}{eBR}\frac{cT_e}{\omega rB}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_\imath}{\tau}\right)\left(\frac{\cos\theta}{r}\frac{\partial^2\tilde{\phi}}{\partial\theta^2} + \sin\theta\frac{\partial^2\tilde{\phi}}{\partial r\partial\theta}\right)$$

$$= -\frac{\imath 2c}{eBR}\frac{cT_e}{\omega B}\frac{\partial n_0}{\partial r}\left(\frac{-\imath m}{r}\right)\left(\frac{1+\eta_\imath}{\tau}\right)\left(\frac{\cos\theta}{r}\frac{\partial\tilde{\phi}}{\partial\theta} + \sin\theta\frac{\partial\tilde{\phi}}{\partial r}\right)$$

$$= -\frac{2c}{eBR}\frac{cT_e\kappa}{\omega B}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_\imath}{\tau}\right)\left(\frac{\cos\theta}{r}\frac{\partial\tilde{\phi}}{\partial\theta} + \sin\theta\frac{\partial\tilde{\phi}}{\partial r}\right)$$

$$= -\frac{2cT_e}{eBR}\frac{c\kappa}{\omega B}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_\imath}{\tau}\right)\left(\frac{\cos\theta}{r}\frac{\partial\tilde{\phi}}{\partial\theta} + \sin\theta\frac{\partial\tilde{\varphi}}{\partial r}\right)$$

$$\vec{\nabla}_\perp.(N\vec{v}_P)$$

$$= \vec{\nabla}_\perp.N\left[-\frac{c^2 m_\imath}{eB^2}\left\{\frac{\partial}{\partial t} + (v_0 + \vec{v}_E + \vec{v}_{D\imath}).\vec{\nabla}\right\}\vec{\nabla}_\perp\tilde{\varphi}\right]$$

We eliminate $(v_0.\nabla)\vec{\nabla}_\perp\tilde{\varphi}$ by performing Galilean transformation in $\hat{e}_\parallel$ direction

$$-\frac{c^2 m_\imath}{e}\vec{\nabla}_\perp.\frac{N}{B^2}\left[\frac{\partial}{\partial t} + \left\{\frac{c}{B}\hat{b}\times\vec{\nabla}_\perp\tilde{\varphi} + \frac{c}{eBN}\hat{b}\times\vec{\nabla}_\perp(P_{\imath 0}(r) + \tilde{p}_\imath)\right\}.\vec{\nabla}_\perp\right]\vec{\nabla}_\perp\tilde{\varphi}$$

Here the terms involving $\rightarrow\ \tilde{p}_\imath$ and $\frac{c}{B}\hat{b}\times\vec{\nabla}_\perp\tilde{\varphi}$ are neglected as they are higher order terms.

$$= \frac{\imath\omega c^2 m_\imath}{e}\left(\vec{\nabla}_\perp.\left(\frac{n_0}{B^2}\vec{\nabla}_\perp\tilde{\varphi}\right)\right) - \frac{c^3 m_\imath}{e^2}\left(\vec{\nabla}_\perp.\left(\frac{\hat{b}\times\vec{\nabla}_\perp P_{\imath 0}}{B^3}\right).\vec{\nabla}_\perp\right)\vec{\nabla}_\perp\tilde{\varphi}$$

Let us evaluate the first quantity.

$$\frac{\imath\omega c^2 m_\imath}{e}\left(\vec{\nabla}_\perp.\left(\frac{n_0}{B^2}\vec{\nabla}_\perp\tilde{\varphi}\right)\right)$$

$$= \frac{\imath\omega c^2 m_\imath}{e}\left\{\frac{n_0}{B^2}\nabla_\perp.\nabla_\perp\tilde{\varphi} + \nabla_\perp\tilde{\varphi}.\nabla_\perp\left(\frac{n_0}{B^2}\right)\right\}$$

