EXPLICIT EVALUATIONS OF RAMANUJAN'S CONTINUED FRACTIONS AND THETA-FUNCTIONS

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Dedicated to my beloved parents

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ABSTRACT

In this thesis, we deal with explicit evaluations of Ramanujan's continued fractions and theta-functions.

Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b)=\sum_{k=-\infty}^{\infty} a^{k(k+1)/2}b^{k(k-1)/2}$$

where $|ab|<1$. If we set $a=qe^{2iz}$, $b=qe^{-2iz}$, and $q=e^{\pi i\tau}$, where z is complex and $\operatorname{Im}(\tau)>0$, then $f(a,b)=\vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta-functions in its standard notation.

Three special cases of $f(a,b)$ are

$$\phi(q):=f(q,q)=\sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\psi(q):=f(q,q^3)=\sum_{k=0}^{\infty} q^{k(k+1)/2},$$

and

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}.$$

If $q=e^{2\pi iz}$ with $\operatorname{Im}(z)>0$, then $f(-q)=q^{-1/24}\eta(z)$, where $\eta(z)$ denotes the classical Dedekind eta-function.

In her thesis, J. Yi (2001) considered two parameterizations $r_{k,n}$ and $r'_{k,n}$ of $f(-q)$, defined as

$$r_{k,n}:=\frac{f(-q)}{k^{1/4}q^{(k-1)/24}f(-q^k)},\quad q=e^{-2\pi\sqrt{n/k}},$$

and

$$r'_{k,n}:=\frac{f(q)}{k^{1/4}q^{(k-1)/24}f(q^k)},\quad q=e^{-\pi\sqrt{n/k}},$$

where n and k are positive real numbers. Using these parameters, she then evaluated many old and new explicit values of the famous Rogers-Ramanujan continued fraction $R(q)$, defined by

$$R(q):=\frac{q^{1/5}}{1}\,\frac{q}{+1}\,\frac{q^2}{+1}\,\frac{q^3}{+1}\,{}_{+\cdots},\quad |q|<1.$$

In this thesis, by using a method similar to that of Yi, we find some general theorems for the explicit evaluations of Ramanujan's cubic continued fraction $G(q)$, defined by

$$G(q):=\frac{q^{1/3}}{1}\,\frac{q+q^2}{+\;1}\,\frac{q^2+q^4}{+\;1}\,\frac{q^3+q^6}{+\;1}\,{}_{+\cdots},\quad |q|<1.$$

In the unorganized portions of his second notebook (published by TIFR in 1957), Ramanujan recorded, without proofs, 23 beautiful identities involving quotients of only eta-functions and no other theta-functions. The identities can be divided into two categories. In the first category, each identity involves four arguments and the second category involves eight arguments. Unlike the first category, the second category identities have not been applied before. In this thesis, we apply some newly proved and some old eta-function identities involving eight arguments to find some new values of the Rogers-Ramnujan continued fraction and the parameters μ_n and λ_n connected with Ramanujan's cubic continued fraction. The new values of μ_n and λ_n also lead to some new Ramanujan-type series for $1/\pi$. In this thesis, we show how the new values μ_n and λ_n combined with some old and newly found modular equations in cubic theory can be applied to find some new series for $1/\pi$ by appealing to a formula established by J. M. and P. B. Borwein (1987) and later modified by H. H. Chan and W. C. Liaw (2000).

Next, Ramanujan-Selberg continued fraction $Z(q)$ is defined by

$$Z(q) := \frac{q^{1/8}}{1} \, \frac{q}{+1+q} \, \frac{q^2}{+1+q^2} \, \frac{q^3}{+1+q^3} \, \frac{q^4}{+1+q^4} {}_{+\dots}, \quad |q|<1,$$

which is closely related to continued fraction $H(q)$, defined by

$$H(q) := \frac{f(-q)}{q^{1/8} f(-q^4)} = q^{1/8} - \frac{q^{7/8}}{1-q} \, \frac{q^2}{+1+q^2} \, \frac{q^3}{-1-q^3} \, \frac{q^4}{+1+q^4} {}_{-\dots}. \quad |q|<1.$$

In this thesis, by using some transformation formulas and modular equations, we present several relations connecting the continued fractions $H(\pm q)$ and $H(\pm q^n)$, $Z(\pm q)$ and $Z(\pm q^n)$, and $H(\pm q^n)$ and $Z(\pm q^n)$, for some positive integers n. We also prove some general theorems for the explicit evaluations of $H(q)$ and $Z(q)$ and find some explicit values.

Ramanujan-Göllnitz-Gordon continued fraction $K(q)$ is defined by

$$K(q) := \frac{q^{1/2}}{1+q} \, \frac{q^2}{+1+q^3} \, \frac{q^4}{+1+q^5} {}_{+\dots} \quad |q|<1.$$

In 1997, Chan and Huang (1997), derived many identities involving Ramanujan-Göllnitz-Gordon continued fraction $K(q)$, which are analogous to $R(q)$ and $G(q)$. In particular, they found explicit values of $K(e^{-\pi\sqrt{n}/2})$, for several positive integers n, by using Weber-Ramanujan class invariants G_n and g_n, defined by

$$G_n = 2^{-1/4} q^{-1/24} \chi(q) \quad \text{and} \quad g_n = 2^{-1/4} q^{-1/24} \chi(-q),$$

where $\chi(q) = \prod_{n=0}^{\infty}(1+q^{2n+1})$.

In this thesis, we establish formulas for the explicit evaluations of $K(e^{-\pi\sqrt{n}/2})$ and $K(e^{-\pi\sqrt{n}/4})$ by using parameterizations $h_{2,n}$ and $s_{4,n}$, respectively, where $h_{2,n}$, for $k = 2$, is a special case of $h_{k,n}$ and $s_{4,n}$, for $k = 4$, is a special case of the parameter $s_{k,n}$ introduced by Yi (2001) and Bruce C. Berndt (2000), respectively, and defined by

$$h_{k,n} = \frac{\phi(q)}{k^{1/4}\phi(q^k)} \text{ and } s_{k,n} = \frac{f(q)}{k^{1/4}q^{(k-1)/24}f(-(-1)^k q^k)}, \text{ where } q = e^{-\pi\sqrt{n/k}}.$$

We find several explicit values of the parameter $s_{4,n}$ by establishing general formulas. We also evaluate some new values of the parameter $h_{2,n}$ by establishing some new theta-function identities.

Yi (2001) also introduced one more parameter $h'_{k,n}$, defined by

$$h'_{k,n} := \frac{\phi(-q)}{k^{1/4}\phi(-q^k)}, \quad q = e^{-2\pi\sqrt{n/k}},$$

where k and n are positive real numbers. She then evaluated several values of $\phi(q)$, $f(q)$ and their quotients. Motivated by Yi's work, we introduce the following two new parameterizations of the theta-function $\psi(q)$. For any positive real numbers k and n, we define

$$g_{k,n} := \frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)}, \text{ and } g'_{k,n} := \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)}, \text{ where } q = e^{-\pi\sqrt{n/k}}.$$

We prove several properties of the parameterizations $g_{k,n}$ and $g'_{k,n}$ and show how they are connected to Yi's parameters $r_{k,n}$, $r'_{k,n}$, $h_{k,n}$, $h'_{k,n}$, and Weber-Ramanujan class-invariants G_n and g_n. By employing some old and newly established theta-function identities, we present some general theorems for the explicit evaluations of $g_{k,n}$, $g'_{k,n}$, $h_{k,n}$, and $h'_{k,n}$ and find many explicit values. We also offer explicit formulas for $\psi(e^{-n\pi})$ and $\psi(-e^{-n\pi})$ for positive real number n and deduce some explicit values. In addition, we establish formulas for the explicit evaluations of the Rogers-Ramanujan continued fraction and Ramanujan's cubic continued fraction in terms of parameterizations $g_{k,n}$, $g'_{k,n}$, $h_{k,n}$, and $h'_{k,n}$ from which particular values can be obtained.

Ramanujan's class invariants G_n and g_n were often applied for the explicit evaluations of continued fractions, theta-functions etc.. In his notebooks, Ramanujan recorded several Schläfli-type modular equations for prime as well as composite degrees. These were proved by Berndt(1998). Baruah (2003) also found three new equations for composite degrees. In this thesis, we use some Schläfli-type modular equations to evaluate some class invariants.

In Ramanujan's cubic theory of elliptic functions, or in the theory of signature 3, the theta-functions $a(q)$, $b(q)$, and $c(q)$, are defined by

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \quad b(q) = \sum_{m,n=-\infty}^{\infty} w^{m-n} q^{m^2+mn+n^2}, \quad (w = e^{2\pi i/3}),$$

and

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.$$

These functions were first introduced by J.M. and P.B. Borwein (1987). Similarly, in Ramanujan's quartic theory, or in the theory of signature 4, the theta-functions $A(q)$, $B(q)$, and $C(q)$, are defined as

$$A(q) = \phi^4(q) + 16q\psi^4(q^2), \ B(q) = \phi^4(q) - 16q\psi^4(q^2), \ \text{and} \ C(q) = 8\sqrt{q}\phi^2(q)\psi^2(q^2),$$

which were first introduced by Berndt, Bhargava and Garvan (1995). While proving the explicit values of $\phi(q)$ and $\psi(q)$, recorded by Ramanujan in his notebook, Berndt and Chan (1995), explicitly determined the value of cubic theta-function $a(e^{-2\pi})$, namely

$$\frac{a(e^{-2\pi})}{\phi^2(e^{-\pi})} = \frac{1}{(12)^{1/8}\sqrt{\sqrt{3}-1}},$$

where $\phi(e^{-\pi}) = \pi^{1/4}/\Gamma(\frac{3}{4})$ is well known. Berndt, Chan and Liaw (2001) evaluated some quotients of quartic theta-functions by using Weber-Ramanujan class invariants. In this thesis, we find some general formulas for the explicit evaluations of cubic and quartic theta-functions and their quotients. We also give some explicit values. In the process, we also establish several transformation formulas of theta-functions in cubic and quartic theory.

DECLARATION

I, Nipen Saikia, hereby declare that the subject matter in this thesis is the record of work done by me during my Ph.D course and that the contents of this thesis has not been submitted previously for any degree whatsoever by me or to the best of my knowledge to anyone else.

The thesis is being submitted to Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

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Certificate of the Supervisor

This is to certify that the thesis entitled **"EXPLICIT EVALAUTIONS OF RAMANUJAN'S CONTINUED FRACTIONS AND THETA-FUNCTIONS,"** submitted to Tezpur University in the **Department of Mathematical Sciences** under the **School of Science and Technology** in the partial fulfillment for the award of the degree of Doctor of Philosophy in **Mathematical Sciences** is a record of research work carried out by Mr. **NIPEN SAIKIA** under my personal supervision and guidance.

All helps received by him/ her from various sources have been duly acknowledged.

No part of this thesis has been reproduced elsewhere for award of any other degree.

Date: 05/09/2005

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Chapter 1

Introduction

1.1 Introduction

It is tacitly assumed throughout the thesis that $|q| < 1$ always. Also, as usual, for any complex number a, we define

$$(a;q)_0 = 1, \quad (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}) \text{ for } n \geq 1, \quad \text{and} \quad (a;q)_\infty := \prod_{k=1}^{\infty}(1-aq^{k-1}). \tag{1.1.1}$$

Now, Ramanujan's general theta-function $f(a,b)$ is given by

$$f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2}b^{k(k-1)/2}, \tag{1.1.2}$$

where $|ab| < 1$. If we set $a = qe^{2iz}$, $b = qe^{-2iz}$, and $q = e^{\pi i\tau}$, where z is complex and $\mathrm{Im}(\tau) > 0$, then$f(a,b) = \vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta-functions in its standard notation [65, p. 464].

We also define the following three special cases of $f(a,b)$:

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{1.1.3}$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.1.4}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.1.5}$$

If $q = e^{2\pi iz}$ with $\mathrm{Im}(z) > 0$, then $f(-q) = q^{-1/24}\eta(z)$, where $\eta(z)$ denotes the classical Dedekind eta-function.

The above theta-functions satisfy the following five transformation formulas. In these formulas it is assumed that α and β are such that the modulus of each exponential argument is less than 1.

Theorem 1.1.1. *(Berndt [15, p. 43, Entry 27 (i)]). If $\alpha\beta = \pi$ then*

$$\sqrt{\alpha}\phi(e^{-\alpha^2}) = \sqrt{\beta}\phi(e^{-\beta^2}).$$

Theorem 1.1.2. *(Berndt [15, p. 43, Entry 27 (ii)]). If $\alpha\beta = \pi$ then*

$$2\sqrt{\alpha}\psi(e^{-2\alpha^2}) = \sqrt{\beta}e^{\alpha^2/4}\phi(-e^{-\beta^2}).$$

Theorem 1.1.3. *(Berndt [15, p. 43, Entry 27 (iii)]). If $\alpha\beta = \pi^2$ then*

$$e^{-\alpha/12}\sqrt[4]{\alpha}f(-e^{-2\alpha}) = e^{-\beta/12}\sqrt[4]{\beta}f(-e^{-2\beta}).$$

Theorem 1.1.4. *(Berndt [15, p. 43, Entry 27 (iv)]). If $\alpha\beta = \pi^2$ then*

$$e^{-\alpha/24}\sqrt[4]{\alpha}f(e^{-\alpha}) = e^{-\beta/24}\sqrt[4]{\beta}f(e^{-\beta}).$$

Theorem 1.1.5. *(Adiga et al. [3]. If $\alpha\beta = \pi^2$ then*

$$e^{-\alpha/8}\sqrt[4]{\alpha}\psi(-e^{-\alpha}) = e^{-\beta/8}\sqrt[4]{\beta}\psi(-e^{-\beta}).$$

Ramanujan recorded several continued fractions and some of their explicit values in his second notebook [54] and his lost notebook [56]. Some of his continued fractions can be expressed in terms theta-functions. The best known continued fraction is the Rogers-Ramanujan continued fraction $R(q)$, defined by

$$R(q) := \frac{q^{1/5}}{1}\,\frac{q}{+1+}\,\frac{q^2}{1}\,\frac{q^3}{+1+\cdots}, \quad |q| < 1. \tag{1.1.6}$$

This continued fraction satisfy the following beautiful relations discovered by Ramanujan [15, p. 267] and proved by Watson [62]:

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \tag{1.1.7}$$

and

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{1.1.8}$$

In his notebooks [54], lost notebook [56] as well as in his first two letters to Hardy [30], Ramanujan also recorded several explicit values of $R(q)$ and $S(q) := -R(-q)$. We refer to a paper by

S.-Y. Kang [45], in which she recorded a table of all known values of the Rogers-Ramanujan's continued fraction up until the time of her paper was published in 1999. More recently, J. Yi [66, 68] found many values, including several new, of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ by using (1.1.7) and (1.1.8), and finding the explicit values of her new parameters $r_{k,n}$ and $r'_{k,n}$, defined by

$$r_{k,n} = \frac{f(-q)}{k^{1/4}q^{(k-1)/24}f(-q^k)}, \tag{1.1.9}$$

where $q = e^{-2\pi\sqrt{n/k}}$, and

$$r'_{k,n} = \frac{f(q)}{k^{1/4}q^{(k-1)/24}f(q^k)}, \tag{1.1.10}$$

where $q = e^{-\pi\sqrt{n/k}}$.

Motivated by her work, in Chapter 2 of this thesis, we use her method to find some general theorems for the explicit evaluations of Ramanujan's cubic continued fraction $G(q)$, defined by

$$G(q) = \frac{q^{1/3}}{1} \, \frac{}{+} \, \frac{q+q^2}{1} \, \frac{}{+} \, \frac{q^2+q^4}{1} \, \frac{}{+} \, \frac{q^3+q^6}{1} \, \frac{}{+} \, \cdots, \quad |q| < 1. \tag{1.1.11}$$

We do this by first defining several parameters of quotients of theta-functions $\phi(q)$, $\psi(q)$ and $f(q)$ for special values of q. For example, after K. G. Ramanathan [51], we define the parameter μ_n as

$$\mu_n = \frac{1}{3\sqrt{3}}\frac{f^6(-q)}{\sqrt{q}f^6(-q^3)}, \quad q = e^{-2\pi\sqrt{n/3}}. \tag{1.1.12}$$

From the definitions of $r_{k,n}$ and μ_n, it is to be noted that $r^6_{3,n}=\mu_n$. The modular transformation formula in Theorem 1.1.3 then implies that $\mu_{1/n} = 1/\mu_n$, and we evaluate many values of μ_n by appealing to theta-function identities, specializing the value of q and solving the resulting polynomial equations. This chapter is almost identical to our paper [12]. It is worthwhile to mention that Ramanujan recorded this continued fraction on page 366 of his lost notebook [56] and remarked that there are many results of $G(q)$, which are analogous to $R(q)$. Motivated by Ramanujan's remark, several results including explicit values were found by Chan [35], [26], Yi [66], N. D. Baruah [9], C. Adiga et al. [1, 3], and S. Bhargava et al. [32].

