# CONTRIBUTIONS TO $t$-CORE PARTITIONS FOR SOME SMALL $t$ BY USING RAMANUJAN'S THETA FUNCTIONS

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
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*Dedicated to my parents*

*(Maa and Baba)*

# Abstract

In this thesis, we use various dissections of Ramanujan's theta functions to obtain infinite families of arithmetic identities for $t$-cores as well as self-conjugate and doubled distinct $t$-core partitions for some small $t$. More precisely, we find infinite families of arithmetic identities for 3-, 4- and 5-cores, for self-conjugate 3-, 4-, 5- and 7-cores, for doubled distinct $t$-cores for $t = 3, ..., 8$, and some arithmetic identities involving doubled distinct 9- and 10-cores. We also give some interesting relations between doubled distinct $t$-cores and self-conjugate $t$-cores for some small $t$. By using Ramanujan's theta functions and a classical result by L. Lorenz in 1871 [43], we find a simple proof of a result on 3-cores found earlier by Granville and Ono [27] using the theory of modular forms. We give alternative simple proofs of some results of Hirschhorn and Sellers [32], Garvan, Kim and Stanton [24], Baruah and Berndt [4], and Baruah and Sarmah [7]. In the process, we also obtain new infinite families of arithmetic identities for $r_3(n)$ and $t_3(n)$, where $r_3(n)$ and $t_3(n)$ represent the number of representations of $n$ as a sum of three squares and as a sum of three triangular numbers, respectively. In the course of our study on 5-cores, we find a new proof of Ramanujan's so-called "most beautiful identity".

# DECLARATION BY THE CANDIDATE

I, Kallol Nath, hereby declare that the subject matter in this thesis entitled, **"Contributions to $t$-Core Partitions for Some Small $t$ by Using Ramanujan's Theta Functions"**, is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/institute.

This thesis is being submitted to the Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

Place: Tezpur.

Date: 15|2|2013

Kallol Nath

**Signature of the candidate**

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## CERTIFICATE OF THE SUPERVISOR

This is to certify that the thesis entitled **"Contributions to $t$-Core Partitions for Some Small $t$ by Using Ramanujan's Theta Functions"** submitted to the School of Sciences of Tezpur University in partial fulfilment for the award of the degree of Doctor of Philosophy in Mathematical Sciences is a record of research work carried out by **Mr. Kallol Nath** under my supervision and guidance.

All help received by him from various sources have been duly acknowledged.

No part of this thesis have been submitted elsewhere for award of any other degree.

Place: Tezpur
Date: March 15, 2013

**(Nayandeep Deka Baruah)**

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Kallol Nath

Date: 15/3/2013 (Kallol Nath)

Place: Tezpur

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# Chapter 1

# Introduction

## 1.1 Introduction

A *partition* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of a natural number $n$ is a finite sequence of non-increasing positive integer *parts* $\lambda_i$ such that $n = \sum_{i=1}^{k} \lambda_i$. The number of partitions of $n$ is denoted by $p(n)$. For example, the partitions of 4 are (4), (3,1), (2,2), (2,1,1), (1,1,1,1), and hence, $p(4) = 5$. By convention, $p(0) = 1$. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.1.1}$$

where, here and throughout the thesis, we assume that $|q| < 1$ and use the standard notations

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n),$$

and

$$(a_1, a_2, \ldots, a_k; q)_\infty := (a_1;q)_\infty(a_2;q)_\infty \cdots (a_k;q)_\infty.$$

In 1919, Ramanujan [52], [55, pp. 210-213] announced that he had found three simple congruences satisfied by $p(n)$, namely,

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1.2}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.1.3}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.1.4}$$

He gave proofs of (1.1.2) and (1.1.3) in [52] and later in a short one page note [53], [54, p. 230] announced that he had also found a proof of (1.1.4). In a posthumously published paper [54], [55, pp. 232–238], Hardy extracted different proofs of (1.1.2)–(1.1.4) from an unpublished manuscript of Ramanujan [56, pp. 133–177], [18]. In [52], Ramanujan also offered a more general conjecture which states that if $\delta = 5^a 7^b 11^c$ and $\lambda$ is an integer such that $24\lambda \equiv 1 \pmod{\delta}$ then

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta}. \tag{1.1.5}$$

Although Ramanujan gave a proof of this conjecture in his unpublished manuscript [56, pp. 133-177], [18] for arbitrary $a$ and $b = c = 0$, later on, his conjecture was needed to be corrected as

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta'}, \tag{1.1.6}$$

where, $\delta' = 5^a 7^{b'} 11^c$ with $b' = b$ if $b = 0,\ 1,\ 2$ and $b' = [(b+2)/2]$ if $b > 2$. In 1938, G. N. Watson [59] published a proof of (1.1.6) for $a = c = 0$ and gave a more detailed version of Ramanujan's proof of (1.1.6) in the case $b = c = 0$. It was not untill 1967 that A. O. L. Atkin [2] proved (1.1.6) for arbitrary $c$ and $a = b = 0$.

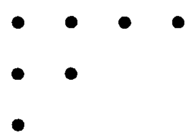
A partition is very often represented with the help of a diagram called Ferrers–Young diagram. The Ferrers–Young diagram of the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of $n$ is formed by arranging $n$ nodes in $k$ rows so that the $i$th row has $\lambda_i$ nodes. For example, the Ferrers–Young diagram of the partition $\lambda = (3, 2, 1, 1)$ of 7 is

$$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \\ \bullet & & \end{array}$$

The conjugate of a partition $\lambda$, denoted $\lambda'$, is the partition whose Ferrers–Young diagram is the reflection along the main diagonal of the diagram of $\lambda$. Therefore, the conjugate of the partition $(3, 2, 1, 1)$ is the partition $(4, 2, 1)$. A partition $\lambda$ is *self-conjugate* if $\lambda = \lambda'$. For example, the partition $(4, 2, 1, 1)$ of 8 is self-conjugate.

The nodes in the Ferrers–Young diagram of a partition are labeled by row and column coordinates as one would label the entries of a matrix. Let $\lambda'_j$ denote the number of nodes in column $j$. The *hook number* $H(i,j)$ of the $(i,j)$ node is defined as the number of nodes directly below and to the right of the node including the node itself. That is, $H(i,j) = \lambda_i + \lambda'_j - j - i + 1$. A partition $\lambda$ is said to be a $t$-core if and only if it has no hook numbers that are multiples of $t$.

**Example.** The Ferrers–Young diagram of the partition $\lambda = (4,2,1)$ of 7 is



The nodes $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, $(2,2)$ and $(3,1)$ have hook numbers 6, 4, 2, 1, 3, 1 and 1, respectively. Therefore, $\lambda$ is a 5-core. Obviously, it is a $t$-core for $t \geq 7$.

If $a_t(n)$ denotes the number of partitions of $n$ that are $t$-cores, then the generating function for $a_t(n)$ is given by [24, Equation (2.1)] , [47, Proposition (3.3)]

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}. \tag{1.1.7}$$

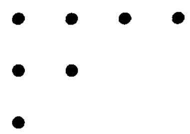
Now, if $\mathrm{asc}_t(n)$ denotes the number of self-conjugate $t$-cores of $n$ then Garvan, Kim and Stanton [24, Eqs. (7.1a) and (7.1b)], and Olsson [48, Eq. (2.37)] found the generating function for $\mathrm{asc}_t(n)$ as

$$\sum_{n=0}^{\infty} \mathrm{asc}_t(n)q^n = (-q;q^2)_\infty (q^{2t};q^{2t})_\infty^{t/2}, \quad \text{for } t \text{ even}, \tag{1.1.8}$$

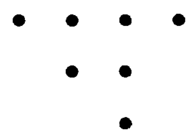
and

$$\sum_{n=0}^{\infty} \mathrm{asc}_t(n)q^n = \frac{(-q;q^2)_\infty (q^{2t};q^{2t})_\infty^{(t-1)/2}}{(-q^t;q^{2t})_\infty}, \quad \text{for } t \text{ odd}. \tag{1.1.9}$$

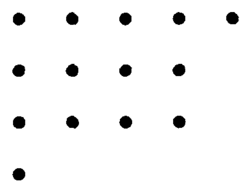
Next, given a partition $\lambda = (\lambda_1, \ \lambda_2, \ \cdots, \ \lambda_k)$ of $n$ with distinct parts, the shifted Ferrers-Young diagram of $\lambda$, $S(\lambda)$, is the Ferrers-Young diagram of $\lambda$ with each row shifted to the right by one node than the previous row. The doubled distinct partition of $\lambda$ is the partition of $2n$ obtained by adding $\lambda_i$ nodes to the $(i-1)$st column of $S(\lambda)$. For example, we consider the partition $(4,2,1)$ of 7 whose Ferrers-Young diagram is as follows:



The shifted Ferrers-Young diagram of the above partition is given by the following diagram:



Now adding 4, 2, and 1 nodes respectively to the null, first, and second columns of the above diagram we obtain the Ferrers-Young diagram



which represents the doubled distinct partition $(5,4,4,1)$ of 14 corresponding to the partition $(4,2,1)$ of 7.

The study of self-conjugate partitions and $t$-core partitions have played roles in variety of areas. The study of $t$-cores for $t$ prime first arose in connection with Nakayama's conjecture [37, 57] in representation theory. At the turn of the last century, Young discovered that partitions of $n$ label the irreducible characters of the symmetric group $S_n$. At about the same time, Frobenius also found that the hook lengths on the diagonal of a self-conjugate partition determine the irrationalities that occur in the character table of the alternating groups $A_n$. On account of these connections, $t$-core partitions find its place in the study of the representation theorists such as in [44, 45, 57, 60]. Hanusa and Jones [28] observed that self-conjugate partitions and $t$-core partitions intersect in several important ways. Also,

R. Nath [46] found that self-conjugate $t$-core partitions are central to prove some representation theoretic conjectures in case of alternating groups.

Garvan, Kim and stanton [24, 26] found that $t$-cores are useful in establishing cranks, which are used to show a combinatorial proof of Ramanujan's famous congruences for the partition function. Garvan [25] also proved some "Ramanujan-type" congruences for $a_p(n)$ for certain special small primes $p$. Hirschhorn and Sellers [32] proved multiplicative formulas for $a_4(n)$ and also conjectured similar multiplicative properties for $a_p(n)$ for other primes $p$.

The $t$-core conjecture has been the topic of a number of papers [23, 25, 27, 40, 41, 49, 50]. This conjecture asserted that every natural number has a $t$-core partition for every integer $t \geq 4$. Granville and Ono [27, 49, 50] have successfully completed the proof of this conjecture using the theory of modular forms and quadratic forms, and the proof has been simplified by Kiming [40].

Again, Baldwin, Depweg, Ford, Kunin and Sze [3] proved that every integer $n > 2$ has a self-conjugate $t$-core partition for $t > 7$, with the exception of $t = 9$, for which infinitely many integers do not have such a partition. We also refer to [31], [35], [32], [51], [14], [15], [4], [38], [9], [10] for further results and generalizations on $t$-cores.

In 1999, Stanton [58] conjectured the monotonicity proposition that, if $n$ and $t$ are natural numbers such that $t \geq 4$ and $n \neq t+1$, then $a_{t+1}(n) \geq a_t(n)$. This conjecture was proved for certain $t$ by Craven [22] and for large $n$ by Anderson [1]. More precisely, Craven [22] proved that if $n$ and $t$ are integers such that $t > 4$ and $n/2 < t < n-1$, then $a_t(n) < a_{t+1}(n)$, and Anderson [1] found that if $t_1$ and $t_2$ are fixed integers satisfying $4 \leq t_1 < t_2$, then $a_{t_1}(n) < a_{t_2}(n)$, for sufficiently large $n$. Also, Kim and Rouse [39] use combination of techniques to find explicit bounds for $a_t(n)$ and as an application prove that for all $n \geq 0$, $n \neq t+1$, $a_{t+1}(n) \geq a_t(n)$ provided $4 \leq t \leq 198$.

Although the monotonicity criterion is conjectured for $t$-core partitions in general, the set of self-conjugate $t$-cores are not found to satisfy a monotonicity criterion for any $n \geq 5$.

However, C. R. H. Hanusa and R. Nath [29] conjectured that

$$\mathrm{asc}_{2t+2}(n) > \mathrm{asc}_{2t}(n), \text{ for all } n \geq 20 \text{ and } 6 \leq 2t \leq 2\lfloor n/4 \rfloor - 4,$$

and

$$\mathrm{asc}_{2t+3}(n) > \mathrm{asc}_{2t+1}(n), \text{ for all } n \geq 56 \text{ and } 9 \leq 2t+1 \leq n-17.$$

They also provide the following partial answers to the conjectures:

$$\mathrm{asc}_{2t+2}(n) > \mathrm{asc}_{2t}(n), \text{when } n/4 < 2t \leq 2\lfloor n/4 \rfloor - 4,$$

and

$$\mathrm{asc}_{2t+3}(n) > \mathrm{asc}_{2t+1}(n), \text{ for all } n \geq 48 \text{ and } n/3 < 2t+1 \leq n-17.$$

In this thesis, we use various dissections of Ramanujan's general theta function in obtaining infinite families of arithmetic identities for the partitions which are $t$-cores, self-conjugate $t$-cores, and doubled distinct $t$-cores for some small $t$. In the following subsection, we state Ramanujan's general theta function and a few of its important properties.

## 1.2 Ramanujan's theta functions and some preliminary results

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{1.2.1}$$

Jacobi's famous triple product identity [16, p. 35, Entry 19] takes the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.2.2}$$

It is easy to verify that

$$f(a,b)=f(b,a),$$
$$f(1,a)=2f(a,a^3),$$
$$f(-1,a)=0,$$

and, if $n$ is an integer,

$$f(a,b)=a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n,b(ab)^{-n}).$$

The three most important special cases of $f(a,b)$ are

$$\varphi(q):=f(q,q)=1+2\sum_{n=1}^{\infty}q^{n^2}=(-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{1.2.3}$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.2.4}$$

and

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}=(q;q)_\infty. \tag{1.2.5}$$

The product representation in (1.2.3)–(1.2.5) arise from (1.2.2).

In the following lemmas we state a few elementary results which will be used in the subsequent chapters of the thesis.

**Lemma 1.2.1.** [16, p. 40, Entries 25(i) and (ii)] *We have*

$$\varphi(q)=\varphi(q^4)+2q\psi(q^8). \tag{1.2.6}$$

**Lemma 1.2.2.** [16, p. 40, Entries 25(iv), (i) and (ii) and (v), (vi) ] *We have*

$$\psi^2(q)=\psi(q^2)(\varphi(q^4)+2q\psi(q^8)) \tag{1.2.7}$$

*and*

$$\varphi^2(q)=\varphi^2(q^2)+4q\psi^2(q^4). \tag{1.2.8}$$

**Lemma 1.2.3.** [16, p. 45, Entries 29(i) and (ii)] *If $ab = cd$, then*

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f\left(ac,bd\right)f\left(ad,bc\right) \tag{1.2.9}$$

*and*

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(b/c,ac^2d\right)f\left(b/d,acd^2\right), \tag{1.2.10}$$

*and, adding* (1.2.9) *and* (1.2.10), *we have*

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af(b/c,ac^2d)f(b/d,acd^2). \tag{1.2.11}$$

**Lemma 1.2.4.** [16, p. 47, Corollary] *If $ab = cd$ then*

$$\begin{aligned} &f(a,b)f(c,d)f\left(an,b/n\right)f\left(cn,d/n\right) \\ &\qquad - f(-a,-b)f(-c,-d)f\left(-an,-b/n\right)f\left(-cn,-d/n\right) \\ &\quad = 2af(c/a,ad)f(d/an,acn)f(n,ab/n)\psi(ab). \end{aligned} \tag{1.2.12}$$

**Lemma 1.2.5.** [16, p. 48, Entry 31] *If $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$ for each integer $n$, then*

$$f(U_1,V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r},\frac{V_{n-r}}{U_r}\right). \tag{1.2.13}$$

**Lemma 1.2.6.** [16, p. 49, Corollaries(i) and (ii)] *We have*

$$\varphi(q) = \varphi(q^9) + 2qf(q^3,q^{15}), \tag{1.2.14}$$

$$\varphi(q) = \varphi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4f(q^5,q^{45}), \tag{1.2.15}$$

$$\psi(q) = f(q^3,q^6) + q\psi(q^9), \tag{1.2.16}$$

*and*

$$\psi(q) = f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25}). \tag{1.2.17}$$

**Lemma 1.2.7.** [16, p. 39, Entry 24(iii) and p. 51, Example(v)] *We have*

$$f(q,q^5) = \psi(-q^3)\sqrt[3]{\frac{\varphi(q)}{\psi(-q)}}. \tag{1.2.18}$$

## 1.3 Work carried out in this thesis

The thesis consists of six chapters including this introductory chapter. In the following few paragraphs we briefly introduce the work done in our research.

By using the theory of modular forms, Granville and Ono [27] proved that

$$a_3(n) = d_{1,3}(3n+1) - d_{2,3}(3n+1), \tag{1.3.1}$$

where $d_{r,3}(n)$ is the number of divisors of $n$ congruent to $r \pmod 3$.

Again, Baruah and Berndt [4] used a modular equation of Ramanujan to prove that

$$a_3(4n+1) = a_3(n), \text{ for all } n \geq 0. \tag{1.3.2}$$

Though not explicitly written in [4], from the same modular equation it follows that

$$a_3(4n+3) = 0, \text{ for all } n \geq 0. \tag{1.3.3}$$

Hirschhorn and Sellers [35] used some elementary generating function manipulations to prove the result (1.3.1) and then employed that to derive an explicit formula for $a_3(n)$ in terms of the factorization of $3n+1$. As corollaries, they derived several infinite families of arithmetic results involving $a_3(n)$, including the generalizations of (1.3.2) and (1.3.3).

In Chapter 2, we use Ramanujan's theta function identities to prove that $u_1(12n+4) = 6a_3(n)$, where $u_1(n)$ denotes the number of representations of a nonnegative integer $n$ in the form $x^2 + 3y^2$ with $x, y \in \mathbb{Z}$. With the help of a classical result by L. Lorenz [43] in 1871, we then deduce (1.3.1). We also show that different proofs of the results by Hirschhorn and Sellers [35] can also be found without considering the factorization of $3n+1$.

Baruah and Berndt [4] also proved that if $\mathrm{asc}_3(n)$ denotes the number of self-conjugate 3-cores of $n$ then $\mathrm{asc}_3(4n+1) = \mathrm{asc}_3(n)$, which is analogous to (1.3.2). We generalize this result and prove the following.

*Let $p \equiv 2 \pmod 3$ be prime and $k$ be a positive even integer. If $\mathrm{asc}_3(n)$ denotes the number of 3-cores of $n$ that are self-conjugates, then for any positive integer $n$,* *we have*

$$\mathrm{asc}_3(n) = \mathrm{asc}_3\left(p^k n + \left(\frac{p^k-1}{3}\right)\right).$$

Chapter 3 of this thesis deals with infinite families of arithmetic identities involving 4-Cores.

In [31, 32], Hirschhorn and Sellers used some elementary generating function manipulations to find certain congruences and the following infinite families of arithmetic relations involving 4-cores: for $k \geq 1$,

$$3^k a_4(3n) = a_4\left(3^{2k+1}n + \frac{5\times 3^{2k}-5}{8}\right), \tag{1.3.4}$$

$$\left(2\times 3^k - 1\right) a_4(3n+1) = a_4\left(3^{2k+1}n + \frac{13\times 3^{2k}-5}{8}\right), \tag{1.3.5}$$

$$\left(\frac{3^{k+1}-1}{2}\right) a_4(9n+2) = a_4\left(3^{2k+2}n + \frac{7\times 3^{2k+1}-5}{8}\right), \tag{1.3.6}$$

$$\left(\frac{3^{k+1}-1}{2}\right) a_4(9n+8) = a_4\left(3^{2k+2}n + \frac{23\times 3^{2k+1}-5}{8}\right). \tag{1.3.7}$$

Again, if $h(-D)$ denotes the class number of primitive binary quadratic forms with discriminant $-D$ and $a_4(n)$ denotes the number of 4-cores of $n$, then, for a square-free integer $8n+5$, Ono and Sze [51] proved that

$$a_4(n) = \frac{1}{2}h(-32n-20). \tag{1.3.8}$$

Employing (1.3.8) and the index formulae for class numbers, Ono and Sze [51] proved (1.3.4)–(1.3.7) and some general identities conjectured by Hirschhorn and Sellers [32].

In Section 3.2 of the thesis, we use Ramanujan's theta function identities to prove that

$$u(8n+5) = 8a_4(n) = v(8n+5) = \frac{1}{3}r_3(8n+5),$$

where $u(n)$ and $v(n)$ denote the number of representations of a nonnegative integer $n$ in the forms $x^2+4y^2+4z^2$ and $x^2+2y^2+2z^2$, respectively, with $x, y, z \in \mathbb{Z}$ and

$r_3(n)$ denotes the number of representations of n as a sum of three squares. With the help of this and a classical result of Gauss, we find a simple proof of (1.3.8).

We also find new proofs of (1.3.4)–(1.3.7) as well as the following analogous new infinite families of identities for $a_4(n)$.

For $k \geq 1$, we have

$$\begin{aligned}
7a_4(5n+1) &= a_4(125n+40),\\
5a_4(5n+2) &= a_4(125n+65),\\
5a_4(5n+3) &= a_4(125n+90),\\
7a_4(5n+4) &= a_4(125n+115),\\
\left(\frac{5^{k+1}-1}{4}\right) a_4(25n) &= a_4\left(5^{2k+2}n+\frac{5^{2k+1}-5}{8}\right),\\
\left(\frac{5^{k+1}-1}{4}\right) a_4(25n+5) &= a_4\left(5^{2k+2}n+\frac{9\times 5^{2k+1}-5}{8}\right),\\
\left(\frac{5^{k+1}-1}{4}\right) a_4(25n+10) &= a_4\left(5^{2k+2}n+\frac{17\times 5^{2k+1}-5}{8}\right),\\
6a_4(25n+15) &= a_4(625n+390)+5a_4(n),\\
\left(\frac{5^{k+1}-1}{4}\right) a_4(25n+20) &= a_4\left(5^{2k+2}n+\frac{33\times 5^{2k+1}-5}{8}\right),
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{3^{k+1}-1}{2}\right) a_4(9n+5) &= a_4\left(3^{2k+2}n+\frac{15\times 3^{2k+1}-5}{8}\right)\\
&\quad+\frac{3^{k+1}-1}{2}a_4(n).
\end{aligned}$$

In Section 3.4 of the thesis, we also present several infinite families of new arithmetic identities for $r_3(n)$ and $t_3(n)$, and some new proofs for the infinite families of arithmetic identities earlier given by Hirschhorn and Sellers [33, 30].

Chapter 4 of our thesis is devoted to obtaining two infinite families of arithmetic identities for 5-Cores.

Garvan, Kim and Stanton [24] gave one analytic and another bijective proofs of

$$a_5(5n+4) = 5a_5(n). \tag{1.3.9}$$

By using a modular equation of degree 5 recorded by Ramanujan in his second notebook [16, p. 280, Entry 13(iii)], Baruah and Berndt [4, Theorem 2.5] proved that

$$a_5(4n+3) = a_5(2n+1) + 2a_5(n). \tag{1.3.10}$$

From the same modular equation it follows that

$$a_5(4n+1) = a_5(2n), \tag{1.3.11}$$

which was missed by Baruah and Berndt [4]. From the above two identities and with the help of mathematical induction, we also deduce the following two infinite families of arithmetic identities for 5-cores.

For any positive integers $n$ and $k$,

$$a_5(2^{2k}n + 2^{2k-1} - 1) = \left(\sum_{r=1}^{k} 2^{2k-2r}\right) a_5(2n) = \frac{2^{2k}-1}{3}\, a_5(2n) \tag{1.3.12}$$

and

$$a_5(2^{2k+1}n + 2^{2k} - 1) = \left(1 + \sum_{r=1}^{k} 2^{(2k+1)-2r}\right) a_5(2n) = \frac{2^{2k+1}+1}{3}\, a_5(2n). \tag{1.3.13}$$

The following two infinite families of congruences for $a_5(n)$ are apparent from (1.3.12) and (1.3.13).

*For any positive integers $n$ and $k$,*

$$a_5(2^{2k}n + 2^{2k-1} - 1) \equiv 0 \left(\mathrm{mod}\ \frac{2^{2k}-1}{3}\right)$$

*and*

$$a_5(2^{2k+1}n + 2^{2k} - 1) \equiv 0 \left(\mathrm{mod}\ \frac{2^{2k+1}+1}{3}\right).$$

In the same chapter, in Section 4.3, we use Ramanujan's theta function identities to find unified proofs of (1.3.9), (1.3.10) and (1.3.11).

Ramanujan found that, if $p(n)$ is the number of partitions of $n$, then

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\,\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{1.3.14}$$

which immediately implies one of Ramanujan's famous partition congruences $p(5n+4) \equiv 0 \pmod 5$. Hardy [55, p. xxxv] says of (1.3.14): *"It would be difficult to find more beautiful formulae than the 'Rogers-Ramanujan' identities, but here Ramanujan must take second place to Rogers; and, if I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting* [(1.3.14)]*."* Hence, (1.3.14) is referred as "Ramanujan's most beautiful identity".

We find a new proof of (1.3.14) arising from the analytic version of (1.3.9). We refer to [36] for another elementary proof of "Ramanujan's most beautiful identity".

In Chapter 5, we deal with infinite families of arithmetic identities for self-conjugate 5-Cores and 7-Cores.