$$=\frac{i\omega c^2 m_i}{e}\frac{n_0}{B^2}\left[\frac{1}{rR}\left\{\frac{\partial}{\partial r}\left(rR\frac{\partial\tilde{\varphi}}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(R\frac{1}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\right)\right\}\right]+\frac{i\omega c^2 m_i}{e}\left(\hat{r}\frac{\partial\tilde{\varphi}}{\partial r}+\frac{\hat{\theta}}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\right).$$
$$\left\{\hat{r}\frac{\partial}{\partial r}\left(\frac{n_0}{B^2}\right)+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{n_0}{B^2}\right)\right\}$$

$$=\frac{i\omega c^2 m_i n_0}{B^2 e}\left[\frac{1}{rR}\left\{rR\frac{\partial^2\tilde{\varphi}}{\partial r^2}+R\frac{\partial\tilde{\varphi}}{\partial r}+r\frac{\partial\tilde{\varphi}}{\partial r}\cos\theta+\frac{R}{r}\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}+\frac{1}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}(-r\sin\theta)\right\}\right]+$$
$$\frac{i\omega c^2 m_i}{e}\left\{\frac{\partial\tilde{\varphi}}{\partial r}\left(\frac{1}{B^2}\frac{\partial n_0}{\partial r}+\frac{n_0 2R\cos\theta}{B^2R^2}\right)+\frac{1}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\frac{1}{r}\frac{n_0 2R(-r\sin\theta)}{B^2R^2}\right\}$$

$$=\frac{i\omega c^2 m_i}{e}\frac{n_0}{B^2}\left(\frac{\partial^2\tilde{\varphi}}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{\varphi}}{\partial r}+\frac{\cos\theta}{R}\frac{\partial\tilde{\varphi}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}-\frac{\sin\theta}{rR}\frac{\partial\tilde{\varphi}}{\partial\theta}\right)+$$
$$\frac{i\omega c^2 m_i}{e}\left(\frac{1}{B^2}\frac{\partial n_0}{\partial r}\frac{\partial\tilde{\varphi}}{\partial r}+\frac{2n_0}{B^2R}\cos\theta\frac{\partial\tilde{\varphi}}{\partial r}-\frac{2n_0\sin\theta}{rB^2R}\frac{\partial\tilde{\varphi}}{\partial\theta}\right)$$

$$=\frac{i\omega c^2 m_i}{e}\frac{n_0}{B^2}\left(\frac{\partial^2\tilde{\varphi}}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}\right)+\frac{i\omega c^2 m_i}{e}\left(\frac{1}{B^2}\frac{\partial n_0}{\partial r}\frac{\partial\tilde{\varphi}}{\partial r}+\frac{3n_0}{B^2R}\cos\theta\frac{\partial\tilde{\varphi}}{\partial r}-\frac{3n_0\sin\theta}{rB^2R}\frac{\partial\tilde{\varphi}}{\partial\theta}+\frac{n_0}{B^2r}\frac{\partial\tilde{\varphi}}{\partial r}\right)$$

$$=\frac{i\omega c^2 m_i}{e}\frac{n_0}{B^2}\left(\frac{\partial^2\tilde{\varphi}}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}\right)+\ldots. \quad \text{(III.7)}$$

The other quantities are negligibly small.

Now the second term

$$-\frac{c^3 m_i}{e^2}\nabla_\perp\cdot\left(\frac{\hat{b}\times\vec{\nabla}_\perp P_{i0}(r)}{B^3}.\vec{\nabla}_\perp\right)\nabla_\perp\tilde{\varphi}$$

$$=-\frac{c^3 m_i}{e^2}\nabla_\perp\cdot\left(\frac{\hat{b}\times\hat{r}\frac{\partial P_{i0}(r)}{\partial r}}{B^3}.\nabla_\perp\right)\nabla_\perp\tilde{\varphi}$$

$$=-\frac{c^3 m_i}{e^2}\nabla_\perp\cdot\left\{\frac{1}{B^3}\left(-\hat{\phi}\frac{\partial P_{i0}}{\partial r}+\hat{\theta}\frac{\partial P_{i0}}{\partial r}\right).\nabla_\perp\right\}\nabla_\perp\tilde{\varphi}$$

$$=-\frac{c^3 m_i}{e^2}\nabla_\perp\cdot\frac{1}{B^3}\left(\frac{\partial P_{i0}}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\nabla_\perp\tilde{\varphi}$$