In the unorganized portions of his second notebook, Ramanujan [54] recorded, without proofs, 23 beautiful identities involving quotients of only eta-functions and no other theta-functions. The identities can be divided into two categories. In the first category, each identity involves four arguments and in the second category, each identity involves eight arguments. The first category identities have been used to find explicit values of the famous Rogers-Ramanujan continued fraction [22], Ramanujan's class invariants [29], a certain quotient of eta-functions [24]. These types of identities were also used to find explicit values of Ramanujan's cubic continued fraction in [1] and by us in Chapter 2. Unlike the first category the second category identities have not been applied before. In Chapter 3, we apply some new and old eta-function identities involving eight arguments to find some new values of the Rogers-Ramnujan continued fraction and the parameters μ_n and λ_n connected with Ramanujan's cubic continued fraction, where μ_n is defined in (1.1.12) and λ_n is defined by

$$\lambda_n = \frac{1}{3\sqrt{3}}\frac{f^6(q)}{\sqrt{q}f^6(q^3)}; \quad q = e^{-\pi\sqrt{n/3}}. \tag{1.1.13}$$

From the definitions of $r'_{k,n}$ and μ_n, we note that $r'^6_{3,n}=\lambda_n$. In fact, λ_n was defined by Ramanujan on page 212 of his lost notebook [56]. He also provided a list of eleven recorded values of λ_n and ten unrecorded values of λ_n. All 21 values of λ_n and several new were established by Berndt, Chan, Kang, and L.-C. Zhang [24]. Yi [66] also found several values of parameters λ_n and μ_n.

The new values of μ_n and λ_n evaluated by us also leads to some new Ramanujan-type series for $1/\pi$. In Sections 3.5-3.6 of Chapter 3 of this thesis, we show how the new values of μ_n and λ_n combined with some old and newly found modular equations in cubic theory can be applied to find some new series for $1/\pi$ by appealing to the formula established by J. M. Borwein and P. B. Borwein [33] and later modified by Chan and W.-C. Liaw [40]. This chapter is almost identical to our paper [13].

Another well-known continued fraction of Ramanujan is Ramanujan-Selberg continued fraction $Z(q)$ [15, p. 221, Entry 1(i)], defined by

$$Z(q) := \frac{q^{1/8}(-q^2;q^2)_\infty}{(-q;q^2)_\infty} = \frac{q^{1/8}}{1}\,\frac{q}{+\,1+q}\,\frac{q^2}{+\,1+q^2}\,\frac{q^3}{+\,1+q^3}\,\frac{q^4}{+\,1+q^4}\,{}_{+\cdots}\qquad |q|<1. \tag{1.1.14}$$

Ramanathan [48] also proved the above equality (1.1.14). If we define

$$T(q) := \frac{q^{1/8}}{1}\,\frac{-q}{+\,1}\,\frac{-q+q^2}{+\,1}\,\frac{-q^3}{+\,1}\,{}_{+\cdots},\quad |q|<1, \tag{1.1.15}$$

then $T^8(q) = -Z^8(-q)$, which is easily deducible from [70, equations (1.7) and (1.9)]. Zhang [70] also established general formulas for explicit evaluations of the continued fractions $Z(q)$ and $T(q)$ in terms of Ramanujan's singular modulus α_n, which is that unique positive number between 0 and 1 satisfying

$$\sqrt{n} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha_n\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha_n\right)},$$

where n is a positive rational number. Closely related to continued fraction $Z(q)$ is the continued fraction $H(q)$ [59, p. 82], defined by

$$H(q) := \frac{f(-q)}{q^{1/8} f(-q^4)} = q^{1/8} - \frac{q^{7/8}}{1-q}{}_{+}\frac{q^2}{1+q^2}{}_{-}\frac{q^3}{1-q^3}{}_{+}\frac{q^4}{1+q^4}{}_{-\dots}. \tag{1.1.16}$$

In Chapter 4, we establish several relations connecting the continued fractions $H(q)$ and $H(q^n)$, $Z(q)$ and $Z(q^n)$, and $H(\pm q)$, $Z(q)$, and $T(q)$ by using some transformation formulas and modular equations. It is obvious that by evaluating $H(q)$, we can easily evaluate $Z(q)$ and $T(q)$ also. Employing modular equations and modular transformation formulas, K. R. Vasuki and K. Shivashankar [59] found explicit values of $H(e^{-\pi\sqrt{n}})$ for $n = 3, 1/3, 5, 1/5, 7, 1/7, 13$ and $1/13$. In Section 4.7 of Chapter 4, we establish general formulas for finding the explicit values $H(e^{-\pi\sqrt{n}})$, for any positive real number n, in terms of the parameter J_n, defined as

$$J_n = \frac{f(-q)}{\sqrt{2} q^{1/8} f(-q^4)}; \quad q := e^{-\pi\sqrt{n}}, \tag{1.1.17}$$

We note here that $J_n = r_{4,n}$. We prove some general theorems for the explicit evaluation of J_n by appealing to Ramanujan's modular equations. We find some specific values of J_n to arrive at some new explicit values of $H(q)$. In addition, we prove formulas for the explicit evaluations of $Z(e^{-\pi\sqrt{n}})$ and $Z(e^{-\pi/\sqrt{n}})$ and present some examples.

Next, the Ramanujan-Göllnitz-Gordon continued fraction $K(q)$ is defined as

$$K(q) := \frac{q^{1/2}}{1+q}{}_{+}\frac{q^2}{1+q^3}{}_{+}\frac{q^4}{1+q^5}{}_{+\dots} \quad |q| < 1. \tag{1.1.18}$$

Chan and S.-S. Huang [37], derived many identities involving the continued fraction $K(q)$, which are analogous to $R(q)$ and $G(q)$. They also evaluated explicitly $K(e^{-\pi\sqrt{n}/2})$ for several positive integers n by using Weber-Ramanujan class invariants G_n and g_n, defined by

$$G_n := 2^{-1/4}q^{-1/24}(-q;q^2)_\infty, \qquad g_n := 2^{-1/4}q^{-1/24}(q;q^2)_\infty, \tag{1.1.19}$$

where $q = e^{-\pi\sqrt{n}}$. In Chapter 5, we establish formulas for the explicit evaluations of $K(e^{-\pi\sqrt{n}/2})$ and $K(e^{-\pi\sqrt{n}/4})$ by using parameterizations $h_{2,n}$ and $s_{4,n}$, respectively, where $h_{2,n}$, for $k = 2$ is a special case of $h_{k,n}$ defined in (1.1.21) below and $s_{4,n}$, for $k = 4$ is a special case of the parameter $s_{k,n}$, where $s_{k,n}$ is introduced by Berndt [18], and defined by

$$s_{k,n} = \frac{f(q)}{k^{1/4}q^{(k-1)/24}f(-(-1)^k q^k)}; \qquad q := e^{-\pi\sqrt{n/k}}. \tag{1.1.20}$$

By establishing some general formulas, we calculate several explicit values of the parameter $s_{4,n}$. Also, we evaluate some new values of the parameter $h_{2,n}$ by establishing new theta-function identities.

In his first notebook, Ramanujan [54, Vol. I, p. 248] recorded many elementary values of $\psi(q)$, $\phi(q)$, and $f(q)$. Particularly, he recorded $\psi(e^{-n\pi})$ for n=1, 2, 4, 8, 1/2, and 1/4, $\phi(e^{-n\pi})$ and $\phi(-e^{-n\pi})$ for n=1, 2, 4, 8, 1/2, and 1/4, and $f(-e^{-n\pi})$ for n=1,2,4, and 8. All these values were proved by B. C. Berndt [17, p. 325]. Ramanujan also recorded non-elementary values of $\phi(e^{-n\pi})$ for n= 3, 5, 7, 9, and 45. Berndt and H. H. Chan [20] found proofs for these. In [66], Yi also introduced the following two parameterizations $h_{k,n}$ and $h'_{k,n}$ along with $r_{k,n}$ and $r'_{k,n}$:

$$h_{k,n} := \frac{\phi(q)}{k^{1/4}\phi(q^k)}, \quad q = e^{-\pi\sqrt{n/k}}, \tag{1.1.21}$$

and

$$h'_{k,n} := \frac{\phi(-q)}{k^{1/4}\phi(-q^k)}, \quad q = e^{-2\pi\sqrt{n/k}}, \tag{1.1.22}$$

where k and n are positive real numbers. Employing modular transformation formulas Theorems 1.1.1, 1.1.3-1.1.4, and some theta-function identities, she evaluated several values of $\phi(q)$, $f(q)$ and their quotients. In particular, she evaluated $\phi(e^{-n\pi})$ for n=1, 2, 3, 4, 5, and 6 and $\phi(-e^{-n\pi})$ for n=1, 2, 4, 6, 8, 10, and 12, $f(-e^{-n\pi})$ for n=3, 5, 6, 7, 8, 10, 12, 1/3, and 2/3, and $f(e^{-n\pi})$ for n=1, 2, 3, 4, 5, 6, and 7. Motivated by her work, for any positive real numbers k and n, in Chapter 6, we define the parameters $g_{k,n}$ and $g'_{k,n}$ by

$$g_{k,n} := \frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)} \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.1 23}$$

and

$$g'_{k,n} := \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)}, \qquad q = e^{-\pi\sqrt{n/k}}. \tag{1.1.24}$$

We prove many properties of the parameterizations $g_{k,n}$ and $g'_{k,n}$ defined in (1.1.23) and (1.1.24) and show how they are connected to Yi's parameters $r_{k,n}$, $r'_{k,n}$, $h_{k,n}$, $h'_{k,n}$, and Weber-Ramanujan class-invariants G_n and g_n. By employing some old and newly established theta-function identities, we present some general theorems for the explicit evaluations of $g_{k,n}$, $g'_{k,n}$, $h_{k,n}$, and $h'_{k,n}$ and find several explicit values. We also offer explicit formulas for $\psi(e^{-n\pi})$ and $\psi(-e^{-n\pi})$ for positive real number n and deduce some explicit values. In addition, we provide formulas for the explicit evaluations of Rogers-Ramanujan continued fraction and Ramanujan'c cubic continued fraction in terms of the parameterizations $g_{k,n}$, $g'_{k,n}$, $h_{k,n}$, and $h'_{k,n}$, from which particular values can be determined. This chapter is almost identical to our paper [14].

There are many applications of Weber-Ramanujan class invariants G_n and g_n defined in (1.1.19). H. Weber [64], was motivated to calculate class invariant so that he could construct Hilbert class fields. On the other hand Ramanujan calculated class invariants to approximate π, and probably for the finding explicit values of Rogers-Ramanujan continued fractions, theta-functions, etc.. Berndt et al. utilized class invariants for the explicit evaluations of Ramanujan's cubic continued fraction, Rogers-Ramanujan continued fraction, theta-functions, and quotients of eta-functions λ_n etc. For details, we refer to [22], [25], [24], [26], and [28]. An account of this work can also be found in [17]. In his notebooks, Ramanujan recorded several Schlafli-type modular equations of prime as well as of composite degrees. Berndt [17] proved all these modular equations via modular form. Baruah [10], gave elementary proofs of seven of these equations and also found three new modular equations of the same nature. Also, Baruah [8], had used some of these modular equations of composite degrees, combined with the prime degree modular equations, recorded in [15, p. 231, 282, 315], to find class invariant G_n.

In Chapter 7, we use some Schläfli-type modular equations of composite as well as of prime degrees to find some new and old class invariants G_n and g_n.

In his famous paper [53], [55, p. 23-39], and on the pages 257-262 of his second notebook [54] Ramanujan gave a outline of of the theories of elliptic functions to alternative bases. The results in these theories were first proved by Berndt et al. [19] in 1995, who gave these an appellation, the theory of signature r (r =3, 4, 6). An account of this work may also be found in Berndt's book [17]. Some of the results in alternative theories were also previously examined by K. Venkatachalienger [61, p. 89-95] and Borweins [33, 34].

In classical theory, the theta-functions $\phi(q)$ and $\psi(q)$ play key roles. In cubic theory, or in the theory of signature of 3, the corresponding theta-functions are $a(q)$, $b(q)$, and $c(q)$, and are defined as follows:

For $w = \exp(2\pi i/3)$,

$$a(q) = \sum_{mn=-\infty}^{\infty} q^{m^2+mn+n^2}, \tag{1.1.25}$$

$$b(q) = \sum_{mn=-\infty}^{\infty} w^{m-n} q^{m^2+mn+n^2}, \tag{1.1.26}$$

and

$$c(q) = \sum_{mn=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}. \tag{1.1.27}$$

The functions defined in (1.1.25) - (1.1.27) are called cubic theta-functions, first introduced by Borweins [34].

In the theory of signature of 4 or in the quartic theory, taking place of $a(q)$, $b(q)$, and $c(q)$ in cubic theory are $A(q)$, $B(q)$, and $C(q)$ [23] and are defined, respectively, as

$$A(q) = \phi^4(q) + 16q\psi^4(q^2). \qquad B(q) = \phi^4(q) - 16q\psi^4(q^2), \tag{1.1.28}$$

and

$$C(q) = 8\sqrt{q}\phi^2(q)\psi^2(q^2), \tag{1.1.29}$$

where $\phi(q)$ and $\psi(q)$ are defined in (1.1.3) and (1.1.4), respectively.

Berndt and Chan [17, p. 328, Corollary 3], explicitly determined the value of cubic theta-function $a(e^{-2\pi})$, namely

$$\frac{a(e^{-2\pi})}{\phi^2(e^{-\pi})} = \frac{1}{(12)^{1/8}\sqrt{\sqrt{3}-1}},$$

where $\phi(e^{-\pi}) = \pi^{1/4}/\Gamma(\frac{3}{4})$ is well known. Also, Berndt et al. [23] evaluated some quotients of quartic theta-functions by using Weber-Ramanujan class invariants while deriving the series for $1/\pi$ associated with the theory of signature 4.

In our last chapter, we find some new explicit values of cubic and quartic theta-functions and their quotients by parameterizations. We establishing some general formulas for the explicit evaluations of these theta-functions and then find their special values. In the process, we also establish some transformation properties of theta-functions in cubic and quartic theory.

In the next two sections, we record all the values of the parameters $r_{k,n}$ and $r'_{k,n}$ evaluated by Yi [66], which we will use in this thesis. We also note that $r_{k,1} = 1$, $r_{k,1/n} = 1/r_{k,n}$, $r_{k,n} = r_{n,k}$, $r'_{k,1} = 1$, $r'_{k,1/n} = 1/r'_{k,n}$, and $r'_{k,n} = r'_{n,k}$.

1.2 Values of $r_{k,n}$

$$r_{1,1} = 1$$

$$r_{2,1} = 1$$

$$r_{2,2} = 2^{1/8}$$

$$r_{2,3} = \left(1+\sqrt{2}\right)^{1/6}$$

$$r_{2,4} = 2^{1/8}\left(1+\sqrt{2}\right)^{1/8}$$

$$r_{2,5} = \sqrt{\frac{1+\sqrt{5}}{2}}$$

$$r_{2,6} = 2^{1/24}\left(\sqrt{3}+1\right)^{1/4}$$

$$r_{2,7} = \left(\frac{\sqrt{2}+1+\sqrt{2\sqrt{2}-1}}{2}\right)^{1/2}$$

$$r_{2,8} = 2^{3/16}\left(1+\sqrt{2}\right)^{1/4}$$

$$r_{2,9} = \left(\sqrt{2}+\sqrt{3}\right)^{1/3}$$

$$r_{2,10} = \left(\frac{1}{2}\left(1+\sqrt{5}\right)\left(\sqrt{\sqrt{5}+1}+\sqrt{2}\right)\right)^{1/4}$$

$$r_{2,12} = \left(1+\sqrt{2}\right)^{5/24}\left(2\left(1+\sqrt{2}+\sqrt{6}\right)\right)^{1/8}$$

$$r_{2,16} = 2^{1/8}\left(1+\sqrt{2}\right)^{1/4}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8}$$

$$r_{2,18} = \frac{\left(1+\sqrt{3}\right)^{1/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}{2^{11/24}}$$

$$r_{2,20} = \frac{\left(1+\sqrt{5}\right)^{5/8}\left(2+3\sqrt{2}+\sqrt{5}\right)^{1/8}}{\sqrt{2}}$$

$$r_{2,27} = \left(1+\sqrt{2}\right)^{5/18}\left\{\sqrt{2}+\sqrt{2}\left(1+\sqrt{2}\right)^{1/3}+\left(1+\sqrt{2}\right)^{2/3}\right\}^{1/3}$$

$$r_{2,32} = 2^{3/16}\left(1+\sqrt{2}\right)^{1/4}\left(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4}\right)^{1/8}$$

$$r_{2,36} = \frac{\left\{2\left(1+35\sqrt{2}-28\sqrt{3}\right)\right\}^{1/8}}{\left(\sqrt{3}-\sqrt{2}\right)^{2/3}}$$

$$r_{2,49} = \frac{1+\sqrt{7+2\sqrt{14}}}{2\sqrt{2}}+\frac{\sqrt{\sqrt{14}+\sqrt{7+2\sqrt{14}}}}{2}$$