Let us recall that $\mathrm{asc}_5(n)$ denotes the number of 5-cores of $n$ that are self-conjugates. Garvan, Kim and Stanton [24] gave bijective proofs of

$$\mathrm{asc}_5(2n+1) = \mathrm{asc}_5(n), \tag{1.3.15}$$

$$\mathrm{asc}_5(5n+4) = \mathrm{asc}_5(n), \tag{1.3.16}$$

$$\mathrm{asc}_7(4n+6) = \mathrm{asc}_7(n), \tag{1.3.17}$$

$$\mathrm{asc}_7(n) = 0, \quad \text{if } n+2 = 4^k(8m+1). \tag{1.3.18}$$

Baruah and Berndt [4] proved (1.3.15). Recently, by applying some deep theorems developed by Cao [19], Baruah and Sarmah [7] proved that

$$\mathrm{asc}_7(8m-1) = 0. \tag{1.3.19}$$

Now, let $r_2(n)$ and $r_3(n)$ denote the number of representations of $n$ as a sum of two squares and three squares, respectively. In our work, we use Ramanujan's theta function identities to find relations between $\mathrm{asc}_5(n)$ and $r_2(n)$, and between $\mathrm{asc}_7(n)$ and $r_3(n)$. We then deduce (1.3.15)–(1.3.19). Interestingly, it turns out that (1.3.18) and (1.3.19) are equivalent [see Corollary 5.5.4]. We also find the following new infinite families of arithmetic properties of self-conjugate 5-cores and 7-cores.

For $k \geq 1$ and prime $p \equiv 3 \pmod 4$, we have

$$\mathrm{asc}_5(n) = \mathrm{asc}_5\left(p^{2k}n + (p^{2k}-1)\right),$$

$$
\begin{aligned}
(2\times 3^k-1)\mathrm{asc}_7(3n+2) &= \mathrm{asc}_7\left(3^{2k+1}n+2(2\times 3^{2k}-1)\right),\\
3^k\mathrm{asc}_7(3n) &= \mathrm{asc}_7\left(3^{2k+1}n+2(3^{2k}-1)\right),\\
\left(\frac{3^{k+1}-1}{2}\right)\mathrm{asc}_7(9n+1) &= \mathrm{asc}_7\left(3^{2k+2}n+(3^{2k+1}-2)\right),\\
\left(\frac{3^{k+1}-1}{2}\right)\mathrm{asc}_7(9n+4) &= \mathrm{asc}_7\left(3^{2k+2}n+2(3^{2k+1}-1)\right),\\
\left(\frac{3^{k+1}-1}{2}\right)\mathrm{asc}_7(9n+7) &= \mathrm{asc}_7\left(3^{2k+2}n+(3^{2k+2}-2)\right)\\
&\quad+\left(\frac{3^{k+1}-3}{2}\right)\mathrm{asc}_7(n-1),
\end{aligned}
$$

and

$$
\begin{aligned}
5\mathrm{asc}_7(5n) &= \mathrm{asc}_7(175n+48),\\
5\mathrm{asc}_7(5n+1) &= \mathrm{asc}_7(125n+73),\\
7\mathrm{asc}_7(5n+2) &= \mathrm{asc}_7(125n+98),\\
7\mathrm{asc}_7(5n+4) &= \mathrm{asc}_7(125n+148),\\
\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7(25n+3) &= \mathrm{asc}_7\left(5^{2k+2}n+5^{2k+1}-2\right),\\
\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7(25n+8) &= \mathrm{asc}_7\left(5^{2k+2}n+2\times 5^{2k+1}-2\right),\\
\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7(25n+13) &= \mathrm{asc}_7\left(5^{2k+2}n+3\times 5^{2k+1}-2\right),\\
\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7(25n+18) &= \mathrm{asc}_7\left(5^{2k+2}n+2(2\times 5^{2k+1}-1)\right),\\
6\mathrm{asc}_7(25n+23) &= \mathrm{asc}_7(625n+623)+5\mathrm{asc}_7(n-1).
\end{aligned}
$$

In the final chapter we discuss identities for doubled distinct $t$-cores for $t = 3, \ldots, 10$.

If $\mathrm{add}_t(n)$ denotes the number of doubled distinct partitions of $n$ that are $t$-cores then the generating function for $\mathrm{add}_t(n)$ is given by Garvan, Kim and Stanton [24, Eq. (8.1a)] as

$$
\sum_{n=0}^{\infty}\mathrm{add}_t(n)q^n = \frac{(-q^2;q^2)_\infty(q^{2t};q^{2t})_\infty^{(t-2)/2}}{(-q^t;q^t)_\infty}, \quad \text{for } t \text{ even},
$$

and

$$\sum_{n=0}^{\infty} \mathrm{add}_t(n)q^n = \frac{(-q^2;q^2)_\infty (q^{2t};q^{2t})_\infty^{(t-1)/2}}{(-q^{2t};q^{2t})_\infty}, \quad \text{for } t \text{ odd.}$$

We note that $\mathrm{add}_t(n) = 0$ if $n$ is odd.

Baruah and Sarmah [7] proved that

$$\mathrm{asc}_9(8n+10) = \mathrm{asc}_9(2n), \tag{1.3.20}$$

and as 2 has no self-conjugate 9-core, there is an infinite sequence of positive integers having no self-conjugate 9-cores.

Among several results on $\mathrm{asc}_t(n)$ and $\mathrm{add}_t(n)$, Baruah and Sarmah [7] proved that

$$\mathrm{add}_3(n) = \mathrm{asc}_3(4n), \tag{1.3.21}$$

and

$$\mathrm{add}_5(n) = \mathrm{asc}_5(2n). \tag{1.3.22}$$

Now, let $t_2(n)$ and $t_3(n)$ denote the number of representations of $n$ as a sum of two triangular numbers and three triangular numbers, respectively, and $r_2(n)$ and $r_3(n)$ denote the number of representations of $n$ as a sum of two squares and three squares, respectively. We present simple alternative proofs of (1.3.20)–(1.3.22). Furthermore, we find several other relations involving $t_2(n)$, $t_3(n)$, $r_2(n)$, $r_3(n)$, $\mathrm{add}_t(n)$ and $\mathrm{asc}_t(n)$, for some small $t$. For example, we deduce the following:

$$\begin{aligned} r_2(24n+5) &= 8\mathrm{asc}_4(3n) = 8\mathrm{add}_6(4n), \\ r_3(16n+14) &= 48\mathrm{add}_8(2n), \\ \mathrm{add}_6(4n) &= \mathrm{asc}_4(3n), \\ \mathrm{add}_4(3n) &= \mathrm{add}_3(n), \end{aligned}$$

$$2\mathrm{add}_5(2n) = \begin{cases} t_2(5n+1), & \text{if } n \equiv 0,2,3,4 \pmod 5; \\ t_2(5n+1) - t_2((n-1)/5), & \text{if } n \equiv 1 \pmod 5. \end{cases},$$

$$6\mathrm{add}_7(2n) = \begin{cases} t_3(7n+4), & \text{if } n \equiv 0,1,3,4,5,6 \pmod 7; \\ t_3(7n+4) - t_3((n-2)/7), & \text{if } n \equiv 2 \pmod 7. \end{cases}.$$

As one of the corollaries, we find the following result.

*If $h(-D)$ denotes the class number of primitive binary quadratic forms with discriminant $-D$ and* $\mathrm{add}_8(n)$ *denotes the number of doubled distinct 8-cores of n, then, for a square-free integer* $16n+14$*, we have*

$$\mathrm{add}_8(2n) = \frac{1}{4}h(-64n-56).$$

Finally, we present several infinite families of new arithmetic identities for $\mathrm{add}_3(n)$, $\mathrm{add}_4(n)$, $\mathrm{add}_5(n)$, $\mathrm{asc}_4(n)$, $\mathrm{add}_6(n)$, $\mathrm{add}_7(n)$, and $\mathrm{add}_8(n)$ along with a new arithmetic identity for $\mathrm{add}_{10}(n)$. For example, for any positive integer $k$, we have the following infinite families of arithmetic identities for $\mathrm{add}_8(n)$.

$$\begin{aligned}
3^k \mathrm{add}_8(6n) &= \mathrm{add}_8\left(2 \times 3^{2k+1}n + \frac{7(3^{2k}-1)}{4}\right), \\
\left(2 \times 3^k - 1\right) \mathrm{add}_8(6n+4) &= \mathrm{add}_8\left(2 \times 3^{2k+1}n + \frac{23 \times 3^{2k}-7}{4}\right), \\
\left(\frac{3^{k+1}-1}{2}\right) \mathrm{add}_8(18n+2) &= \mathrm{add}_8\left(2 \times 3^{2k+2}n + \frac{5 \times 3^{2k+1}-7}{4}\right), \\
\left(\frac{3^{k+1}-1}{2}\right) \mathrm{add}_8(18n+8) &= \mathrm{add}_8\left(2 \times 3^{2k+2}n + \frac{13 \times 3^{2k+1}-7}{4}\right), \\
\left(\frac{3^{k+1}-1}{2}\right) \mathrm{add}_8(18n+14) &= \mathrm{add}_8\left(2 \times 3^{2k+2}n + \frac{21 \times 3^{2k+1}-7}{4}\right) \\
&\quad + \left(\frac{3^{k+1}-3}{2}\right) \mathrm{add}_8(2n),
\end{aligned}$$

and

$$\begin{aligned}
5\mathrm{add}_8(10n) &= \mathrm{add}_8(250n+42), \\
5\mathrm{add}_8(10n+4) &= \mathrm{add}_8(250n+142), \\
7\mathrm{add}_8(10n+6) &= \mathrm{add}_8(250n+192), \\
7\mathrm{add}_8(10n+8) &= \mathrm{add}_8(250n+242), \\
\left(\frac{5^{k+1}-1}{4}\right) \mathrm{add}_8(50n+2) &= \mathrm{add}_8\left(2 \times 5^{2k+2}n + \frac{3 \times 5^{2k+1}-7}{4}\right), \\
\left(\frac{5^{k+1}-1}{4}\right) \mathrm{add}_8(50n+12) &= \mathrm{add}_8\left(2 \times 5^{2k+2}n + \frac{11 \times 5^{2k+1}-7}{4}\right),
\end{aligned}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_8(50n+22)=\mathrm{add}_8\left(2\times 5^{2k+2}n+\frac{19\times 5^{2k+1}-7}{4}\right),$$
$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_8(50n+32)=\mathrm{add}_8\left(2\times 5^{2k+2}n+\frac{27\times 5^{2k+1}-7}{4}\right),$$
$$6\mathrm{add}_8(50n+42)=\mathrm{add}_8(1250n+1092)+5\mathrm{add}_8(2n).$$

# Chapter 2

# Some Results on 3-Cores

## 2.1 Introduction

We stated in the introductory chapter that by using the theory of modular forms, Granville and Ono [49] proved that

$$a_3(n) = d_{1,3}(3n+1) - d_{2,3}(3n+1), \tag{2.1.1}$$

where $d_{r,3}(n)$ is the number of divisors of $n$ congruent to $r \pmod 3$.

Again, Baruah and Berndt [4] used a modular equation of Ramanujan to prove that

$$a_3(4n+1) = a_3(n), \text{ for all } n \geq 0. \tag{2.1.2}$$

Though not explicitly written in [4], from the same modular equation it follows that

$$a_3(4n+3) = 0, \text{ for all } n \geq 0. \tag{2.1.3}$$

---

Note: The contents of this chapter has been accepted in *Proceedings of the American Mathematical Society* [9].

Hirschhorn and Sellers [35] used some elementary generating function manipulations to prove the result (2.1.1) and then employed that to derive an explicit formula for $a_3(n)$ in terms of the factorization of $3n+1$. As corollaries, they derived several infinite families of arithmetic results involving $a_3(n)$, including the generalizations of (2.1.2) and (2.1.3).

We recall that if $a_3(n)$ denotes the number of partitions of $n$ that are 3-cores, then from (1.1.7) the generating function for $a_3(n)$ is given by

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}. \tag{2.1.4}$$

We note that, by (1.2.5), the formula (2.1.4) reduces to

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{f^3(-q^3)}{f(-q)}. \tag{2.1.5}$$

In the next section, we use Ramanujan's theta function identities to prove that $u_1(12n+4) = 6a_3(n)$, where $u_1(n)$ denotes the number of representations of a nonnegative integer $n$ in the form $x^2+3y^2$ with $x,y \in \mathbb{Z}$. We then deduce (2.1.1) with the help of a classical result by L. Lorenz [43] in 1871.

We further show that different proofs of the results by Hirschhorn and Sellers [35] can also be found without considering the factorization of $3n+1$.

We also generalize a result of Baruah and Berndt [4] which states that if $\mathrm{asc}_3(n)$ is the number of self-conjugate 3-cores of $n$, then $\mathrm{asc}_3(4n+1) = \mathrm{asc}_3(n)$.

## 2.2 Main theorems

**Theorem 2.2.1.** *If $u_1(n)$ denotes the number of representations of a nonnegative integer $n$ in the form $x^2+3y^2$ with $x,y \in \mathbb{Z}$, and $a_3(n)$ is the number of 3-cores of $n$, then*

$$u_1(12n+4) = 6a_3(n). \tag{2.2.1}$$

In the following process of proving (2.2.1), we also find some other results involving $u_1(n)$.

*Proof.* We have

$$\sum_{n=0}^{\infty} u_1(n)q^n = \varphi(q)\varphi(q^3), \tag{2.2.2}$$

which we rewrite with the help of (1.2.6) as

$$\begin{aligned}\sum_{n=0}^{\infty} u_1(n)q^n &= \left(\varphi(q^4) + 2q\psi(q^8)\right)\left(\varphi(q^{12}) + 2q^3\psi(q^{24})\right) \\ &= \left(\varphi(q^4)\varphi(q^{12}) + 4q^4\psi(q^8)\psi(q^{24})\right) + 2q\left(\psi(q^8)\varphi(q^{12}) + q^2\varphi(q^4)\psi(q^{24})\right).\end{aligned} \tag{2.2.3}$$

Extracting the terms involving $q^{2n}$ and $q^{2n+1}$, respectively, in (2.2.3), we find that

$$\sum_{n=0}^{\infty} u_1(2n)q^n = \varphi(q^2)\varphi(q^6) + 4q^2\psi(q^4)\psi(q^{12}) \tag{2.2.4}$$

and

$$\sum_{n=0}^{\infty} u_1(2n+1)q^n = 2\psi(q^4)\varphi(q^6) + 2q\varphi(q^2)\psi(q^{12}) = 2\psi(q)\psi(q^3), \tag{2.2.5}$$

where the last equality of (2.2.5) is proved by several authors, for examples, in [17, p. 356], [5], [42], and [20].

Now, with the help of (2.2.2) and (2.2.5) we can rewrite the identity (2.2.4) in the form

$$\sum_{n=0}^{\infty} u_1(2n)q^n = \sum_{n=0}^{\infty} u_1(n)q^{2n} + 2q^2\sum_{n=0}^{\infty} u_1(2n+1)q^{4n}. \tag{2.2.6}$$

Equating the coefficients of $q^{2n+1}$, $q^{4n}$, and $q^{4n+2}$, respectively, from both sides of (2.2.6), we obtain

$$u_1(4n+2) = 0, \tag{2.2.7}$$

$$u_1(8n) = u_1(2n), \tag{2.2.8}$$

$$u_1(8n+4) = 3u_1(2n+1). \tag{2.2.9}$$

Again employing (1.2.14) in (2.2.2), we have

$$\sum_{n=0}^{\infty} u_1(n)q^n = \varphi(q^3)\left(\varphi(q^9) + 2qf(q^3, q^{15})\right). \tag{2.2.10}$$

Either extracting the terms or equating the coefficients of the terms involving $q^{3n}$, $q^{3n+2}$, and $q^{3n+1}$, respectively, from both sides of (2.2.10), we find that

$$u_1(3n) = u_1(n), \tag{2.2.11}$$

$$u_1(3n+2) = 0, \tag{2.2.12}$$

$$\sum_{n=0}^{\infty} u_1(3n+1)q^n = 2\varphi(q)f(q,q^5). \tag{2.2.13}$$

Now, we find a 2-dissection of $\varphi(q)f(q,q^5)$. To this end, setting $a = -q\omega$ and $b = -q\omega^2$, where $\omega$ is a nonreal cube root of unity, in Jacobi triple product identity (1.2.2), we find that

$$\begin{aligned}\frac{f(-q\omega,-q\omega^2)}{(q^2\omega,q^2\omega^2;q^4)_\infty} &= \frac{(q\omega;q^2)_\infty(q\omega^2;q^2)_\infty(q^2;q^2)_\infty}{(q^2w,q^2w^2;q^4)_\infty}\\ &= \frac{(q\omega^2;q^2)_\infty(q^2;q^2)_\infty}{(q^2w;q^4)_\infty(-q\omega;q^2)_\infty}\\ &= \frac{(q^2;q^2)_\infty}{(-qw^2;q^2)_\infty(-q\omega;q^2)_\infty}.\end{aligned} \tag{2.2.14}$$

Changing the base of the $q$-products $(-qw^2;q^2)_\infty$ and $(-q\omega;q^2)_\infty$ in (2.2.14), we deduce that

$$\begin{aligned}\frac{f(-q\omega,-q\omega^2)}{(q^2\omega,q^2\omega^2;q^4)_\infty} &= (q^2;q^2)_\infty(-q;q^6)_\infty(-q^5;q^6)_\infty\\ &= \frac{(q^2;q^2)_\infty f(q,q^5)}{(q^6;q^6)_\infty},\end{aligned}$$

where we also used (1.2.2).

Thus,

$$\begin{aligned}\varphi(q)f(q,q^5) &= \frac{\varphi(q)f(-q\omega,-q\omega^2)(q^6;q^6)_\infty}{(q^2\omega,q^2\omega^2;q^4)_\infty(q^2;q^2)_\infty}\\ &= \frac{\varphi(q)f(-q\omega,-q\omega^2)(q^{12};q^{12})_\infty}{(q^4;q^4)_\infty}\\ &= \frac{f(q,q)f(-q\omega,-q\omega^2)f(-q^{12})}{f(-q^4)},\end{aligned} \tag{2.2.15}$$

where we have used (1.2.3) and (1.2.5).

Now, setting $a = b = q$, $c = -q\omega$, and $d = -q\omega^2$ in (1.2.11), we find that

$$f(q,q)f(-q\omega,-q\omega^2) = f^2(-q^2\omega,-q^2\omega^2) + qf(-\omega^2,-\omega q^4)f(-\omega,-q^4\omega^2). \quad (2.2.16)$$

Employing (1.2.2) in the expressions on the right side of (2.2.16), manipulating the $q$-products, and then using the resulting identity in (2.2.15), we obtain

$$\varphi(q)f(q,q^5) = \psi(q^2)f(q^2,q^4) + 3q\frac{f^3(-q^{12})}{f(-q^4)}. \quad (2.2.17)$$

Now, employing (2.2.17) in (2.2.13), we find that

$$\sum_{n=0}^{\infty} u_1(3n+1)q^n = 2\psi(q^2)f(q^2,q^4) + 6q\frac{f^3(-q^{12})}{f(-q^4)}. \quad (2.2.18)$$

Extracting the terms involving $q^{4n+1}$ from both sides of (2.2.18) and then using (2.1.5), we obtain

$$\sum_{n=0}^{\infty} u_1(12n+4)q^n = 6\frac{f^3(-q^3)}{f(-q)} = 6\sum_{n=0}^{\infty} a_3(n)q^n,$$

from which we readily deduce (2.2.1) to complete the proof. □

Now the identity (2.1.3) can easily be deduced.

**Corollary 2.2.2.** *The identity* (2.1.3) *holds good.*

*Proof.* Replacing $n$ by $3n+2$ in (2.2.7), we have

$$u_1(12n+10) = 0.$$

By (2.2.8), the above identity is equivalent to

$$u_1(48n+40) = 0,$$

which, with the help of (2.2.1), can easily be reduced to (2.1.3). □

With the aid of a result of Lorenz [43], the identity (2.1.1) can also be deduced easily.

**Corollary 2.2.3.** *The identity* (2.1.1) *holds good.*

*Proof.* A classical result of Lorenz [43] states that

$$u_1(n) = 2(d_{1,3}(n) - d_{2,3}(n)) + 4(d_{4,12}(n) - d_{8,12}(n)), \tag{2.2.19}$$

where $d_{r,3}(n)$ is the number of divisors of $n$ congruent to $r$ (mod 3).

Hirschhorn [34] proved the equivalent form of (2.2.19) as

$$\varphi(q)\varphi(q^3) = 1 + 2\sum_{n=1}^{\infty}(d_{1,3}(n) - d_{2,3}(n))q^n + 4\sum_{n=1}^{\infty}(d_{4,12}(n) - d_{8,12}(n))q^n,$$

which can be rewritten in the form

$$\sum_{n=0}^{\infty} u_1(n)q^n = 1 + 2\sum_{n=1}^{\infty}(d_{1,3}(n) - d_{2,3}(n))q^n + 4\sum_{n=1}^{\infty}(d_{1,3}(n) - d_{2,3}(n))q^{4n}. \tag{2.2.20}$$

Equating the coefficients of $12n+4$ from both sides of (2.2.20), we find that

$$u_1(12n+4) = 2(d_{1,3}(12n+4) - d_{2,3}(12n+4)) + 4(d_{1,3}(3n+1) - d_{2,3}(3n+1)).$$

Noting that $d_{1,3}(12n+4) - d_{2,3}(12n+4) = d_{1,3}(3n+1) - d_{2,3}(3n+1)$, we immediately arrive at (2.1.1). □

Employing (2.1.1) and a standard counting argument, Hirschhorn and Sellers [35] found an explicit formula for $a_3(n)$ in terms of the factorization of $3n+1$ and deduced several infinite families of arithmetic results involving $a_3(n)$. Now we can also find different proofs of their results without considering the factorization of $3n+1$. We demonstrate this by proving one of their results in Theorem 2.2.5. First we prove the following lemma.

**Lemma 2.2.4.** *If $u_1(n)$ denotes the number of representations of a nonnegative integer $n$ in the form $x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $p \equiv 2 \pmod 3$ is an odd prime, then*

$$u_1(p^2 n) = u_1(n). \tag{2.2.21}$$

*Proof.* Let $p \equiv 2 \pmod 3$ be an odd prime. Setting $n = p$ and $a = b = q$ in (1.2.13), we obtain

$$\varphi(q) = \varphi(q^{p^2}) + \sum_{r=1}^{p-1} q^{r^2} f(q^{p(p-2r)}, q^{p(p+2r)}). \tag{2.2.22}$$

With successive use of the trivial identity $f(a,b) = af(a^{-1}, a^2 b)$, we can rewrite the above identity in the form

$$\begin{aligned}\varphi(q) &= \varphi(q^{p^2}) + 2qf(q^{p(p-2)}, q^{p(p+2)}) + 2q^{2^2} f(q^{p(p-2\cdot 2)}, q^{p(p+2\cdot 2)}) \\ &\quad + 2q^{3^2} f(q^{p(p-2\cdot 3)}, q^{p(p+2\cdot 3)}) + \cdots + 2q^{\left(\frac{p-1}{2}\right)^2} f(q^p, q^{p(2p-1)}).\end{aligned} \tag{2.2.23}$$

Replacing $q$ by $q^3$ in (2.2.23), we have

$$\begin{aligned}\varphi(q^3) &= \varphi(q^{3p^2}) + 2q^3 f(q^{3p(p-2)}, q^{3p(p+2)}) + 2q^{3\cdot 2^2} f(q^{3p(p-2\cdot 2)}, q^{3p(p+2\cdot 2)}) \\ &\quad + 2q^{3\,3^2} f(q^{3p(p-2\cdot 3)}, q^{3p(p+2\cdot 3)}) + \cdots + 2q^{3\left(\frac{p-1}{2}\right)^2} f(q^{3p}, q^{3p(2p-1)}).\end{aligned} \tag{2.2.24}$$

Employing (2.2.23) and (2.2.24) in (2.2.2) and then extracting the terms involving $q^{pn}$ from both sides of the resulting identity by noting that prime $p \equiv 2 \pmod 3$ and squares are always congruent to 0 or 1 modulo 3, we find that

$$\sum_{n=0}^{\infty} u_1(pn) q^n = \varphi(q^p)\varphi(q^{3p}) = \sum_{n=0}^{\infty} u_1(n) q^{pn}. \tag{2.2.25}$$

Equating the coefficients of $q^{pn}$ from both sides of (2.2.25), we readily arrive at (2.2.21) to complete the proof. □

**Theorem 2.2.5.** [35, Corollary 8] *Let $p \equiv 2 \pmod 3$ be prime and $k$ be a positive even integer. If $a_3(n)$ denotes the number of 3-cores of $n$, then for any positive integer $n$, we have*

$$a_3(n) = a_3\left(p^k n + \left(\frac{p^k - 1}{3}\right)\right). \tag{2.2.26}$$

*Proof.* First we prove the theorem for $p = 2$.

Replacing $n$ by $6n + 2$ in (2.2.8), we obtain

$$u_1(12n + 4) = u_1(12(4n + 1) + 4). \tag{2.2.27}$$

Employing (2.2.1) in (2.2.27), we arrive at (2.1.2), from which (2.2.26) for $p = 2$ can be readily deduced by induction.

Next, we prove the theorem for an odd prime $p \equiv 2 \pmod 3$.

Replacing $n$ by $12n + 4$ in (2.2.21), we have

$$u_1(12n+4) = u_1(p^2(12n+4)) = u\left(12\left(p^2n + \frac{p^2-1}{3}\right) + 4\right). \tag{2.2.28}$$

Employing (2.2.1) in (2.2.28), we arrive at

$$a_3(n) = a_3\left(p^2n + \frac{p^2-1}{3}\right),$$

which implies (2.2.26) by induction. □

If $\mathrm{asc}_3(n)$ denotes the number of 3-cores of $n$ that are self-conjugates, then Baruah and Berndt [4] proved that $\mathrm{asc}_3(4n+1) = \mathrm{asc}_3(n)$. In the following theorem, we generalize this result by proving a theorem analogous to Theorem 2.2.5, where $a_3(n)$ is replaced by $\mathrm{asc}_3(n)$.