$$=-\frac{c^3 m_i}{e^2}\nabla_\perp\cdot\frac{1}{B^3}\left(\frac{1}{r}\frac{\partial P_{i0}}{\partial r}\right)\nabla_\perp\frac{\partial\tilde{\varphi}}{\partial\theta}$$

$$=-\frac{c^3 m_i}{e^2}\left\{\frac{1}{B^3}\frac{1}{r}\frac{\partial P_{i0}}{\partial r}\nabla_\perp.\nabla_\perp\frac{\partial\tilde{\varphi}}{\partial\theta}+\nabla_\perp\frac{\partial\tilde{\varphi}}{\partial\theta}.\nabla_\perp\left(\frac{1}{B^3 r}\frac{\partial P_{i0}}{\partial r}\right)\right\}$$

$$=-\frac{c^3m_i}{e^2}\left[\frac{1}{rB^3}\frac{\partial P_{i0}}{\partial r}\nabla_\perp\cdot\left(\hat{r}\frac{\partial^2\tilde{\varphi}}{\partial r\partial\theta}+\frac{\hat{\theta}}{r}\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}\right)+\left(\hat{r}\frac{\partial^2\tilde{\varphi}}{\partial r\partial\theta}+\frac{\hat{\theta}}{r}\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}\right)\cdot\left(\hat{r}\frac{\partial}{\partial r}+\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{1}{B^3r}\frac{\partial P_{i0}}{\partial r}\right)\right]$$

$$=-\frac{c^3m_i}{e^2}\left[\frac{1}{rB^3}\frac{\partial P_{i0}}{\partial r}\frac{1}{rR}\left\{\frac{\partial}{\partial r}\left(rR\frac{\partial^2\tilde{\varphi}}{\partial r\partial\theta}\right)+\frac{\partial}{\partial\theta}\left(\frac{R}{r}\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}\right)\right\}\right]$$
$$-\frac{c^3m_i}{e^2}\left[\frac{\partial^2\tilde{\varphi}}{\partial r\partial\theta}\frac{\partial}{\partial r}\left(\frac{1}{B^3r}\frac{\partial P_{i0}}{\partial r}\right)+\frac{1}{r}\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{B^3r}\frac{\partial P_{i0}}{\partial r}\right)\right]$$

$$=-\frac{c^3m_i}{e^2}\left[\frac{1}{rB^3}\frac{\partial P_{i0}}{\partial r}\frac{1}{rR}\left\{rR\frac{\partial^3\tilde{\varphi}}{\partial r^2\partial\theta}+R\frac{\partial^2\tilde{\varphi}}{\partial r\partial\theta}+r\frac{\partial^2\tilde{\varphi}}{\partial r\partial\theta}\cos\theta+\frac{R}{r}\frac{\partial^3\tilde{\varphi}}{\partial\theta^3}+\frac{1}{r}\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}(-r\sin\theta)\right\}\right]$$
$$-\frac{c^3m_i}{e^2}\left[\frac{\partial^2\tilde{\varphi}}{\partial r\partial\theta}\left\{\frac{1}{B^3r}\frac{\partial^2P_{i0}}{\partial r^2}+\frac{1}{B^3}\frac{\partial P_{i0}}{\partial r}\left(-\frac{1}{r^2}\right)+\frac{1}{r}\frac{\partial P_{i0}}{\partial r}\frac{(-3R^2\cos\theta)}{B^3R^3}\right\}+\frac{1}{r^3}\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}\frac{\partial P_{i0}}{\partial r}\frac{\{-3R^2(-r\sin\theta)\}}{B^3R^3}\right]$$

$$=-\frac{c^3m_iT_e}{e^2B^3r}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\left(\frac{\partial^3\tilde{\varphi}}{\partial r^2\partial\theta}+\frac{1}{r^2}\frac{\partial^3\tilde{\varphi}}{\partial\theta^3}\right)$$
$$-\frac{c^3m_i}{e^2}\left[\frac{1}{B^3r}\frac{\partial^2P_{i0}}{\partial r^2}\frac{\partial^2\tilde{\varphi}}{\partial r\partial\theta}-\frac{2\cos\theta}{rB^3R}\frac{\partial P_{i0}}{\partial r}\frac{\partial^2\tilde{\varphi}}{\partial r\partial\theta}+\frac{2\sin\theta}{r^2B^3R}\frac{\partial P_{i0}}{\partial r}\frac{\partial^2\tilde{\varphi}}{\partial r\partial\theta}\right]$$