$$r_{2,50} = \frac{2^{5/8}}{5^{1/4}-1}$$

$$r_{2,72} = \frac{\left(\sqrt{2}+\sqrt{3}\right)^{1/3}\left(-\sqrt{2}+4+2\sqrt{3}+3^{3/4}\left(\sqrt{3}+1\right)\right)^{1/3}}{2^{13/48}\left(\sqrt{2}-1\right)^{5/12}}$$

$$r_{2,3/2} = \frac{\left(1+\sqrt{3}\right)^{1/4}}{2^{7/24}}$$

$$r_{2,5/2} = \frac{\left(\sqrt{\sqrt{5}+1}+\sqrt{2}\right)^{1/4}}{2^{1/4}}$$

$$r_{2,7/2} = \frac{\left(3+\sqrt{7}\right)^{1/4}}{2^{3/8}}$$

$$r_{2\,9/2} = \frac{\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}{2^{13/24}}$$

$$r_{2,25/2} = \frac{5^{1/4}+1}{2^{5/8}}$$

$$r_{2,27/2} = \frac{\left(1+\sqrt{3}\right)^{1/12}\left(1-\sqrt{3}+2^{2/3}\sqrt{3}\right)^{1/3}}{2^{3/8}\left(2^{1/3}-1\right)^{1/3}}$$

$$r_{2,63/2} = \frac{\left(7 - 2\sqrt{3} + \sqrt{21} + (3+\sqrt{3})\sqrt{3 + 16\sqrt{21} - 27\sqrt{7}}\right)^{1/3}}{2^{13/24}\left(\sqrt{3}-1\right)^{2/3}\left(3-\sqrt{7}\right)^{1/12}}$$

$$r_{2,9/4} = \frac{\left(-1 + 35\sqrt{2} + 28\sqrt{3}\right)^{1/8}}{2^{1/8}\left(\sqrt{2}+\sqrt{3}\right)^{1/3}}$$

$$r_{2,9/8} = \frac{2^{5/48}\left(\sqrt{2}-1\right)^{5/12}\left(\sqrt{3}-1\right)^{1/3}\left(\sqrt{3}-\sqrt{2}\right)^{1/3}}{\left(-1-\sqrt{2}+\sqrt{3}+3^{3/4}\sqrt{2}-\sqrt{6}\right)^{1/3}}$$

$$r_{3,3} = 3^{1/12}\left(3+2\sqrt{3}\right)^{1/12} = \frac{3^{1/8}\left(1+\sqrt{3}\right)^{1/6}}{2^{1/12}}$$

$$r_{3,4} = \sqrt{\frac{\sqrt{3}+1}{\sqrt{2}}}$$

$$r_{3,5} = \left(\frac{\sqrt{5}+1}{2}\right)^{5/6} = \left(\frac{11+5\sqrt{5}}{2}\right)^{1/6}$$

$$r_{3,7} = \left(\frac{\sqrt{3}+\sqrt{7}}{2\left(2-\sqrt{3}\right)}\right)^{1/4}$$

$$r_{3,8} = \left(\sqrt{2}+1\right)^{1/3}\left(\sqrt{2}+\sqrt{3}\right)^{1/4}$$

$$r_{3,9} = 3^{1/6}\left(1+2^{1/3}+2^{2/3}\right)^{1/3} = \frac{3^{1/6}}{\left(2^{1/3}-1\right)^{1/3}}$$

$$r_{3,18} = 3^{1/6}\left(1+\sqrt{2}\right)^{5/18}\left(2+\sqrt{2}\left(1+\sqrt{2}\right)^{1/3}+\left(1+\sqrt{2}\right)^{2/3}\right)^{1/3}$$

$$r_{3,25} = \frac{1}{2}\left(1+\sqrt[3]{10}+\sqrt{5+2\sqrt[3]{10}+\sqrt[3]{10^2}}\right)$$

$$r_{3,49} = \frac{3+\sqrt[3]{2^2}\sqrt[3]{7}+\sqrt[3]{2}\sqrt[3]{7^2}+\sqrt{49+13\sqrt[3]{2^2}\sqrt[3]{7}+8\sqrt[3]{2}\sqrt[3]{7^2}}}{2\sqrt{3}}$$

$$r_{4,4} = 2^{5/16}\left(1+\sqrt{2}\right)^{1/4}$$

$$r_{4,8} = 2^{1/4}\left(1+\sqrt{2}\right)^{3/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8}$$

$$r_{4,9} = \frac{1}{2}\left(1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}\right)$$

$$r_{5,2} = \left(2+\sqrt{5}\right)^{1/6}$$

$$r_{5,1/2} = \left(\sqrt{5}-2\right)^{1/6}$$

$$r_{5,1/3} = \left(\frac{-11+5\sqrt{5}}{2}\right)^{1/6}$$

$$r_{5,4} = \left(\frac{1+\sqrt{5}+\sqrt{2}+\sqrt{1+\sqrt{5}}}{2}\right)^{1/2}$$

$$r_{5,1/4} = \left(\frac{1+\sqrt{5}-\sqrt{2}+\sqrt{1+\sqrt{5}}}{2}\right)^{1/2}$$

$$r_{5,5} = \left(25+10\sqrt{5}\right)^{1/6} = \sqrt{\frac{5+\sqrt{5}}{2}}$$

$$r_{5,1/5} = \left(\frac{5-2\sqrt{5}}{25}\right)^{1/6}$$

$$r_{5,7} = \left(\frac{1}{216}\left(3\sqrt{5}+a_3+b_3+\sqrt{57+6\sqrt{5}(a_3+b_3)+a_3^2+b_3^2}\right)\right)^{1/2}$$

$$r_{5,1/7} = \left(\frac{1}{216}\left(-3\sqrt{5}-a_3-b_3+\sqrt{57+6\sqrt{5}(a_3+b_3)+a_3^2+b_3^2}\right)\right)^{1/2}$$

$$\textit{where} \quad a_3 = \left(54\sqrt{5}-6\sqrt{21}\right)^{1/3} \quad \textit{and} \quad b_3 = \left(54\sqrt{5}+6\sqrt{21}\right)^{1/3}$$

$$r_{5,8} = \left\{\frac{(3+\sqrt{5})\,(1+\sqrt{2})}{2}\right\}^{1/2} = \left(63+45\sqrt{2}+28\sqrt{5}+20\sqrt{10}\right)^{1/6}$$

$$r_{5,1/8} = \left\{\frac{(3-\sqrt{5})\,(1-\sqrt{2})}{2}\right\}^{1/2} = \left(-63+45\sqrt{2}+28\sqrt{5}-20\sqrt{10}\right)^{1/6}$$

$$r_{5,9} = \left(104+60\sqrt{3}+45\sqrt{5}+26\sqrt{15}\right)^{1/6}$$

$$r_{5,1/9} = \left(104-60\sqrt{3}+45\sqrt{5}-26\sqrt{15}\right)^{1/6}$$

$$r_{5,20} = \frac{\sqrt{5+\sqrt{5}}}{5^{1/4}-1}$$

$$r_{5,4/5} = \left\{\frac{2(3+2\ \ 5^{1/4})}{5(1+\sqrt{5})}\right\}^{1/4}$$

$$r_{6,6} = \frac{3^{1/8}\sqrt{\sqrt{3}+1}\left(1+\sqrt{3}+\sqrt{2}\ \ 3^{3/4}\right)^{1/3}}{2^{13/24}}$$

$$r_{13,3} = \frac{\sqrt{11+\sqrt{13}}+\sqrt{3+\sqrt{13}}}{2\sqrt{2}}$$

$$r_{13,9} = \frac{1}{4}\left(\left(\sqrt{3}+1\right)\left(\sqrt{3}+\sqrt{13}\right)+2\sqrt{\left(3+2\sqrt{3}\right)\left(4+\sqrt{13}\right)}\right)$$

$$r_{25,2} = \frac{1}{2}\left(a+b+\sqrt{a^2+b^2-\frac{2}{3}}\right)$$

$$r_{25,1/2} = \frac{1}{2}\left(a + b - \sqrt{a^2 + b^2 - \frac{2}{3}}\right),$$

where $a = \left(\sqrt{5} + \frac{\sqrt{30}}{9}\right)^{1/3}$ *and* $b = \left(\sqrt{5} - \frac{\sqrt{30}}{9}\right)^{1/3}$

$$r_{25,3} = \frac{1}{2}\left(1 + \sqrt[3]{10} + \sqrt{5 + 2\sqrt[3]{10} + \sqrt[3]{10^2}}\right)$$

$$r_{25,1/3} = \frac{1}{2}\left(-1 - \sqrt[3]{10} + \sqrt{5 + 2\sqrt[3]{10} + \sqrt[3]{10^2}}\right)$$

$$r_{25,4} = \frac{1}{2}\left(3 + \sqrt[4]{5} + \sqrt{5} + \sqrt[4]{5^3}\right) = \frac{\sqrt[4]{5} + 1}{\sqrt[4]{5} - 1}$$

$$r_{25,1/4} = \frac{1}{2}\left(3 - \sqrt[4]{5} + \sqrt{5} - \sqrt[4]{5^3}\right) = \frac{\sqrt[4]{5} - 1}{\sqrt[4]{5} + 1}$$

$$r_{25,7} = \frac{1}{6}\left(4\sqrt{5} + a_1 + b_1 + \sqrt{\left(4\sqrt{5} + a_1 + b_1\right)^2 - 36}\right)$$

$$r_{25,1/7} = \frac{1}{6}\left(4\sqrt{5} + a_1 + b_1 - \sqrt{\left(4\sqrt{5} + a_1 + b_1\right)^2 - 36}\right),$$

where $a_1 = \left(\frac{1}{2}\left(2251\sqrt{5} + 9\sqrt{105}\right)\right)^{1/3}$ *and* $b_1 = \left(\frac{1}{2}\left(2251\sqrt{5} - 9\sqrt{105}\right)\right)^{1/3}$

$$r_{25,9} = \frac{\sqrt[4]{60} + 2 - \sqrt{3} + \sqrt{5}}{\sqrt[4]{60} - 2 + \sqrt{3} - \sqrt{5}}$$

$$r_{25,1/9} = \frac{\sqrt[4]{60} - 2 + \sqrt{3} - \sqrt{5}}{\sqrt[4]{60} + 2 - \sqrt{3} + \sqrt{5}}$$

$$r_{25,16} = \frac{1}{4}\left(2 + \sqrt[4]{20}\right)\left(17 + 11\sqrt[4]{5} + 7\sqrt{5} + 5\sqrt[4]{5^3}\right)$$

$$r_{25,1/16} = \frac{1}{4}\left(2 - \sqrt[4]{20}\right)\left(17 - 11\sqrt[4]{5} + 7\sqrt{5} - 5\sqrt[4]{5^3}\right)$$

$$r_{25,49} = \frac{1}{8}\left(a_2 + \sqrt{5b_2} + \sqrt{\left(a_2 + 2\sqrt{5b_2}\right)^2 - 64}\right)$$

$$r_{25,1/49} = \frac{1}{8}\left(a_2 + \sqrt{5b_2} - \sqrt{\left(a_2 + 2\sqrt{5b_2}\right)^2 - 64}\right)$$

where $a_2 = 1497 + 651\sqrt{5} + 565\sqrt{7} + 247\sqrt{35}$

$b_2 = 437430 + 195566\sqrt{5} + 165333\sqrt{7} + 73917\sqrt{35}$

$$r_{3/2,\,3/2} = \frac{2^{11/24}3^{1/8}\left(1 + \sqrt{3}\right)^{1/6}}{\left(1 + \sqrt{3} + \sqrt{2}\ 3^{3/4}\right)}$$

$$r_{5/2,\,5/2} = \left\{\frac{5\left(1 + \sqrt{5}\right)}{\sqrt{2}\left(3 + 2\ 5^{1/4}\right)}\right\}^{1/4}$$

1.3 Values of $r'_{k,n}$

$$r'_{1,1} = 1$$

$$r'_{2,2} = 2^{5/16}\left(\sqrt{2}-1\right)^{1/4}$$

$$r'_{3,3} = \frac{3^{1/8}\left(\sqrt{3}-1\right)^{1/6}}{2^{1/12}}$$

$$r'_{3,5} = \left(\frac{1+\sqrt{5}}{2}\right)^{1/6}$$

$$r'_{3,7} = \left(\frac{2\left(2+\sqrt{3}\right)}{\sqrt{3}+\sqrt{7}}\right)^{1/4}$$

$$r'_{3,9} = 3^{1/6}$$

$$r'_{3,25} = \frac{1+\sqrt{5}}{2}$$

$$r'_{3,49} = \frac{\sqrt{3}+\sqrt{7}}{2}$$

$$r'_{4,4} = \frac{2^{9/16}}{\left(9\ 2^{1/4}+4\sqrt{2}-3\ 2^{3/4}\right)^{1/8}}$$

$$r'_{5,3} = \left(\frac{\sqrt{5}+1}{2}\right)^{1/6}$$

$$r'_{5,1/3} = \left(\frac{\sqrt{5}-1}{2}\right)^{1/6}$$

$$r'_{5,5} = \sqrt{\frac{5-\sqrt{5}}{2}}$$

$$r'_{5,7} = \left(2+\sqrt{5}\right)^{1/6}$$

$$r'_{5,1/7} = \left(\sqrt{5}-2\right)^{1/6}$$

$$r'_{5\ 9} = \left(104+60\sqrt{3}-45\sqrt{5}-26\sqrt{15}\right)^{1/6}$$

$$r'_{5\ 1/9} = \left(104-60\sqrt{3}-45\sqrt{5}+26\sqrt{15}\right)^{1/6}$$

$$r'_{6\ 6} = \frac{2^{11/16}3^{1/8}\left(\sqrt{2}-1\right)^{1/12}\left(\sqrt{3}+1\right)^{1/6}}{\left(2-3\sqrt{2}+3\ 3^{1/4}+3^{3/4}\right)^{1/3}}$$

$$r'_{13,3} = \frac{\sqrt{5+\sqrt{13}}+\sqrt{\sqrt{13}-3}}{2\sqrt{2}}$$

$$r'_{13\,9} = \frac{1}{4}\left(\left(\sqrt{3}+1\right)\left(\sqrt{13}-\sqrt{3}\right)+2\sqrt{\left(3+2\sqrt{3}\right)\left(4-\sqrt{13}\right)}\right)$$

$$r'_{25\,3} = \frac{\sqrt{5}+1}{2}$$

$$r'_{25,1/3} = \frac{\sqrt{5}-1}{2}$$

$$r'_{25,7} = \frac{1}{6}\left(2\sqrt{5}+a_4+b_4+\sqrt{\left(2\sqrt{5}+a_4+b_4\right)^2-36}\right)$$

$$r'_{25,1/7} = \frac{1}{6}\left(2\sqrt{5}+a_4+b_4-\sqrt{\left(2\sqrt{5}+a_4+b_4\right)^2-36}\right),$$

$$\textit{where} \quad a_4 = \left(\frac{1}{2}\left(17\sqrt{5}+3\sqrt{105}\right)\right)^{1/3} \quad \textit{and} \quad b_4 = \left(\frac{1}{2}\left(17\sqrt{5}-3\sqrt{105}\right)\right)^{1/3}$$

$$r'_{25,9} = \frac{\sqrt[4]{60}+2+\sqrt{3}-\sqrt{5}}{\sqrt[4]{60}-2-\sqrt{3}+\sqrt{5}}$$

$$r'_{25,1/9} = \frac{\sqrt[4]{60}-2-\sqrt{3}+\sqrt{5}}{\sqrt[4]{60}+2+\sqrt{3}-\sqrt{5}}$$

$$r'_{25,27} = 2+\sqrt{5}+\left(1+\sqrt[6]{5}\right)\left(20+9\sqrt{5}\right)^{1/3}$$

$$r'_{25,1/27} = -2+\sqrt{5}+\left(-20+9\sqrt{5}\right)^{1/3}-\frac{1}{2}\left(3+\sqrt{5}\right)\left(-20+9\sqrt{5}\right)^{2/3}$$

$$r'_{25,49} = \frac{1}{8}\left(a_5+2\sqrt{5b_5}+\sqrt{\left(a_5+2\sqrt{5b_5}\right)^2-64}\right)$$

$$r'_{25,1/49} = \frac{1}{8}\left(a_5+2\sqrt{5b_5}-\sqrt{\left(a_5+2\sqrt{5b_5}\right)^2-64}\right),$$

$$\textit{where} \quad a_5 = 1497-651\sqrt{5}+565\sqrt{7}-247\sqrt{35}$$

$$\textit{and} \quad b_5 = 437430-195566\sqrt{5}+165333\sqrt{7}-73917\sqrt{35}$$

Chapter 2

Some General Theorems on the Explicit Evaluations of Ramanujan's Cubic Continued Fraction

2.1 Introduction

From (1.1.11) recall the definition of Ramanujan's cubic continued fraction $G(q)$,

$$G(q) := \frac{q^{1/3}}{1} \, \frac{}{+} \, \frac{q+q^2}{1} \, \frac{}{+} \, \frac{q^2+q^4}{1} \, \frac{}{+} \, \frac{q^3+q^6}{1} \, \frac{}{+} \cdots, \tag{2.1.1}$$

where $|q| < 1$. This continued fraction was recorded by Ramanujan in his second letter to Hardy [30] and on page 366 of his lost notebook [56], and claimed that there are many results of $G(q)$ which are analogous to the famous Rogers-Ramanujan continued fraction $R(q)$, defined by

$$R(q) := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots, \quad |q| < 1.$$

Motivated by Ramanuajan's claims, Chan [35] proved three identities giving relations between $G(q)$ and the three continued fractions $G(-q)$, $G(q^2)$, and $G(q^3)$. Baruah [9] found two new identities giving relations between $G(q)$ and the two continued fractions $G(q^5)$ and $G(q^7)$. Chan [35] also found three reciprocity theorems for $G(q)$. He also evaluated $G(-e^{-\pi\sqrt{n}})$ for $n = 1$ and $n = 5$ and $G(e^{-\pi\sqrt{n}})$ for $n = 1, 2, 4$, and $2/9$. Berndt, Chan and Zhang [26] have found general formulas for $G(-e^{-\pi\sqrt{n}})$ and $G(e^{-\pi\sqrt{n}})$ in terms of Weber-Ramanujan class invariants

G_n and g_n, defined by

$$G_n := 2^{-1/4}q^{-1/24}(-q;q^2)_\infty \qquad \text{and} \qquad g_n := 2^{-1/4}q^{-1/24}(q;q^2)_\infty, \quad q = e^{-\pi\sqrt{n}}.$$

where $(a;q)_\infty := \Pi_{n=0}^{\infty}(1-aq^n), |q| < 1$.