**Theorem 2.2.6.** *Let $p \equiv 2 \pmod 3$ be prime and $k$ be a positive even integer. If $\mathrm{asc}_3(n)$ denotes the number of 3-cores of $n$ that are self-conjugates, then for any positive integer $n$, we have*

$$\mathrm{asc}_3(n) = \mathrm{asc}_3\left(p^kn + \left(\frac{p^k-1}{3}\right)\right). \tag{2.2.29}$$

*Proof.* Let us define $s(n)$ by

$$\varphi(q) = \sum_{n\geq 0} s(n)q^n. \tag{2.2.30}$$

Employing (1.2.14) in (2.2.30) and extracting the terms involving $q^{3n+1}$, we find that

$$\sum_{n\geq 0} s(3n+1)q^n = 2f(q,q^5). \tag{2.2.31}$$

Again, by [16, p. 51, Example (v)], (1.2.4) and (1.1.9) with $t = 3$, we have

$$f(q,q^5) = \sum_{n\geq 0} \mathrm{asc}_3(n)q^n. \tag{2.2.32}$$

From (2.2.31) and (2.2.32), we arrive at

$$s(3n+1) = 2\mathrm{asc}_3(n). \tag{2.2.33}$$

Again, employing (2.2.30) in (2.2.23) and then proceeding as in the proof of Theorem 2.2.5, we obtain

$$s(p^2n) = s(n). \tag{2.2.34}$$

From (2.2.33) and (2.2.34), we find that

$$\begin{aligned}2\mathrm{asc}_3(n) = s(3n+1) = s(p^2(3n+1)) &= s\left(3\left(p^2n+\frac{p^2-1}{3}\right)+1\right)\\ &= 2\mathrm{asc}_3\left(p^2n+\frac{p^2-1}{3}\right),\end{aligned}$$

from which we readily arrive at (2.2.29) by induction. □

# Chapter 3

# Infinite Families of Arithmetic Identities for 4-Cores

## 3.1 Introduction and preliminary results

If $a_4(n)$ denotes the number of partitions of $n$ that are 4-cores, then the generating function for $a_4(n)$ is given by (1.1.7) as

$$\sum_{n=0}^{\infty} a_4(n)q^n = \frac{(q^4;q^4)_\infty^4}{(q;q)_\infty}. \tag{3.1.1}$$

Manipulating the $q$-products, and then using (1.2.4), we have

$$\sum_{n=0}^{\infty} a_4(n)q^n = \psi(q)\psi^2(q^2). \tag{3.1.2}$$

Let $u(n)$ and $v(n)$ be the number of representations of a nonnegative integer $n$ in the forms $x^2 + 4y^2 + 4z^2$ and $x^2 + 2y^2 + 2z^2$, respectively, with $x, y, z \in \mathbb{Z}$ and $r_3(n)$ be the number of representations of $n$ as a sum of three squares. By employing simple theta function identities of Ramanujan, in Section 3.2, we prove that

$$u(8n+5) = 8a_4(n) = v(8n+5) = \frac{1}{3}r_3(8n+5).$$

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Note: A portion of this chapter has appeared in *Bulletin of the Australian Mathematical Society* [10].

Again, with the help of this and a classical result of Gauss, we find a simple proof of a result on $a_4(n)$ proved earlier by K. Ono and L. Sze [51], which states that if $h(-D)$ denotes the class number of primitive binary quadratic forms with discriminant $-D$ and $a_4(n)$ denotes the number of 4-cores of $n$, then, for a square-free integer $8n+5$,

$$a_4(n) = \frac{1}{2}h(-32n-20). \tag{3.1.3}$$

In Section 3.3, we find new proofs for the results on $a_4(n)$ by Hirschhorn and Sellers [32] along with some analogous new infinite families of identities. We mention here that Hirschhorn and Sellers also proved the identity $24a_4(n) = r_3(8n+5)$ from which earlier results of Hirschhorn and Sellers [32] can easily be deduced with the help of the other results in [33].

In Section 3.4 of the thesis, we also present several infinite families of new arithmetic identities for $r_3(n)$ and $t_3(n)$ along with some new proofs for the infinite families of arithmetic identities earlier given by Hirschhorn and sellers [33, 30].

In Section 3.5, we prove some more infinite families of arithmetic identities that actually missed by Hirschhorn and Sellers [32] and by us [10].

## 3.2 Identities connecting $u(n)$, $v(n)$, $r_3(n)$ and $a_4(n)$

In this section, we present the relations among $u(n)$, $v(n)$, $r_3(n)$ with $a_4(n)$.

**Theorem 3.2.1.** *If $u(n)$ and $v(n)$ denote the number of representations of a non-negative integer $n$ in the forms $x^2+4y^2+4z^2$ and $x^2+2y^2+2z^2$, respectively, where $x, y, z \in \mathbb{Z}$, and $a_4(n)$ is the number of 4-cores of $n$, then*

$$u(8n+5) = 8a_4(n) = v(8n+5). \tag{3.2.1}$$

In the following process of proving (3.2.1), we also find some other results involving $u(n)$ and $v(n)$.

*Proof.* First we prove the first equality in (3.2.1). Clearly, the generating function for $u(n)$ is given by

$$\sum_{n=0}^{\infty} u(n)q^n = \varphi(q)\varphi^2(q^4).$$

With the aid of (1.2.6), we rewrite the above as

$$\begin{aligned}\sum_{n=0}^{\infty} u(n)q^n &= \varphi^2(q^4)\left(\varphi(q^4) + 2q\psi(q^8)\right) \\ &= \varphi^3(q^4) + 2q\varphi^2(q^4)\psi(q^8). \qquad (3.2.2)\end{aligned}$$

Extracting the terms involving $q^{4n}$, $q^{4n+1}$, $q^{4n+2}$ and $q^{4n+3}$ respectively, in (3.2.2), we find that

$$\sum_{n=0}^{\infty} u(4n)q^n = \varphi^3(q), \qquad (3.2.3)$$

$$\sum_{n=0}^{\infty} u(4n+1)q^n = 2\varphi^2(q)\psi(q^2), \qquad (3.2.4)$$

$$u(4n+2) = 0,$$

$$u(4n+3) = 0.$$

Now, with the help of (1.2.6), we can rewrite the identity (3.2.3) in the form

$$\sum_{n=0}^{\infty} u(4n)q^n = \varphi^3(q^4) + 6q\varphi^2(q^4)\psi(q^8) + 12q^2\varphi(q^4)\psi^2(q^8) + 8q^3\psi(q^8). \qquad (3.2.5)$$

Equating the coefficients of $q^{4n}$, $q^{4n+1}$, $q^{4n+2}$, $q^{4n+3}$ respectively, from both sides of (3.2.5), we obtain

$$u(16n) = u(4n),$$

$$\sum_{n=0}^{\infty} u(16n+4) = 6\varphi^2(q)\psi(q^2), \qquad (3.2.6)$$

$$\sum_{n=0}^{\infty} u(16n+8) = 12\varphi(q)\psi^2(q^2), \qquad (3.2.7)$$

$$\sum_{n=0}^{\infty} u(16n+12) = 8\psi^3(q^2). \qquad (3.2.8)$$

From (3.2.8), it further follows that

$$\sum_{n=0}^{\infty} u(32n+12) = 8\psi^3(q), \tag{3.2.9}$$

$$u(32n+28) = 0.$$

Again, from (3.2.4) and (3.2.6) it follows that

$$3u(4n+1) = u(16n+4). \tag{3.2.10}$$

Now, employing (1.2.8) in (3.2.4), we have

$$\sum_{n=0}^{\infty} u(4n+1)q^n = 2\psi(q^2)\varphi^2(q^2) + 8q\psi(q^2)\psi^2(q^4). \tag{3.2.11}$$

Extracting the terms involving $q^{2n}$ and $q^{2n+1}$ from both sides of (3.2.11), we respectively find that

$$\sum_{n=0}^{\infty} u(8n+1)q^n = 2\psi(q)\varphi^2(q), \tag{3.2.12}$$

$$\sum_{n=0}^{\infty} u(8n+5)q^n = 8\psi(q)\psi^2(q^2). \tag{3.2.13}$$

Employing (3.1.2) in (3.2.13) and then equating the coefficients of $q^n$ from both sides, we readily deduce the first equality of (3.2.1).

Now we prove the second equality of (3.2.1). To this end, we note that the generating function for $v(n)$ is given by

$$\sum_{n=0}^{\infty} v(n)q^n = \varphi(q)\varphi^2(q^2). \tag{3.2.14}$$

With the help of (1.2.6), we rewrite (3.2.14) as

$$\begin{aligned}\sum_{n=0}^{\infty} v(n)q^n &= \varphi^2(q^2)\left(\varphi(q^4) + 2q\psi(q^8)\right)\\ &= \varphi^2(q^2)\varphi(q^4) + 2q\varphi^2(q^2)\psi(q^8).\end{aligned}$$

Extracting the even and odd terms of the above, we obtain

$$\sum_{n=0}^{\infty} v(2n)q^n = \varphi^2(q)\varphi(q^2), \tag{3.2.15}$$

$$\sum_{n=0}^{\infty} v(2n+1)q^n = 2\varphi^2(q)\psi(q^4). \tag{3.2.16}$$

Now, applying (1.2.8) in (3.2.15), and then extracting the even and odd terms, we find that

$$\sum_{n=0}^{\infty} v(4n)q^n = \varphi^3(q), \tag{3.2.17}$$

$$\sum_{n=0}^{\infty} v(4n+2)q^n = 4\varphi(q)\psi^2(q^2). \tag{3.2.18}$$

Next, employing (1.2.6) in (3.2.17) and then extracting the terms involving $q^{4n}$, $q^{4n+1}$, $q^{4n+2}$, and $q^{4n+3}$, respectively, we find that

$$v(16n) = v(4n),$$

$$\sum_{n=0}^{\infty} v(16n+4)q^n = 6\varphi^2(q)\psi(q^2), \tag{3.2.19}$$

$$\sum_{n=0}^{\infty} v(16n+8)q^n = 12\varphi(q)\psi^2(q^2), \tag{3.2.20}$$

$$\sum_{n=0}^{\infty} v(16n+12)q^n = 8\psi^3(q^2). \tag{3.2.21}$$

It follows from (3.2.21) that

$$\sum_{n=0}^{\infty} v(32n+12)q^n = 8\psi^3(q), \tag{3.2.22}$$

$$v(32n+28) = 0.$$

Now, employing (1.2.8) in (3.2.16), and then extracting the even and odd terms, we find that

$$\sum_{n=0}^{\infty} v(4n+1)q^n = 2\varphi^2(q)\psi(q^2), \tag{3.2.23}$$

$$\sum_{n=0}^{\infty} v(4n+3)q^n = 8\psi^3(q^2). \tag{3.2.24}$$

It follows from (3.2.24) that

$$\sum_{n=0}^{\infty} v(8n+3)q^n = 8\psi^3(q), \tag{3.2.25}$$

$$v(8n+7) = 0.$$

Also, from (3.2.19) and (3.2.23), we have

$$3v(4n+1) = v(16n+4). \tag{3.2.26}$$

On the other hand, employing (1.2.8) in (3.2.23) and then extracting the odd and even terms of the resulting identity, and with the aid of (3.1.2), we find that

$$\sum_{n=0}^{\infty} v(8n+1)q^n = 2\psi(q)\varphi^2(q), \tag{3.2.27}$$

$$\sum_{n=0}^{\infty} v(8n+5)q^n = 8\psi(q)\psi^2(q^2) = 8\sum_{n=0}^{\infty} a_4(n)q^n. \tag{3.2.28}$$

From (3.2.28), we easily deduce the second equality of (3.2.1) to finish the proof.

□

**Corollary 3.2.2.** *If $r_3(n)$ denotes the number of representations of $n$ as a sum of three squares, then*

$$r_3(8n+5) = 3u(8n+5) = 3v(8n+5) = 24a_4(n). \tag{3.2.29}$$

*Proof.* We note that

$$\sum_{n=0}^{\infty} r_3(n)q^n = \varphi^3(q). \tag{3.2.30}$$

From (3.2.3), (3.2.17) and (3.2.30), we deduce that

$$r_3(n) = u(4n) = v(4n). \tag{3.2.31}$$

Now, replacing $n$ by $2n+1$ in (3.2.10) and (3.2.26), then employing (3.2.31), we obtain

$$3u(8n+5) = u(32n+20) = r_3(8n+5) \text{ and } 3v(8n+5) = v(32n+20) = r_3(8n+5),$$

from which, with the help of (3.2.1), we easily deduce (3.2.29). □

Next, we deduce the formula given above as (3.1.3) due to Ono and Sze [51, Theorem 2].

**Corollary 3.2.3.** (Ono and Sze [51, Theorem 2]). *Formula* (3.1.3) *holds.*

*Proof.* A classical result due to Gauss states that if $n$ is square-free and $n > 4$, then

$$r_3(n) = \begin{cases} 24h(-n), & \text{for } n \equiv 3 \pmod 8; \\ 12h(-4n), & \text{for } n \equiv 1,2,5,6 \pmod 8; \\ 0, & \text{for } n \equiv 7 \pmod 8). \end{cases}$$

Now (3.1.3) readily follows from Corollary 3.2.2. □

We end this section by giving two more corollaries arising from the proof of the above theorem.

**Corollary 3.2.4.** *We have*

$$u(8n+1) = v(8n+1), \tag{3.2.32}$$

$$u(16n+8) = v(16n+8) = 3v(4n+2). \tag{3.2.33}$$

*Proof.* Identity (3.2.32) follows from (3.2.12) and (3.2.27), and (3.2.33) follows from (3.2.7), (3.2.18) and (3.2.20). □

**Corollary 3.2.5.** *We have*

$$u(32n+12) = r_3(8n+3) = v(32n+12) = v(8n+3) = 8t_3(n), \tag{3.2.34}$$

*where $t_3(n)$ is the number of representations of $n$ as a sum of three triangular numbers.*

*Proof.* We note that

$$\sum_{n=0}^{\infty} t_3(n)q^n = \psi^3(q). \tag{3.2.35}$$

Now (3.2.34) follows easily from (3.2.35), (3.2.9), (3.2.22), (3.2.25), (3.2.3), and (3.2.30).

□

## 3.3 Infinite families of arithmetic properties of $a_4(n)$

In this section, we prove some infinite families of arithmetic identities for $a_4(n)$ by using the results from the previous section. First, we deduce the following infinite families of arithmetic identities.

**Theorem 3.3.1.** [Hirschhorn-Sellers [32]]. *If* $a_4(n)$ *denotes the number of* 4*-cores of* $n$*, and* $k \geq 1$ *then*

$$3^k a_4(3n) = a_4\left(3^{2k+1}n + \frac{5 \times 3^{2k} - 5}{8}\right), \tag{3.3.1}$$

$$\left(2 \times 3^k - 1\right) a_4(3n+1) = a_4\left(3^{2k+1}n + \frac{13 \times 3^{2k} - 5}{8}\right), \tag{3.3.2}$$

$$\left(\frac{3^{k+1}-1}{2}\right) a_4(9n+2) = a_4\left(3^{2k+2}n + \frac{7 \times 3^{2k+1} - 5}{8}\right), \tag{3.3.3}$$

$$\left(\frac{3^{k+1}-1}{2}\right) a_4(9n+8) = a_4\left(3^{2k+2}n + \frac{23 \times 3^{2k+1} - 5}{8}\right). \tag{3.3.4}$$

*Proof.* Cooper and Hirschhorn [21] found the following arithmetic properties of $r_3(n)$.

For any nonnegative integer $n$ and any integer $k \geq 1$, we have

$$3^k r_3(6n+5) = r_3(9^k(6n+5)), \tag{3.3.5}$$

$$\left(2 \times 3^k - 1\right) r_3(24n+13) = r_3\left(9^k(24n+13)\right), \tag{3.3.6}$$

$$\left(\frac{3^{k+1}-1}{2}\right) r_3(72n+21) = r_3\left(9^k(72n+21)\right), \tag{3.3.7}$$

$$\left(\frac{3^{k+1}-1}{2}\right) r_3(72n+69) = r_3\left(9^k(72n+69)\right). \tag{3.3.8}$$

Replacing $n$ by $4n$ in (3.4.1), we have

$$3^k r_3\left(8(3n)+5\right) = r_3\left(8\left(3^{2k+1}n + \frac{5 \times 3^{2k} - 5}{8}\right) + 5\right),$$

from which we readily deduce (3.3.1) by employing (3.2.29).

In a similar fashion, (3.3.2)–(3.3.4) follow from (3.4.2)–(3.4.4), respectively. □

In the next theorem we give some more infinite families of arithmetic identities for $a_4(n)$.

**Theorem 3.3.2.** *If $a_4(n)$ denotes the number of 4-cores of $n$, and $k \geq 1$ then*

$$5a_4(5n+2) = a_4(125n+65), \tag{3.3.9}$$

$$5a_4(5n+3) = a_4(125n+90), \tag{3.3.10}$$

$$\left(\frac{5^{k+1}-1}{4}\right) a_4(25n) = a_4\left(5^{2k+2}n + \frac{5^{2k+1}-5}{8}\right), \tag{3.3.11}$$

$$\left(\frac{5^{k+1}-1}{4}\right) a_4(25n+5) = a_4\left(5^{2k+2}n + \frac{9\times 5^{2k+1}-5}{8}\right), \tag{3.3.12}$$

$$\left(\frac{5^{k+1}-1}{4}\right) a_4(25n+10) = a_4\left(5^{2k+2}n + \frac{17\times 5^{2k+1}-5}{8}\right), \tag{3.3.13}$$

$$\left(\frac{5^{k+1}-1}{4}\right) a_4(25n+20) = a_4\left(5^{2k+2}n + \frac{33\times 5^{2k+1}-5}{8}\right). \tag{3.3.14}$$

Before proving the theorem, we prove the following lemma concerning $r_3(n)$.

**Lemma 3.3.3.** *If $r_3(n)$ denotes the number of representations of $n$ as a sum of three squares, then*

$$5r_3(5n+1) = r_3\left(25(5n+1)\right), \tag{3.3.15}$$

$$5r_3(5n+4) = r_3\left(25(5n+4)\right), \tag{3.3.16}$$

$$\left(\frac{5^{k+1}-1}{4}\right) r_3(25n+5) = r_3\left(25^k(25n+5)\right), \tag{3.3.17}$$

$$\left(\frac{5^{k+1}-1}{4}\right) r_3(25n+10) = r_3\left(25^k(25n+10)\right), \tag{3.3.18}$$

$$\left(\frac{5^{k+1}-1}{4}\right) r_3(25n+15) = r_3\left(25^k(25n+15)\right), \tag{3.3.19}$$

$$\left(\frac{5^{k+1}-1}{4}\right) r_3(25n+20) = r_3\left(25^k(25n+20)\right). \tag{3.3.20}$$

*Proof.* Employing the five dissection of $\varphi(q)$ from (1.2.15) in (3.2.30) and then extracting the terms involving $q^{5l+r}$ for $r = 0, 1, 2, 3, 4$, respectively, we find that

$$\sum_{n=0}^{\infty} r_3(5n)q^n = \varphi^3(q^5) + 24q\varphi(q^5)f(q,q^9)f(q^3,q^7), \tag{3.3.21}$$

$$\sum_{n=0}^{\infty} r_3(5n+1)q^n = 6\varphi^2(q^5)f(q^3,q^7) + 24qf^2f(q,q^9)(q^3,q^7), \tag{3.3.22}$$

$$\sum_{n=0}^{\infty} r_3(5n+2)q^n = 12\varphi(q^5)f^2(q^3,q^7) + 8q^2f^3(q,q^9),$$

$$\sum_{n=0}^{\infty} r_3(5n+3)q^n = 8f^3(q^3,q^7) + 12q\varphi(q^5)f^2(q,q^9),$$

$$\sum_{n=0}^{\infty} r_3(5n+4)q^n = 6\varphi^2(q^5)f(q,q^9) + 24qf^2(q,q^9)f(q^3,q^7). \tag{3.3.23}$$

Now, from [16, p. 262, Entry 10(iv)], we note that

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q,q^9)f(q^3,q^7). \tag{3.3.24}$$

Employing (3.3.24) in (3.3.21), we obtain

$$\sum_{n=0}^{\infty} r_3(5n)q^n = 6\varphi^2(q)\varphi(q^5) - 5\varphi^3(q^5), \tag{3.3.25}$$

which we rewrite , with the aid of (3.2.30), as

$$\sum_{n=0}^{\infty} r_3(5n)q^n = 6\varphi^2(q)\varphi(q^5) - 5\sum_{n=0}^{\infty} r_3(n)q^{5n}. \tag{3.3.26}$$

Similarly, employing (3.3.24) in (3.3.22) and (3.3.23), we obtain

$$\sum_{n=0}^{\infty} r_3(5n+1)q^n = 6\varphi^2(q)f(q^3,q^7), \tag{3.3.27}$$

and

$$\sum_{n=0}^{\infty} r_3(5n+4)q^n = 6\varphi^2(q)f(q,q^9), \tag{3.3.28}$$

respectively.

Again, using (3.3.24) in (3.3.25), and then extracting the terms involving $q^{5n}$, we deduce that

$$\sum_{n=0}^{\infty} r_3(25n)q^n = 6\varphi(q)\varphi^2(q^5) + 48q\varphi(q)f(q,q^9)f(q^3,q^7) - 5\varphi^3(q), \tag{3.3.29}$$

Employing (3.3.24) once again in (3.3.29), we find that

$$\sum_{n=0}^{\infty} r_3(25n)q^n = 7\varphi^3(q) - 6\varphi(q)\varphi^2(q^5),$$

which we rewrite, with the help of (1.2.15) as

$$\sum_{n=0}^{\infty} r_3(25n)q^n = 7\varphi^3(q) - 6\varphi^2(q^5)\left\{\varphi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4f(q^5,q^{45})\right\}. \tag{3.3.30}$$

Now, employing (3.2.30) in (3.3.30), and then extracting the terms involving $q^{5n}$, we find that

$$\sum_{n=0}^{\infty} r_3(125n)q^n = 7\sum_{n=0}^{\infty} r_3(5n)q^n - 6\varphi^2(q)\varphi(q^5). \tag{3.3.31}$$

Employing (3.3.26) in (3.3.31), we arrive at

$$5\sum_{n=0}^{\infty} r_3(n)q^{5n} = 6\sum_{n=0}^{\infty} r_3(5n)q^n - \sum_{n=0}^{\infty} r_3(125n)q^n. \tag{3.3.32}$$

we are now in a position to prove (3.3.15)–(3.3.20). First we prove (3.3.15) and (3.3.16). Equating the terms involving $q^{5n+1}$ and $q^{5n+4}$, respectively, from both sides of (3.3.30), we obtain

$$\sum_{n=0}^{\infty} r_3(125n+25)q^n = 7\sum_{n=0}^{\infty} r_3(5n+1)q^n - 12\varphi^2(q)f(q^3,q^7), \tag{3.3.33}$$

and

$$\sum_{n=0}^{\infty} r_3(125n+100)q^n = 7\sum_{n=0}^{\infty} r_3(5n+4)q^n - 12\varphi^2(q)f(q,q^9), \tag{3.3.34}$$

respectively. Employing (3.3.27) and (3.3.28) in (3.3.33) and (3.3.34), respectively, and then equating the coefficients of $q^n$ from both sides of the resulting identities, we readily deduce (3.3.15) and (3.3.16).

Next, we prove (3.3.17). Equating the coefficients of $q^{5n+1}$ from both sides of (3.3.32), we deduce that

$$6r_3(25n+5) = r_3(25(25n+5)), \tag{3.3.35}$$

which is (3.3.17) for $k = 1$.

Again, equating the coefficients of $q^{25(5n+1)}$ from both sides of (3.3.32), we find that

$$5r_3(25n+5) = 6r_3(5^2(25n+5) - r_3(25^2(25n+5)),$$

which, with an aid from (3.3.35), reduces to

$$31r_3(25n+5) = r_3(25^2(25n+5)),$$

which is nothing but (3.3.17) with $k = 2$. We complete the proof of (3.3.17) by mathematical induction.

We now prove (3.3.18). Equating the coefficients of $q^{5n+2}$ from both sides of (3.3.32), we obtain

$$6r_3(25n+10) = r_3(25(25n+10)), \tag{3.3.36}$$

which is (3.3.18) for $k = 1$.

Again, equating the coefficients of $q^{25(5n+2)}$ from both sides of (3.3.32), we find that

$$5r_3(25n+10) = 6r_3(5^2(25n+10) - r_3(25^2(25n+10)),$$

which, by (3.3.36), reduces to

$$31r_3(25n+10) = r_3(25^2(25n+10)),$$

which is (3.3.18) with $k = 2$. Now the proof of (3.3.17) can be completed by mathematical induction.

In a similar fashion, equating the respective coefficients of $q^{5n+3}$ and $q^{5n+4}$ from both sides of (3.3.32), and proceeding as in the proofs of (3.3.17) and (3.3.18), we can prove (3.3.19) and (3.3.20). Thus, we complete the proof of the lemma. □

*Proof of Theorem* 3.3.2. Replacing $n$ by $8n+4$ in (3.3.15), we find that

$$5r_3(8(5n+2)+5) = r_3(8(125n+65)+5). \tag{3.3.37}$$

Employing (3.2.29) in (3.3.37), we readily deduce (3.3.9).

Next, replacing $n$ by $8n+5$ in (3.3.16), and then using (3.2.29), we deduce (3.3.10).

Again, replacing $n$ by $8n$ in (3.3.17), we have

$$\left(\frac{5^{k+1}-1}{4}\right) r_3(8(25n)+5) = r_3\left(8\left(5^{2k+2}+\frac{5^{2k+1}-5}{8}\right)+5\right),$$

which implies (3.3.11) with the aid of (3.2.29).

Similarly, replacing $n$ by $8n+3$, $8n+6$, and $8n+1$ in (3.3.18), (3.3.19), and (3.3.20), respectively, and then employing (3.2.29), we deduce (3.3.13), (3.3.14), and (3.3.12), respectively, to finish the proof. □

## 3.4 Infinite families of results involving $r_3(n)$ and $t_3(n)$

In this section, we give new proofs for the infinite families of arithmetic identities involving sum of three squares given by Hirschhorn and Sellers [33], and by Baruah and Boruah [8].