$$=-\frac{c^3m_iT_e}{e^2B^3r}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\left(\frac{\partial^3\tilde{\varphi}}{\partial r^2\partial\theta}+\frac{1}{r^2}\frac{\partial^3\tilde{\varphi}}{\partial\theta^3}\right)\qquad\text{(III.8)}$$

Let us define:

$$\kappa=\frac{m}{r}$$

$$\omega_*=\frac{cT_e\kappa}{eBr_n}$$

$$r_n^{-1}=\frac{1}{n_0}\frac{\partial n_0}{\partial r}$$

Now we rewrite equation (III.6) with the relevant evaluated terms.

$$\Rightarrow -i\omega n_0\frac{e}{T_e}(1-i\delta)\tilde{\varphi}-\frac{2cn_0}{BR}\left(\sin\theta\frac{\partial\tilde{\varphi}}{\partial r}+\frac{\cos\theta}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\right)-\frac{c}{Br}\frac{\partial n_0}{\partial r}\frac{\partial\tilde{\varphi}}{\partial\theta}$$
$$-\frac{2cT_e}{eBR}\frac{c\kappa}{\omega B}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\left(\frac{\cos\theta}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}+\sin\theta\frac{\partial\tilde{\phi}}{\partial r}\right)+\frac{i\omega c^2m_i}{e}\left(\frac{n_0}{B^2}\left(\frac{\partial^2\tilde{\varphi}}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}\right)\right)$$
$$-\frac{c^3m_iT_e(-im)}{e^2B^3r}\frac{\partial n_0}{\partial r}\left(\frac{1+\eta_i}{\tau}\right)\left(\frac{\partial^2\tilde{\varphi}}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\tilde{\varphi}}{\partial\theta^2}\right)$$
$$-\frac{in_0e}{q^2R^2\omega^2m_i}\left[\left\{\omega+\omega_*\left(\frac{1+\eta_i}{\tau}\right)\right\}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)^2\tilde{\varphi}\right]+\frac{n_0cm}{qR\omega Br}\left(\frac{\partial}{\partial\theta}+q\frac{\partial}{\partial\phi}\right)\frac{\partial v_{\parallel 0}(r)}{\partial r}\tilde{\varphi}=0$$

Let us divide by $\frac{in_0e}{T_e}$

$$\Rightarrow \frac{c^2 m_i T_e}{e^2 B^2}\left(\omega + \frac{cT_e\kappa}{eBr_n}\left(\frac{1+\eta_i}{\tau}\right)\right)\frac{\partial^2\tilde{\varphi}}{\partial r^2} - \frac{c^2 m_i T_e \kappa^2}{e^2 B^2}\left(\omega + \frac{cT_e\kappa}{eBr_n}\left(\frac{1+\eta_i}{\tau}\right)\right)\tilde{\varphi}$$
$$+ \omega_*\tilde{\varphi} - \omega(1-i\delta)\tilde{\varphi} - \frac{T_e}{q^2R^2\omega^2 m_i}\left(\omega + \omega_*\left(\frac{1+\eta_i}{\tau}\right)\right)\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\phi}\right)^2\tilde{\varphi}$$
$$+ \frac{2icT_e}{eBR\omega}\left(\omega + \omega_*\left(\frac{1+\eta_i}{\tau}\right)\right)\left(\sin\theta\frac{\partial\tilde{\varphi}}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\right) + \frac{n_0 cm}{qR\omega Br}\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\phi}\right) \times \frac{T_e}{in_0e}\frac{\partial v_{\|0}(r)}{\partial r}\tilde{\varphi} = 0$$