They evaluated $G(-e^{-\pi\sqrt{n}})$ for $n = 1, 5, 13$, and 37 and $G(e^{-\pi\sqrt{n}})$ for $n = 2, 10, 22$, and 58. Ramanathan [47] has also found $G(e^{-\pi\sqrt{10}})$ by using Kronecker's limit formula. This value was recorded by Ramanujan on page 366 of his lost notebook [56]. By using modular equations and transformation formulas for theta-functions Adiga et al. [1] and [3], Vasuki and Shivashankara [59] have recently found $G(-e^{-\pi\sqrt{n}})$ for $n = 1/147, 1/75, 1/27, 1/13, 1/9, 1/7, 1/5, 1/3, 1, 3, 5$, and $25/3$, and $G(e^{-\pi\sqrt{n}})$ for $n = 1/3, 1, 4/3, 4, 16/3$, and 16. Other values of $G(q)$ can be found by using the reciprocity theorems given by Chan [35] and Adiga et al. [3].

In this chapter, we present some general theorems for evaluating $G(-e^{-\pi\sqrt{n}})$ and $G(e^{-\pi\sqrt{n}})$ by using modular equations and transformation formulas for theta-functions. Our theorems are motivated by Yi's paper [67], in which she evaluates many new explicit values $R(q)$.

Since modular equations are key in our evaluations of $G(q)$, so we give the definition of a modular equation. The complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^2}{(n!)^2}k^{2n} = \frac{\pi}{2}{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2), \tag{2.1.2}$$

where $0 < k < 1$, ${}_2F_1$ denotes the ordinary or Gaussian hypergeometric function and

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1).$$

The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L, and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{2.1.3}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is implied by (2.1.3).

If we set

$$q = exp\left(-\pi\frac{K'}{K}\right) \text{ and } q' = exp\left(-\pi\frac{L'}{L}\right), \tag{2.1.4}$$

we see that (2.1.3) is equivalent to the relation $q^n = q'$. Thus, a modular equation can be viewed as an identity involving theta-functions at the arguments q and q^n. Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α. The multiplier m connecting α and β is defined by

$$m = \frac{K}{L}, \tag{2.1.5}$$

where $z_r = \phi^2(q)$. Ramanujan also established many "mixed "modular equations in which four distinct moduli appear. We will define " mixed "modular equation in next chapter.

We shall make use some new and old eta-function and theta-function identities in our work. We record these results in next section for further reference. Proofs of the new identities are also given.

2.2 Modular equations

In this section, we state and prove some modular equations which will be used in in finding theorems for the explicit evaluations of $G(q)$.

Theorem 2.2.1. *(Berndt [16, p. 204, Entry 51]) If*

$$P = \frac{f^2(-q)}{q^{1/6}f^2(-q^3)} \quad and \quad Q = \frac{f^2(-q^2)}{q^{1/3}f^2(-q^6)},$$

then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3. \tag{2.2.1}$$

Theorem 2.2.2. *(Berndt [15, p. 246, Entry 1(iv)]) If*

$$P = \frac{f^3(-q)}{q^{1/4}f^3(-q^3)} \quad and \quad Q = \frac{f^3(-q^3)}{q^{3/4}f^3(-q^9)},$$

then

$$\left(1 + \frac{9}{PQ}\right)^3 = 1 + \frac{27}{P^4}. \tag{2.2.2}$$

Theorem 2.2.3. *(Berndt [16, p. 221, Entry 62]) If*

$$P = \frac{f(-q)}{q^{1/12}f(-q^3)} \quad and \quad Q = \frac{f(-q^5)}{q^{5/12}f(-q^{15})},$$

then

$$(PQ)^2 + 5 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3. \tag{2.2.3}$$

Theorem 2.2.4. *(Berndt [16, p. 236, Entry 69]) If*

$$P = \frac{f(-q)}{q^{1/12}f(-q^3)} \quad and \quad Q = \frac{f(-q^7)}{q^{7/12}f(-q^{21})},$$

then

$$(PQ)^3 + \frac{27}{(PQ)^3} = \left(\frac{Q}{P}\right)^4 - 7\left(\frac{Q}{P}\right)^2 + 7\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^4. \tag{2.2.4}$$

Theorem 2.2.5. *(Berndt [17, p. 127]) If*

$$P = \frac{f(-q)}{q^{1/12}f(-q^3)} \quad and \quad Q = \frac{f(-q^{11})}{q^{11/12}f(-q^{33})},$$

then

$$(PQ)^5 + \left(\frac{3}{PQ}\right)^5 + 11\left\{(PQ)^4 + \left(\frac{3}{PQ}\right)^4\right\} + 66\left\{(PQ)^3 + \left(\frac{3}{PQ}\right)^3\right\}$$

$$+253\left\{(PQ)^2 + \left(\frac{3}{PQ}\right)^2\right\} + 693\left\{PQ + \frac{3}{PQ}\right\} + 1386 = \left(\frac{Q}{P}\right)^6 + \left(\frac{P}{Q}\right)^6. \tag{2.2.5}$$

Theorem 2.2.6. *(Berndt [16, p. 210, Entry 56]) If*

$$P = \frac{f(-q^{1/3})}{q^{1/9}f(-q^3)} \quad and \quad Q = \frac{f(-q^{2/3})}{q^{2/9}f(-q^6)},$$

then

$$P^3 + Q^3 = P^2Q^2 + 3PQ. \tag{2.2.6}$$

Theorem 2.2.7. *If*

$$P = \frac{f(-q^{1/3})}{q^{1/9}f(-q^3)} \quad and \quad Q = \frac{f(-q)}{q^{2/9}f(-q^9)},$$

then

$$(PQ)^3 + \left(\frac{9}{PQ}\right)^3 + 27\left(\left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3\right) + 9(P^3 + Q^3)$$

$$+243\left(\frac{1}{P^3} + \frac{1}{Q^3}\right) + 81 = \left(\frac{Q}{P}\right)^6. \tag{2.2.7}$$

Proof. This easily follows from Theorem 2.2.2. □

Theorem 2.2.8. *If*

$$P = \frac{\phi(q^{1/3})}{\phi(q^3)} \quad \text{and} \quad Q = \frac{\phi(q)}{\phi(q^9)},$$

then

$$PQ + \frac{9}{PQ} + 3\left(\frac{P}{Q} + \frac{Q}{P}\right) - 9\left(\frac{1}{P} + \frac{1}{Q}\right) - 3(P+Q) + 9 = \left(\frac{Q}{P}\right)^2. \tag{2.2.8}$$

Proof. We use the first two identities of Entry 1 (iii) [15, p. 345]. □

Theorem 2.2.9. *If*

$$P = \frac{\phi(q^{1/3})}{\phi(q^3)} \quad \text{and} \quad Q = \frac{\phi(q^{5/3})}{\phi(q^{15})},$$

then

$$\left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3 + 15\left(\frac{P^2}{Q} + \frac{Q^2}{P} + 3\frac{P}{Q^2} + 3\frac{Q}{P^2}\right) + 5(P+Q)(6+PQ)$$

$$+45\left(\frac{1}{P} + \frac{1}{Q}\right)\left(2 + \frac{3}{PQ}\right) = (PQ)^2 + \frac{81}{(PQ)^2} + 10(P+Q)^2 + 90\left(\frac{1}{P} + \frac{1}{Q}\right)^2$$

$$+15\left(\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right) + 45\left(\frac{P}{Q} + \frac{Q}{P}\right) + 40. \tag{2.2.9}$$

Proof. We use the first two identities of Entry 1 (iii) [15, p. 345] and Remark 1 of Theorem 2.1 in [9, p. 245, 247]. □

Theorem 2.2.10. *If*

$$P = \frac{\psi(-q^{1/3})}{q^{1/3}\psi(-q^3)} \quad \text{and} \quad Q = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)},$$

then

$$PQ + \frac{9}{PQ} + 3\left(\frac{P}{Q} + \frac{Q}{P}\right) + 9\left(\frac{1}{P} + \frac{1}{Q}\right) + 3(P+Q) + 9 = \left(\frac{Q}{P}\right)^2. \tag{2.2.10}$$

Proof. We use the first and last identities of Entry 1 (ii) [15, p. 345]. □

Theorem 2.2.11. *(Baruah[9, p. 253]) If*

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \text{and} \quad Q = \frac{\psi(q^5)}{q^{5/4}\psi(q^{15})},$$

then

$$(PQ)^4 + \frac{81}{(PQ)^4} + 15\left(\left(\frac{Q}{P}\right)^4 + \left(\frac{P}{Q}\right)^4\right) + 120 - 10(P^4 + Q^4)$$

$$-90\left(\frac{1}{P^4} + \frac{1}{Q^4}\right) = \left(\frac{P}{Q}\right)^2\left(\left(\frac{Q}{P}\right)^8 + \left(\frac{P}{Q}\right)^4 + 15\left(\frac{Q}{P}\right)^4 + 15\right) \tag{2.2.11}$$

Theorem 2.2.12. *(Baruah [9, p. 250]) If*

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \text{and} \quad Q = \frac{\psi(q^7)}{q^{7/4}\psi(q^{21})},$$

then

$$k_1(PQ)^3 + k_2(PQ) = k_3(PQ)^2 + k_4\left(\frac{P}{Q}\right)^2 - k_5, \tag{2.2.12}$$

where

$$k_1 = \left(\frac{P}{Q}\right)^8 - 1, \quad k_2 = 14P^4\left(\left(\frac{P}{Q}\right)^4 - 1\right), \quad k_3 = P^4(7 - P^4),$$

$$k_4 = 7P^4(P^4 - 3), \quad \text{and} \quad k_5 = 27\left(\frac{P}{Q}\right)^4 - 7P^4\left(3 + 3\left(\frac{P}{Q}\right)^4 - P^4\right).$$

2.3 Explicit values of $G(q)$

Theorem 2.3.1. *We have*

(i) *For* $q = e^{-\pi\sqrt{n/3}}$, *let*

$$\lambda_n = \frac{1}{3\sqrt{3}}\frac{f^6(q)}{\sqrt{q}f^6(q^3)}.$$

Then

$$3\left(1 - \lambda_n^2\right)^{1/3} = 4w^2 + \frac{1}{w},$$

where $w = G(-q)$.

(ii) *For* $q = e^{-2\pi\sqrt{n/3}}$, *let*

$$\mu_n = \frac{1}{3\sqrt{3}}\frac{f^6(-q)}{\sqrt{q}f^6(-q^3)}.$$

Then

$$3\left(1 + \mu_n^2\right)^{1/3} = 4v^2 + \frac{1}{v},$$

where $v = G(q)$.

Proof. We use the first identity of Entry 1(iv) [15, p. 345]. □

Several values of λ_n were recorded by Ramanujan on page 212 of his lost notebook [56] All of those values were proved by Berndt et al [24]. They also evaluated many new values by using modular j-invariants, Weber-Ramanujan class invariants, modular equations, Kronecker's limit formula, and an empirical process. Thus, one can use Theorem 2.3.1 to find the values of $G(-e^{-\pi\sqrt{n/3}})$ and $G(e^{-\pi\sqrt{n/3}})$ if the corresponding values of μ_n and λ_n are known.

Theorem 2.3.2. *If μ_n and λ_n are as defined in Theorem 2.3.1, then*

$$\mu_{1/n} = \frac{1}{\mu_n} \quad and \quad \lambda_{1/n} = \frac{1}{\lambda_n}.$$

Proof. We use the definitions of μ_n and λ_n, and Theorems 1.1.3 and 1.1.4, respectively. □

Corollary 2.3.3. $\mu_1 = 1$ *and* $\lambda_1 = 1$.

Theorem 2.3.4. *If μ_n is as defined in Theorem 2.3.1, then*

(i) $3\left((\mu_n\mu_{4n})^{1/3} + \dfrac{1}{(\mu_n\mu_{4n})^{1/3}}\right) = \dfrac{\mu_n}{\mu_{4n}} + \dfrac{\mu_{4n}}{\mu_n}$,

(ii) $3\left(1 + \left(\dfrac{3}{\mu_n\mu_{9n}}\right)^{1/2}\right)^3 = 1 + \dfrac{1}{\mu_n^2}$,

(iii) $3\left((\mu_n\mu_{25n})^{1/3} + \dfrac{1}{(\mu_n\mu_{25n})^{1/3}}\right) + 5 = \left(\dfrac{\mu_{25n}}{\mu_n}\right)^{1/2} - \left(\dfrac{\mu_n}{\mu_{25n}}\right)^{1/2}$,

(iv) $3\sqrt{3}\left((\mu_n\mu_{49n})^{1/2} + \dfrac{1}{(\mu_n\mu_{49n})^{1/2}}\right) = \left(\dfrac{\mu_{49n}}{\mu_n}\right)^{2/3} - \left(\dfrac{\mu_n}{\mu_{49n}}\right)^{2/3} - 7\left(\dfrac{\mu_{49n}}{\mu_n}\right)^{1/3} + 7\left(\dfrac{\mu_n}{\mu_{49n}}\right)^{1/3}$,

(v) $9\sqrt{3}\left((\mu_n\mu_{121n})^{5/6} + \dfrac{1}{(\mu_n\mu_{121n})^{5/6}}\right) + 99\left((\mu_n\mu_{121n})^{2/3} + \dfrac{1}{(\mu_n\mu_{121n})^{2/3}}\right)$
$+ 198\sqrt{3}\left((\mu_n\mu_{121n})^{1/2} + \dfrac{1}{(\mu_n\mu_{121n})^{1/2}}\right) + 759\left((\mu_n\mu_{121n})^{1/3} + \dfrac{1}{(\mu_n\mu_{121n})^{1/3}}\right)$
$+ 693\sqrt{3}\left((\mu_n\mu_{121n})^{1/6} + \dfrac{1}{(\mu_n\mu_{121n})^{1/6}}\right) + 1386 = \left(\dfrac{\mu_{121n}}{\mu_n}\right) + \left(\dfrac{\mu_n}{\mu_{121n}}\right)$.

Proof. The theorem follows from the definition of μ_n and Theorems 2.2.1-2.2.5. Theorem 2.3.4 (i)-(iv) were also found by Yi [66]. □

Theorem 2.3.5. *We have*

(i) $\mu_2 = \sqrt{2} + 1$,

(ii) $\mu_4 = \dfrac{3\sqrt{3} + 5}{\sqrt{2}}$,

(iii) $\mu_3 = \sqrt{6\sqrt{3} + 9}$,

(iv) $\mu_9 = \dfrac{3}{(\sqrt[3]{2} - 1)^2}$,

(v) $\mu_5 = \dfrac{11 + 5\sqrt{5}}{2}$,

(vi) $\mu_{25} = \dfrac{1}{2^6}\left(1 + \sqrt[3]{10} + \sqrt{5 + 2\sqrt[3]{10} + \sqrt[3]{10^2}}\right)^6$,

(vii) $\mu_7 = \left(\dfrac{3 + 2\sqrt{3} + 2\sqrt{7} + \sqrt{21}}{2}\right)^{3/2}$,

(viii) $\mu_{49} = \left(\frac{a+\sqrt{a^2-4}}{2}\right)^3$ *where* $a = \frac{43}{3} + \frac{13}{3}\sqrt[3]{4}\sqrt[3]{7} + \frac{8}{3}\sqrt[3]{2}\sqrt[3]{49}$,

(ix) $\mu_{11} = \left(1551 + 900\sqrt{3} + 470\sqrt{11} + 270\sqrt{33}\right)^{1/2}$.

Proof. We set $n = 1/2$ and 1, $n = 1/3$ and 1, $n = 1/5$ and 1, $n = 1/7$ and 1, and $n = 1/11$, in Theorems 2.3.4 (i)-(v), respectively. We obtain the results by appealing to Theorem 2.3.2 and Corollary 2.3.3, and then solving the resulting polynomial equations.