**Theorem 3.4.1.** *For any nonnegative integer n and any integer $k \geq 1$, if $r_3(n)$ denotes the number of representation of n as a sum of three squares, then*

$$\left(2\times 3^k-1\right) r_3(3n+1) = r_3(9^k(3n+1)), \tag{3.4.1}$$

$$3^k r_3(3n+2) = r_3\left(9^k(3n+2)\right), \tag{3.4.2}$$

$$\left(\frac{3^{k+1}-1}{2}\right) r_3(9n+3) = r_3\left(9^k(9n+3)\right), \tag{3.4.3}$$

$$\left(\frac{3^{k+1}-1}{2}\right) r_3(9n+6) = r_3\left(9^k(9n+6)\right), \tag{3.4.4}$$

$$\left(\frac{3^{k+1}-1}{2}\right) r_3(9n) = r_3\left(9^k(9n)\right) + \left(\frac{3^{k+1}-3}{2}\right) r_3(n). \tag{3.4.5}$$

*Proof.* Employing (1.2.14) in (3.2.30), and then extracting the terms involving $q^{3n}$,

$q^{3n+1}$, $q^{3n+2}$ from both sides of the resulting identity, we find that

$$\sum_{n=0}^{\infty} r_3(3n)q^n = \varphi^3(q^3) + 8qf^3(q,q^5), \tag{3.4.6}$$

$$\sum_{n=0}^{\infty} r_3(3n+1)q^n = 6\varphi^2(q^3)f(q,q^5), \tag{3.4.7}$$

$$\sum_{n=0}^{\infty} r_3(3n+2)q^n = 12\varphi(q^3)f^2(q,q^5). \tag{3.4.8}$$

Employing (1.2.18) and (3.2.30) in (3.4.6), we obtain

$$\sum_{n=0}^{\infty} r_3(3n)q^n = \sum_{n=0}^{\infty} r_3(n)q^{3n} + 8q\psi^3(-q^3)\frac{\varphi(q)}{\psi(-q)}. \tag{3.4.9}$$

Now, from a result of Baruah and Ojah [6, Theorem 2.1], we note that

$$\frac{1}{\psi(-q)} = \frac{\psi(-q^9)}{\psi^4(-q^3)}\left(\frac{\varphi^2(q^9)}{\chi^2(q^3)} + q\frac{\varphi(q^9)\psi(-q^9)}{\chi(q^3)} + q^2\psi^2(-q^9)\right). \tag{3.4.10}$$

Employing (3.4.10) and (1.2.14) in (3.4.9) and then extracting the terms involving $q^{3n}$ only from both sides of the resulting identity, we find that

$$\sum_{n=0}^{\infty} r_3(9n)q^n = \sum_{n=0}^{\infty} r_3(n)q^n + 24q\frac{\psi^3(-q^3)\varphi(q^3)}{\psi(-q)}, \tag{3.4.11}$$

where we have also used (1.2.18). Employing again (3.4.10) in (3.4.11) and then extracting the terms involving $q^{3n}$, $q^{3n+1}$, $q^{3n+2}$ from both sides of the resulting identity, we deduce that

$$\sum_{n=0}^{\infty} r_3(27n)q^n = \sum_{n=0}^{\infty} r_3(3n)q^n + 24qf^3(q,q^5), \tag{3.4.12}$$

$$\sum_{n=0}^{\infty} r_3(27n+9)q^n = 5\sum_{n=0}^{\infty} r_3(3n+1)q^n, \tag{3.4.13}$$

$$\sum_{n=0}^{\infty} r_3(27n+18)q^n = 3\sum_{n=0}^{\infty} r_3(3n+2)q^n, \tag{3.4.14}$$

where (1.2.18), (3.4.7) and (3.4.8) are also used.

Employing (3.4.6) in (3.4.12), we obtain

$$\sum_{n=0}^{\infty} r_3(27n)q^n = 4\sum_{n=0}^{\infty} r_3(3n)q^n - 3\sum_{n=0}^{\infty} r_3(n)q^{3n}. \tag{3.4.15}$$

Again, equating the terms involving $q^{3n+1}$, $q^{3n+2}$ and $q^{3n}$, from both sides of (3.4.15), we find that

$$r_3(81n+27) = 4r_3(9n+3), \tag{3.4.16}$$

$$r_3(81n+54) = 4r_3(9n+6) \tag{3.4.17}$$

$$r_3(81n) = 4r_3(9n) - 3r_3(n). \tag{3.4.18}$$

From (3.4.13)–(3.4.18) and by mathematical induction, we complete the proof. □

We now show that the results of Hirschhorn and Sellers [30] and Baruah and Boruah [8] on sum of three triangular numbers given in the following theorem easily follow from Theorem 3.4.1.

**Theorem 3.4.2.** *For any nonnegative integer $n$ and any integer $k \geq 1$, if $t_3(n)$ denotes the number of representation of $n$ as a sum of three triangular numbers, then*

$$3^k t_3(3n+1) = t_3\left(3^{2k+1}n + \frac{11\times 3^{2k}-3}{8}\right), \tag{3.4.19}$$

$$\left(2\times 3^k - 1\right) t_3(3n+2) = t_3\left(3^{2k+1} + \frac{19\times 3^{2k}-3}{8}\right), \tag{3.4.20}$$

$$\left(\frac{3^{k+1}-1}{2}\right) t_3(9n) = t_3\left(3^{2k+2} + \frac{3^{2k+1}-3}{8}\right), \tag{3.4.21}$$

$$\left(\frac{3^{k+1}-1}{2}\right) t_3(9n+6) = t_3\left(3^{2k+2} + \frac{17\times 3^{2k+1}-3}{8}\right) \tag{3.4.22}$$

$$\left(\frac{3^{k+1}-1}{2}\right) t_3(9n+3) = t_3\left(3^{2k+2}n + \frac{3(3^{2k+2}-1)}{8}\right) + \left(\frac{3^{k+1}-3}{2}\right) t_3(n). \tag{3.4.23}$$

*Proof.* Employing (1.2.6) in (3.2.30) and then equating the terms involving $q^{4n+3}$ from both sides of the resulting identity, we find that

$$\sum_{n=0}^{\infty} r_3(4n+3)q^n = 8\psi^3(q^2),$$

Hence, we have

$$\sum_{n=0}^{\infty} r_3(8n+3)q^n = 8\psi^3(q) = 8\sum_{n=0}^{\infty} t_3(n)q^n,$$

and consequently,

$$r_3(8n+3) = 8t_3(n). \tag{3.4.24}$$

Now, replacing $n$ by $8n+6$ in (3.4.1), we obtain

$$\left(2\times 3^k - 1\right) r_3\left(8\left(3n+2\right)+3\right) = r_3\left(8\left(3^{2k+1}n + \frac{19\times 9^k - 3}{8}\right)+3\right). \tag{3.4.25}$$

Employing (3.4.24)in (3.4.25), we easily arrive at (3.4.20).

In a similar fashion, replacing $n$ by $8n+3$, $8n$, $8n+5$ and $8n+3$ in (3.4.2)–(3.4.5), respectively, we deduce (3.4.19), (3.4.21)–(3.4.23). □

Some more infinite families of arithmetic identities for $t_3(n)$ are presented in the following theorem.

**Theorem 3.4.3.** *For any nonnegative integer $n$ and any integer $k \geq 1$, we have*

$$5t_3(5n+1) = t_3(125n+34), \tag{3.4.26}$$

$$5t_3(5n+2) = t_3(125n+59), \tag{3.4.27}$$

$$7t_3(5n+3) = t_3(125n+84), \tag{3.4.28}$$

$$7t_3(5n) = t_3(125n+9), \tag{3.4.29}$$

$$\frac{5^{k+1}-1}{4}t_3(25n+4) = t_3\left(5^{2k+2}n + \frac{7\times 5^{2k+1}-3}{8}\right), \tag{3.4.30}$$

$$6t_3(25n+9) = t_3\left(625n+234\right) + 5t_3(n), \tag{3.4.31}$$

$$\frac{5^{k+1}-1}{4}t_3(25n+14) = t_3\left(5^{2k+2}n + \frac{23\times 5^{2k+1}-3}{8}\right), \tag{3.4.32}$$

$$\frac{5^{k+1}-1}{4}t_3(25n+19) = t_3\left(5^{2k+2}n + \frac{31\times 5^{2k+1}-3}{8}\right), \tag{3.4.33}$$

$$\frac{5^{k+1}-1}{4}t_3(25n+24) = t_3\left(5^{2k+2}n + \frac{39\times 5^{2k+1}-3}{8}\right). \tag{3.4.34}$$

*Proof.* From (3.3.15)–(3.3.20), we note that, for any nonnegative integer $n$ and any integer $k \geq 1$,

$$5r_3(5n+1) = r_3\left(25(5n+1)\right), \tag{3.4.35}$$

$$5r_3(5n+4) = r_3\left(25(5n+4)\right), \tag{3.4.36}$$

$$\left(\frac{5^{k+1}-1}{4}\right) r_3(25n+5) = r_3\left(25^k(25n+5)\right), \tag{3.4.37}$$

$$\left(\frac{5^{k+1}-1}{4}\right) r_3(25n+10) = r_3\left(25^k(25n+10)\right), \tag{3.4.38}$$

$$\left(\frac{5^{k+1}-1}{4}\right) r_3(25n+15) = r_3\left(25^k(25n+15)\right), \tag{3.4.39}$$

$$\left(\frac{5^{k+1}-1}{4}\right) r_3(25n+20) = r_3\left(25^k(25n+20)\right). \tag{3.4.40}$$

Again, from (3.3.30) and (3.3.32), we note that

$$\sum_{n=0}^{\infty} r_3(25n)q^n = 7\varphi^3(q) - 6\varphi^2(q^5)\left(\varphi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4 f(q^5,q^{45})\right), \tag{3.4.41}$$

$$5\sum_{n=0}^{\infty} r_3(n)q^{5n} = 6\sum_{n=0}^{\infty} r_3(5n)q^n - \sum_{n=0}^{\infty} r_3(125n)q^n. \tag{3.4.42}$$

Using (3.2.30) and equating the coefficients of $q^{5n+2}$ and $q^{5n+3}$, respectively, from both sides of (3.4.41), we deduce that

$$7r_3(5n+2) = r_3\left(25(5n+2)\right), \tag{3.4.43}$$

$$7r_3(5n+3) = r_3\left(25(5n+3)\right). \tag{3.4.44}$$

On the other hand, equating the coefficients of $q^{5n}$ from both sides of (3.4.42), we obtain

$$6r_3(25n) = r_3\left(625n\right) + 5r_3(n). \tag{3.4.45}$$

Now, replacing $n$ by $8n+2$, $8n+3$, $8n+5$, $8n$, $8n+3$, $8n+6$, $8n+1$, $8n+4$, and $8n+7$ in (3.4.35), (3.4.36), (3.4.43)–(3.4.45), (3.4.37)–(3.4.40), respectively, we deduce (3.4.26)–(3.4.29), (3.4.31), (3.4.33), (3.4.30), (3.4.32),(3.4.34). □

## 3.5 Some other new arithmetic identities for $a_4(n)$

We found new proofs for the identities earlier given by Hirschhorn and Sellers [32] and also obtain some analogous new infinite families of arithmetic identities for $a_4(n)$. We notice that the identities stated in the following theorem were missed by Hirschhorn and Sellers [32] and by us [10]. Here we prove these by using the results from Section 3.4.

**Theorem 3.5.1.** *For any nonnegative integer n and positive integers k, we have*

$$\left(\frac{3^{k+1}-1}{2}\right)a_4(9n+5) = a_4\left(3^{2k+2}n + \frac{15\times 3^{2k+1}-5}{8}\right) + \frac{3^{k+1}-1}{2}a_4(n). \quad (3.5.1)$$

*Proof.* Replacing $n$ by $8n+5$ in (3.4.18), we have

$$4r_3(8(9n+5)+5) = r_3\left(8(81n+50)+5\right) + 3r_3(8n+5),$$

which can be rewritten, with the aid of (3.2.29), as

$$4a_4(9n+5) = a_4\left(81n+50\right) + 3a_4(n). \quad (3.5.2)$$

Iterating (3.5.2) and by mathematical induction we easily arrive at (3.5.1). □

The identities stated in the following theorem were missed by us in [10].

**Theorem 3.5.2.** *For any nonnegative integer n, we have*

$$7a_4(5n+1) = a_4(125n+40), \quad (3.5.3)$$

$$7a_4(5n+4) = a_4(125n+115), \quad (3.5.4)$$

$$6a_4(25n+15) = a_4\left(625n+390\right) + 5a_4(n). \quad (3.5.5)$$

*Proof.* Replacing $n$ by $8n+2$ in (3.4.44), we find that

$$7r_3(8(5n+1)+5) = r_3(8(125n+40)+5). \quad (3.5.6)$$

Employing (3.2.29) in (3.5.6), we readily deduce (3.5.3). Next, replacing $n$ by $8n+7$ in (3.4.43), and then using (3.2.29), we deduce (3.5.4). Again, replacing $n$ by $8n+5$ in (3.4.45), we have

$$6r_3(8(25n+15)+5) = r_3\left(8(625n+390)+5\right) + 5r_3(8n+5),$$

which implies (3.5.5) with the aid of (3.2.29). □

# Chapter 4

# Two Infinite Families of Arithmetic Identities for 5-Cores

## 4.1 Introduction

If $a_5(n)$ denotes the number of partitions of $n$ that are 5-cores, then from (1.1.7) the generating function for $a_5(n)$ is given by

$$\sum_{n=0}^{\infty} a_5(n)q^n = \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}. \tag{4.1.1}$$

We note that, by (1.2.5), the formula (4.1.1) reduces to

$$\sum_{n=0}^{\infty} a_5(n)q^n = \frac{f^5(-q^5)}{f(-q)}. \tag{4.1.2}$$

By using a modular equation of degree 5 recorded by Ramanujan in his second notebook [16, p. 280, Entry 13(iii)], Baruah and Berndt [4, Theorem 2.5] proved that

$$a_5(4n+3) = a_5(2n+1) + 2a_5(n). \tag{4.1.3}$$

In fact, they first transcribed the said modular equation into the equivalent form

$$\frac{f^5(-q^5)}{f(-q)} - 4q^3\frac{f^5(-q^{20})}{f(-q^4)} = \frac{f^5(q^5)}{f(q)} + 2q\frac{f^5(-q^{10})}{f(-q^2)},$$

which can be rewritten, with the aid of (4.1.2), as

$$\frac{1}{2}\left(\sum_{n=0}^{\infty} a_5(n)q^n - \sum_{n=0}^{\infty}(-1)^n a_5(n)q^n\right) = q\sum_{n=0}^{\infty} a_5(n)q^{2n} + 2q^3\sum_{n=0}^{\infty} a_5(n)q^{4n}. \tag{4.1.4}$$

Then by equating the coefficients of $q^{4n+3}$ on both sides of (4.1.4), they readily arrived at (4.1.3).

Now, by equating the coefficients of $q^{4n+1}$ on both sides of (4.1.4), we deduce that

$$a_5(4n+1) = a_5(2n). \tag{4.1.5}$$

The above identity was missed by Baruah and Berndt [4].

In the next section, we give some preliminary results, which will be used in Section 4.3 to prove our main results on $a_5(n)$.

## 4.2 Preliminary results

In the following lemmas, we state some properties satisfied by Ramanujan's theta functions, which will be used in the subsequent section.

**Lemma 4.2.1.** [16, p. 278] *We have*

$$\varphi(q^5)\varphi(-q) - \varphi(-q^5)\varphi(q) = -4qf(-q^4)f(-q^{20}). \tag{4.2.1}$$

**Theorem 4.2.2.** *We have*

$$\psi^3(q)\psi(q^5) - q\psi(q)\psi^3(q^5) = \sum_{n=0}^{\infty} a_5(n)q^n + q\sum_{n=0}^{\infty} a_5(n)q^{2n}. \tag{4.2.2}$$

*Proof.* From [16, p. 262, Entry 10(v)], we note that

$$\psi^2(q) - q\psi^2(q^5) = f(q,q^4)f(q^2,q^3). \tag{4.2.3}$$

Multiplying both sides by $\dfrac{\psi^3(q^5)}{\psi(q)}$ and employing (1.2.2), we find that

$$\psi(q)\psi^3(q^5) - q\frac{\psi^5(q^5)}{\psi(q)} = \frac{f^5(-q^{10})}{f(-q^2)}. \tag{4.2.4}$$

Again, squaring both sides of (4.2.3), and then multiplying by $\dfrac{\psi(q^5)}{\psi(q)}$, we obtain

$$\psi^3(q)\psi(q^5) + q^2\frac{\psi^5(q^5)}{\psi(q)} - 2q\psi(q)\psi^3(q^5) = \frac{f^5(-q^5)}{f(-q)}, \tag{4.2.5}$$

where we have repeatedly used (1.2.2).

Multiplying (4.2.4) by $q$, adding with (4.2.5), and then using (4.1.2), we arrive at (4.2.2). □

**Theorem 4.2.3.** *We have*

$$(-q;q^2)^3_\infty(-q^5;q^{10})_\infty - (q;q^2)^3_\infty(q^5;q^{10})_\infty = \frac{4q}{(q^{10};q^{20})_\infty(q^2;q^4)^3_\infty} + 2q\frac{(q^2;q^4)^2_\infty}{(q^{10};q^{20})^2_\infty}. \tag{4.2.6}$$

*Proof.* Setting $a=q$, $b=q^9$, $c=q^3$, $d=q^7$, and $n=q^2$ in (1.2.12), we have

$$f(q,q^9)f^2(q^3,q^7)f(q^5,q^5) - f(-q,-q^9)f^2(-q^3,-q^7)f(-q^5,-q^5) = 2qf^2(q^2,q^8)f(q^4,q^6)\psi(q^{10}). \tag{4.2.7}$$

Applying (1.2.2) in (4.2.7), and then manipulating the $q$-products, we find that

$$(-q;q^2)_\infty(-q^3,-q^5,-q^7;q^{10})_\infty - (q;q^2)_\infty(q^3,q^5,q^7;q^{10})_\infty = 2q(-q^2;q^2)_\infty(-q^2,-q^8,-q^{10};q^{10})_\infty. \tag{4.2.8}$$

Again, setting $a=q^{-1}$, $b=q^{11}$, $c=q^3$, $d=q^7$, and $n=q^2$ in (1.2.12), and then using the trivial identity $f(a,b)=af(a^{-1},a^2b)$, we obtain

$$(-q;q^2)_\infty(-q,-q^5,-q^9;q^{10})_\infty + (q;q^2)_\infty(q,q^5,q^9;q^{10})_\infty = 2(-q^2;q^2)_\infty(-q^4,-q^6,-q^{10};q^{10})_\infty. \tag{4.2.9}$$

Furthermore, setting $a=-q$, $b=-q^9$, $c=q^3$, and $d=q^7$, in (1.2.10), we find that

$$(q,q^9,-q^3,-q^7;q^{10})_\infty - (-q,-q^9,q^3,q^7;q^{10})_\infty = -2q(-q^{10};q^{10})^2_\infty(q^2,q^6,q^{14},q^{18};q^{20})_\infty. \tag{4.2.10}$$

Multiplying (4.2.8), and (4.2.9) and then using (4.2.10), we arrive at (4.2.6) to finish the proof. □

**Remark 4.2.4.** *The q-series identity* (4.2.6) *is equivalent to the modular equation* [16, p. 281, Entry 13(vii)]

$$(\alpha^3\beta)^{1/8} + \{(1-\alpha)^3(1-\beta)\}^{1/8} = 1 - 2^{1/3}\left\{\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right\}^{1/24}$$

*where $\beta$ has degree* 5 *over $\alpha$.*

## 4.3 Main results on $a_5(n)$

At the beginning of this section, we prove (4.3.1) by showing the equivalence of their generating functions.

**Theorem 4.3.1.** *If $a_5(n)$ denotes the number of* 5*-cores of $n$, then*

$$5\sum_{n=0}^{\infty} a_5(n)q^n = \sum_{n=0}^{\infty} a_5(5n+4)q^n. \tag{4.3.1}$$

Here we present two proofs of the above theorem.

*First Proof of Theorem* 4.3.1. Let $t(n)$ be defined by

$$\psi^3(q)\psi(q^5) - q\psi(q)\psi^3(q^5) = \sum_{n=0}^{\infty} t(n)q^n. \tag{4.3.2}$$

Then it is clear from (4.2.2) that

$$\sum_{n=0}^{\infty} t(2n)q^n = \sum_{n=0}^{\infty} a_5(2n)q^n \tag{4.3.3}$$

and

$$\sum_{n=0}^{\infty} t(2n+1)q^n = \sum_{n=0}^{\infty} a_5(2n+1)q^n + \sum_{n=0}^{\infty} a_5(n)q^n. \tag{4.3.4}$$

Now, employing (1.2.17) in (4.3.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} t(n)q^n =& \psi(q^5)\left(f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25})\right)^3 \\ &- q\psi^3(q^5)\left(f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25})\right).\end{aligned} \tag{4.3.5}$$

Extracting the terms involving $q^{5n+4}$ from both sides of (4.3.5), we find that

$$\sum_{n=0}^{\infty} t(5n+4)q^n = 6\psi(q)\psi(q^5)f(q^2,q^3)f(q,q^4) - (\psi^3(q)\psi(q^5) - q\psi(q)\psi^3(q^5)). \tag{4.3.6}$$

Employing (4.2.3) in (4.3.6), we obtain

$$\sum_{n=0}^{\infty} t(5n+4)q^n = 5(\psi^3(q)\psi(q^5) - q\psi(q)\psi^3(q^5)) = 5\sum_{n=0}^{\infty} t(n)q^n. \tag{4.3.7}$$

From (4.3.3) and (4.3.7), we find that

$$5\sum_{n=0}^{\infty} a_5(2n)q^n = \sum_{n=0}^{\infty} a_5(10n+4)q^n. \tag{4.3.8}$$

To complete the proof, we need to show that (4.3.8) also holds when $2n$ is replaced by $2n+1$. To this end, extracting the odd parts from both sides of (4.3.7), we have

$$\sum_{n=0}^{\infty} t(2(5n+4)+1)q^n = 5\sum_{n=0}^{\infty} t(2n+1)q^n. \tag{4.3.9}$$

From (4.3.9) and (4.3.4), we find that

$$5\sum_{n=0}^{\infty} a_5(2n+1)q^n + 5\sum_{n=0}^{\infty} a_5(n)q^n = \sum_{n=0}^{\infty} a_5(10n+9)q^n + \sum_{n=0}^{\infty} a_5(5n+4)q^n. \tag{4.3.10}$$

Extracting the even parts from both sides of (4.3.10), and then using (4.3.8), we obtain

$$5\sum_{n=0}^{\infty} a_5(4n+1)q^n = \sum_{n=0}^{\infty} a_5(5(4n+1)+4)q^n. \tag{4.3.11}$$

Again, extracting the terms involving $q^{4n+1}$ from both sides of (4.3.10), and then using (4.3.11), we find that

$$5\sum_{n=0}^{\infty} a_5(8n+3)q^n = \sum_{n=0}^{\infty} a_5(5(8n+3)+4)q^n. \tag{4.3.12}$$

We continue the process, and find by mathematical induction that, for any integer $k \geq 2$,

$$5\sum_{n=0}^{\infty} a_5(2^kn+2^{k-1}-1)q^n = \sum_{n=0}^{\infty} a_5\left(5(2^kn+2^{k-1}-1)+4\right)q^n. \tag{4.3.13}$$

Since any odd integer can always be written in the form $2^k n + 2^{k-1} - 1$, $n \geq 0$, $k \geq 2$, we conclude from (4.3.13) that

$$5\sum_{n=0}^{\infty} a_5(2n+1)q^n = \sum_{n=0}^{\infty} a_5(10n+9)q^n. \tag{4.3.14}$$

From (4.3.8) and (4.3.14), we arrive at (4.3.1) to finish the first proof of Theorem 4.3.1. □

*Second proof of Theorem* 4.3.1. Let $w(n)$ be defined by

$$\sum_{n=0}^{\infty} w(n)q^n = \varphi(-q)\varphi^3(-q^5). \tag{4.3.15}$$

Replacing $q$ by $-q$ in (1.2.15), using it in (4.3.15), and then extracting the terms involving $q^{5n+r}$ for $r = 0, 1, 2, 3, 4$, respectively, from both sides of the resulting identity, we obtain

$$\sum_{n=0}^{\infty} w(5n)q^n = \varphi^3(-q)\varphi(-q^5), \tag{4.3.16}$$

$$\sum_{n=0}^{\infty} w(5n+1)q^n = -2\varphi^3(-q)f(-q^3,-q^7), \tag{4.3.17}$$

$$w(5n+2) = 0, \tag{4.3.18}$$

$$w(5n+3) = 0, \tag{4.3.19}$$

and

$$\sum_{n=0}^{\infty} w(5n+4)q^n = 2\varphi^3(-q)f(-q,-q^9), \tag{4.3.20}$$

respectively.