Again let us divide by $\omega + \omega^*\left(\frac{1+\eta_i}{\tau}\right)$

$$\Rightarrow a_i^2\frac{\partial^2\tilde{\varphi}}{\partial r^2} - b\tilde{\varphi} - \frac{(\omega-\omega_*-i\omega\delta)}{(\omega+\omega_*\left(\frac{1+\eta_i}{\tau}\right))}\tilde{\varphi} - \left(\frac{\omega_*\epsilon_c}{\omega\kappa a_i}\right)^2\left(\frac{\partial}{\partial\theta} + i\kappa sr\right)^2\tilde{\varphi}$$
$$- 2\epsilon_n\frac{\omega_*}{\omega}\left(\cos\theta + \frac{i\sin\theta}{\kappa}\frac{\partial}{\partial r}\right)\tilde{\varphi} + \frac{mcT_e}{\left(\omega+\omega^*\left(\frac{1+\eta_i}{\tau}\right)\right)reB\omega qRi}\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\phi}\right)\frac{\partial v_{\|0}(r)}{\partial r}\tilde{\varphi} = 0$$

$$\Rightarrow a_i^2\frac{\partial^2\tilde{\varphi}}{\partial r^2} - b\tilde{\varphi} + \frac{(\omega_*-\omega+i\omega\delta)}{(\omega+\omega_*\left(\frac{1+\eta_i}{\tau}\right))}\tilde{\varphi} - \left(\frac{\omega_*\epsilon_c}{\omega\kappa a_i}\right)^2\left(\frac{\partial}{\partial\theta} + i\kappa sr\right)^2\tilde{\varphi}$$
$$- 2\epsilon_n\frac{\omega_*}{\omega}\left(\cos\theta + \frac{i\sin\theta}{\kappa}\frac{\partial}{\partial r}\right)\tilde{\varphi} + \frac{\kappa a_i c_s k_\|}{(\omega+\omega_*\left(\frac{1+\eta_i}{\tau}\right))\omega}\frac{\partial v_{\|0}(r)}{\partial r}\tilde{\varphi} = 0 \qquad \text{(III.9)}$$

where

$\kappa = nq/r = m/r$

$s = (r/q)(dq/dr)$

$a_i^2 = c^2 m_i T_e / e^2 B^2$

$c_s^2 = T_e/m_i$

$b = \kappa^2 a_i^2$

$\epsilon_n = q\epsilon_c = r_n/R$

$\tau = T_e/T_i$

$\omega_* = (\kappa c T_e / eBr_n)$

$\nabla_\|\tilde{\varphi} = ik_\|\tilde{\varphi}$

$$\nabla_{\|} = \frac{1}{qR}\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\phi}\right).$$

The equation (III.9) is reported as equation (4.1) in Chapter 4.

Let us define:

$$v_{\|0}(r) = v_{\|00} - \left(\frac{v_{\|00}r}{L_{v1}} + \frac{v_{\|00}r^2}{2L_{v2}}\right)$$

$$\Rightarrow \frac{dv_{\|0}(r)}{dr} = -\frac{v_{\|00}}{L_{v1}} - \frac{v_{\|00}r}{L_{v2}}.$$

Let us put

$$x = \kappa r s \Rightarrow r = \frac{x}{\kappa s} \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{\kappa s}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} \Rightarrow \frac{1}{\kappa s}\frac{\partial}{\partial r} \Rightarrow \frac{\partial}{\partial r} = \kappa s\frac{\partial}{\partial x}$$

$$\frac{\partial^2}{\partial x^2} = \frac{1}{(\kappa s)^2}\frac{\partial^2}{\partial r^2}$$

$$\frac{\partial^2}{\partial r^2} = (\kappa s)^2\frac{\partial^2}{\partial x^2}$$