The values of $\mu_{1/n}$ for n =2, 3, 4, 5, 7, 9, 25, and 49 can easily be found by applying Theorems 2.3.2 and 2.3.5. □

Theorems 2.3.5 (i)-(viii) can also be found in [66].

Theorem 2.3.6. *We have*

(i) *For* $q = e^{-2\pi\sqrt{n}}$, *let*

$$C_n = \frac{f(-q^{1/3})}{\sqrt{3}q^{1/9}f(-q^3)}.$$

Then

$$3 + 3\sqrt{3}C_n^3 = 4v^2 + \frac{1}{v},$$

where $v = G(q)$.

(ii) *For* $q = e^{-\pi\sqrt{n}}$, *let*

$$D_n = \frac{f(q^{1/3})}{\sqrt{3}q^{1/9}f(q^3)}.$$

Then

$$3 - 3\sqrt{3}D_n^3 = 4w^2 + \frac{1}{w},$$

where $v = G(-q)$.

Proof We use Entry 1 (iv) [15, p. 345]. □

Theorem 2.3.7. *If* C_n *and* D_n *are as defined in Theorem 2.3.6, then*

$$C_{1/n} = \frac{1}{C_n} \quad and \quad D_{1/n} = \frac{1}{D_n}.$$

Proof. We use the definitions of C_n and D_n, and then Theorem 1.1 3 and Theorem 1 1.4. □

Corollary 2.3.8. $C_1 = 1$ *and* $D_1 = 1$.

Theorem 2.3.9. *If C_n and D_n are as defined in Theorem 2.3.6, then*

(i) $C_n^3 + C_{4n}^3 = \sqrt{3}C_nC_{4n}(C_nC_{4n} + 1)$,

(ii) $C_n^3 + D_n^3 = \sqrt{3}C_nD_n(C_nD_n - 1)$,

(iii) $$(C_nC_{9n})^3 + \frac{1}{(C_nC_{9n})^3} + \left(\frac{C_n}{C_{9n}}\right)^3 + \left(\frac{C_{9n}}{C_n}\right)^3 + \sqrt{3}\left(C_n^3 + C_{9n}^3 + \frac{1}{C_n^3} + \frac{1}{C_{9n}^3}\right) + 3 = \left(\frac{C_{9n}}{\sqrt{3}C_n}\right),$$

(iv) $$(D_nD_{9n})^3 + \frac{1}{(D_nD_{9n})^3} + \left(\frac{D_n}{D_{9n}}\right)^3 + \left(\frac{C_{9n}}{C_n}\right)^3 - \sqrt{3}\left(D_n^3 + D_{9n}^3 + \frac{1}{D_n^3} + \frac{1}{D_{9n}^3}\right) + 3 = \left(\frac{D_{9n}}{\sqrt{3}D_n}\right).$$

Proof. We use the definitions of C_n and D_n and Theorems 2.2.6 and 2.2.7. □

Theorem 2.3.10. *We have*

(i) $C_2 = (\sqrt{3} + \sqrt{2})^{1/3}$,

(ii) $D_2 = \dfrac{\sqrt{3}a^3 - \sqrt{2} - \sqrt{3}}{\sqrt{2}a}$ *where* $a = (\sqrt{2} + \sqrt{3})^{1/3}$

(iii) $C_4 = \dfrac{\sqrt{3} + 1 + \sqrt{2\sqrt{3}}}{2}$

(iv) $$D_4 = \frac{2\sqrt{30 + 15\sqrt{2\sqrt{3}} + 16\sqrt{3} + 9\sqrt{2\sqrt{27}}} - 4 - 3\sqrt{2\sqrt{3}} - 2\sqrt{3} - \sqrt{2\sqrt{27}}}{4},$$

(v) $C_3 = \left(\sqrt{3}(1 + 2^{1/3} + 2^{2/3})\right)$,

(vi) $D_3 = 3^{1/6}$,

(vii) $C_9 = \left(3\left(6 + 3\sqrt{3} + (738 + 426\sqrt{3})^{1/3} + (776 + 448\sqrt{3})^{1/3}\right)\right)^{1/3}$,

(viii) $D_9 = \left(3\left(6 - 3\sqrt{3} + (738 - 426\sqrt{3})^{1/3} + (776 - 448\sqrt{3})^{1/3}\right)\right)^{1/3}$.

Proof. Putting $n = 1/2$ and 1 in Theorem 2.3.9(i) and then solving the polynomial equations we obtain C_2 and C_4. Again setting n =1/2, 1 in Theorem 2.3.9(ii)and then solving the polynomial equations we obtain D_2 and D_4. Setting n =1/3, 1 in Theorem 2.3.9(iii) and then again solving the resulting polynomial equations we obtain C_3, D_3, C_9, and D_9. □

The values of $C_{1/n}$ and $D_{1/n}$ for n =2, 3, 4, and 9 can easily be calculated by applying Theorems 2.3 7 and 2.3.10.

Theorem 2.3.11. *For $q = e^{-\pi n}$, let*

$$S_n = \frac{\phi(q^{1/3})}{\sqrt{3}\phi(q^3)}.$$

Then

$$2G(-q) = 1 - \sqrt{3}S_n.$$

Proof. We use Entry 1 (ii) [15, p. 345]. □

Theorem 2.3.12. *If S_n is as defined in Theorem 2.3.11, then*

$$S_{1/n} = \frac{1}{S_n}.$$

Proof. We use Theorem 1.1.1 and the definition of S_n. □

Corollary 2.3.13. $S_1 = 1$.

Theorem 2.3.14. *If S_n is as defined in Theorem 2.3.11 then*

(i) $$3\left(S_nS_{3n} + \frac{1}{S_nS_{3n}}\right) + 3\left(\frac{S_n}{S_{3n}} + \frac{S_{3n}}{S_n}\right) - 3\sqrt{3}\left(\frac{1}{S_n} + \frac{1}{S_{3n}} + S_n + S_{3n}\right) + 9 = \left(\frac{S_{3n}}{S_n}\right)^2,$$

and

(ii) $$\left(\frac{S_n}{S_{5n}}\right)^3 + \left(\frac{S_{5n}}{S_n}\right)^3 + 15\sqrt{3}\left(S_n^2 + \frac{1}{S_n^2}\right)\left(S_{5n} + \frac{1}{S_{5n}} - 10\right) + 15\sqrt{3}$$
$$\times\left(S_n + \frac{1}{S_n}\right)\left(S_{5n}^2 + \frac{1}{S_{5n}^2} + 2\right) + 60\left(S_nS_{5n} + \frac{1}{S_nS_{5n}}\right) = 15\left(\left(\frac{S_n}{S_{5n}}\right)^2 + \left(\frac{S_{5n}}{S_n}\right)^2\right)$$
$$+9\left(S_n^2S_{5n}^2 + \frac{1}{S_n^2S_{5n}^2}\right) + 45\left(\frac{S_n}{S_{5n}} + \frac{S_{5n}}{S_n}\right) + 30\left(S_{5n}^2 + \frac{1}{S_{5n}^2}\right) + 30\sqrt{3}\left(S_{5n} + \frac{1}{S_{5n}}\right) + 40.$$

Proof. We use the definition of S_n and Theorems 2.2.8 and 2.2.9. □

Theorem 2.3.15. *We have*

(i) $S_3 = 2 - \sqrt{3} - \dfrac{2(-5+3\sqrt{3})}{a^{1/3}} + a^{1/3}$, *where* $a = 8\left(7 - 4\sqrt{3}\right)$,

(ii) $$S_5 = \frac{\left(28 - 15\sqrt{3} + 7\sqrt{15} - 12\sqrt{5} + \sqrt{40530 - 23400\sqrt{3} - 18138\sqrt{5} + 10472\sqrt{15}}\right)}{2(2-\sqrt{3})}.$$

Proof. We set $n = 1$ in above theorem and then solve the resulting polynomial equations to obtain the results. □

The values of $S_{1/3}$ and $S_{1/5}$ follow from Theorems 2.3.12 and 2.3.15.

Theorem 2.3.16. *For $q = e^{-\pi\sqrt{n}}$, let*

$$L_n = \frac{\psi(-q^{1/3})}{\sqrt{3}q^{1/3}\psi(-q^3)}.$$

Then

$$-G(-q) = \frac{1}{1+\sqrt{3}L_n}.$$

Proof. We use Entry 1 (i) [15, p. 345] □

It is clear from the above theorem that to evaluate $-G(-e^{-\pi\sqrt{n}})$, we need the value of L_n.

Theorem 2.3.17. *If L_n is as defined in Theorem 2.3.16, then*

$$L_{1/n} = \frac{1}{L_n}.$$

Proof. We use Theorem 1.1.5 and the definition of L_n. □

Corollary 2.3.18. $L_1 = 1$.

Theorem 2.3.19. *If L_n is as defined in Theorem 2.3.16, then*

$$3\left(L_nL_{9n} + \frac{1}{L_nL_{9n}}\right) + 3\left(\frac{L_n}{L_{9n}} + \frac{L_{9n}}{L_n}\right) + 3\sqrt{3}\left(\frac{1}{L_n} + \frac{1}{L_{9n}} + L_n + L_{9n}\right) + 9 = \left(\frac{L_{9n}}{L_n}\right)^2.$$

Proof. We use the definition of L_n and Theorem 2.2.10. □

Theorem 2.3.20. *We have*

(i) $L_3 = \dfrac{1}{\sqrt{3}} + \dfrac{2\cdot 2^{1/3}}{\sqrt{3}} + \dfrac{2^{2/3}}{\sqrt{3}}$,

(ii) $L_9 = 2 + \sqrt{3} + (38 + 22\sqrt{3})^{1/3} + 2(2 + \sqrt{3})^{2/3}$.

Proof. Setting $n = 1/3$ and $n = 1$ in the above theorem and then solving the resulting polynomial equations, we obtain the results. □

The values of $L_{1/3}$ and $L_{1/9}$ follow from Theorems 2.3.17 and 2.3.20.

Theorem 2.3.21. *For $q = e^{-\pi\sqrt{n/3}}$, let*

$$B_n = \frac{\psi^4(-q)}{3q\psi^4(-q^3)}.$$

Then

$$-G^3(-q) = \frac{1}{1 + 3B_n}.$$

Proof. We use Entry 1 (i) [15, p. 345]. □

Theorem 2.3.22. *If B_n is as defined in Theorem 2.3.21, then*

$$B_{1/n} = \frac{1}{B_n}.$$

Proof. We use Theorem 1.1.5 and the definition of B_n. □

Corollary 2.3.23. $B_1 = 1$.

Theorem 2.3.24. *If B_n is as defined in Theorem 2.3.21, then*

(i) $$\sqrt{3}\left((B_nB_{9n})^{1/4}+\frac{1}{(B_nB_{9n})^{1/4}}\right)+3=\left(\frac{B_{9n}}{B_n}\right)^{1/2},$$

(ii) $$9\left(B_nB_{25n}+\frac{1}{B_nB_{25n}}\right)+15\left(\frac{B_n}{B_{25n}}+\frac{B_{25n}}{B_n}\right)+30\left(\frac{1}{B_n}+\frac{1}{B_{25n}}+B_n+B_{25n}\right)+120$$
$$=\left(\frac{B_{25n}}{B_n}\right)^{3/2}+\left(\frac{B_n}{B_{25n}}\right)^{3/2}+15\left(\left(\frac{B_{25n}}{B_n}\right)^{1/2}+\left(\frac{B_n}{B_{25n}}\right)^{1/2}\right),$$

(iii) $$a_1(B_nB_{49n})^{3/4}-a_2(B_nB_{49n})^{1/4}+a_3(B_nB_{49n})^{1/2}+a_4\left(\frac{B_n}{B_{49n}}\right)^{1/2}+a_5=0,$$

where $$a_1=\left(\frac{B_n}{B_{49n}}\right)^2-1,\quad a_2=14B_n\left(\frac{B_n}{B_{49n}}-1\right),\quad a_3=\sqrt{3}B_n(7+3B_n),$$
$$a_4=7\sqrt{3}B_n(B_n+1),\quad \textit{and}\quad a_5=3\sqrt{3}\frac{B_n}{B_{49n}}+7\sqrt{3}B_n\left(\frac{B_n}{B_{49n}}+B_n+1\right).$$

Proof. We replace q by $-q$ in Theorem 2.2.11 and 2.2.12 and use the definition of B_n. □

Theorem 2.3.25. *We have*

(i) $B_3=\sqrt{3}(2+\sqrt{3})$,

(ii) $B_9=\dfrac{(1+\sqrt[3]{2})^2}{\sqrt{3}}$,

(iii) $B_5=9+4\sqrt{5}$,

(iv) $B_{25}=\left(\dfrac{2(a-17)}{a+b}\right)^2$ *where* $a=(5761+\sqrt{421121})^{1/3}$ *and* $b=\sqrt{68-4a+a^2}$,

(v) $B_7=\dfrac{1}{9-6\sqrt{3}+2\sqrt{49-28\sqrt{3}}}$.

Proof. Setting $n=1/3$ and 1, $n=1/5$ and 1, and $n=1/7$, in Theorem 2.3.24 (i), (ii), and (iii), respectively, using Theorem 2.3.22 and Corollary 2.3.23, and solving the resulting polynomial equations, we obtain the results. □

The values of $B_{1/n}$ for $n=3$, 5, 7, 9, and 25 follow from Theorems 2.3.22 and 2.3.25.

Remark 2.3.1. (i) Theorem 2.3.4 implies that if we know μ_n, then we can evaluate μ_{4n}, $\mu_{n/4}$, μ_{9n}, $\mu_{n/9}$, μ_{25n}, $\mu_{n/25}$, μ_{49n}, $\mu_{n/49}$, μ_{121n}, or $\mu_{n/121}$. Thus, by Theorem 2.3.1(ii), if we know $G(e^{-2\pi\sqrt{n/3}})$ then we can also evaluate $G(e^{-4\pi\sqrt{n/3}})$, $G(e^{-\pi\sqrt{n/3}})$, $G(e^{-2\pi\sqrt{3n}})$, $G(e^{-2\pi\sqrt{n/27}})$, $G(e^{-10\pi\sqrt{n/3}})$, $G(e^{-2\pi\sqrt{n/75}})$, $G(e^{-14\pi\sqrt{n/3}})$, $G(e^{-2\pi\sqrt{n/147}})$, $G(e^{-22\pi\sqrt{n/3}})$, or $G(e^{-2\pi\sqrt{n/363}})$.

(ii) Using cubic Russell-type modular equations of degrees $p=13$, 17, 19, 23, 29, 41, 47, 53, and 59, derived by Chan and Liaw [39] and Liaw [46] (see also [30]), one can also find relations connecting μ_n and μ_{p^2n}.

(iii)Theorem 2.3.9 implies that if we know C_n, then we can evaluate C_{4n}, $C_{n/4}$, D_n, D_{4n}, $D_{n/4}$, C_{9n}, $C_{n/9}$, D_{9n}, or $D_{n/9}$. So using Theorem 2.3.4, if we know $G(e^{-2\pi\sqrt{n}})$, then we can evaluate $G(e^{-4\pi\sqrt{n}})$, $G(e^{-\pi\sqrt{n}})$, $G(-e^{-\pi\sqrt{n}})$, $G(-e^{-2\pi\sqrt{n}})$, $G(-e^{-2\pi\sqrt{n}/2})$, $G(e^{-6\pi\sqrt{n}})$, $G(e^{-2\pi\sqrt{n/9}})$, $G(-e^{-3\pi\sqrt{n}})$, or $G(-e^{-\pi\sqrt{n/9}})$.

(iv)Theorem 2.3.14 implies that if we know that S_n, then we can evaluate S_{3n}, $S_{n/3}$, S_{5n}, or $S_{n/5}$. Thus, by Theorem 2.3.11, if we know $G(-e^{-\pi n})$, then we can evaluate $G(-e^{-3\pi n})$, $G(-e^{-\pi n/3})$, $G(-e^{-5\pi n})$, or $G(-e^{-\pi n/5})$.

(v) Theorem 2.3.19 implies that if we know L_n, then we can compute L_{9n} or $L_{n/9}$, that is by Theorem 2.3.16, if we know $G(e^{-\pi\sqrt{n}})$, then we can also evaluate $G(e^{-3\pi\sqrt{n}})$ or $G(e^{-\pi\sqrt{n}/3})$.

(vi)Theorem 2.3.24 implies that if we know B_n, then we can compute B_{9n}, $B_{n/9}$, B_{25n}, $B_{n/25}$, B_{49n}, or $B_{n/49}$, that is, by Theorem 2.3.21, if we know $G(-e^{-\pi\sqrt{n/3}})$, then we can also evaluate $G(-e^{-\pi\sqrt{3n}})$, $G(-e^{-\pi\sqrt{n/27}})$, $G(-e^{-5\pi\sqrt{3n}})$, $G(-e^{-\pi\sqrt{n/75}})$, $G(-e^{-7\pi\sqrt{n/3}})$, or $G(-e^{-\pi\sqrt{n/147}})$.