Now, employing (1.2.15) in (4.3.16), we have

$$\sum_{n=0}^{\infty} w(5n)q^n = \varphi(-q^5)\left(\varphi(-q^{25}) - 2qf(-q^{15},-q^{35}) + 2q^4f(-q^5,-q^{45})\right)^3. \tag{4.3.21}$$

Extracting the terms involving $q^{5n}$ from both sides of (4.3.21), we find that

$$\sum_{n=0}^{\infty} w(25n)q^n = \varphi(-q)\varphi^3(-q^5) - 24q\varphi(-q)\varphi(-q^5)f(-q^3,-q^7)f(-q,-q^9). \tag{4.3.22}$$

Now, from [16, p. 262, Entry 10(iv)], we note that

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q,q^9)f(q^3,q^7). \tag{4.3.23}$$

Replacing $q$ by $-q$ in (4.3.23), we have

$$\varphi^2(-q) - \varphi^2(-q^5) = -4qf(-q,-q^9)f(-q^3,-q^7). \tag{4.3.24}$$

Employing (4.3.24) in (4.3.22),

$$\sum_{n=0}^{\infty} w(25n)q^n = 6\sum_{n=0}^{\infty} w(5n)q^n - 5\sum_{n=0}^{\infty} w(n)q^n, \tag{4.3.25}$$

which may also be written as

$$\sum_{n=0}^{\infty} w(25n)q^n - \sum_{n=0}^{\infty} w(5n)q^n = 5\left(\sum_{n=0}^{\infty} w(5n)q^n - \sum_{n=0}^{\infty} w(n)q^n\right). \tag{4.3.26}$$

Next, multiplying both sides of (4.3.24) by $\dfrac{\varphi^3(-q^5)}{\varphi(-q)}$, and then employing (1.2.2), we find that

$$\left(\varphi^2(-q) - \varphi^2(-q^5)\right)\frac{\varphi^3(-q^5)}{\varphi(-q)} = -4q\frac{f^5(-q^5)}{f(-q)}. \tag{4.3.27}$$

Furthermore, squaring both sides of (4.3.24), and then multiplying by $\dfrac{\varphi(-q^5)}{\varphi(-q)}$, we obtain

$$\left(\varphi^2(-q) - \varphi^2(-q^5)\right)^2\frac{\varphi(-q^5)}{\varphi(-q)} = 16q^2\frac{f^5(-q^{10})}{f(-q^2)}, \tag{4.3.28}$$

where we have repeatedly used (1.2.2). Adding (4.3.27) and (4.3.28), and then using (4.1.2), we arrive at

$$\varphi^3(-q)\varphi(-q^5) - \varphi(-q)\varphi^3(-q^5) = 16q^2\sum_{n=0}^{\infty} a_5(n)q^{2n} - 4q\sum_{n=0}^{\infty} a_5(n)q^n. \tag{4.3.29}$$

Employing (4.3.15) and (4.3.16) in (4.3.29), we have

$$\sum_{n=0}^{\infty} w(n)q^n - \sum_{n=0}^{\infty} w(5n)q^n = 4q\sum_{n=0}^{\infty} a_5(n)q^n - 16q^2\sum_{n=0}^{\infty} a_5(n)q^{2n}. \tag{4.3.30}$$

Extracting the even and odd terms from both sides of (4.3.30), we find that

$$\sum_{n=0}^{\infty} w(2n)q^n - \sum_{n=0}^{\infty} w(10n)q^n = 4\sum_{n=0}^{\infty} a_5(2n-1)q^n - 16\sum_{n=0}^{\infty} a_5(n-1)q^n, \tag{4.3.31}$$

$$\sum_{n=0}^{\infty} w(2n+1)q^n - \sum_{n=0}^{\infty} w(10n+5)q^n = 4\sum_{n=0}^{\infty} a_5(2n)q^n. \tag{4.3.32}$$

Now, extracting the terms involving $q^{2n+1}$ from both sides of (4.3.26), and then employing (4.3.32), we arrive at (4.3.8).

As in the case of the first proof, to complete the proof, we need to show that (4.3.8) also holds when $2n$ is replaced by $2n+1$.

To this end, extracting the terms involving $q^{2n}$ from both sides of (4.3.26) and then using (4.3.31), we have

$$5\sum_{n=0}^{\infty} a_5(2n-1)q^n - \sum_{n=0}^{\infty} a_5(10n-1)q^n = 4\left(5\sum_{n=0}^{\infty} a_5(n-1)q^n - \sum_{n=0}^{\infty} a_5(5n-1)q^n\right). \tag{4.3.33}$$

Replacing $n$ by (2n+1) in (4.3.33), and then employing (4.3.8), we find that

$$5\sum_{n=0}^{\infty} a_5(4n+1)q^n = \sum_{n=0}^{\infty} a_5(20n+9)q^n = \sum_{n=0}^{\infty} a_5(5(4n+1)+4)q^n. \tag{4.3.34}$$

Again, replacing $n$ by $4n+2$ in (4.3.33), and employing (4.3.34), we arrive at (4.3.12). The remaining part of the proof is similar to that of the first proof. □

We now deduce Ramanujan's "Most Beautiful Identity."

**Corollary 4.3.2.** *The following identity*

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\,\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6} \tag{4.3.35}$$

*holds.*

*Proof.* Since

$$\frac{1}{f(-q)} = \sum_{n=0}^{\infty} p(n)q^n,$$

by (4.1.2), we have

$$\sum_{n=0}^{\infty} a_5(n)q^n = f^5(-q^5)\sum_{n=0}^{\infty} p(n)q^n. \tag{4.3.36}$$

Extracting the terms involving $q^{5n+4}$ from both sides of (4.3.36), we find that

$$\sum_{n=0}^{\infty} a_5(5n+4)q^n = f^5(-q)\sum_{n=0}^{\infty} p(5n+4)q^n. \tag{4.3.37}$$

Employing (4.3.1) in (4.3.37), we obtain

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\,\frac{1}{f^5(-q)}\sum_{n=0}^{\infty} a_5(n)q^n,$$

from which, with the aid of (4.1.2) again, (4.3.35) follows readily. □

In the next theorem, we present various arithmetic properties of $a_5(n)$.

**Theorem 4.3.3.** *If* $a_5(n)$ *denotes the number of* 5*-cores of* $n$*, then*

$$5a_5(4n+3) - a_5(8n+7) = 8a_5(n) + 2a_5(2n+1), \tag{4.3.38}$$

$$5a_5(4n+1) - a_5(8n+3) = 2a_5(2n), \tag{4.3.39}$$

$$a_5(2n) = 4a_5(4n+1) - a_5(8n+3), \tag{4.3.40}$$

$$a_5(4n+1) - a_5(16n+7) + 4a_5(8n+3) = 8a_5(2n), \tag{4.3.41}$$

$$a_5(4n+3) - a_5(16n+15) + 4a_5(8n+7) = 8a_5(2n+1) + 4a_5(n). \tag{4.3.42}$$

*Proof.* Let $v_1(n)$ be defined by

$$\sum_{n=0}^{\infty} v_1(n)q^n = 4q\varphi(q)\varphi(q^5)f(q,q^9)f(q^3,q^7). \tag{4.3.43}$$

Applying (1.2.2) and (1.2.3) in (4.3.43) and also using (1.2.5), we obtain

$$\sum_{n=0}^{\infty} v_1(n)q^n = 4q(-q;q^2)_\infty^3(-q^5;q^{10})_\infty f^3(-q^{10})f(-q^2). \tag{4.3.44}$$

Replacing $q$ by $-q$ in (4.3.44),

$$\sum_{n=0}^{\infty} (-1)^n v_1(n)q^n = -4q(q;q^2)_\infty^3(q^5;q^{10})_\infty f^3(-q^{10})f(-q^2). \tag{4.3.45}$$

Adding (4.3.44) and (4.3.45), we have

$$\sum_{n=0}^{\infty} v_1(n)q^n + \sum_{n=0}^{\infty}(-1)^n v_1(n)q^n$$
$$= 4qf(-q^2)f^3(-q^{10})\left((-q;q^2)_\infty^3(-q^5;q^{10})_\infty - (q;q^2)_\infty^3(q^5;q^{10})_\infty\right). \qquad (4.3.46)$$

Employing (4.2.6) in (4.3.46), we find that

$$\sum_{n=0}^{\infty} v_1(2n)q^n = \frac{8qf(-q)f^3(-q^5)}{(q^5;q^{10})_\infty(q;q^2)_\infty^3} + \frac{4qf(-q)f^3(-q^5)(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2}. \qquad (4.3.47)$$

Manipulating the $q$-products and recalling the product representation of $\varphi(q)$ from (1.2.3), we rewrite the above in the form

$$\sum_{n=0}^{\infty} v_1(2n)q^n = 8qf^2(-q^2)f^2(-q^{10})\frac{\varphi(-q^5)}{\varphi(-q)} + 4q(q;q^2)_\infty^3(q^5;q^{10})f(-q^2)f^3(-q^{10}),$$

which, with the aid of (4.3.45), implies

$$\sum_{n=0}^{\infty} v_1(2n)q^n + \sum_{n=0}^{\infty}(-1)^n v_1(n)q^n = 8qf^2(-q^2)f^2(-q^{10})\frac{\varphi(-q^5)}{\varphi(-q)}. \qquad (4.3.48)$$

Replacing $q$ by $-q$ in (4.3.48), we have

$$\sum_{n=0}^{\infty}(-1)^n v_1(2n)q^n + \sum_{n=0}^{\infty} v_1(n)q^n = -8qf^2(-q^2)f^2(-q^{10})\frac{\varphi(q^5)}{\varphi(q)}. \qquad (4.3.49)$$

Adding (4.3.48) and (4.3.49), and then using (4.2.1) and the trivial identity $\varphi(q)\varphi(-q) = \varphi^2(-q^2)$, we find that

$$\sum_{n=0}^{\infty} v_1(2n)q^n + \sum_{n=0}^{\infty}(-1)^n v_1(2n)q^n + \sum_{n=0}^{\infty} v_1(n)q^n + \sum_{n=0}^{\infty}(-1)^n v_1(n)q^n$$
$$= 32q^2f^2(-q^2)f^2(-q^{10})\frac{f(-q^4)f(-q^{20})}{\varphi^2(-q^2)}. \qquad (4.3.50)$$

Extracting the terms involving $q^{2n}$ from both sides of (4.3.50) and then replacing $q^2$ by $q$, we deduce that

$$\sum_{n=0}^{\infty} v_1(4n)q^n + \sum_{n=0}^{\infty} v_1(2n)q^n = 16qf^2(-q^2)f^2(-q^{10})\frac{\varphi(-q^5)}{\varphi(-q)},$$

which, by (4.3.48), reduces to

$$\sum_{n=0}^{\infty} v_1(4n)q^n = \sum_{n=0}^{\infty} v_1(2n)q^n + 2\sum_{n=0}^{\infty}(-1)^n v_1(n)q^n. \tag{4.3.51}$$

Now, employing (4.3.23) in (4.3.43), we have

$$\sum_{n=0}^{\infty} v_1(n)q^n = \varphi^3(q)\varphi(q^5) - \varphi(q)\varphi^3(q^5). \tag{4.3.52}$$

From (4.3.52) and (4.3.29), we obtain

$$\sum_{n=0}^{\infty} v_1(n)q^n = 16q^2\sum_{n=0}^{\infty} a_5(n)q^{2n} + 4q\sum_{n=0}^{\infty}(-1)^n a_5(n)q^n. \tag{4.3.53}$$

Using (4.3.53) in (4.3.51), we deduce that

$$5\sum_{n=0}^{\infty} a_5(2n-1)q^n - \sum_{n=0}^{\infty} a_5(4n-1)q^n = 4\sum_{n=0}^{\infty} a_5(n-1)q^n + 8q^2\sum_{n=0}^{\infty} a_5(n)q^{2n} - 2q\sum_{n=0}^{\infty} a_5(n)q^n. \tag{4.3.54}$$

Equating the coefficients of even and odd terms, respectively, from both sides of (4.3.54), we arrive at (4.3.38) and (4.3.39), respectively.

Next, employing (1.2.6) in (4.3.52), and then extracting the terms involving $q^{4n}$ we find that

$$\sum_{n=0}^{\infty} v_1(4n)q^n = \sum_{n=0}^{\infty} v_1(n)q^n + 16q^2\left(\psi^3(q^2)\psi(q^{10}) - q^2\psi(q^2)\psi^3(q^{10})\right). \tag{4.3.55}$$

Extracting the terms involving $q^{2n}$ and $q^{2n+1}$, respectively, from both sides of (4.3.55), we obtain

$$\sum_{n=0}^{\infty} v_1(8n)q^n - \sum_{n=0}^{\infty} v_1(2n)q^n = 16q\left(\psi^3(q)\psi(q^5) - q\psi(q)\psi^3(q^5)\right) \tag{4.3.56}$$

and

$$\sum_{n=0}^{\infty} v_1(2n+1)q^n = \sum_{n=0}^{\infty} v_1(8n+4)q^n, \tag{4.3.57}$$

respectively. From (4.3.57) and (4.3.53), we easily deduce (4.3.40).

Again, employing (4.2.2) and (4.3.53) in (4.3.56), we find that

$$\left(\sum_{n=0}^{\infty} a_5(2n-1)q^n - 4\sum_{n=0}^{\infty} a_5(n-1)q^n\right) - \left(\sum_{n=0}^{\infty} a_5(8n-1)q^n - 4\sum_{n=0}^{\infty} a_5(4n-1)q^n\right)$$
$$= 4q\left(\sum_{n=0}^{\infty} a_5(n)q^n + q\sum_{n=0}^{\infty} a_5(n)q^{2n}\right). \tag{4.3.58}$$

Equating the coefficients of $q^{2n+1}$ and $q^{2n+2}$, respectively, from both sides of (4.3.58), we arrive at (4.3.41) and (4.3.42), respectively, to finish the proof.

□

Finally, we are in a position to prove (4.1.3) and (4.1.5).

**Theorem 4.3.4.** *Identities* (4.1.3) *and* (4.1.5) *hold.*

*Proof.* Employing (4.3.40) in (4.3.39), we find that

$$a_5(8n+3) = 3a_5(4n+1). \tag{4.3.59}$$

Employing (4.3.59) in (4.3.40) we readily arrive at (4.1.3).

Next, replacing $n$ by $2n+1$ in (4.3.38), we have

$$a_5(16n+15) = 5a_5(8n+7) - 8a_5(2n+1) - 2a_5(4n+3) \tag{4.3.60}$$

Employing (4.3.60) in (4.3.42), we obtain

$$3a_5(4n+3) - a_5(8n+7) = 4a_5(n). \tag{4.3.61}$$

Using (4.3.38) in (4.3.61), we easily deduce (4.1.5) to complete the proof. □

With the aid of (4.1.3), (4.1.5), and mathematical induction, we easily prove the following two infinite families of arithmetic identities for $a_5(n)$.

**Theorem 4.3.5.** *Let $a_5(n)$ denote the number of 5-cores of $n$. Then, for any positive integers $n$ and $k$, we have*

$$a_5(2^{2k}n + 2^{2k-1} - 1) = \left(\sum_{r=1}^{k} 2^{2k-2r}\right) a_5(2n) = \frac{2^{2k}-1}{3}\, a_5(2n) \tag{4.3.62}$$

*and*

$$a_5(2^{2k+1}n + 2^{2k} - 1) = \left(1 + \sum_{r=1}^{k} 2^{(2k+1)-2r}\right) a_5(2n) = \frac{2^{2k+1}+1}{3}\, a_5(2n). \tag{4.3.63}$$

From (4.3.62) and (4.3.63), we readily arrive at the following two infinite families of congruences for $a_5(n)$.

**Corollary 4.3.6.** *For any positive integers $n$ and $k$, we have*

$$a_5(2^{2k}n + 2^{2k-1} - 1) \equiv 0 \left(\bmod\ \frac{2^{2k}-1}{3}\right)$$

*and*

$$a_5(2^{2k+1}n + 2^{2k} - 1) \equiv 0 \left(\bmod\ \frac{2^{2k+1}+1}{3}\right).$$

# Chapter 5

# Infinite Families of Arithmetic Identities for Self-Conjugate 5-Cores and 7-Cores

## 5.1 Introduction

In the introductory chapter, we have discussed in detail self-conjugate $t$-core partitions and indicated the contributions of Garvan, Kim and Stanton [24], Baruah and Berndt [4], Baldwin, Depweg, Ford, Kunin and Sze [3], Baruah and Sarmah [7], and Hanusa and Nath [29].

By employing (1.2.2) and manipulating the $q$-products, and then using (1.1.9), we have

$$\sum_{n=0}^{\infty} \mathrm{asc}_5(n)q^n = f(q,q^9)f(q^3,q^7), \tag{5.1.1}$$

$$\sum_{n=0}^{\infty} \mathrm{asc}_7(n)q^n = f(q,q^{13})f(q^3,q^{11})f(q^5,q^9). \tag{5.1.2}$$

In Sections 5.2 and 5.3 of this chapter, we use Ramanujan's theta function identities to find relations between $\mathrm{asc}_5(n)$ and $r_2(n)$, and between $\mathrm{asc}_7(n)$ and $r_3(n)$. We then deduce several results proved earlier by Garvan, Kim and Stanton [24], Baruah and Berndt [4], and Baruah and Sarmah [7].

In Sections 5.4 and 5.5, we also find new infinite families of arithmetic identities for self-conjugate 5-cores and 7-cores.

## 5.2 Relations between $\mathrm{asc}_5(n)$ and $r_2(n)$

**Theorem 5.2.1.** *If $r_2(n)$ is the number of representations of a nonnegative integer $n$ as a sum of two squares and $\mathrm{asc}_5(n)$ is the number of self-conjugate 5-cores of $n$, then*

$$8\mathrm{asc}_5(n) = \begin{cases} r_2(5(n+1)), & \text{if } n \equiv 0,1,2,3 \pmod 5; \\ r_2(5(n+1)) - r_2((n+1)/5), & \text{if } n \equiv 4 \pmod 5. \end{cases} \tag{5.2.1}$$

*Proof.* We have

$$\sum_{n=0}^{\infty} r_2(n)q^n = \varphi^2(q). \tag{5.2.2}$$

Employing (1.2.8) in (5.2.2), we obtain

$$\sum_{n=0}^{\infty} r_2(n)q^n = \sum_{n=0}^{\infty} r_2(n)q^{2n} + 4q\psi^2(q^4). \tag{5.2.3}$$

Equating the coefficients of $q^{2n}$ from both sides of (5.2.3), we immediately deduce the elementary identity

$$r_2(2n) = r_2(n). \tag{5.2.4}$$

Again, with the help of (1.2.15), we may rewrite (5.2.2) as

$$\sum_{n=0}^{\infty} r_2(n)q^n = (\varphi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4 f(q^5,q^{45}))^2. \tag{5.2.5}$$

Extracting the terms involving $q^{5n}$ from both sides of (5.2.5), we find that

$$\sum_{n=0}^{\infty} r_2(5n)q^n = \varphi^2(q^5) + 8qf(q^3,q^7)f(q,q^9), \tag{5.2.6}$$

which, with the help of (5.1.1), can be written as

$$\sum_{n=0}^{\infty} r_2(5n)q^n = \sum_{n=0}^{\infty} r_2(n)q^{5n} + 8q\sum_{n=0}^{\infty} \mathrm{asc}_5(n)q^n. \tag{5.2.7}$$

Equating the coefficients of the terms $q^{5n+r}$, $r = 1,2,3,4$ and 0, respectively, from both sides of (5.2.7), we have

$$r_2(25n+5) = 8\mathrm{asc}_5(5n), \tag{5.2.8}$$

$$r_2(25n+10) = 8\mathrm{asc}_5(5n+1), \tag{5.2.9}$$

$$r_2(25n+15) = 8\mathrm{asc}_5(5n+2), \tag{5.2.10}$$

$$r_2(25n+20) = 8\mathrm{asc}_5(5n+3), \tag{5.2.11}$$

$$r_2(25n) - r_2(n) = 8\mathrm{asc}_5(5n-1). \tag{5.2.12}$$

From (5.2.8)–(5.2.12), we readily finish our proof.

□

## 5.3 Relations between $\mathrm{asc}_7(n)$ and $r_3(n)$

**Theorem 5.3.1.** *If $r_3(n)$ is the number of representations of a nonnegative integer $n$ as a sum of three squares and $\mathrm{asc}_7(n)$ is the number of self-conjugate 7-cores of $n$, then*

$$48\mathrm{asc}_7(n) = \begin{cases} r_3(7(n+2)), & \text{if } n \equiv 0,1,2,3,4,6 \pmod 7; \\ r_3(7(n+2)) - r_3((n+2)/7), & \text{if } n \equiv 5 \pmod 7. \end{cases} \tag{5.3.1}$$

*Proof.* We have

$$\sum_{n=0}^{\infty} r_3(n)q^n = \varphi^3(q), \tag{5.3.2}$$

which we rewrite with the aid of (1.2.6) as

$$\sum_{n=0}^{\infty} r_3(n)q^n = (\varphi(q^4) + 2q\psi(q^8))^3. \tag{5.3.3}$$

Equating the coefficients of $q^{4n}$ from both sides of (5.3.3), we find that

$$r_3(4n) = r_3(n), \tag{5.3.4}$$

which, of course, is a well-known classical result.

Again, setting $n = 7$ and $a = b = q$ in (1.2.13), we obtain

$$\varphi(q) = \varphi(q^{49}) + 2qf(q^{35}, q^{63}) + 2q^4 f(q^{21}, q^{77}) + 2q^9 f(q^7, q^{91}). \tag{5.3.5}$$

Employing (5.3.5) in (5.3.2) and equating the terms involving $q^{7n}$, and then replacing $q^7$ by $q$, we find that

$$\sum_{n=0}^{\infty} r_3(7n)q^n = \varphi^3(q^7) + 48q^2 f(q^5, q^9) f(q^3, q^{11}) f(q, q^{13}), \tag{5.3.6}$$

which, with the help of (5.1.2), can be written as

$$\sum_{n=0}^{\infty} r_3(7n)q^n = \sum_{n=0}^{\infty} r_3(n)q^{7n} + 48q^2 \sum_{n=0}^{\infty} \mathrm{asc}_7(n)q^n. \tag{5.3.7}$$

Equating the coefficients of $q^{7n+r}$, $r = 1, 2, 3, 4, 5, 6$, and 0, respectively, from both sides of (5.3.7), we have

$$r_3(49n + 7) = 48\mathrm{asc}_7(7n - 1), \tag{5.3.8}$$

$$r_3(49n + 14) = 48\mathrm{asc}_7(7n), \tag{5.3.9}$$

$$r_3(49n + 21) = 48\mathrm{asc}_7(7n + 1), \tag{5.3.10}$$

$$r_3(49n + 28) = 48\mathrm{asc}_7(7n + 2), \tag{5.3.11}$$

$$r_3(49n + 35) = 48\mathrm{asc}_7(7n + 3), \tag{5.3.12}$$

$$r_3(49n + 42) = 48\mathrm{asc}_7(7n + 4), \tag{5.3.13}$$

$$r_3(49n) - r_3(n) = 48\mathrm{asc}_7(7n - 2). \tag{5.3.14}$$

From (5.3.8)–(5.3.14), we arrive at (5.3.1) to finish the proof. □

## 5.4 Infinite families of arithmetic properties of $\mathrm{asc}_5(n)$

**Theorem 5.4.1.** (Garvan, Kim and Stanton [24]). *The identity*

$$\mathrm{asc}_5(2n + 1) = \mathrm{asc}_5(n) \tag{5.4.1}$$

*holds.*

*Proof.* Replacing $n$ by $25n+5$ in (5.2.4), we have

$$r_2(50n+10) = r_2(25n+5). \tag{5.4.2}$$

Employing (5.2.8) and (5.2.9) in (5.4.2), we deduce that

$$\mathrm{asc}_5(5n) = \mathrm{asc}_5(10n+1). \tag{5.4.3}$$

Again, from (5.2.4) and (5.2.8)–(5.2.11), we find that

$$\mathrm{asc}_5(5n+1) = \mathrm{asc}_5(10n+3), \tag{5.4.4}$$

$$\mathrm{asc}_5(5n+2) = \mathrm{asc}_5(10n+5), \tag{5.4.5}$$

$$\mathrm{asc}_5(5n+3) = \mathrm{asc}_5(10n+7). \tag{5.4.6}$$

Furthermore, with the aid of (5.2.4), we write

$$r_2(25n) - r_2(n) = r_2(50n) - r_2(2n). \tag{5.4.7}$$

Applying (5.2.12) in (5.4.7), we obtain

$$\mathrm{asc}_5(5n+4) = \mathrm{asc}_5(10n+9). \tag{5.4.8}$$

Now from (5.4.3)–(5.4.6), and (5.4.8), we arrive at (5.4.1) to complete the proof. □

Iterating (5.4.1) and by mathematical induction, we immediately have the following result.

**Corollary 5.4.2.** *For any positive integer $k$, we have* $\mathrm{asc}_5(2^k n + (2^k - 1)) = \mathrm{asc}_5(n)$.

**Theorem 5.4.3.** (Garvan, Kim and Stanton [24]). *The identity*

$$\mathrm{asc}_5(5n+4) = \mathrm{asc}_5(n) \tag{5.4.9}$$

*holds.*

*Proof.* From [16, p. 262, Entry 10(iv)], we note that

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q,q^9)f(q^3,q^7). \tag{5.4.10}$$

Employing (5.4.10) in (5.2.6), we find that

$$\sum_{n=0}^{\infty} r_2(5n)q^n = 2\varphi^2(q) - \varphi^2(q^5), \tag{5.4.11}$$

which can be rewritten, with the aid of (5.2.2), as

$$\sum_{n=0}^{\infty} r_2(5n)q^n = 2\sum_{n=0}^{\infty} r_2(n)q^n - \sum_{n=0}^{\infty} r_2(n)q^{5n}. \tag{5.4.12}$$

Equating the coefficients of $q^{5n}$ from both sides of (5.4.12), we find that

$$r_2(25n) = 2r_2(5n) - r_2(n). \tag{5.4.13}$$

Again, adding (5.2.7) and (5.4.12), and then equating the coefficients of $q^n$ from both sides of the resulting identity, we obtain

$$r_2(5n) = r_2(n) + 4\mathrm{asc}_5(n-1). \tag{5.4.14}$$

Employing (5.2.12) and (5.4.14) in (5.4.13), we deduce that

$$\mathrm{asc}_5(5n-1) = \mathrm{asc}_5(n-1), \tag{5.4.15}$$

which is equivalent to (5.4.9). □

**Theorem 5.4.4.** *Let $p \equiv 3 \pmod 4$ be a prime. If $\mathrm{asc}_5(n)$ denotes the number of self-conjugate 5-cores of $n$, then for any positive integer $n$ and any positive even integer $k$, we have*

$$\mathrm{asc}_5(n) = \mathrm{asc}_5\left(p^k n + (p^k - 1)\right). \tag{5.4.16}$$

First we prove the following lemma.