With these substitutions, we will get equation (III.9) as

$$\left[\frac{\partial^2}{\partial x^2} - \sigma^2\left(\frac{\partial}{\partial\theta} + ix\right)^2 - \epsilon\left(\cos\theta + is\sin\theta\frac{\partial}{\partial x}\right) + \lambda\right]\tilde{\varphi}$$

$$+ \left(\frac{\kappa a_i c_s x}{bs^2 qR\omega(\omega + \omega^*(\frac{1+\eta_i}{\tau}))}\left(-\frac{v_{\|00}}{L_{v1}} - \frac{v_{\|00}x}{\kappa s L_{v2}^2}\right)\frac{\partial V_{\|0}(x)}{\partial x}\right)\tilde{\varphi} = 0 \qquad \text{(III.10)}$$

where

$$x = \kappa r s$$

$$\sigma = \frac{\epsilon_c}{bs}$$

$\epsilon = \frac{2\epsilon_n}{bs^2}$

$\lambda = \frac{1}{a_i^2\kappa^2 s^2}\left[\frac{(\omega-\omega_* - \imath\delta)}{\omega\tau + \omega_*\tau\left(\frac{1+\eta_i}{\tau}\right)} - b\right]$

With these substitutions equation (III.10) will become

$$\left[\frac{\partial^2}{\partial x^2} - \sigma^2\left(\frac{\partial}{\partial\theta} + ix\right)^2 - \epsilon\left(\cos\theta + \imath s\sin\theta\frac{\partial}{\partial x}\right) + \lambda - p_1 x - p_2 x^2\right]\tilde{\varphi} = 0 \qquad \text{(III.11)}$$

where

$p_1 = \frac{\kappa a_i c_s V_{\|00}}{bs^2 L_{v1} q R\omega(\omega+\omega^*\left(\frac{1+\eta_i}{\tau}\right))}$

$p_2 = \frac{a_i c_s V_{\|00}}{bs^3 L_{v2} q R\omega(\omega+\omega^*\left(\frac{1+\eta_i}{\tau}\right))}.$

This is our eigenvalue equation.

In the usual theory of high n ballooning mode, one maps the poloidal angle θ on to an extended coordinate χ with $-\infty < \chi < \infty$ and writes the perturbation in the form

$\tilde{\varphi}(\theta, x) = \sum_m e^{-\imath m\theta}\int_{-\infty}^{\infty} e^{\imath m\chi}\hat{\phi}(\chi, x)d\chi,$

where

$\hat{\phi} = A(x)F(\chi, x)\exp[-ix(\chi + \chi_0)].$

Here χ_0 is an arbitrary phase of the eikonal A(x) is assumed to vary on some scale intermediate between the equilibrium scale length and the perpendicular wavelength. Now to leading order (in $n^{-1/2}$ expansion), the ballooning equation becomes

$$\left(\sigma^2\frac{d^2}{d\chi^2} + (\chi+\chi_0)^2 + \epsilon[\cos\chi + s(\chi+\chi_0)\sin\chi] + p_1 x + p_2 x^2 - \lambda\right)F(\chi, x) = 0 \text{(III.12)}$$

To explore its implication for radial mode structure and stability of toroidal ITG waves one need the higher order ballooning theory. In the higher order theory χ_0 is obtained from the equation $(\partial\lambda/\partial\chi_0)(x, \chi_0) = 0$ and A(x) satisfies

$$\frac{\partial^2\lambda}{\partial\chi_0^2} \cdot \frac{d^2A}{dx^2} + [2(\lambda - \lambda_0) - 2p_1x - 2p_2x^2]A = 0, \quad \text{(III.13)}$$

where

$$\lambda_0 = i\frac{\epsilon_c}{bs},$$

χ_0 is an arbitrary phase of the ikonal. Equation (III.12) is a simple Weber equation. When p_2 is positive and $\partial^2\lambda/\partial\chi_0^2 > 0(\partial^2\lambda/\partial\chi_0^2 > 0$ is necessary in order that the mode be most unstable), $A(x)$ is localized Gaussian function. However, an important change is introduced by the velocity term for the case of negative magnetic shear. $A(x)$ is then given by

$$A(x) = \exp\left[-i\frac{1}{2}\left(|p_2|/|\frac{\partial^2\lambda}{\partial\chi_0^2}|\right)^{1/2}(x + x_0)^2\right], \quad \text{(III.14)}$$

where $x_0 = p_1/|p_2|$.

The eigenvalue is given by

$$\lambda = \lambda_0 + \frac{p_1^2}{2|p_2|} - i\frac{1}{2}\left(|p_2||\frac{\partial^2\lambda}{\partial\chi_0^2}|\right)^{1/2}. \quad \text{(III.15)}$$

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