Chapter 3

Some More Explicit Values of Ramanujan's Continued Fractions

3.1 Introduction

The classical Dedekind eta-function $\eta(z)$ is defined by

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}), \quad \text{Im} z > 0.$$

Following Ramanujan's notations, we set $q = \exp(2\pi i z)$ and

$$f(-q) = q^{-1/24} \eta(z).$$

In the unorganized portions of his second notebook, Ramanujan [54] recorded without proofs 23 beautiful identities involving quotients of only eta-functions and no other theta-functions. Proofs of these can be found in [31], [16] and [7]. The identities can be divided into two categories. In the first category, each identity involves four arguments and the second category involves eight arguments. The first category identities have been used to find explicit values of the famous Rogers-Ramanujan continued fraction [22], Ramanujan's cubic continued fraction [12], [1], Ramanujan's class invariants [29], and a certain quotient of eta-functions [24]. Unlike the first category the second category identities have not been applied before. In this chapter, we use these identities and some new identities of the same nature to find many new explicit values of the famous Rogers-Ramanujan continued fraction $R(q)$ as defined in (1.1.6). We also find some new values of λ_n and μ_n defined in (1.1.13) and (1.1.12), respectively, which can be used to find the explicit values Ramanujan's cubic continued fraction $G(q)$ as defined in (1.1.11).

In Section 3.2, we state 10 eta-function identities involving eight arguments. We also give proofs of the new identities.

In Section 3.3, we define the parameter s_n as defined by Yi [67] and find new explicit values of $R(q)$ by using some identities in Section 3.2.

In Section 3.4, we find some values of λ_n and μ_n by using the identities recorded in Section 3.2. The corresponding values of $G(q)$ can be found by solving a cubic equation as given in Theorem 2.3.1.

The parameters λ_n and μ_n are connected with Ramanujan's cubic theory of elliptic functions. In Sections 3.5 and 3.6, we show how the new values of μ_n and λ_n combined with some old and newly found modular equations in cubic theory can be applied to find some new series for $1/\pi$ by appealing to the formula established by J. M. Borwein and P. B. Borwein [33] and later modified by Chan and W.-C. Liaw [40].

We end this introduction by recalling from Berndt's book [15, p. 325], the definition of Ramanujan's "mixed" modular equation or modular equation of composite degrees. Let K, K', L_1, L_1', L_2, L_2', L_3, and L_3' denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, and $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2, and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1\frac{K'}{K} = \frac{L_1'}{L_1}, \quad n_2\frac{K'}{K} = \frac{L_2'}{L_2}, \quad \text{and} \quad n_3\frac{K'}{K} = \frac{L_3'}{L_3} \tag{3.1.1}$$

hold. Then a "mixed" modular equation is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, and $\sqrt{\delta}$ that is induced by (3.1.1). In such an instance, we say that β, γ, and δ are of degrees n_1, n_2, and n_3, respectively, over α or α, β, γ, and δ have degrees 1, n_1, n_2, and n_3, respectively. Denoting $z_r = \phi^2(q^r)$, where

$$q = \exp(-\pi K'/K), \quad \phi(q) = f(q,q), \quad |q| < 1;$$

the multipliers m, and m' associated with α; β, and γ, δ, respectively are defined by

$$m = \frac{z_1}{z_{n_1}}, \quad m' = \frac{z_{n_2}}{z_{n_3}}. \tag{3.1.2}$$

3.2 Eta-function identities

In this section, we state and prove some eta-function identities involving eight arguments which will be used in finding explicit values of $R(q)$, μ_n, and λ_n.

Theorem 3.2.1. *(Berndt [16], p. 214, Entry 59) If*

$$P = \frac{f(-q^3)f(-q^5)}{q^{1/3}f(-q)f(-q^{15})} \quad \text{and} \quad Q = \frac{f(-q^6)f(-q^{10})}{q^{2/3}f(-q^2)f(-q^{30})},$$

then

$$PQ + \frac{1}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 + 4. \tag{3.2.1}$$

Theorem 3.2.2. *(Berndt [16], p. 230, Entry 65, Baruah [7], Theorem 2.3) If*

$$P = \frac{f(-q)f(-q^2)}{q^{1/2}f(-q^5)f(-q^{10})} \quad \text{and} \quad Q = \frac{f(-q^3)f(-q^6)}{q^{3/2}f(-q^{15})f(-q^{30})},$$

then

$$PQ + \frac{25}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 - 3\left(\frac{Q}{P} + \frac{P}{Q} + 2\right). \tag{3.2.2}$$

Theorem 3.2.3. *If*

$$P = \frac{f(-q^5)f(-q^7)}{qf(-q)f(-q^{35})} \quad \text{and} \quad Q = \frac{f(-q^{10})f(-q^{14})}{q^2f(-q^2)f(-q^{70})},$$

then

$$P^3 + Q^3 + PQ(2P + 2Q + 1) = P^2Q^2. \tag{3.2.3}$$

Proof. We set

$$R := \frac{f(q^5)f(q^7)}{qf(q)f(q^{35})}.$$

Employing Entries 12 (i) and (iii) of Chapter 17 of Berndt's book [15, p. 124], we find that

$$R = \sqrt{\frac{z_5z_7}{z_1z_{35}}}\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24} \tag{3.2.4}$$

and

$$Q = \sqrt{\frac{z_5z_7}{z_1z_{35}}}\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12}, \tag{3.2.5}$$

where β, γ, and δ have degrees 5, 7, and 35, respectively, over α.
From (3.2.4) and (3.2.5), it readily follows that

$$\frac{Q}{R}=\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24} \tag{3.2.6}$$

and

$$\frac{R^2}{Q}=\sqrt{\frac{m'}{m}}, \tag{3.2.7}$$

where $m=z_1/z_5$ and $m'=z_7/z_{35}$.

Now, by Entries 18 (vi) and (vii) of Chapter 20 of [15, p. 423], we note that

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8}-\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}+2\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/12}=\sqrt{\frac{m'}{m}} \tag{3.2.8}$$

and

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8}+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8}-\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}+2\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12}=-\sqrt{\frac{m}{m'}}. \tag{3.2.9}$$

Multiplying both sides of (3.2.8) and (3.2.9) by $(\beta\gamma(1-\beta)(1-\gamma))^{1/8}$ and $(\alpha\delta(1-\alpha)(1-\delta))^{1/8}$, respectively, and then combining the two results, we find that

$$\begin{aligned}(\beta\gamma(1-\beta)(1-\gamma))^{1/8}&\left\{\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}-2\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/12}+\sqrt{\frac{m'}{m}}\right\}\\ =(\alpha\delta(1-\alpha)(1-\delta))^{1/8}&\left\{\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}-2\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12}-\sqrt{\frac{m}{m'}}\right\}\end{aligned} \tag{3.2.10}$$

Dividing both sides of (3.2.10) by $(\alpha\delta(1-\alpha)(1-\delta))^{1/8}$ and then employing (3.2.4) and (3.2.5), we deduce that

$$Q^2R^2+R^3-2R^2Q=Q^3-QR-2Q^2R. \tag{3.2.11}$$

If we replace q by $-q$ then R is converted to $-P$ and Q remains unaltered. Thus, (3.2.11) is transformed into

$$P^2Q^2-P^3-2P^2Q=PQ+Q^3+2Q^2P, \tag{3.2.12}$$

which immediately implies (3.2 3). □

Theorem 3.2.4. *(Berndt [16], p. 186, Entry 34, Baruah [7], Theorem 2.1) If*

$$u=\frac{f(-q^3)f(-q^6)}{q^{3/4}f(-q^9)f(-q^{18})} \quad \textit{and} \quad v=\frac{f(-q)f(-q^2)}{qf(-q^9)f(-q^{18})},$$

then

$$u^4=v^3+3v^2+9v. \tag{3.2.13}$$

Theorem 3.2.5. *(Berndt [16], p. 192, Entry 39) If*

$$u = \frac{f(-q^3)f(-q^{15})}{q^{3/2}f(-q^9)f(-q^{45})} \quad \textit{and} \quad v = \frac{f(-q)f(-q^5)}{q^2 f(-q^9)f(-q^{45})},$$

then

$$u^4 - 3u^2v = v^3 + 3v^2 + 9v. \tag{3.2.14}$$

Theorem 3.2.6. *(Berndt [16], p. 218, Entry 61, Baruah [7], Theorem 2.2) If*

$$P = \frac{f(-q^6)f(-q^5)}{q^{1/4}f(-q^2)f(-q^{15})} \quad \textit{and} \quad Q = \frac{f(-q^3)f(-q^{10})}{q^{3/4}f(-q)f(-q^{30})},$$

then

$$PQ + 1 + \frac{1}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2. \tag{3.2.15}$$

Theorem 3.2.7. *(Berndt [16], p. 215, Entry 60) If*

$$P = \frac{f(-q)f(-q^5)}{q^{1/2}f(-q^3)f(-q^{15})} \quad \textit{and} \quad Q = \frac{f(-q^2)f(-q^{10})}{qf(-q^6)f(-q^{30})},$$

then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 - 4\frac{Q}{P} - 4\frac{P}{Q}. \tag{3.2.16}$$

Next three theorems are new.

Theorem 3.2.8. *If*

$$P = \frac{f(-q^3)f(-q^7)}{q^{1/2}f(-q)f(-q^{21})} \quad \textit{and} \quad Q = \frac{f(-q^6)f(-q^{14})}{qf(-q^2)f(-q^{42})},$$

then

$$PQ + \frac{1}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 + 4\left(\frac{Q}{P} + \frac{P}{Q}\right). \tag{3.2.17}$$

Proof. We employ the modular equations in [15, p. 401, Entries 13 (i) and (ii)] and proceed as in Theorem 3.2.3. □

Theorem 3.2.9. *If*

$$P = \frac{f(-q)f(-q^{11})}{qf(-q^3)f(-q^{33})} \quad \textit{and} \quad Q = \frac{f(-q^2)f(-q^{22})}{q^2 f(-q^6)f(-q^{66})},$$

then

$$P^3 + Q^3 = (PQ)^2 + PQ(2P + 2Q + 3). \tag{3.2.18}$$

Proof. To prove the theorem we employ the modular equations [15, p. 408, Entries 14 (i) and (ii)] and proceed as in Theorem 3.2.3. □

Theorem 3.2.10. *If*

$$P = \frac{f(-q^3)f(-q^{13})}{qf(-q)f(-q^{39})} \quad and \quad Q = \frac{f(-q^6)f(-q^{26})}{q^2f(-q^2)f(-q^{78})},$$

then

$$P^3 + Q^3 = (PQ)^2 + PQ(1 - 2P - 2Q). \tag{3.2.19}$$

Proof. We employ the modular equations in the first case of [15, p. 426, Entry 19 (iv)] and proceed as in the proof of Theorem 3.2.3. □

3.3 Explicit values of $R(q)$

Recently, Yi [67] has found many explicit values of $R(q)$ by using eta-function identities and transformation formulas given in Theorem 1.1.3 and Theorem 1.1.4. In this section, we use some of the eta-function identities given in Section 3.2 to find many new explicit values of $R(q)$.

The following relation was stated by Ramanujan [15, p. 267] and first proved by Watson [62]

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{3.3.1}$$

Theorem 3.3.1. *(Berndt et al. [26], Proposition 2.4, Yi [67], Theorem 2.3.1(i))*
For $q = e^{-2\pi\sqrt{n/5}}$, let

$$s_n = \frac{f^6(-q)}{5\sqrt{5}qf^6(-q^5)}.$$

Then if $2a = 5\sqrt{5}s_n + 11$,

$$R^5(e^{-2\pi\sqrt{n/5}}) = \sqrt{a^2+1} - a.$$

Using the transformation formula given in Theorem 1.1.3, we also have the following theorem.

Theorem 3.3.2. *(Yi [67], Theorem 4.2.(i)) We have*

$$s_{1/n} = 1/s_n.$$

Yi [67] found the values of s_n and $s_{1/n}$ for n = 1, 2, 3, 4, 5, 7, 8, and 9. In this chapter, we find the values of s_n and $s_{1/n}$ for n = 6, 3/2, 14, 7/2, 18, 9/2 and found some new values of $R(q)$ by using Theorem 3.3.1. We will use the Theorems 3 2.1 - 3.2.3 stated in Section 3.2.

Theorem 3.3.3. *(Yi [67], Theorem 4.4) We have*

$$\sqrt{5}\left\{(s_n s_{4n})^{1/6} + (s_n s_{4n})^{-1/6}\right\} = \left(\frac{s_n}{s_{4n}}\right)^{1/2} + \left(\frac{s_{4n}}{s_n}\right)^{1/2}.$$

Theorem 3.3.4. *We have*

$$\left(\frac{s_{9n}s_{36n}}{s_n s_{4n}}\right)^{1/6} + \left(\frac{s_n s_{4n}}{s_{9n}s_{36n}}\right)^{1/6} = \left(\frac{s_n s_{36n}}{s_{4n}s_{9n}}\right)^{1/2} + \left(\frac{s_{4n}s_{9n}}{s_n s_{36n}}\right)^{1/2} + 4.$$

Proof. Setting $q = e^{-2\pi\sqrt{n/5}}$ in Theorem 3.2.1 and using the definition of s_n in Theorem 3.3.1, we complete the proof. □

Theorem 3.3.5. *We have*

$$s_6 = (\sqrt{2}+1)^3(\sqrt{10}+3), \qquad s_{1/6} = (\sqrt{2}-1)^3(\sqrt{10}-3),$$

$$s_{3/2} = (\sqrt{2}+1)^3(\sqrt{10}-3), \qquad s_{2/3} = (\sqrt{2}-1)^3(\sqrt{10}+3).$$

Proof. Setting $n = 1/6$ and using Theorem 3.3.2, we find that

$$(s_6 s_{3/2})^{1/3} + (s_6 s_{3/2})^{-1/3} = 6. \tag{3.3.2}$$

Solving (3.3.2), we deduce that

$$(s_6 s_{3/2})^{1/3} = 3 + 2\sqrt{2}. \tag{3.3.3}$$

Thus,

$$(s_6 s_{3/2})^{1/6} = \sqrt{2} + 1. \tag{3.3.4}$$

Again, setting $n = 1/6$ in Theorem 3.3.3, we find that

$$\left(\frac{s_6}{s_{3/2}}\right)^{1/2} + \left(\frac{s_6}{s_{3/2}}\right)^{-1/2} = \sqrt{5}\{(s_6 s_{3/2})^{1/6} + (s_6 s_{3/2})^{-1/6}\}. \tag{3.3.5}$$

Using (3.3.4) in (3.3.5), we obtain

$$\left(\frac{s_6}{s_{3/2}}\right)^{1/2} + \left(\frac{s_6}{s_{3/2}}\right)^{-1/2} = 2\sqrt{10}. \tag{3.3.6}$$

Solving for $(s_6/s_{3/2})^{1/2}$, we find that

$$\left(\frac{s_6}{s_{3/2}}\right)^{1/2} = \sqrt{10} + 3. \tag{3.3.7}$$

Combining (3.3.4) and (3.3.7), we derive the values of s_6 and $s_{3/2}$. Then, the values of $s_{1/6}$ and $s_{2/3}$ follow from Theorem 3.3.2. □

Corollary 3.3.6. *We have*

$$R^5(e^{-2\pi\sqrt{6/5}}) = \frac{1}{2}\left\{-261 - 175\sqrt{2} - 105\sqrt{5} - 75\sqrt{10} + \sqrt{1885 + 1330\sqrt{2} + 840\sqrt{5} + 594\sqrt{10}}\right\},$$

$$R^5(e^{-\pi\sqrt{2/15}}) = \frac{1}{2}\left\{-261 + 175\sqrt{2} - 105\sqrt{5} + 75\sqrt{10} + \sqrt{1885 - 1330\sqrt{2} + 840\sqrt{5} - 594\sqrt{10}}\right\},$$

$$R^5(e^{-\pi\sqrt{6/5}}) = \frac{1}{2}\left\{-261 - 175\sqrt{2} + 105\sqrt{5} + 75\sqrt{10} + \sqrt{1885 + 1330\sqrt{2} - 840\sqrt{5} - 594\sqrt{10}}\right\},$$

$$R^5(e^{-2\pi\sqrt{2/15}}) = \frac{1}{2}\left\{-261 + 175\sqrt{2} + 105\sqrt{5} - 75\sqrt{10} + \sqrt{1885 - 1330\sqrt{2} - 840\sqrt{5} + 594\sqrt{10}}\right\}.$$

Proof. These results follow from Theorems 3.3.1 and 3.3.5. □

Theorem 3.3.7. *We have*

$$5\left\{(s_n s_{4n} s_{9n} s_{36n})^{1/6} + (s_n s_{4n} s_{9n} s_{36n})^{-1/6}\right\}$$
$$= \left(\frac{s_{9n}s_{36n}}{s_n s_{4n}}\right)^{1/3} + \left(\frac{s_n s_{4n}}{s_{9n}s_{36n}}\right)^{1/3} - 3\left\{\left(\frac{s_{9n}s_{36n}}{s_n s_{4n}}\right)^{1/6} + \left(\frac{s_n s_{4n}}{s_{9n}s_{36n}}\right)^{1/6} + 2\right\}.$$