**Lemma 5.4.5.** *If $r_2(n)$ denotes the number of representations of a nonnegative integer $n$ as a sum of two squares and $p \equiv 3 \pmod 4$ is a prime, then*

$$r_2(p^2 n) = r_2(n). \tag{5.4.17}$$

*Proof.* Setting $n = p$ and $a = b = q$ in (1.2.13), we obtain

$$\varphi(q) = \varphi(q^{p^2}) + \sum_{r=1}^{p-1} q^{r^2} f(q^{p(p-2r)}, q^{p(p+2r)}). \tag{5.4.18}$$

With successive use of the trivial identity $f(a,b) = af(a^{-1}, a^2b)$, we can rewrite the above identity in the form

$$\begin{aligned}\varphi(q) = \varphi(q^{p^2}) &+ 2qf(q^{p(p-2)}, q^{p(p+2)}) + 2q^{2^2} f(q^{p(p-2\cdot 2)}, q^{p(p+2\cdot 2)}) \\ &+ 2q^{3^2} f(q^{p(p-2\cdot 3)}, q^{p(p+2\cdot 3)}) + \cdots + 2q^{\left(\frac{p-1}{2}\right)^2} f(q^p, q^{p(2p-1)}).\end{aligned} \tag{5.4.19}$$

Employing (5.4.19) in (5.2.2) and then extracting the terms involving $q^{pn}$ from both sides of the resulting identity by noting that prime $p \equiv 3 \pmod 4$ and $r_2(4n+3) = 0$, we find that

$$\sum_{n=0}^{\infty} r_2(pn)q^n = \varphi^2(q^p) = \sum_{n=0}^{\infty} r_2(n)q^{pn}. \tag{5.4.20}$$

Equating the coefficients of $q^{pn}$ from both sides of (5.4.20), we readily arrive at (5.4.17) to complete the proof of the lemma. □

*Proof of Theorem 5.4.4.* Since prime $p \equiv 3 \pmod 4$, we note that $p^2$ is either of the form $5m+1$ or $5m+4$. At first, let $p^2$ be of the form $5m+1$. Replacing $n$ by $25n+5$ in (5.4.17), we find that

$$r_2\left(25\left(p^2n + \frac{p^2-1}{5}\right) + 5\right) = r_2(25n+5). \tag{5.4.21}$$

Employing (5.2.8) in (5.4.21), we obtain

$$\mathrm{asc}_5\left(5p^2n + p^2 - 1\right) = \mathrm{asc}_5(5n). \tag{5.4.22}$$

Similarly, replacing $n$ by $25n+10$, $25n+15$ and $25n+20$, in turn, in (5.4.17), and then employing (5.2.9)-(5.2.11), respectively, we deduce that

$$\mathrm{asc}_5\left(5p^2n + 2p^2 - 1\right) = \mathrm{asc}_5(5n+1), \tag{5.4.23}$$

$$\mathrm{asc}_5\left(5p^2n + 3p^2 - 1\right) = \mathrm{asc}_5(5n+2), \tag{5.4.24}$$

$$\mathrm{asc}_5\left(5p^2n + 4p^2 - 1\right) = \mathrm{asc}_5(5n+3). \tag{5.4.25}$$

Again, from (5.4.17), we have

$$r_2(p^2(25n)) - r_2(p^2n) = r_2(25n) - r_2(n),$$

which, with the aid of (5.2.12), reduces to

$$\mathrm{asc}_5\left(5p^2n - 1\right) = \mathrm{asc}_5(5n-1). \tag{5.4.26}$$

From (5.4.22)–(5.4.26), we find that

$$\mathrm{asc}_5\left(p^2n + p^2 - 1\right) = \mathrm{asc}_5(n). \tag{5.4.27}$$

Iterating (5.4.27) and by applying mathematical induction, we arrive at (5.4.16) when $p^2$ is of the form $5m+1$.

Next, let $p^2$ be of the form $5m+4$.

Replacing $n$ by $25n+5$ in (5.4.17), we find that

$$r_2\left(25\left(p^2n + \frac{p^2-4}{5}\right) + 20\right) = r_2(25n+5). \tag{5.4.28}$$

Employing (5.2.8) and (5.2.11) in (5.4.28), we obtain

$$\mathrm{asc}_5\left(5p^2n + p^2 - 1\right) = \mathrm{asc}_5(5n). \tag{5.4.29}$$

Again, replacing $n$ by $25n+10$ in (5.4.17), we find that

$$r_2\left(25\left(p^2n + \frac{2(p^2-4)}{5} + 1\right) + 15\right) = r_2(25n+10). \tag{5.4.30}$$

Employing (5.2.9) and (5.2.10) in (5.4.30), we deduce that

$$\mathrm{asc}_5\left(5p^2n + 2p^2 - 1\right) = \mathrm{asc}_5(5n+1). \tag{5.4.31}$$

Similarly, replacing $n$ by $25n+15$ and $25n+20$, in turn, in (5.4.17), and then using (5.2.9), (5.2.10) and (5.2.11), and (5.2.8) in the resulting identities, we obtain

$$\mathrm{asc}_5\left(5p^2n + 3p^2 - 1\right) = \mathrm{asc}_5(5n+2), \tag{5.4.32}$$

$$\mathrm{asc}_5\left(5p^2n + 4p^2 - 1\right) = \mathrm{asc}_5(5n+3). \tag{5.4.33}$$

Finally, we note that (5.4.26) also holds when $p^2$ is of the form $5m+4$.

Now, from (5.4.26), (5.4.29), and (5.4.31)–(5.4.33), we arrive at

$$\mathrm{asc}_5\left(p^2n+p^2-1\right)=\mathrm{asc}_5(n), \tag{5.4.34}$$

which, upon iteration implies (5.4.16) by mathematical induction, when $p^2$ is of the form $5m+4$. Thus, we complete the proof.

□

## 5.5 Infinite families of arithmetic properties of $\mathrm{asc}_7(n)$

In this section, at first, we present a simple proof of an identity given by Baruah and Sarmah [7].

**Theorem 5.5.1.** (Baruah and Sarmah [7]). *The identity*

$$\mathrm{asc}_7(8m-1)=0. \tag{5.5.1}$$

*holds.*

*Proof.* For any nonnegative integer $n$, it is well known that

$$r_3(8n+7)=0. \tag{5.5.2}$$

Replacing $n$ by $8n+7$, with $n\equiv\ 0,1,2,3,4,6 \pmod 7$, in (5.3.1), we find that

$$48\mathrm{asc}_7(8n+7)=r_3(8(7(n+1))+7). \tag{5.5.3}$$

Employing (5.5.2) in (5.5.3), we readily deduce that

$$\mathrm{asc}_7(8n+7)=0,\ \text{ for } n\equiv\ 0,1,2,3,4,6 \pmod 7. \tag{5.5.4}$$

Again, replacing $n$ by $8n+7$ in (5.3.14), we find that

$$48\mathrm{asc}_7(8(7n+5)+7)=r_3(8(49n+42)+7)-r_3(8n+7). \tag{5.5.5}$$

Employing (5.5.2) in (5.5.5), we have $\mathrm{asc}_7(8(7n+5)+7)=0$, which means

$$\mathrm{asc}_7(8n+7)=0, \text{ for } n \equiv 5 \pmod 7. \tag{5.5.6}$$

From (5.5.4) and (5.5.6), we immediately arrive at (5.5.1). □

**Theorem 5.5.2.** (Garvan, Kim and Stanton [24]). *The identity*

$$\mathrm{asc}_7(4n+6)=\mathrm{asc}_7(n), \tag{5.5.7}$$

*holds.*

*Proof.* Replacing $n$ by $49n+7$ in (5.3.4), we have

$$r_3(4(49n+7))=r_3(49n+7). \tag{5.5.8}$$

Employing (5.3.11) and (5.3.8) in (5.5.8), we obtain

$$\mathrm{asc}_7(7n-1)=\mathrm{asc}_7(28n+2) \tag{5.5.9}$$

Again, from (5.3.4) and (5.3.8)–(5.3.13), we find that

$$\mathrm{asc}_7(7n)=\mathrm{asc}_7(28n+6), \tag{5.5.10}$$

$$\mathrm{asc}_7(7n+1)=\mathrm{asc}_7(28n+10), \tag{5.5.11}$$

$$\mathrm{asc}_7(7n+2)=\mathrm{asc}_7(28n+14), \tag{5.5.12}$$

$$\mathrm{asc}_7(7n+3)=\mathrm{asc}_7(28n+18), \tag{5.5.13}$$

$$\mathrm{asc}_7(7n+4)=\mathrm{asc}_7(28n+22). \tag{5.5.14}$$

Next, employing (5.3.4), we may write

$$r_3(49n)-r_3(n)=r_3(4(49n))-r_3(4n). \tag{5.5.15}$$

Applying (5.3.14) in (5.5.15), we find that

$$\mathrm{asc}_7(7n-2)=\mathrm{asc}_7(7(4n)-2)=\mathrm{asc}_7(28n-2). \tag{5.5.16}$$

Now, from (5.5.9)–(5.5.14) and (5.5.16), we deduce (5.5.7) to complete the proof.

□

**Corollary 5.5.3.** *If* $\mathrm{asc}_7(n)$ *denotes the number of self-conjugate* 7*-cores of* $n$ *and* $k$ *is a positive integer, then*

$$\mathrm{asc}_7(n) = \mathrm{asc}_7(4^k(n+2)-2). \tag{5.5.17}$$

*Proof.* Iterating (5.5.7) and by mathematical induction, we deduce (5.5.17). □

**Corollary 5.5.4.** (Garvan, Kim and Stanton [24]). *The identity*

$$\mathrm{asc}_7(n) = 0, \quad \text{if } n+2 = 4^k(8m+1) \tag{5.5.18}$$

*holds.*

*Proof.* Replacing $n$ by $8m-1$ in (5.5.17), we find that

$$\mathrm{asc}_7(8m-1) = \mathrm{asc}_7(4^k(8m+1)-2). \tag{5.5.19}$$

Employing (5.5.1) in (5.5.19), we arrive at (5.5.18). □

**Theorem 5.5.5.** *If* $\mathrm{asc}_7(n)$ *denotes the number of self-conjugate* 7*-cores of* $n$*, then for any integer* $k \geq 1$*, we have*

$$(2\times 3^k - 1)\mathrm{asc}_7(3n+2) = \mathrm{asc}_7\left(3^{2k+1}n + 2(2\times 3^{2k}-1)\right), \tag{5.5.20}$$

$$3^k\mathrm{asc}_7(3n) = \mathrm{asc}_7\left(3^{2k+1}n + 2(3^{2k}-1)\right), \tag{5.5.21}$$

$$\left(\frac{3^{k+1}-1}{2}\right)\mathrm{asc}_7(9n+1) = \mathrm{asc}_7\left(3^{2k+2}n + (3^{2k+1}-2)\right), \tag{5.5.22}$$

$$\left(\frac{3^{k+1}-1}{2}\right)\mathrm{asc}_7(9n+4) = \mathrm{asc}_7\left(3^{2k+2}n + 2(3^{2k+1}-1)\right), \tag{5.5.23}$$

$$\left(\frac{3^{k+1}-1}{2}\right)\mathrm{asc}_7(9n+7) = \mathrm{asc}_7\left(3^{2k+2}n + (3^{2k+2}-2)\right) + \left(\frac{3^{k+1}-3}{2}\right)\mathrm{asc}_7(n-1). \tag{5.5.24}$$

*Proof.* Hirschhorn and Sellers [33] found the following arithmetic properties of $r_3(n)$. For any nonnegative integer $n$ and any integer $k \geq 1$, we have

$$(2\times 3^k - 1)\, r_3(3n+1) = r_3(9^k(3n+1)), \tag{5.5.25}$$

$$3^k r_3(3n+2) = r_3\left(9^k(3n+2)\right), \tag{5.5.26}$$

$$\left(\frac{3^{k+1}-1}{2}\right) r_3(9n+3) = r_3\left(9^k(9n+3)\right), \tag{5.5.27}$$

$$\left(\frac{3^{k+1}-1}{2}\right) r_3(9n+6) = r_3\left(9^k(9n+6)\right). \tag{5.5.28}$$

Replacing $n$ by $49n+2$ in (5.5.25), we have

$$(2\times 3^k - 1) r_3\left(7(3\times 7n+1)\right) = r_3\left(7\left(3^{2k+1}\times 7n + 3^{2k}\right)\right),$$

which on employing (5.3.1) may be written as

$$(2\times 3^k - 1)\mathrm{asc}_7\left(3(7n+6)+2\right) = \mathrm{asc}_7\left(3^{2k+1}(7n+6) + 2(2\times 3^{2k}-1)\right). \tag{5.5.29}$$

In a similar way, replacing $n$ by $49n+9$, $49n+23$, $49n+30$, $49n+37$, $49n+44$, respectively, in (5.5.25), we find that

$$(2\times 3^k - 1)\mathrm{asc}_7\left(3(7n)+2\right) = \mathrm{asc}_7\left(3^{2k+1}(7n) + 2(2\times 3^{2k}-1)\right), \tag{5.5.30}$$

$$(2\times 3^k - 1)\mathrm{asc}_7\left(3(7n+2)+2\right) = \mathrm{asc}_7\left(3^{2k+1}(7n+2) + 2(2\times 3^{2k}-1)\right), \tag{5.5.31}$$

$$(2\times 3^k - 1)\mathrm{asc}_7\left(3(7n+3)+2\right) = \mathrm{asc}_7\left(3^{2k+1}(7n+3) + 2(2\times 3^{2k}-1)\right), \tag{5.5.32}$$

$$(2\times 3^k - 1)\mathrm{asc}_7\left(3(7n+4)+2\right) = \mathrm{asc}_7\left(3^{2k+1}(7n+4) + 2(2\times 3^{2k}-1)\right), \tag{5.5.33}$$

$$(2\times 3^k - 1)\mathrm{asc}_7\left(3(7n+5)+2\right) = \mathrm{asc}_7\left(3^{2k+1}(7n+5) + 2(2\times 3^{2k}-1)\right). \tag{5.5.34}$$

Again, by employing (5.5.25), we have

$$(2\times 3^k - 1)\left(r_3\left(49(3n+1)\right) - r_3(3n+1)\right) = r_3\left(49\left(9^k(3n+1)\right)\right) - r_3\left(9^k(3n+1)\right),$$

which on employing (5.3.1) may be written as

$$(2\times 3^k - 1)\mathrm{asc}_7\left(3(7n+1)+2\right) = \mathrm{asc}_7\left(3^{2k+1}(7n+1) + 2(2\times 3^{2k}-1)\right). \tag{5.5.35}$$

From (5.5.29)–(5.5.35), we arrive at (5.5.20).

In a similar way, from (5.5.26)–(5.5.28), respectively, we deduce (5.5.21)–(5.5.23).

Hirschhorn and Sellers [33] also found the following arithmetic properties of $r_3(n)$. For any nonnegative integer $n$, we have

$$4r_3(9n) = r_3(81n) + 3r_3(n). \tag{5.5.36}$$

Replacing $n$ by $7n$, we have

$$4r_3(7(9n)) = r_3(7(81n)) + 3r_3(7n). \tag{5.5.37}$$

For $n \equiv 1, 2, 3, 4, 5, 6 \pmod 7$, by employing (5.3.1), in (5.5.37), we have

$$4\mathrm{asc}_7(9n-2) = \mathrm{asc}_7(81n-2) + 3\mathrm{asc}_7(n-2). \tag{5.5.38}$$

Again, replacing $n$ by $49n$ in (5.5.36), we have

$$4r_3(49(9n)) = r_3(49(81n)) + 3r_3(49n). \tag{5.5.39}$$

Subtracting (5.5.37) from (5.5.39), we have

$$\begin{aligned} 4\left(r_3(49(9n)) - r_3(9n)\right) &= \left(r_3(49(81n)) - r_3(81n)\right) \\ &\quad + 3\left(r_3(49n) - r_3(n)\right). \end{aligned} \tag{5.5.40}$$

Employing (5.3.1), the above can be written as

$$4\mathrm{asc}_7(63n-2) = \mathrm{asc}_7(561n-2) + 3\mathrm{asc}_7(7n-2), \tag{5.5.41}$$

which is (5.5.38) for $n \equiv 0 \pmod 7$. From (5.5.38) and (5.5.41), we readily arrive at

$$4\mathrm{asc}_7(9n+7) = \mathrm{asc}_7(81n+79) + 3\mathrm{asc}_7(n-1). \tag{5.5.42}$$

Iterating (5.5.42), and by mathematical induction we arrive at (5.5.24) □

**Theorem 5.5.6.** *If* $\mathrm{asc}_7(n)$ *denotes the number of self-conjugate* 7*-cores of* $n$ *and* $k \geq 1$, *then*

$$5\mathrm{asc}_7(5n) = \mathrm{asc}_7(125n+48), \tag{5.5.43}$$

$$5\mathrm{asc}_7(5n+1) = \mathrm{asc}_7(125n+73), \tag{5.5.44}$$

$$7\mathrm{asc}_7(5n+2) = \mathrm{asc}_7(125n+98), \tag{5.5.45}$$

$$7\mathrm{asc}_7(5n+4) = \mathrm{asc}_7(125n+148), \tag{5.5.46}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7(25n+3) = \mathrm{asc}_7\left(5^{2k+2}n + 5^{2k+1} - 2\right), \tag{5.5.47}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7(25n+8)=\mathrm{asc}_7\left(5^{2k+2}n+2\times 5^{2k+1}-2\right), \tag{5.5.48}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7(25n+13)=\mathrm{asc}_7\left(5^{2k+2}n+3\times 5^{2k+1}-2\right), \tag{5.5.49}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7(25n+18)=\mathrm{asc}_7\left(5^{2k+2}n+2(2\times 5^{2k+1}-1)\right), \tag{5.5.50}$$

$$6\mathrm{asc}_7(25n+23)=\mathrm{asc}_7(625n+623)+5\mathrm{asc}_7(n-1). \tag{5.5.51}$$

*Proof.* We recall the following results involving $r_3(n)$ from Chapter 3 [Eqs. (3.3.15)–(3.3.20)].

$$5r_3(5n+1)=r_3\left(25(5n+1)\right), \tag{5.5.52}$$

$$5r_3(5n+4)=r_3\left(25(5n+4)\right), \tag{5.5.53}$$

$$\left(\frac{5^{k+1}-1}{4}\right)r_3(25n+5)=r_3\left(25^k(25n+5)\right), \tag{5.5.54}$$

$$\left(\frac{5^{k+1}-1}{4}\right)r_3(25n+10)=r_3\left(25^k(25n+10)\right), \tag{5.5.55}$$

$$\left(\frac{5^{k+1}-1}{4}\right)r_3(25n+15)=r_3\left(25^k(25n+15)\right), \tag{5.5.56}$$

$$\left(\frac{5^{k+1}-1}{4}\right)r_3(25n+20)=r_3\left(25^k(25n+20)\right). \tag{5.5.57}$$

Replacing $n$ by $49n+4$ in (5.5.52), we find that

$$5r_3(7(35n+3))=r_3(7(875n+75)),$$

which on employing (5.3.1) may be written as

$$5\mathrm{asc}_7\left(5(7n)+1\right)=\mathrm{asc}_7\left(125(7n)+73\right). \tag{5.5.58}$$

In a similar way, replacing $n$ by $49n+11$, $49n+18$, $49n+25$, $49n+32$, $49n+46$, respectively, in (5.5.52), we obtain

$$5\mathrm{asc}_7\left(5(7n+1)+1\right)=\mathrm{asc}_7\left(125(7n+1)+73\right), \tag{5.5.59}$$

$$5\mathrm{asc}_7\left(5(7n+2)+1\right)=\mathrm{asc}_7\left(125(7n+2)+73\right), \tag{5.5.60}$$

$$5\mathrm{asc}_7\left(5(7n+3)+1\right)=\mathrm{asc}_7\left(125(7n+3)+73\right), \tag{5.5.61}$$

$$5\mathrm{asc}_7\left(5(7n+4)+1\right)=\mathrm{asc}_7\left(125(7n+4)+73\right), \tag{5.5.62}$$

$$5\mathrm{asc}_7\left(5(7n+6)+1\right)=\mathrm{asc}_7\left(125(7n+6)+73\right). \tag{5.5.63}$$

Again, by employing (5.5.52) and (5.5.53), we have

$$5\left(r_3\left(49(5n+4)\right)-r_3(5n+4)\right)$$
$$=r_3\left(49(125n+100)\right)-r_3\left(125n+100\right),$$

which on employing (5.3.1) may be written as

$$5\mathrm{asc}_7\left(5(7n+5)+1\right)=\mathrm{asc}_7\left(125(7n+5)+73\right). \tag{5.5.64}$$

From (5.5.58)–(5.5.64), we readily deduce (5.5.44). In a similar way employing (5.5.52), (5.5.53), and then using (5.3.1), we deduce (5.5.43).

Again, replacing $n$ by $49n+4$ in (5.5.54), we have

$$\left(\frac{5^{k+1}-1}{4}\right)r_3(7(175n+15))=r_3\left(7\left(7\times 5^{2k+2}n+3\times 5^{2k+1}\right)\right),$$

which on employing (5.3.1) may be written as

$$5\mathrm{asc}_7\left(25(7n)+13)\right)=\mathrm{asc}_7\left(5^{2k+2}(7n)+3\times 5^{2k+1}-2\right). \tag{5.5.65}$$

In a similar way, replacing $n$ by $49n+11$, $49n+18$, $49n+25$, $49n+32$, $49n+46$, respectively, in (5.5.54), we find that

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7\left(25(7n+1)+13\right)$$
$$=\mathrm{asc}_7\left(5^{2k+2}(7n+1)+3\times 5^{2k+1}-2\right), \tag{5.5.66}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7\left(25(7n+2)+13\right)$$
$$=\mathrm{asc}_7\left(5^{2k+2}(7n+2)+3\times 5^{2k+1}-2\right), \tag{5.5.67}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7\left(25(7n+3)+13\right)$$
$$=\mathrm{asc}_7\left(5^{2k+2}(7n+3)+3\times 5^{2k+1}-2\right), \tag{5.5.68}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7\left(25(7n+4)+13\right)$$
$$=\mathrm{asc}_7\left(5^{2k+2}(7n+4)+3\times 5^{2k+1}-2\right), \tag{5.5.69}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7\left(25(7n+6)+13\right)$$
$$=\mathrm{asc}_7\left(5^{2k+2}(7n+6)+3\times 5^{2k+1}-2\right). \tag{5.5.70}$$

Again, by employing (5.5.54) and (5.5.57), we have

$$\left(\frac{5^{k+1}-1}{4}\right)\left(r_3\left(49(25n+20)\right)-r_3(25n+20)\right)$$
$$=r_3\left(49(25^k(25n+20))\right)-r_3\left(25^k(25n+20)\right),$$

which on employing (5.3.1) may be written as

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{asc}_7\left(25(7n+5)+13)\right)$$
$$=\mathrm{asc}_7\left(5^{2k+2}(7n+5)+3\times 5^{2k+1}-2\right). \tag{5.5.71}$$

Again, from (5.5.65)–(5.5.71), we readily deduce (5.5.49).

Next, we recall the following results involving $r_3(n)$ from chapter 3 [Eqs. (3.4.43)–(3.4.45)].

$$7r_3(5n+2)=r_3\left(25(5n+2)\right), \tag{5.5.72}$$

$$7r_3(5n+3)=r_3\left(25(5n+3)\right), \tag{5.5.73}$$

$$6r_3(25n)=r_3\left(625n\right)+5r_3(n). \tag{5.5.74}$$

Using (5.5.73) and (5.5.72), and with the aid of (5.3.1), we easily deduce (5.5.45) and (5.5.46). In a similar way, employing (5.5.54)–(5.5.57), and then using (5.3.1), we deduce (5.5.47), (5.5.50) and (5.5.48). Finally, using (5.5.74) and (5.3.1), and proceeding as in the proof of (5.5.24), we arrive at (5.5.51) to finish the proof. □

# Chapter 6

# Infinite Families of Arithmetic Identities for Doubled Distinct $t$-Cores for $t = 3, 4, \cdots, 10$

## 6.1 Introduction

We recall that if $\mathrm{add}_t(n)$ denotes the number of doubled distinct partitions of $n$ that are $t$-cores then the generating function for $\mathrm{add}_t(n)$ is given by Garvan, Kim and Stanton [24, Eq. (8.1a)] as

$$\sum_{n=0}^{\infty} \mathrm{add}_t(n)q^n = \frac{(-q^2;q^2)_\infty (q^{2t};q^{2t})_\infty^{(t-2)/2}}{(-q^t;q^t)_\infty}, \quad \text{for } t \text{ even}, \tag{6.1.1}$$

and

$$\sum_{n=0}^{\infty} \mathrm{add}_t(n)q^n = \frac{(-q^2;q^2)_\infty (q^{2t};q^{2t})_\infty^{(t-1)/2}}{(-q^{2t};q^{2t})_\infty}, \quad \text{for } t \text{ odd}. \tag{6.1.2}$$

Note that $\mathrm{add}_t(n) = 0$ if $n$ is odd.

Baruah and Sarmah [7] proved that

$$\mathrm{asc}_9(8n+10) = \mathrm{asc}_9(2n), \tag{6.1.3}$$

and as 2 has no self-conjugate 9-core, there is an infinite sequence of positive integers having no self-conjugate 9-cores.

By applying some deep theorems developed by Cao [19], Baruah and Sarmah [7] proved that

$$\mathrm{asc}_7(8m-1)=0. \tag{6.1.4}$$

Baruah and Sarmah [7] also found some interesting relations connecting self-conjugate and doubled distint $t$-core partitions. Motivated by these, we put our efforts to see more results in their direction including some of our own.