Proof. Setting $q = e^{-2\pi\sqrt{n/5}}$ in Theorem 3.2.2 and employing the definition of s_n in Theorem 3.3.1, we complete the proof. □

Theorem 3.3.8. *We have*

$$s_{18} = \left(2 + \sqrt{6} + \sqrt{9 + 4\sqrt{6}}\right)^3 \left(2\sqrt{5} + \sqrt{30} + \sqrt{49 + 20\sqrt{6}}\right),$$

$$s_{1/18} = \left(2 + \sqrt{6} - \sqrt{9 + 4\sqrt{6}}\right)^3 \left(2\sqrt{5} + \sqrt{30} - \sqrt{49 + 20\sqrt{6}}\right),$$

$$s_{9/2} = \left(2 + \sqrt{6} + \sqrt{9 + 4\sqrt{6}}\right)^3 \left(2\sqrt{5} + \sqrt{30} - \sqrt{49 + 20\sqrt{6}}\right),$$

$$s_{2/9} = \left(2 + \sqrt{6} - \sqrt{9 + 4\sqrt{6}}\right)^3 \left(2\sqrt{5} + \sqrt{30} + \sqrt{49 + 20\sqrt{6}}\right).$$

Proof. Setting $n = 1/18$ in Theorem 3.3.7 and using Theorem 3.3.2, we find that

$$(s_{18}s_{9/2})^{1/3} + (s_{18}s_{9/2})^{-1/3} - 8\left((s_{18}s_{9/2})^{1/3} + (s_{18}s_{9/2})^{-1/3}\right) - 6 = 0 \qquad (3\,3\,8)$$

From (3.3 8), we deduce that

$$(s_{18}s_{9/2})^{1/6} + (s_{18}s_{9/2})^{-1/6} = 4 + 2\sqrt{6}. \qquad (3\,3\,9)$$

Solving (3.3.9) for $(s_{18}s_{9/2})^{1/6}$, we obtain

$$(s_{18}s_{9/2})^{1/6} = 2 + \sqrt{6} + \sqrt{9 + 4\sqrt{6}}. \qquad (3.3\,10)$$

Again, setting $n = 1/18$ in Theorem 3.3.3, we find that

$$\left(\frac{s_{18}}{s_{9/2}}\right)^{1/2} + \left(\frac{s_{18}}{s_{9/2}}\right)^{-1/2} = \sqrt{5}\{(s_{18}s_{9/2})^{1/6} + (s_{18}s_{9/2})^{-1/6}\}. \tag{3.3.11}$$

Using (3.3.10) in (3.3.11), we find that

$$\left(\frac{s_{18}}{s_{9/2}}\right)^{1/2} + \left(\frac{s_{18}}{s_{9/2}}\right)^{-1/2} = 4\sqrt{5} + 2\sqrt{30}. \tag{3.3.12}$$

Solving this for $(s_{18}/s_{9/2})^{1/2}$, we deduce that

$$\left(\frac{s_{18}}{s_{9/2}}\right)^{1/2} = 2\sqrt{5} + \sqrt{30} + \sqrt{49 + 20\sqrt{6}}. \tag{3.3.13}$$

From (3.3.10) and (3.3.13), we derive the values of s_{18} and $s_{9/2}$. Then the values of $s_{1/18}$ and $s_{2/9}$ follow from Theorem 3.3.2. □

Corollary 3.3.9. *We have*

(i) $R^5(e^{-6\pi\sqrt{2/5}}) = \sqrt{b^2+1} - b$, *where* $2b = 5\sqrt{5}s_{18} + 11$,

(ii) $R^5(e^{-\sqrt{2}\pi/(3\sqrt{5})}) = \sqrt{b^2+1} - b$, *where* $2b = 5\sqrt{5}s_{1/18} + 11$,

(iii) $R^5(e^{-3\pi\sqrt{2/5}}) = \sqrt{b^2+1} - b$, *where* $2b = 5\sqrt{5}s_{9/2} + 11$,

(iv) $R^5(e^{-2\sqrt{2}\pi/(3\sqrt{5})}) = \sqrt{b^2+1} - b$, *where* $2b = 5\sqrt{5}s_{2/9} + 11$,

where s_{18}, $s_{1/18}$, $s_{9/2}$, and $s_{2/9}$ are given in Theorem 3.3.8.

Proof. The proofs of these follow from Theorems 3.3.1 and 3.3.8. □

Theorem 3.3.10. *We have*

$$\left(\frac{s_{49n}}{s_n}\right)^{1/2} + \left(\frac{s_{49n}}{s_n}\right)^{-1/2} + \left(\frac{s_{49n}s_{196n}}{s_n s_{4n}}\right)^{1/6}\left\{2\left(\frac{s_{49n}}{s_n}\right)^{1/6} + \left(\frac{s_{49n}}{s_n}\right)^{-1/6} + 1\right\} = \left(\frac{s_{49n}s_{196n}}{s_n s_{4n}}\right)^{1/3}.$$

Proof. Setting $q = e^{-2\pi\sqrt{n/5}}$ in Theorem 3.2.3 and employing the definition of s_n in Theorem 3.3.1, we complete the proof. □

Theorem 3.3.11. *We have*

$$s_{14} = \left(3+\sqrt{10}\right)^3\left(5\sqrt{2}+7\right),$$

$$s_{1/14} = \left(\sqrt{10}-3\right)^3\left(5\sqrt{2}-7\right),$$

$$s_{7/2} = \left(3+\sqrt{10}\right)^3\left(5\sqrt{2}-7\right),$$

$$s_{2/7} = \left(\sqrt{10}-3\right)^3\left(5\sqrt{2}+7\right).$$

Proof. Setting $n = 1/14$ in Theorem 3.3.10 and using Theorem 3.3.2, we find that

$$(s_{14}s_{7/2})^{1/3} - 6(s_{14}s_{7/2})^{1/6} - 1 = 0. \tag{3.3.14}$$

Solving (3.3.14) for $(s_{14}s_{7/2})^{1/6}$, we find that

$$(s_{14}s_{7/2})^{1/6} = 3 + \sqrt{10}. \tag{3.3.15}$$

Now, setting $n = 1/14$ in Theorem 3.3.3 and applying Theorem 3.3.2, we find that

$$\left(\frac{s_{14}}{s_{7/2}}\right)^{1/2} + \left(\frac{s_{14}}{s_{7/2}}\right)^{-1/2} = \sqrt{5}\{(s_{14}s_{7/2})^{1/6} + (s_{14}s_{7/2})^{-1/6}\}. \tag{3.3.16}$$

Using (3.3.15) in (3.3.16), we obtain

$$\left(\frac{s_{14}}{s_{7/2}}\right)^{1/2} + \left(\frac{s_{14}}{s_{7/2}}\right)^{-1/2} = 10\sqrt{2}. \tag{3.3.17}$$

Solving this for $(s_{14}/s_{7/2})^{1/2}$, we deduce that

$$\left(\frac{s_{14}}{s_{7/2}}\right)^{1/2} = 5\sqrt{2} + 7. \tag{3.3.18}$$

From (3.3.15) and (3.3.18), we easily deduce the values of s_{14} and $s_{7/2}$. Then the values of $s_{1/14}$ and $s_{2/7}$ follows immediately from Theorem 3.3.2. ☐

Corollary 3.3.12. *We have*

(i) $$R^5(e^{-2\pi\sqrt{14/5}}) = \frac{1}{2}\left\{\sqrt{2710525 + 1916530\sqrt{2} + 1212120\sqrt{5} + 857142\sqrt{10}}\right\} - (9261 + 6475\sqrt{2} + 4095\sqrt{5} + 2925\sqrt{10}),$$

(ii) $$R^5(e^{-\pi\sqrt{2/35}}) = \frac{1}{2}\left\{\sqrt{2710525 - 1916530\sqrt{2} + 1212120\sqrt{5} - 857142\sqrt{10}}\right\} - (9261 - 6475\sqrt{2} + 4095\sqrt{5} - 2925\sqrt{10}),$$

(iii) $$R^5(e^{-\pi\sqrt{14/5}}) = \frac{1}{2}\left\{\sqrt{2710525 - 1916530\sqrt{2} - 1212120\sqrt{5} + 857142\sqrt{10}}\right\} - (9261 - 6475\sqrt{2} - 4095\sqrt{5} + 2925\sqrt{10}),$$

(iv) $$R^5(e^{-2\sqrt{2}\pi/\sqrt{35}}) = \frac{1}{2}\left\{\sqrt{2710525 + 1916530\sqrt{2} - 1212120\sqrt{5} - 857142\sqrt{10}}\right\} - (9261 + 6475\sqrt{2} - 4095\sqrt{5} - 2925\sqrt{10}).$$

Proof. These results follow from Theorems 3.3.1 and 3.3.11. ☐

3.4 Explicit values of $G(q)$

We have already mentioned in Section 2.3 of previous chapter that if we know the λ_n or μ_n for a particular values of n then we can evaluate the values of $G(-e^{-2\pi\sqrt{n/3}})$ or $G(e^{-2\pi\sqrt{n/3}})$ by solving a cubic equation. In this section, we find many new values of λ_n and μ_n by using the eta-function identities with eight arguments stated in Section 3.2.

Theorem 3.4.1. *We have*

$$\left(\frac{\mu_{25n}\mu_{100n}}{\mu_n\mu_{4n}}\right)^{1/6}+\left(\frac{\mu_n\mu_{4n}}{\mu_{25n}\mu_{100n}}\right)^{1/6}=\left(\frac{\mu_n\mu_{100n}}{\mu_{4n}\mu_{25n}}\right)^{1/2}+\left(\frac{\mu_{4n}\mu_{25n}}{\mu_n\mu_{100n}}\right)^{1/2}+4.$$

Proof. We set $q=e^{-2\pi\sqrt{n/3}}$ in Theorem 3.2.1 and use the definition of μ_n in Theorem 2.3.1. □

Theorem 3.4.2. *We have*

$$\mu_{10}=(\sqrt{5}+2)(\sqrt{2}+1)^3, \qquad \mu_{1/10}=(\sqrt{5}-2)(\sqrt{2}-1)^3,$$

$$\mu_{5/2}=(\sqrt{5}-2)(\sqrt{2}+1)^3, \qquad \mu_{2/5}=(\sqrt{5}+2)(\sqrt{2}-1)^3.$$

Proof. Setting $n=1/10$ in Theorem 3.4.1 and using Theorem 2.3.2, we find that

$$(\mu_{10}\mu_{5/2})^{1/3}+(\mu_{10}\mu_{5/2})^{-1/3}=6. \tag{3.4.1}$$

Solving for $(\mu_{10}\mu_{5/2})^{1/3}$, we find that

$$(\mu_{10}\mu_{5/2})^{1/3}=3+2\sqrt{2}=(\sqrt{2}+1)^2. \tag{3.4.2}$$

We recall Theorem 2.3.4(i) in Chapter 2 that

$$3\left\{(\mu_n\mu_{4n})^{1/3}+(\mu_n\mu_{4n})^{-1/3}\right\}=\frac{\mu_n}{\mu_{4n}}+\frac{\mu_{4n}}{\mu_n}. \tag{3.4.3}$$

Putting $n=1/10$ in (3.4.3) and using Theorem 2.3.2, we obtain

$$3\left((\mu_{10}\mu_{5/2})^{1/3}+(\mu_{10}\mu_{5/2})^{-1/3}\right)=\frac{\mu_{10}}{\mu_{5/2}}+\frac{\mu_{5/2}}{\mu_{10}}. \tag{3.4.4}$$

Using (3.4.1), we deduce that

$$\frac{\mu_{10}}{\mu_{5/2}}+\frac{\mu_{5/2}}{\mu_{10}}=18. \tag{3.4.5}$$

Solving this for $\mu_{10}/\mu_{5/2}$, we find that

$$\frac{\mu_{10}}{\mu_{5/2}}=9+4\sqrt{5}=(\sqrt{5}+2)^2 \tag{3.4.6}$$

Thus by (3.4.2) and (3.4.6) we easily deduce the values of μ_{10} and $\mu_{5/2}$. The values of $\mu_{1/10}$ and $\mu_{2/5}$ then follow from Theorem 2.3.2. □

Remark 3.4.1. Same values can also be obtained by employing Theorem 3.2.6.

Theorem 3.4.3. *We have*

$$(\mu_{9n}\mu_{36n})^{2/3} = 3\left\{(\mu_n\mu_{4n}\mu_{9n}\mu_{36n})^{1/2} + (\mu_n\mu_{4n}\mu_{9n}\mu_{36n})^{1/3} + (\mu_n\mu_{4n}\mu_{9n}\mu_{36n})^{1/6}\right\}.$$

Proof. We set $q = e^{-2\pi\sqrt{n/3}}$ in Theorem 3.2.4 and use the definition of μ_n in Theorem 2.3.1. □

Theorem 3.4.4. *We have*

$$\mu_6 = 3\sqrt{3}(\sqrt{3}+\sqrt{2}), \qquad \mu_{1/6} = \frac{\sqrt{3}-\sqrt{2}}{3\sqrt{3}},$$

$$\mu_{3/2} = 3\sqrt{3}(\sqrt{3}-\sqrt{2}), \qquad \mu_{2/3} = \frac{\sqrt{3}+\sqrt{2}}{3\sqrt{3}}.$$

Proof. Putting $n = 1/6$ in Theorem 3.4.3 and using Theorem 2.3.2, we deduce that

$$\left(\mu_6\mu_{3/2}\right)^{2/3} = 9. \tag{3.4.7}$$

Thus,

$$\left(\mu_6\mu_{3/2}\right)^{1/6} = \sqrt{3}. \tag{3.4.8}$$

Again, setting $n = 1/6$ in (3.4.3) and using Theorem 2.3.2, we obtain

$$3\left(\left(\mu_6\mu_{3/2}\right)^{1/3} + \left(\mu_6\mu_{3/2}\right)^{-1/3}\right) = \frac{\mu_6}{\mu_{3/2}} + \frac{\mu_{3/2}}{\mu_6}. \tag{3.4.9}$$

Using (3.4.8) in (3.4.9), we obtain

$$\frac{\mu_6}{\mu_{3/2}} + \frac{\mu_{3/2}}{\mu_6} = 10. \tag{3.4.10}$$

Solving this for $\mu_6/\mu_{3/2}$, we find that

$$\frac{\mu_6}{\mu_{3/2}} = 5 + 2\sqrt{6} = (\sqrt{3}+\sqrt{2})^2 \tag{3.4.11}$$

Thus, by (3.4.8) and (3.4.11) we easily deduce the values of μ_6 and $\mu_{3/2}$. The values of $\mu_{1/6}$ and $\mu_{2/3}$ then follow from Theorem 2.3.2. □

Theorem 3.4.5. *We have*

$$\begin{aligned}(\mu_{9n}\mu_{225n})^{2/3} &- 3\left(\mu_{9n}\mu_{225n}\right)^{1/3}\left(\mu_n\mu_{9n}\mu_{25n}\mu_{225n}\right)^{1/6} \\ &= 3\left\{(\mu_n\mu_{9n}\mu_{25n}\mu_{225n})^{1/2} + (\mu_n\mu_{9n}\mu_{25n}\mu_{225n})^{1/3} + (\mu_n\mu_{9n}\mu_{25n}\mu_{225n})^{1/6}\right\}.\end{aligned}$$

Proof. We set $q = e^{-2\pi\sqrt{n/3}}$ in Theorem 3.2.5 and use the definition of μ_n in Theorem 2.3.1. □

Theorem 3.4.6. *We have*

$$\mu_{15} = \frac{3\sqrt{3}}{2}\left(\sqrt{5}+\sqrt{3}\right)\left(3\sqrt{3}+5\right)\sqrt{\sqrt{5}+2},$$
$$\mu_{1/15} = \frac{1}{6\sqrt{3}}\left(\sqrt{5}-\sqrt{3}\right)\left(3\sqrt{3}-5\right)\sqrt{\sqrt{5}-2},$$
$$\mu_{5/3} = \frac{1}{6\sqrt{3}}\left(\sqrt{5}+\sqrt{3}\right)\left(3\sqrt{3}+5\right)\sqrt{\sqrt{5}-2},$$
$$\mu_{3/5} = \frac{3\sqrt{3}}{2}\left(\sqrt{5}-\sqrt{3}\right)\left(3\sqrt{3}-5\right)\sqrt{\sqrt{5}+2}.$$

Proof. Setting $n = 1/15$ in Theorem 3.4.5 and using Theorem 2.3.2, we find that

$$\left(\frac{\mu_{15}}{\mu_{5/3}}\right)^{2/3} - 3\left(\frac{\mu_{15}}{\mu_{5/3}}\right)^{1/3} = 9. \tag{3.4.12}$$

Solving for $(\mu_{15}/\mu_{5/3})^{1/3}$, we find that

$$\left(\frac{\mu_{15}}{\mu_{5/3}}\right)^{1/3} = \frac{3(1+\sqrt{5})}{2}. \tag{3.4.13}$$

Now, from Theorem 2.3.4(iii), we note that

$$3\left\{(\mu_n\mu_{25n})^{1/3} + (\mu_n\mu_{25n})^{-1/3}\right\} + 5 = \left(\frac{\mu_{25n}}{\mu_n}\right)^{1/2} - \left(\frac{\mu_n}{\mu_{25n}}\right)^{1/2}. \tag{3.4.14}$$