By employing (1.2.2) and manipulating the $q$-products, and then using (6.1.2), (6.1.1) and (1.2.4), we have

$$\sum_{n=0}^{\infty}\mathrm{add}_3(n)q^n = f(q^2,q^4), \tag{6.1.5}$$

$$\sum_{n=0}^{\infty}\mathrm{add}_4(n)q^n = \psi(q^2), \tag{6.1.6}$$

$$\sum_{n=0}^{\infty}\mathrm{add}_5(n)q^n = f(q^2,q^8)f(q^4,q^6), \tag{6.1.7}$$

$$\sum_{n=0}^{\infty}\mathrm{add}_6(n)q^n = \psi(q^6)f(q^2,q^4), \tag{6.1.8}$$

$$\sum_{n=0}^{\infty}\mathrm{add}_7(n)q^n = f(q^2,q^{12})f(q^4,q^{10})f(q^6,q^8), \tag{6.1.9}$$

$$\sum_{n=0}^{\infty}\mathrm{add}_8(n)q^n = \psi(q^2)\psi(q^4)\psi(q^8), \tag{6.1.10}$$

$$\sum_{n=0}^{\infty}\mathrm{add}_9(n)q^n = f(q^2,q^{16})f(q^4,q^{14})f(q^6,q^{12})f(q^8,q^{10}), \tag{6.1.11}$$

and

$$\sum_{n=0}^{\infty}\mathrm{add}_{10}(n)q^n = f(q^8,q^{12})f(q^6,q^{14})f(q^4,q^{16})f(q^2,q^{18}). \tag{6.1.12}$$

Again, by employing (1.2.2) and manipulating the $q$-products, and then using (1.1.9), (1.1.8) and (1.2.4), we have

$$\sum_{n=0}^{\infty}\mathrm{asc}_3(n)q^n = f(q,q^5), \tag{6.1.13}$$

$$\sum_{n=0}^{\infty}\mathrm{asc}_4(n)q^n = \psi(q)\psi(q^4), \tag{6.1.14}$$

$$\sum_{n=0}^{\infty} \mathrm{asc}_5(n)q^n = f(q,q^9)f(q^3,q^7), \tag{6.1.15}$$

$$\sum_{n=0}^{\infty} \mathrm{asc}_7(n)q^n = f(q,q^{13})f(q^3,q^{11})f(q^5,q^9), \tag{6.1.16}$$

and

$$\sum_{n=0}^{\infty} \mathrm{asc}_9(n)q^n = f(q,q^{17})f(q^3,q^{15})f(q^5,q^{13})f(q^7,q^{11}). \tag{6.1.17}$$

Now, let $t_2(n)$ and $t_3(n)$ denote the number of representations of $n$ as a sum of two triangular numbers and three triangular numbers, respectively, and $r_2(n)$ and $r_3(n)$ denote the number of representations of $n$ as a sum of two squares and three squares, respectively.

In Sections 6.2 and 6.3 of this chapter, we use Ramanujan's theta function identities to find relations among $\mathrm{add}_5(n)$, $t_2(n)$ and $\mathrm{asc}_5(n)$, and $\mathrm{add}_6(n)$, $r_2(n)$ and $\mathrm{asc}_4(n)$, respectively.

Section 6.4 is devoted to finding relations between $\mathrm{add}_7(n)$ and $t_3(n)$, and between $\mathrm{add}_7(n)$ and $\mathrm{asc}_7(n)$. As a corollary, we also deduce (6.1.4). In Sections 6.5 and 6.6, we find relations between $\mathrm{add}_8(n)$ and $r_3(n)$, and between $\mathrm{add}_9(n)$ and $\mathrm{asc}_9(n)$.

In the process, we also find a simple proof of (6.1.3). In Section 6.7, we find new infinite families of arithmetic identities for doubled distinct 3-cores and 4-cores and also observe a new proof of a result given by Baruah and Sarmah [7] which states that if $\mathrm{add}_3(n)$ and $\mathrm{asc}_3(n)$ denote the number of doubled distinct and self conjugate 3-cores of $n$, respectively, then

$$\mathrm{add}_3(n) = \mathrm{asc}_3(4n). \tag{6.1.18}$$

In Sections 6.8–6.11, we present infinite families of new arithmetic identities for $\mathrm{add}_5(n)$, $\mathrm{asc}_4(n)$, $\mathrm{add}_6(n)$, $\mathrm{add}_7(n)$, and $\mathrm{add}_8(n)$.

In the final section, we find a new arithmetic identity for $\mathrm{add}_{10}(n)$.

## 6.2 Relations between $\text{add}_5(n)$ and $t_2(n)$, and between $\text{add}_5(n)$ and $\text{asc}_5(n)$

**Theorem 6.2.1.** *If $t_2(n)$ is the number of representations of a nonnegative integer $n$ as a sum of two triangular numbers and $\text{add}_5(n)$ is the number of doubled distinct 5-cores of $n$, then*

$$2\text{add}_5(2n) = \begin{cases} t_2(5n+1), & \text{if } n \equiv 0,2,3,4 \pmod 5; \\ t_2(5n+1) - t_2((n-1)/5), & \text{if } n \equiv 1 \pmod 5. \end{cases} \tag{6.2.1}$$

*Proof.* We have

$$\sum_{n=0}^{\infty} t_2(n)q^n = \psi^2(q). \tag{6.2.2}$$

Employing (1.2.17) in (6.2.2) and extracting the terms involving $q^{5n+1}$ from both sides of the resulting identity, dividing both sides by $q$ and then replacing $q^5$ by $q$, we find that

$$\sum_{n=0}^{\infty} t_2(5n+1)q^n = 2f(q^2,q^3)f(q,q^4) + q\psi^2(q^5). \tag{6.2.3}$$

Using (6.1.7) and (6.2.2) in (6.2.3), we have

$$\sum_{n=0}^{\infty} t_2(5n+1)q^n = 2\sum_{n=0}^{\infty} \text{add}_5(2n)q^n + q\sum_{n=0}^{\infty} t_2(n)q^{5n}. \tag{6.2.4}$$

Equating the coefficients of $q^{5n+r}$ for $r = 0$, 2, 3, 4 and 1, respectively, from both sides of (6.2.4), we obtain

$$\begin{aligned} t_2(25n+1) &= 2\text{add}_5(10n), \\ t_2(25n+11) &= 2\text{add}_5(10n+4), \\ t_2(25n+16) &= 2\text{add}_5(10n+6), \\ t_2(25n+21) &= 2\text{add}_5(10n+8), \\ t_2(25n+6) - t_2(n) &= 2\text{add}_5(10n+2), \end{aligned} \tag{6.2.5}$$

which readily implies (6.2.1). □

**Theorem 6.2.2.** (Baruah and Sarmah [7]). *If* $\operatorname{add}_5(n)$ *and* $\operatorname{asc}_5(n)$ *denote the number of doubled distinct and self conjugate* 5*-cores of* $n$*, respectively, then*

$$\operatorname{add}_5(n) = \operatorname{asc}_5(2n). \tag{6.2.6}$$

Here we give an alternative proof of (6.2.6).

*Proof.* Define $u_2(n)$ by

$$\sum_{n=0}^{\infty} u_2(n)q^n = \varphi^2(q) - \varphi^2(q^5). \tag{6.2.7}$$

Now, from [16, p. 262, Entries 10(iv) and 10(v)], we note that

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q,q^9)f(q^3,q^7), \tag{6.2.8}$$

$$\psi^2(q) - q\psi^2(q^5) = f(q,q^4)f(q^2,q^3). \tag{6.2.9}$$

Employing (6.1.15) and (6.2.7) in (6.2.8), we have

$$\sum_{n=0}^{\infty} u_2(n)q^n = 4q\sum_{n=0}^{\infty} \operatorname{asc}_5(n)q^n, \tag{6.2.10}$$

Again, employing (1.2.8) on the right side of (6.2.7), and then extracting the coefficients of $q^{2n+1}$ from both sides of the resulting identity, we obtain

$$\sum_{n=0}^{\infty} u_2(2n+1)q^n = 4\left(\psi^2(q^2) - q^2\psi^2(q^{10})\right),$$

which, with the aid of (6.2.9) and (6.1.7), implies

$$\sum_{n=0}^{\infty} u_2(2n+1)q^n = 4\sum_{n=0}^{\infty} \operatorname{add}_5(n)q^n. \tag{6.2.11}$$

From (6.2.10) and (6.2.11), we deduce (6.2.6). □

## 6.3 Relations among $\operatorname{add}_6(n)$, $\operatorname{asc}_4(n)$ and $r_2(n)$

**Theorem 6.3.1.** *If* $r_2(n)$ *is the number of representations of a nonnegative integer* $n$ *as a sum of two squares and* $\operatorname{add}_6(n)$ *and* $\operatorname{asc}_4(n)$ *are the number of doubled distinct* 6*-cores and self-conjugate* 4*-cores of* $n$*, respectively, then*

$$\frac{1}{8}r_2(24n+5) = \operatorname{asc}_4(3n) = \operatorname{add}_6(4n). \tag{6.3.1}$$

In the following process of proving (6.3.1), we also find some other well-known results involving $r_2(n)$.

*Proof.* Since

$$\sum_{n=0}^{\infty} r_2(n)q^n = \varphi^2(q), \tag{6.3.2}$$

we rewrite (1.2.8) in the form

$$\sum_{n=0}^{\infty} r_2(n)q^n = \sum_{n=0}^{\infty} r_2(n)q^{2n} + 4q\psi^2(q^4).$$

Extracting the terms involving $q^{2n}$ and $q^{2n+1}$ from both sides of the above identity, we obtain

$$r_2(2n) = r_2(n),$$

and

$$\sum_{n=0}^{\infty} r_2(2n+1)q^n = 4\psi^2(q^2). \tag{6.3.3}$$

From (6.3.3), it readily follows that

$$\sum_{n=0}^{\infty} r_2(4n+1)q^n = 4\psi^2(q), \tag{6.3.4}$$

$$r_2(4n+3) = 0. \tag{6.3.5}$$

Now, employing (1.2.7) in (6.3.4), extracting the terms involving $q^{2n+1}$ from both sides of the resulting identity, and using (6.1.14), we find that

$$\sum_{n=0}^{\infty} r_2(8n+5)q^n = 8\psi(q)\psi(q^4) = 8\sum_{n=0}^{\infty} \mathrm{asc}_4(n)q^n. \tag{6.3.6}$$

Next, employing (1.2.16) in (6.3.4), extracting the terms involving $q^{3n+1}$ from both sides of the resulting identity, and using (6.1.8), we obtain

$$\sum_{n=0}^{\infty} r_2(12n+5)q^n = 8\psi(q^3)f(q,q^2) = 8\sum_{n=0}^{\infty} \mathrm{add}_6(2n)q^n. \tag{6.3.7}$$

From (6.3.6) and (6.3.7), we arrive at (6.3.1) to finish the proof. □

## 6.4 Relations between $\mathrm{add}_7(n)$ and $t_3(n)$, and between $\mathrm{add}_7(n)$ and $\mathrm{asc}_7(n)$

**Theorem 6.4.1.** *If $t_3(n)$ is the number of representations of a nonnegative integer $n$ as a sum of three triangular numbers and* $\mathrm{add}_7(n)$ *is the number of doubled distinct 7-cores of $n$, then*

$$6\mathrm{add}_7(2n) = \begin{cases} t_3(7n+4), & \text{if } n \equiv 0,1,3,4,5,6 \pmod 7; \\ t_3(7n+4) - t_3((n-2)/7), & \text{if } n \equiv 2 \pmod 7. \end{cases} \tag{6.4.1}$$

*Proof.* We have

$$\sum_{n=0}^{\infty} t_3(n)q^n = \psi^3(q), \tag{6.4.2}$$

Setting $n = 7$ and $a = 1$, $b = q$ in (1.2.13), we also have

$$\psi(q) = f(q^{21}, q^{28}) + qf(q^{14}, q^{35}) + q^3 f(q^7, q^{42}) + q^6\psi(q^{49}). \tag{6.4.3}$$

Now we employ (6.4.3) in (6.4.2), extract the terms involving $q^{7n+4}$ from both sides of the resulting identity, divide both sides by $q^4$ and then replace $q^7$ by $q$, to find that

$$\sum_{n=0}^{\infty} t_3(7n+4)q^n = q^2 \sum_{n=0}^{\infty} t_3(n)q^{7n} + 6f(q,q^6)f(q^2,q^5)f(q^3,q^4).$$

With the help of (6.1.9), the above identity can be written as

$$\sum_{n=0}^{\infty} t_3(7n+4)q^n = q^2 \sum_{n=0}^{\infty} t_3(n)q^{7n} + 6\sum_{n=0}^{\infty} \mathrm{add}_7(2n)q^n. \tag{6.4.4}$$

Equating the coefficients of $q^{7n+r}$ with $r = 0,1,3,4,5,6$, and 2, respectively, from both sides of (6.4.4), we arrive at the identities

$$t_3(49n+4) = 6\mathrm{add}_7(14n),$$

$$t_3(49n+11) = 6\mathrm{add}_7(14n+2),$$

$$t_3(49n+25) = 6\mathrm{add}_7(14n+6),$$

$$t_3(49n+32) = 6\mathrm{add}_7(14n+8),$$

$$t_3(49n+39) = 6\text{add}_7(14n+10),$$
$$t_3(49n+46) = 6\text{add}_7(14n+12),$$
$$t_3(49n+18) - t_3(n) = 6\text{add}_7(14n+4),$$

which readily implies (6.4.1). □

**Theorem 6.4.2.** *If* $\text{add}_7(n)$ *and* $\text{asc}_7(n)$ *denote the number of doubled distinct* 7-*cores and self-conjugate* 7-*cores, respectively, of* $n$, *then*

$$\text{add}_7(n) = \text{asc}_7(4n+3). \tag{6.4.5}$$

*Proof.* First recall that the generating function of $\text{asc}_7(n)$ is given by (6.1.16).

Next, from [16, p. 46, Entries 30(ii) and 30(iii)], we have

$$f(a,b) = f(a^3b, ab^3) + af\left(\frac{b}{a}, \frac{a}{b}a^4b^4\right). \tag{6.4.6}$$

Setting, in turn, $a = q$ and $b = q^{13}$; $a = q^3$ and $b = q^{11}$; and $a = q^5$ and $b = q^9$; in (6.4.6), we find that

$$f(q,q^{13}) = f(q^{16},q^{40}) + qf(q^{12},q^{44}), \tag{6.4.7}$$
$$f(q^3,q^{11}) = f(q^{20},q^{36}) + q^3f(q^8,q^{48}), \tag{6.4.8}$$
$$f(q^5,q^9) = f(q^{24},q^{32}) + q^5f(q^4,q^{52}). \tag{6.4.9}$$

Employing (6.4.7)–(6.4.9) in (6.1.16), and then extracting the terms involving $q^{4n+3}$ from both sides of the resulting identity, we obtain

$$\sum_{n=0}^{\infty} \text{asc}_7(4n+3)q^n = f(q^2,q^{12})f(q^4,q^{10})f(q^6,q^8). \tag{6.4.10}$$

Employing (6.1.9) in (6.4.10), and then comparing the coefficients of $q^n$, we readily arrive at (6.4.5). □

**Corollary 6.4.3.** *Identity* (6.1.4) *holds.*

*Proof.* Since $\text{add}_7(2n+1) = 0$, (6.1.4) readily follows from (6.4.5). □

## 6.5 Relations between $\mathrm{add}_8(n)$ and $r_3(n)$

**Theorem 6.5.1.** *If $r_3(n)$ is the number of representations of a nonnegative integer $n$ as a sum of three squares and $\mathrm{add}_8(n)$ is the number of doubled distinct 8-cores of $n$, then*

$$r_3(16n+14) = 48\mathrm{add}_8(2n). \tag{6.5.1}$$

*Proof.* We have

$$\sum_{n=0}^{\infty} r_3(n)q^n = \varphi^3(q). \tag{6.5.2}$$

Employing (1.2.6) in (6.5.2) and then equating the terms involving $q^{4n+2}$ from both sides of the resulting identity, we find that

$$\sum_{n=0}^{\infty} r_3(4n+2)q^n = 12\varphi(q)\psi^2(q^2). \tag{6.5.3}$$

Employing (1.2.6) once again in (6.5.3) and then extracting the terms involving $q^{2n+1}$ from the resulting identity, we obtain

$$\sum_{n=0}^{\infty} r_3(8n+6)q^n = 24\psi^2(q)\psi(q^4). \tag{6.5.4}$$

Employing (1.2.7) in (6.5.4) and then equating the terms involving $q^{2n+1}$, and also using (6.1.10), we arrive at (6.5.1) to complete the proof. □

**Corollary 6.5.2.** *If $h(-D)$ denotes the class number of primitive binary quadratic forms with discriminant $-D$ and $\mathrm{add}_8(n)$ denotes the number of doubled distinct 8-cores of $n$, then, for a square-free integer $16n+14$, we have*

$$\mathrm{add}_8(2n) = \frac{1}{4}h(-64n-56). \tag{6.5.5}$$

*Proof.* A classical result due to Gauss states that if $n$ is square-free and $n > 4$, then

$$r_3(n) = \begin{cases} 24h(-n), & \text{for } n \equiv 3 \pmod 8; \\ 12h(-4n), & \text{for } n \equiv 1,2,5,6 \pmod 8; \\ 0, & \text{for } n \equiv 7 \pmod 8). \end{cases}$$

Now (6.5.5) readily follows from (6.5.1). □

## 6.6 Relations between $\mathrm{asc}_9(n)$ and $\mathrm{add}_9(n)$

The following theorem was found by Baruah and Sarmah [7, Theorem 5.8]. Here we give another simple proof.

**Theorem 6.6.1.** *If* $\mathrm{add}_9(n)$ *and* $\mathrm{asc}_9(n)$ *represent the number of doubled distinct 9-cores and self-conjugate 9-cores, respectively, of* $n$, *then*

$$\mathrm{add}_9(n) = \mathrm{asc}_9(4n+6) - \mathrm{asc}_9(n-1). \tag{6.6.1}$$

*Proof.* First recall that the generating functions of $\mathrm{add}_9(n)$ and $\mathrm{asc}_9(n)$ are given by (6.1.11) and (6.1.17).

Next, setting, in turn, $a = q$ and $b = q^{17}$; $a = q^3$ and $b = q^{15}$; $a = q^5$ and $b = q^{13}$; and $a = q^7$ and $b = q^{11}$; in (6.4.6), we find that

$$\begin{aligned} f(q, q^{17}) &= f(q^{20}, q^{52}) + qf(q^{16}, q^{56}), \\ f(q^3, q^{15}) &= f(q^{24}, q^{48}) + q^3 f(q^{12}, q^{60}), \\ f(q^5, q^{13}) &= f(q^{28}, q^{44}) + q^5 f(q^{8}, q^{64}), \\ f(q^7, q^{11}) &= f(q^{32}, q^{40}) + q^7 f(q^{4}, q^{68}). \end{aligned}$$

Employing the above identities in (6.1.17), extracting the terms involving $q^{4n+2}$ from both sides of the resulting identity, and then using again (6.1.17) and (6.1.11), we deduce that

$$\sum_{n=0}^{\infty} \mathrm{asc}_9(4n+2)q^n = q\sum_{n=0}^{\infty} \mathrm{add}_9(n)q^n + q^2 \sum_{n=0}^{\infty} \mathrm{asc}_9(n)q^n.$$

Equating the coefficients of $q^{n+1}$ from both sides of the above, we easily arrive at (6.6.1) □

**Corollary 6.6.2.** *Identity* (6.1.3) *holds.*

*Proof.* Since $\mathrm{add}_9(2n+1) = 0$, identity (6.1.3) follows easily from (6.6.1) when $n$ is replaced by $2n+1$. □

## 6.7 Infinite families of results on $\mathrm{add}_3(n)$ and $\mathrm{add}_4(n)$

In this section, we find relations among $\mathrm{add}_3(n)$, $\mathrm{add}_4(n)$ and $\mathrm{asc}_3(n)$ from the coefficients of $\varphi(q)$. We also find infinite families of arithmetic identities for $\mathrm{add}_3(n)$ and $\mathrm{add}_4(n)$.

**Theorem 6.7.1.** *Let $p > 3$ be prime and $k$ be a positive even integer. If* $\mathrm{add}_3(n)$ *denotes the number of doubled distinct* 3*-cores of $n$, then*

$$\mathrm{add}_3(n) = \mathrm{add}_3\left(p^k n + \frac{p^k - 1}{12}\right). \tag{6.7.1}$$

*Proof.* Let us define $s(n)$ by

$$\varphi(q) = \sum_{n \geq 0} s(n) q^n. \tag{6.7.2}$$

Employing (6.7.2) in (1.2.14), and then extracting the terms involving $q^{3n+1}$ from both sides of the resulting identity, we find that

$$\sum_{n \geq 0} s(3n+1) q^n = 2f(q, q^5). \tag{6.7.3}$$

Employing (6.4.6), with $a = q$, $b = q^5$, on the right hand side of (6.7.3), and then equating the terms involving $q^{4n}$ from the resulting identity, we obtain

$$\sum_{n \geq 0} s(12n+1) q^n = 2f(q^2, q^4),$$

which, by (6.1.5), implies

$$s(12n+1) = 2\mathrm{add}_3(n). \tag{6.7.4}$$

Now, setting $n = p$ and $a = b = q$ in (1.2.13), we have

$$\varphi(q) = \varphi(q^{p^2}) + \sum_{r=1}^{p-1} q^{r^2} f(q^{p(p-2r)}, q^{p(p+2r)}). \tag{6.7.5}$$

With successive use of the trivial identity $f(a, b) = af(a^{-1}, a^2 b)$, we can rewrite (6.7.5) in the form

$$\varphi(q) = \varphi(q^{p^2}) + 2qf(q^{p(p-2)}, q^{p(p+2)}) + 2q^{2^2} f(q^{p(p-2\cdot 2)}, q^{p(p+2\cdot 2)})$$
$$+ 2q^{3^2} f(q^{p(p-2\cdot 3)}, q^{p(p+2\cdot 3)}) + \cdots + 2q^{\left(\frac{p-1}{2}\right)^2} f(q^p, q^{p(2p-1)}). \tag{6.7.6}$$

Employing (6.7.2) in (6.7.6) and equating the coefficients of $q^{p^2n}$, we find that

$$s(n) = s(p^2n). \tag{6.7.7}$$

Replacing $n$ by $12n+1$ in (6.7.7), we have

$$s(12n+1) = s\left(12\left(p^2n + \frac{p^2-1}{12}\right)+1\right).$$

Employing (6.7.4) in the above and by mathematical induction, we readily arrive at (6.7.1) to complete the proof. □

**Corollary 6.7.2.** (Baruah and Sarmah [7]) *Identity* (6.1.18) *holds.*

*Proof.* Identity (6.1.18) follows easily from (6.1.13), (6.7.3) and (6.7.4). □

**Theorem 6.7.3.** *If* $\mathrm{add}_4(n)$ *denotes the number of doubled distinct* 4*-cores of* $n$*, then for any odd prime* $p$ *and any positive even integer* $k$*, we have*

$$\mathrm{add}_4(n) = \mathrm{add}_4\left(p^kn + \frac{p^k-1}{4}\right). \tag{6.7.8}$$

*Proof.* From (1.2.6) and (6.7.2), we have

$$\sum_{n\geq 0} s(n)q^n = \varphi(q^4) + 2q\psi(q^8). \tag{6.7.9}$$

Extracting the terms involving $q^{4n+1}$ from both sides of (6.7.9), we find that

$$\sum_{n\geq 0} s(4n+1)q^n = 2\psi(q^2), \tag{6.7.10}$$

which, by (6.1.6), implies

$$s(4n+1) = 2\mathrm{add}_4(n). \tag{6.7.11}$$

Now, replacing $n$ by $4n+1$ in (6.7.7), and then employing (6.7.11) and mathematical induction, we deduce (6.7.8). □

**Corollary 6.7.4.** *We have*

$$\mathrm{add}_4(3n) = \mathrm{add}_3(n). \tag{6.7.12}$$

*Proof.* Replacing $n$ by $3n$ in (6.7.11) and then employing (6.7.4), we easily arrive at (6.7.12). □

## 6.8 Infinite families of arithmetic identities for $\mathrm{add}_5(n)$

**Theorem 6.8.1.** *If* $\mathrm{add}_5(n)$ *is the number of doubled distinct* 5*-cores of* $n$*, then*

$$\mathrm{add}_5(2n) = \mathrm{add}_5\left(2 \cdot 5^k n + \frac{5^k - 1}{2}\right). \tag{6.8.1}$$

*Proof.* Employing (6.2.9) in (6.2.3), we find that

$$\sum_{n=0}^{\infty} t_2(5n+1)q^n = 2\psi^2(q) - q\psi^2(q^5),$$

which can be rewritten, with the aid of (6.2.2), as

$$\sum_{n=0}^{\infty} t_2(5n+1)q^n = 2\sum_{n=0}^{\infty} t_2(n)q^n - q\sum_{n=0}^{\infty} t_2(n)q^{5n}. \tag{6.8.2}$$

Equating the coefficients of $q^{5n+1}$ from both sides of (6.8.2), we find that

$$t_2(25n+6) = 2t_2(5n+1) - t_2(n). \tag{6.8.3}$$

Again, adding (6.2.4) and (6.8.2), and then equating the coefficients of $q^n$ from both sides of the resulting identity, we obtain

$$t_2(5n+1) = t_2(n) + \mathrm{add}_5(2n). \tag{6.8.4}$$

Employing (6.2.5) and (6.8.4) in (6.8.3), we find

$$\mathrm{add}_5(2n) = \mathrm{add}_5(10n+2).$$

Iterating the above and using mathematical induction, we arrive at (6.8.1) to finish the proof. □

**Theorem 6.8.2.** *Let* $p \equiv 3 \pmod 4$ *be a prime. If* $\mathrm{add}_5(n)$ *denotes the number of doubled distinct* 5*-cores of* $n$*, then for any positive integer* $n$ *and any positive even integer* $k$*, we have*

$$\mathrm{add}_5(2n) = \mathrm{add}_5\left(2p^k n + \frac{p^k - 1}{2}\right). \tag{6.8.5}$$

First we prove the following lemma.

**Lemma 6.8.3.** *If $t_2(n)$ denotes the number of representations of a nonnegative integer $n$ as a sum of two triangular numbers and $p \equiv 3 \pmod 4$ is a prime, then*

$$t_2(p^2n + \frac{p^2-1}{4}) = t_2(n). \tag{6.8.6}$$

*Proof.* First we note from (6.3.4) that

$$\sum_{n=0}^{\infty} r_2(4n+1)q^n = 4\psi^2(q) = 4\sum_{n=0}^{\infty} t_2(n)q^n,$$

from which we arrive at

$$r_2(4n+1) = 4t_2(n). \tag{6.8.7}$$

Next, employing (6.7.6) in (6.3.2) and then extracting the terms involving $q^{pn}$ from both sides of the resulting identity by noting that prime $p \equiv 3 \pmod 4$ and $r_2(4n+3) = 0$ from (6.3.5), we find that

$$\sum_{n=0}^{\infty} r_2(pn)q^n = \varphi^2(q^p) = \sum_{n=0}^{\infty} r_2(n)q^{pn}. \tag{6.8.8}$$

Equating the coefficients of $q^{pn}$ from both sides of (6.8.8), we readily arrive at

$$r_2(p^2n) = r_2(n). \tag{6.8.9}$$

Replacing $n$ by $4n+1$ in (6.8.9), we have

$$r_2\left(4\left(p^2n + \frac{p^2-1}{4}\right) + 1\right) = r_2(4n+1). \tag{6.8.10}$$

Employing (6.8.7) in (6.8.10), we arrive at (6.8.6) to finish the proof of the lemma. □

*Proof of Theorem 6.8.2.* Replacing $n$ by $25n+1$ in (6.8.6), we find that

$$t_2\left(5\left(5p^2n + \frac{p^2-1}{4}\right) + 1\right) = t_2(25n+1). \tag{6.8.11}$$

Employing (6.2.1) in (6.8.11), we obtain

$$\text{add}_5\left(2p^2(5n) + \frac{p^2-1}{2}\right) = \text{add}_5(10n). \tag{6.8.12}$$

Similarly, replacing $n$ by $25n+11$, $25n+16$ and $25n+21$, in turn, in (6.8.6), and then employing (6.2.1),respectively, we deduce that

$$\mathrm{add}_5\left(2p^2(5n+2)+\frac{p^2-1}{2}\right)=\mathrm{add}_5(10n+4), \qquad (6.8.13)$$

$$\mathrm{add}_5\left(2p^2(5n+3)+\frac{p^2-1}{2}\right)=\mathrm{add}_5(10n+6), \qquad (6.8.14)$$

$$\mathrm{add}_5\left(2p^2(5n+4)+\frac{p^2-1}{2}\right)=\mathrm{add}_5(10n+8). \qquad (6.8.15)$$

Again, from (6.8.6), we have

$$t_2(25(p^2n+\frac{p^2-1}{4})+6)-t_2(p^2n+\frac{p^2-1}{4})=t_2(25n+6)-t_2(n),$$

which, with the aid of (6.2.1), reduces to

$$\mathrm{add}_5\left(2p^2(5n+1)+\frac{p^2-1}{2}\right)=\mathrm{add}_5(10n+2). \qquad (6.8.16)$$

From (6.8.12)–(6.8.16), we find that

$$\mathrm{add}_5\left(2p^2n+\frac{p^2-1}{2}\right)=\mathrm{add}_5(2n). \qquad (6.8.17)$$

Iterating (6.8.17) and by applying mathematical induction, we arrive at (6.8.5). □

## 6.9 Infinite families of results on $\mathrm{asc}_4(n)$ and $\mathrm{add}_6(n)$

**Theorem 6.9.1.** *Let $p \equiv 3 \pmod 4$ be prime and $k$ be a positive even integer. If $\mathrm{asc}_4(n)$ denotes the number of self-conjugate 4-cores of $n$, then*

$$\mathrm{asc}_4(n)=\mathrm{asc}_4\left(p^kn+\frac{5(p^k-1)}{8}\right). \qquad (6.9.1)$$

*Proof.* Replacing $n$ by $8n+5$ in (6.8.9), we have

$$r_2(8n+5)=r_2\left(8\left(p^2n+\frac{5p^2-5}{8}\right)+5\right). \qquad (6.9.2)$$

Employing (6.3.6) in (6.9.2), we find that

$$\mathrm{asc}_4(n)=\mathrm{asc}_4\left(p^2n+\frac{5(p^2-1)}{8}\right),$$

from which, by mathematical induction, we arrive at (6.9.1). □

**Theorem 6.9.2.** *Let $p \equiv 3 \pmod 4$ and $p > 3$ be prime. If $\mathrm{add}_6(n)$ denotes the number of doubled distinct 6-cores of $n$, then*

$$\mathrm{add}_6(2n) = \mathrm{add}_6\left(2p^2n + \frac{5(p^2-1)}{6}\right). \tag{6.9.3}$$

*Proof.* Replacing $n$ by $12n+5$ in (6.8.9), we have

$$r_2(12n+5) = r_2\left(12\left(p^2n + \frac{5p^2-5}{12}\right) + 5\right). \tag{6.9.4}$$

Employing (6.3.7) in (6.9.4), we readily arrive at (6.9.3). □

## 6.10 Infinite families of results on $\mathrm{add}_7(n)$

**Theorem 6.10.1.** *If $\mathrm{add}_7(n)$ denotes the number of doubled distinct 7-cores of $n$, then for any positive integer $k$, we have*

$$3^k\mathrm{add}_7(6n) = \mathrm{add}_7\left(2\times 3^{2k+1}n + \frac{(5\times 3^{2k}-5)}{4}\right), \tag{6.10.1}$$

$$(2\times 3^k - 1)\mathrm{add}_7(6n+2) = \mathrm{add}_7\left(2\times 3^{2k+1}n + \frac{(13\times 3^{2k}-5)}{4}\right), \tag{6.10.2}$$

$$\left(\frac{3^{k+1}-1}{2}\right)\mathrm{add}_7(18n+4) = \mathrm{add}_7\left(2\times 3^{2k+2}n + \frac{(7\times 3^{2k+1}-5)}{4}\right), \tag{6.10.3}$$

$$\left(\frac{3^{k+1}-1}{2}\right)\mathrm{add}_7(18n+10) = \mathrm{add}_7\left(2\times 3^{2k+2}n + \frac{15\times 3^{2k+1}-5}{4}\right) + \left(\frac{3^{k+1}-3}{2}\right)\mathrm{add}_7(2n), \tag{6.10.4}$$

$$\left(\frac{3^{k+1}-1}{2}\right)\mathrm{add}_7(18n+16) = \mathrm{add}_7\left(2\times 3^{2k+2}n + \frac{(23\times 3^{2k+1}-5)}{4}\right). \tag{6.10.5}$$

*Proof.* We first recall the results involving $t_3(n)$ from Chapter 3 [Eqs. (3.4.19)–(3.4.23)].

Replacing $n$ by $49n+1$ in (3.4.19), we have

$$3^k t_3\left(7(21n)+4\right) = t_3\left(7\left(3^{2k+1}\times 7n + \frac{5(3^{2k}-1)}{8}\right)+4\right).$$

Since $\frac{5(3^{2k}-1)}{8} \equiv 0, 1, 5 \pmod 7$, on employing (6.4.1), the above may be written as

$$3^k\mathrm{add}_7\left(6(7n)\right) = \mathrm{add}_7\left(2\times 3^{2k+1}(7n) + \frac{5(3^{2k}-1)}{4}\right). \tag{6.10.6}$$

In a similar way, replacing $n$ by $49n+8$, $49n+15$, $49n+29$, $49n+36$, $49n+43$, in turn, in (3.4.19), we find that

$$3^k\mathrm{add}_7\,(6(7n+1)) = \mathrm{add}_7\left(2\times 3^{2k+1}(7n+1)+\frac{5(3^{2k}-1)}{4}\right), \qquad (6.10.7)$$

$$3^k\mathrm{add}_7\,(6(7n+2)) = \mathrm{add}_7\left(2\times 3^{2k+1}(7n+2)+\frac{5(3^{2k}-1)}{4}\right), \qquad (6.10.8)$$

$$3^k\mathrm{add}_7\,(6(7n+4)) = \mathrm{add}_7\left(2\times 3^{2k+1}(7n+4)+\frac{5(3^{2k}-1)}{4}\right), \qquad (6.10.9)$$

$$3^k\mathrm{add}_7\,(6(7n+5)) = \mathrm{add}_7\left(2\times 3^{2k+1}(7n+5)+\frac{5(3^{2k}-1)}{4}\right), \qquad (6.10.10)$$

$$3^k\mathrm{add}_7\,(6(7n+6)) = \mathrm{add}_7\left(2\times 3^{2k+1}(7n+6)+\frac{5(3^{2k}-1)}{4}\right), \qquad (6.10.11)$$

respectively. Furthermore, replacing $n$ by $49n+22$ in (3.4.19) and then subtracting (3.4.19) from the resulting identity, we obtain

$$\begin{aligned}&3^k\left(t_3\left(49(3n+1)+18\right)-t_3(3n+1)\right)\\&= t_3\left(49\left(3^{2k+1}n+\frac{11\times 3^{2k}-3}{8}\right)+18\right)-t_3\left(3^{2k+1}n+\frac{11\times 3^{2k}-3}{8}\right),\end{aligned}$$

which, with the aid of (6.4.1), reduces to

$$3^k\mathrm{add}_7\,(6(7n+3)) = \mathrm{add}_7\left(2\times 3^{2k+1}(7n+3)+\frac{5(3^{2k}-1)}{4}\right). \qquad (6.10.12)$$

From (6.10.6)–(6.10.12), we arrive at (6.10.1).

Next, putting $k=1$ in (3.4.23), we have

$$4t_3(9n+3) = t_3(81n+30)+3t_3(n). \qquad (6.10.13)$$

Replacing $n$ by $7n+4$ in the above identity, we note that

$$4t_3(7(9n+5)+4) = t_3\,(7(81n+50)+4)+3t_3(7n+4). \qquad (6.10.14)$$

For $n\equiv 0,1,3,4,5,6 \pmod 7$, by employing (6.4.1) in (6.10.14), we find that

$$4\mathrm{add}_7(18n+10) = \mathrm{add}_7(162n+100)+3\mathrm{add}_7(2n). \qquad (6.10.15)$$

Again, replacing $n$ by $49n+18$ in (6.10.13), we have

$$4t_3(49(9n+3)+18) = t_3(49(81n+30)+18)+3t_3(49n+18). \qquad (6.10.16)$$

Subtracting (6.10.14) from (6.10.16) and then employing (6.4.1), we find that

$$4\mathrm{add}_7(126n+46) = \mathrm{add}_7(1134n+424) + 3\mathrm{add}_7(14n+4), \tag{6.10.17}$$

which is (6.10.15) for $n \equiv 2 \pmod 7$.

From (6.10.15) and (6.10.17), for any nonnegative integer $n$, we arrive at

$$4\mathrm{add}_7(18n+10) = \mathrm{add}_7(162n+100) + 3\mathrm{add}_7(2n). \tag{6.10.18}$$

Iterating (6.10.18), and by mathematical induction, we deduce (6.10.4).

In a similar way, we can derive (6.10.2), (6.10.3), and (6.10.5) by using (3.4.20)–(3.4.22), respectively. □

**Theorem 6.10.2.** *If* $\mathrm{add}_7(n)$ *denotes the number of doubled distinct 7-cores of* $n$ *and* $k \geq 1$, *then*

$$5\mathrm{add}_7(10n+2) = \mathrm{add}_7(250n+80), \tag{6.10.19}$$

$$7\mathrm{add}_7(10n+4) = \mathrm{add}_7(250n+130), \tag{6.10.20}$$

$$7\mathrm{add}_7(10n+6) = \mathrm{add}_7(250n+180), \tag{6.10.21}$$

$$5\mathrm{add}_7(10n+8) = \mathrm{add}_7(250n+230), \tag{6.10.22}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_7(50n) = \mathrm{add}_7\left(2\times 5^{2k+2}n + \frac{5^{2k+1}-5}{4}\right), \tag{6.10.23}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_7(50n+10) = \mathrm{add}_7\left(2\times 5^{2k+2}n + \frac{9\times 5^{2k+1}-5}{4}\right), \tag{6.10.24}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_7(50n+20) = \mathrm{add}_7\left(2\times 5^{2k+2}n + \frac{17\times 5^{2k+1}-5}{4}\right), \tag{6.10.25}$$

$$6\mathrm{add}_7(50n+30) = \mathrm{add}_7(1250n+780) + 5\mathrm{add}_7(2n), \tag{6.10.26}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_7(50n+40) = \mathrm{add}_7\left(2\times 5^{2k+2}n + \frac{33\times 5^{2k+1}-5}{4}\right). \tag{6.10.27}$$

*Proof.* We recall the results involving $t_3(n)$ from Chapter 3 [Eqs. (3.4.26)–(3.4.34)].

Replacing $n$ by $49n+2$ in (3.4.26), we have

$$5t_3(7(35n+1)+4) = t_3(7(875n+40)+4),$$

which, by (6.4.1), gives

$$5\text{add}_7\,(10(7n)+2)) = \text{add}_7\,(250(7n)+80)\,. \tag{6.10.28}$$

Similarly, replacing $n$ by $49n+9$, $49n+16$, $49n+30$, $49n+37$, $49n+44$, in turn, in (3.4.26), we deduce that

$$5\text{add}_7\,(10(7n+1)+2)) = \text{add}_7\,(250(7n+1)+80)\,, \tag{6.10.29}$$

$$5\text{add}_7\,(10(7n+2)+2)) = \text{add}_7\,(250(7n+2)+80)\,, \tag{6.10.30}$$

$$5\text{add}_7\,(10(7n+4)+2)) = \text{add}_7\,(250(7n+4)+80)\,, \tag{6.10.31}$$

$$5\text{add}_7\,(10(7n+5)+2)) = \text{add}_7\,(250(7n+5)+80)\,, \tag{6.10.32}$$

$$5\text{add}_7\,(10(7n+6)+2)) = \text{add}_7\,(250(7n+6)+80)\,, \tag{6.10.33}$$

respectively.

Furthermore, replacing $n$ by $49n+23$ in (3.4.26) and combining with (3.4.27), we have

$$5\,(t_3\,(49(5n+2)+18) - t_3(5n+2)) = t_3\,(49(125n+59)+18) - t_3\,(125n+59)\,,$$

which, by (6.4.1), may be recast as

$$5\text{add}_7\,(10(7n+3)+2) = \text{add}_7\,(250(7n+3)+80)\,. \tag{6.10.34}$$

From (6.10.28)–(6.10.34), we deduce (6.10.19).

In a similar fashion, employing (3.4.26), (3.4.27), and then using (6.4.1), we deduce (6.10.22). Again, from (3.4.28) and (3.4.29), with the aid of (6.4.1), we easily arrive at (6.10.20) and (6.10.21), respectively.

Next, replacing $n$ by $49n+5$ in (3.4.33), we have

$$\begin{aligned}\left(\frac{5^{k+1}-1}{4}\right)&t_3(7(175n+20)+4)\\ &= t_3\left(7\left(7\times 5^{2k+2}n+\frac{33\times 5^{2k+1}-5}{8}\right)+4\right),\end{aligned}$$

which, with the help of (6.4.1), can be written as

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_7\left(50(7n)+40\right)) \\ =\mathrm{add}_7\left(2\times 5^{2k+2}(7n)+\frac{33\times 5^{2k+1}-5}{4}\right). \qquad (6.10.35)$$

Similarly, replacing $n$ by $49n+12$, $49n+19$, $49n+26$, $49n+33$, $49n+40$, in turn, in (3.4.33), we find that

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_7\left(50(7n+1)+40\right)) \\ =\mathrm{add}_7\left(2\times 5^{2k+2}(7n+1)+\frac{33\times 5^{2k+1}-5}{4}\right), \qquad (6.10.36)$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_7\left(50(7n+2)+40\right)) \\ =\mathrm{add}_7\left(2\times 5^{2k+2}(7n+2)+\frac{33\times 5^{2k+1}-5}{4}\right), \qquad (6.10.37)$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_7\left(50(7n+3)+40\right)) \\ =\mathrm{add}_7\left(2\times 5^{2k+2}(7n+3)+\frac{33\times 5^{2k+1}-5}{4}\right), \qquad (6.10.38)$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_7\left(50(7n+4)+40\right)) \\ =\mathrm{add}_7\left(2\times 5^{2k+2}(7n+4)+\frac{33\times 5^{2k+1}-5}{4}\right), \qquad (6.10.39)$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_7\left(50(7n+5)+40\right)) \\ =\mathrm{add}_7\left(2\times 5^{2k+2}(7n+5)+\frac{33\times 5^{2k+1}-5}{4}\right), \qquad (6.10.40)$$

respectively.

Furthermore, replacing $n$ by $49n+47$ in (3.4.33) and combining with (3.4.34), we have

$$\left(\frac{5^{k+1}-1}{4}\right)\left(t_3\left(49(25n+24)+18\right)-t_3(25n+24)\right) \\ =t_3\left(49\left(5^{2k+2}n+\frac{39\times 5^{2k+1}-3}{8}\right)+18\right) \\ -t_3\left(5^{2k+2}n+\frac{39\times 5^{2k+1}-3}{8}\right),$$

which, with the help of (6.4.1), can be written as

$$\begin{aligned}&\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_7\left(50(7n+6)+40)\right)\\&\qquad=\mathrm{add}_7\left(2\times 5^{2k+2}(7n+6)+\frac{33\times 5^{2k+1}-5}{4}\right).\end{aligned} \tag{6.10.41}$$

From (6.10.35)–(6.10.41), we deduce (6.10.27).

Similarly, we can deduce (6.10.23)–(6.10.25), by employing (3.4.30), (3.4.32)-(3.4.34) and (6.4.1).

Finally, using (3.4.31) and (6.4.1), and proceeding as in the proof of (6.10.4), we arrive at (6.10.26) to finish the proof. □

## 6.11 Infinite families of arithmetic identities of $\mathrm{add}_8(n)$

In this section, we present some infinite families of arithmetic identities of $\mathrm{add}_8(n)$, the number of doubled distinct 8-cores of a nonnegative integer $n$, by employing the corresponding results on $r_3(n)$.

**Theorem 6.11.1.** *For any positive integer $k$, we have*

$$3^k\mathrm{add}_8(6n)=\mathrm{add}_8\left(2\times 3^{2k+1}n+\frac{7(3^{2k}-1)}{4}\right), \tag{6.11.1}$$

$$\left(2\times 3^k-1\right)\mathrm{add}_8(6n+4)=\mathrm{add}_8\left(2\times 3^{2k+1}n+\frac{23\times 3^{2k}-7}{4}\right), \tag{6.11.2}$$

$$\left(\frac{3^{k+1}-1}{2}\right)\mathrm{add}_8(18n+2)=\mathrm{add}_8\left(2\times 3^{2k+2}n+\frac{5\times 3^{2k+1}-7}{4}\right), \tag{6.11.3}$$

$$\left(\frac{3^{k+1}-1}{2}\right)\mathrm{add}_8(18n+8)=\mathrm{add}_8\left(2\times 3^{2k+2}n+\frac{13\times 3^{2k+1}-7}{4}\right), \tag{6.11.4}$$

$$\begin{aligned}\left(\frac{3^{k+1}-1}{2}\right)\mathrm{add}_8(18n+14)=\mathrm{add}_8&\left(2\times 3^{2k+2}n+\frac{21\times 3^{2k+1}-7}{4}\right)\\&+\left(\frac{3^{k+1}-3}{2}\right)\mathrm{add}_8(2n).\end{aligned} \tag{6.11.5}$$

*Proof.* First recall the results involving $r_3(n)$ from Chapter 3 [Eqs. (3.4.1)–(3.4.4), (3.4.18)]. Replacing $n$ by $16n+4$ in (3.4.2), we have

$$3^k r_3\left(16\left(3n\right)+14\right) = r_3\left(16\left(3^{2k+1}n + \frac{7\times 3^{2k}-7}{8}\right)+14\right). \tag{6.11.6}$$

Employing (6.5.1) in (6.11.6), we arrive at (6.11.1).

Similarly, we can deduce (6.11.2)–(6.11.4) by replacing $n$ by $16n+15$, $16n+3$, and $16n+8$ in (3.4.1), (3.4.3), and (3.4.4), respectively, and then using (6.5.1).

Next, replacing $n$ by $16n+14$ in (3.4.18), we have

$$4r_3(16(9n+7)+14) = r_3(16(81n+70)+14) + 3r_3(16n+14). \tag{6.11.7}$$

Employing (6.5.1) in (6.11.7), we find that

$$4\mathrm{add}_8(18n+14) = \mathrm{add}_8(162n+140) + 3\mathrm{add}_8(2n). \tag{6.11.8}$$

Iterating (6.11.8) and by mathematical induction, we arrive at (6.11.5) to complete the proof. □

**Theorem 6.11.2.** *For any positive integer $k$, we have*

$$5\mathrm{add}_8(10n) = \mathrm{add}_8(250n+42), \tag{6.11.9}$$

$$5\mathrm{add}_8(10n+4) = \mathrm{add}_8(250n+142), \tag{6.11.10}$$

$$7\mathrm{add}_8(10n+6) = \mathrm{add}_8(250n+192), \tag{6.11.11}$$

$$7\mathrm{add}_8(10n+8) = \mathrm{add}_8(250n+242), \tag{6.11.12}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_8(50n+2) = \mathrm{add}_8\left(2\times 5^{2k+2}n + \frac{3\times 5^{2k+1}-7}{4}\right), \tag{6.11.13}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_8(50n+12) = \mathrm{add}_8\left(2\times 5^{2k+2}n + \frac{11\times 5^{2k+1}-7}{4}\right), \tag{6.11.14}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_8(50n+22) = \mathrm{add}_8\left(2\times 5^{2k+2}n + \frac{19\times 5^{2k+1}-7}{4}\right), \tag{6.11.15}$$

$$\left(\frac{5^{k+1}-1}{4}\right)\mathrm{add}_8(50n+32) = \mathrm{add}_8\left(2\times 5^{2k+2}n + \frac{27\times 5^{2k+1}-7}{4}\right), \tag{6.11.16}$$

$$7\mathrm{add}_8(50n+42) = \mathrm{add}_8(1250n+1092) + 5\mathrm{add}_8(2n). \tag{6.11.17}$$

*Proof.* We recall the results involving $r_3(n)$ from Chapter 3 [Eqs. (3.4.35)–(3.4.40), and Eqs. (3.4.43)–(3.4.45)].

Replacing $n$ by $16n+9$ in (3.4.35), we have

$$5r_3(16(5n+2)+14) = r_3(16(125n+142)+14). \tag{6.11.18}$$

Employing (6.5.1) in (6.11.18), we easily arrive at (6.11.10).

Similarly, replacing $n$ by $16n+2$, $16n+1$, $16n+4$, $16n+7$, $16n+10$, $16n+12$, $16n+15$, $16n+14$ in (3.4.36)–(3.4.40), (3.4.43)–(3.4.45), respectively, and with the aid of (6.5.1), we deduce (6.11.9), (6.11.13)–(6.11.16), (6.11.11), (6.11.12), (6.11.17). □

## 6.12 An arithmetic identity for $\mathrm{add}_{10}(n)$

**Theorem 6.12.1.** *If* $\mathrm{add}_{10}(n)$ *is the number of doubled distinct* 10*-cores of* $n$*, then*

$$\mathrm{add}_{10}(50n+22) = 4\mathrm{add}_{10}(10n+2) + 5\mathrm{add}_{10}(2n-2). \tag{6.12.1}$$

*Proof.* From (6.1.12), we have

$$\sum_{n=0}^{\infty} \mathrm{add}_{10}(2n)q^n = f(q^4,q^6)f(q^3,q^7)f(q^2,q^8)f(q,q^9). \tag{6.12.2}$$

Now, from [16, p. 46, Entry 30(i)], we note that

$$f(a,ab^2)f(b,a^2b) = f(a,b)\psi(ab). \tag{6.12.3}$$

Setting, in turn, $a=q$, $b=q^4$ and $a=q^3$, $b=q^2$ in (6.12.3), we find that

$$f(q,q^9)f(q^4,q^6) = f(q,q^4)\psi(q^5), \tag{6.12.4}$$

$$f(q^3,q^7)f(q^2,q^8) = f(q^2,q^3)\psi(q^5). \tag{6.12.5}$$

Employing (6.12.4) and (6.12.5) in (6.12.2), we have

$$\sum_{n=0}^{\infty} \mathrm{add}_{10}(2n)q^n = \psi^2(q^5)f(q,q^4)f(q^2,q^3). \tag{6.12.6}$$

Next, from [16, p. 262, Entry 10(v)], we recall that

$$\psi^2(q) - q\psi^2(q^5) = f(q, q^4)f(q^2, q^3). \tag{6.12.7}$$

Using (6.12.7) in (6.12.6), we obtain

$$\sum_{n=0}^{\infty} \text{add}_{10}(2n)q^n = \psi^2(q)\psi^2(q^5) - q\psi^4(q^5). \tag{6.12.8}$$

Employing (1.2.17) in (6.12.8) and then extracting the terms involving $q^{5n+1}$ from both sides of the resulting identity, we find that

$$\sum_{n=0}^{\infty} \text{add}_{10}(10n+2)q^n = \psi^4(q) - q\psi^2(q)\psi^2(q^5). \tag{6.12.9}$$

From (6.12.8) and (6.12.9)

$$\sum_{n=0}^{\infty} \text{add}_{10}(10n+2)q^n + q\sum_{n=0}^{\infty} \text{add}_{10}(2n)q^n = \psi^4(q) - q^2\psi^4(q^5). \tag{6.12.10}$$

Employing (1.2.17) on the first term of the right hand side of (6.12.10) and then equating the terms involving $q^{5n+2}$ from both sides of the resulting identity, we find that

$$\sum_{n=0}^{\infty} \text{add}_{10}(50n+22)q^n + \sum_{n=0}^{\infty} \text{add}_{10}(10n+2)q^n = 5\psi^4(q) - 5q^2\psi^4(q^5), \tag{6.12.11}$$

where we have also used (6.12.7).

Finally, using (6.12.10) in (6.12.11) and then equating the coefficients of $q^n$ from both sides of the resulting identity, we arrive at (6.12.1) to complete the proof. □

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