Setting $n = 1/15$ in (3.4.14) and using Theorem 2.3.2, we deduce that

$$3\left\{\left(\frac{\mu_{15}}{\mu_{5/3}}\right)^{1/3} + \left(\frac{\mu_{5/3}}{\mu_{15}}\right)^{1/3}\right\} + 5 = \left(\mu_{15}\mu_{5/3}\right)^{1/2} - \left(\mu_{15}\mu_{5/3}\right)^{-1/2}. \tag{3.4.15}$$

Using (3.4.13), we obtain

$$\left(\mu_{15}\mu_{5/3}\right)^{1/2} - \left(\mu_{15}\mu_{5/3}\right)^{-1/2} = 5\sqrt{5}+9. \tag{3.4.16}$$

Solving this for $(\mu_{15}\mu_{5/3})^{1/2}$, we find that

$$\left(\mu_{15}\mu_{5/3}\right)^{1/2} = \frac{9+5\sqrt{3}+5\sqrt{5}+3\sqrt{15}}{2}. \tag{3.4.17}$$

Thus by (3.4.13) and (3.4.17) we easily deduce the values of μ_{15} and $\mu_{5/3}$. The values of $\mu_{1/15}$ and $\mu_{3/5}$ then follow from Theorem 2.3.2. □

Theorem 3.4.7. *We have*

$$\left(\frac{\mu_{49n}\mu_{196n}}{\mu_n\mu_{4n}}\right)^{1/6} + \left(\frac{\mu_n\mu_{4n}}{\mu_{49n}\mu_{196n}}\right)^{1/6} = \left(\frac{\mu_{4n}\mu_{49n}}{\mu_n\mu_{196n}}\right)^{1/2} + \left(\frac{\mu_n\mu_{196n}}{\mu_{4n}\mu_{49n}}\right)^{1/2}$$
$$+4\left\{\left(\frac{\mu_{4n}\mu_{49n}}{\mu_n\mu_{196n}}\right)^{1/6} + \left(\frac{\mu_n\mu_{196n}}{\mu_{4n}\mu_{49n}}\right)^{1/6}\right\}.$$

Proof. We set $q = e^{-2\pi\sqrt{n/3}}$ in Theorem 3.2.8 and use the definition of μ_n in Theorem 2.3.1. □

Theorem 3.4.8. *We have*

$$\mu_{14} = (\sqrt{3}+\sqrt{2})^3(2\sqrt{2}+\sqrt{7}), \qquad \mu_{1/14} = (\sqrt{3}-\sqrt{2})^3(2\sqrt{2}-\sqrt{7}),$$
$$\mu_{7/2} = (\sqrt{3}+\sqrt{2})^3(2\sqrt{2}-\sqrt{7}), \qquad \mu_{2/7} = (\sqrt{3}-\sqrt{2})^3(2\sqrt{2}+\sqrt{7}).$$

Proof. We put $n = 1/14$ in Theorem 3.4.7 and (3.4.3) and proceed as in the proof of Theorem 3.4.2 to complete the proof. □

Theorem 3.4.9. *We have*

$$3\left\{(\mu_n\mu_{4n}\mu_{25n}\mu_{100n})^{1/6} + (\mu_n\mu_{4n}\mu_{25n}\mu_{100n})^{-1/6}\right\}$$
$$= \left(\frac{\mu_{4n}\mu_{100n}}{\mu_n\mu_{25n}}\right)^{1/2} + \left(\frac{\mu_n\mu_{25n}}{\mu_{4n}\mu_{100n}}\right)^{1/2} - 4\left\{\left(\frac{\mu_{4n}\mu_{100n}}{\mu_n\mu_{25n}}\right)^{1/6} + \left(\frac{\mu_n\mu_{25n}}{\mu_{4n}\mu_{100n}}\right)^{1/6}\right\}.$$

Proof. We set $q = e^{-2\pi\sqrt{n/3}}$ in Theorem 3.2.7 and use the definition of μ_n in Theorem 2.3.1. □

Theorem 3.4.10. *We have*

$$\mu_{20} = \frac{1}{4\sqrt{2}}\left(\sqrt{5}+\sqrt{3}\right)^3\left(29+13\sqrt{5}\right) \qquad \mu_{1/20} = \frac{1}{4\sqrt{2}}\left(\sqrt{5}-\sqrt{3}\right)^3\left(13\sqrt{5}-29\right)$$
$$\mu_{5/4} = \frac{1}{4\sqrt{2}}\left(\sqrt{5}-\sqrt{3}\right)^3\left(29+13\sqrt{5}\right) \qquad \mu_{4/5} = \frac{1}{4\sqrt{2}}\left(\sqrt{5}+\sqrt{3}\right)^3\left(13\sqrt{5}-29\right)$$

Proof. By setting $n = 1/20$ in Theorem 3.4.9 and (3.4.14) and proceeding as in the proof of Theorem 3.4.2, we complete the proof. □

Theorem 3.4.11. *We have*

$$(\mu_n\mu_{121n})^{1/2} + (\mu_{4n}\mu_{484n})^{1/2} = \sqrt{3}\,(\mu_n\mu_{4n}\mu_{121n}\mu_{484n})^{1/3}$$
$$+ (\mu_n\mu_{4n}\mu_{121n}\mu_{484n})^{1/6}\left\{2\,(\mu_n\mu_{121n})^{1/6} + 2\,(\mu_{4n}\mu_{484n})^{1/6} + \sqrt{3}\right\}.$$

Proof. We set $q = e^{-2\pi\sqrt{n/3}}$ in Theorem 3.2.9 and use the definition of μ_n in Theorem 2.3.1. □

Theorem 3.4.12. *We have*

$$\mu_{22} = \left(\frac{\sqrt{3}+\sqrt{11}+\sqrt{2\sqrt{33}-2}}{4}\right)^3\left(6+\sqrt{33}+\sqrt{68+12\sqrt{33}}\right)^{3/2},$$
$$\mu_{1/22} = \left(\frac{\sqrt{3}+\sqrt{11}-\sqrt{2\sqrt{33}-2}}{4}\right)^3\left(6+\sqrt{33}-\sqrt{68+12\sqrt{33}}\right)^{3/2},$$
$$\mu_{11/2} = \left(\frac{\sqrt{3}+\sqrt{11}-\sqrt{2\sqrt{33}-2}}{4}\right)^3\left(6+\sqrt{33}+\sqrt{68+12\sqrt{33}}\right)^{3/2},$$
$$\mu_{2/11} = \left(\frac{\sqrt{3}+\sqrt{11}+\sqrt{2\sqrt{33}-2}}{4}\right)^3\left(6+\sqrt{33}-\sqrt{68+12\sqrt{33}}\right)^{3/2}.$$

Proof. By setting $n = 1/22$ in Theorem 3.4.11 and (3.4.3) and proceeding as in the proof of Theorem 3.4.2 we complete the proof. □

Theorem 3.4.13. *We have*

$$\mu_{44} = \left(\frac{3+\sqrt{3}+\sqrt{8+6\sqrt{3}}}{2}\right)^3 \left(31020+17910\sqrt{3}+\sqrt{1924544699+1111136400\sqrt{3}}\right)^{1/2},$$

$$\mu_{1/44} = \left(\frac{3+\sqrt{3}-\sqrt{8+6\sqrt{3}}}{2}\right)^3 \left(31020+17910\sqrt{3}-\sqrt{1924544699+1111136400\sqrt{3}}\right)^{1/2},$$

$$\mu_{11/4} = \left(\frac{3+\sqrt{3}-\sqrt{8+6\sqrt{3}}}{2}\right)^3 \left(31020+17910\sqrt{3}+\sqrt{1924544699+1111136400\sqrt{3}}\right)^{1/2},$$

$$\mu_{4/11} = \left(\frac{3+\sqrt{3}+\sqrt{8+6\sqrt{3}}}{2}\right)^3 \left(31020+17910\sqrt{3}-\sqrt{1924544699+1111136400\sqrt{3}}\right)^{1/2}.$$

Proof. By setting $n = 1/11$ in Theorem 3.4.11 and using Theorem 2.3.2, we deduce that

$$\left(\frac{\mu_{44}}{\mu_{11/4}}\right)^{1/6} + \left(\frac{\mu_{11/4}}{\mu_{44}}\right)^{1/6} = 3+\sqrt{3}. \tag{3.4.18}$$

Solving (3.4.18) for $(\mu_{44}/\mu_{11/4})^{1/6}$, we find that

$$\left(\frac{\mu_{44}}{\mu_{11/4}}\right)^{1/6} = \frac{3+\sqrt{3}+\sqrt{8+6\sqrt{3}}}{2}. \tag{3.4.19}$$

Now , we recall from Theorem 2.3.4(v) in Chapter 2 that

$$\begin{aligned}&9\sqrt{3}\left((\mu_n\mu_{121n})^{5/6}+\frac{1}{(\mu_n\mu_{121n})^{5/6}}\right)+99\left((\mu_n\mu_{121n})^{2/3}+\frac{1}{(\mu_n\mu_{121n})^{2/3}}\right)\\&+198\sqrt{3}\left((\mu_n\mu_{121n})^{1/2}+\frac{1}{(\mu_n\mu_{121n})^{1/2}}\right)+759\left((\mu_n\mu_{121n})^{1/3}+\frac{1}{(\mu_n\mu_{121n})^{1/3}}\right)\\&+693\sqrt{3}\left((\mu_n\mu_{121n})^{1/6}+\frac{1}{(\mu_n\mu_{121n})^{1/6}}\right)+1386=\left(\frac{\mu_{121n}}{\mu_n}\right)+\left(\frac{\mu_n}{\mu_{121n}}\right).\end{aligned} \tag{3.4.20}$$

Setting $n = 1/44$ in (3.4.20), we arrive at

$$\begin{aligned}&9\sqrt{3}\left(\left(\frac{\mu_{44}}{\mu_{11/4}}\right)^{5/6}+\left(\frac{\mu_{11/4}}{\mu_{44}}\right)^{5/6}\right)+99\left(\left(\frac{\mu_{44}}{\mu_{11/4}}\right)^{2/3}+\left(\frac{\mu_{11/4}}{\mu_{44}}\right)^{2/3}\right)\\&+198\sqrt{3}\left(\left(\frac{\mu_{44}}{\mu_{11/4}}\right)^{1/2}+\left(\frac{\mu_{11/4}}{\mu_{44}}\right)^{1/2}\right)+759\left(\left(\frac{\mu_{44}}{\mu_{11/4}}\right)^{1/3}+\left(\frac{\mu_{11/4}}{\mu_{44}}\right)^{1/3}\right)\end{aligned}$$

$$+693\sqrt{3}\left(\left(\frac{\mu_{44}}{\mu_{11/4}}\right)^{1/6}+\left(\frac{\mu_{11/4}}{\mu_{44}}\right)^{1/6}\right)+1386=\mu_{44}\mu_{11/4}+\frac{1}{\mu_{44}\mu_{11/4}}. \tag{3.4.21}$$

Using (3.4.19) in (3.4.21), we find that

$$\mu_{44}\mu_{11/4}+\frac{1}{\mu_{44}\mu_{11/4}}=60\left(1034+597\sqrt{3}\right). \tag{3.4.22}$$

Solving (3.4.22) for $\mu_{44}\mu_{11/4}$, we deduce that

$$\mu_{44}\mu_{11/4}=31020+17910\sqrt{3}+\left(1924544699+1111136400\sqrt{3}\right)^{1/2}. \tag{3.4.23}$$

From (3.4.19) and (3.4.23) we deduce the values of μ_{44} and $\mu_{11/4}$. The values of $\mu_{1/44}$ and $\mu_{4/11}$ then follow from Theorem 2.3.2. □

Theorem 3.4.14. *We have*

$$\left(\frac{\mu_{169n}}{\mu_n}\right)^{1/2}+\left(\frac{\mu_{676n}}{\mu_{4n}}\right)^{1/2}=\left(\frac{\mu_{169n}\mu_{676n}}{\mu_n\mu_{4n}}\right)^{1/3}+\left(\frac{\mu_{169n}\mu_{676n}}{\mu_n\mu_{4n}}\right)^{1/6}$$
$$\times\left\{1-2\left(\frac{\mu_{169n}}{\mu_n}\right)^{1/6}-2\left(\frac{\mu_{676n}}{\mu_{4n}}\right)^{1/2}\right\}.$$

Proof. We set $q=e^{-2\pi\sqrt{n/3}}$ in Theorem 3.2.10 and use the definition of μ_n in Theorem 2.3.1. □

Theorem 3.4.15. *We have*

$$\mu_{26}=\left(3+2\sqrt{2}\right)^3\left(\sqrt{26}+5\right),$$
$$\mu_{1/26}=\left(3-2\sqrt{2}\right)^3\left(\sqrt{26}-5\right),$$
$$\mu_{13/2}=\left(3+2\sqrt{2}\right)^3\left(\sqrt{26}-5\right),$$
$$\mu_{2/13}=\left(3-2\sqrt{2}\right)^3\left(\sqrt{26}+5\right).$$

Proof. By setting $n=1/26$ in Theorem 3.4.14 and (3.4.3) and proceeding as in the proof of Theorem 3.4.2, we complete the proof. □

Theorem 3.4.16. *We have*

$$(\lambda_{9n}\lambda_{225n})^{2/3}-3(\lambda_{9n}\lambda_{225n})^{1/3}(\lambda_n\lambda_{9n}\lambda_{25n}\lambda_{225n})^{1/6}$$
$$=3\left\{(\lambda_n\lambda_{9n}\lambda_{25n}\lambda_{225n})^{1/2}+(\lambda_n\lambda_{9n}\lambda_{25n}\lambda_{225n})^{1/3}+(\lambda_n\lambda_{9n}\lambda_{25n}\lambda_{225n})^{1/6}\right\}.$$

Proof. We set $q=-e^{-\pi\sqrt{n/3}}$ in Theorem 3.2.5 and use the definition of λ_n in Theorem 2.3.1(i). □

Theorem 3.4.17. *We have*

$$\lambda_{15} = \frac{3\sqrt{3}}{2}\left(\sqrt{5}-\sqrt{3}\right)\left(3\sqrt{3}+5\right)\sqrt{\sqrt{5}-2},$$
$$\lambda_{1/15} = \frac{1}{6\sqrt{3}}\left(\sqrt{5}+\sqrt{3}\right)\left(3\sqrt{3}-5\right)\sqrt{\sqrt{5}+2},$$
$$\lambda_{5/3} = \frac{1}{6\sqrt{3}}\left(\sqrt{5}-\sqrt{3}\right)\left(3\sqrt{3}+5\right)\sqrt{\sqrt{5}+2},$$
$$\lambda_{3/5} = \frac{3\sqrt{3}}{2}\left(\sqrt{5}+\sqrt{3}\right)\left(3\sqrt{3}-5\right)\sqrt{\sqrt{5}-2}.$$

Proof. Setting $n = 1/15$ in Theorem 3.4.16 and using Theorem 2.3.2, we find that

$$\left(\frac{\lambda_{15}}{\lambda_{5/3}}\right)^{2/3} + 3\left(\frac{\lambda_{15}}{\lambda_{5/3}}\right)^{1/3} = 9. \tag{3.4.24}$$

Solving for $(\lambda_{15}/\lambda_{5/3})^{1/3}$, we find that

$$\left(\frac{\lambda_{15}}{\lambda_{5/3}}\right)^{1/3} = \frac{3(\sqrt{5}-1)}{2}. \tag{3.4.25}$$

Now, by [24, p. 278, Theorem 4.3], we note that

$$3\left\{(\lambda_n\lambda_{25n})^{1/3} + (\lambda_n\lambda_{25n})^{-1/3}\right\} - 5 = \left(\frac{\lambda_{25n}}{\lambda_n}\right)^{1/2} - \left(\frac{\lambda_n}{\lambda_{25n}}\right)^{1/2}. \tag{3.4.26}$$

Setting $n = 1/15$ in (3.4.26) and using Theorem 2.3.2, we deduce that

$$3\left\{\left(\frac{\lambda_{15}}{\lambda_{5/3}}\right)^{1/3} + \left(\frac{\lambda_{5/3}}{\lambda_{15}}\right)^{1/3}\right\} - 5 = \left(\lambda_{15}\lambda_{5/3}\right)^{1/2} - \left(\lambda_{15}\lambda_{5/3}\right)^{-1/2}. \tag{3.4.27}$$

Using (3.4.25) in (3.4.27), we obtain

$$\left(\lambda_{15}\lambda_{5/3}\right)^{1/2} - \left(\lambda_{15}\lambda_{5/3}\right)^{-1/2} = 5\sqrt{5} - 9. \tag{3.4.28}$$

Solving (3.4.28) for $(\lambda_{15}\lambda_{5/3})^{1/2}$, we find that

$$\left(\lambda_{15}\lambda_{5/3}\right)^{1/2} = \frac{(\sqrt{5}-\sqrt{3})(3\sqrt{3}+5)}{2}. \tag{3.4.29}$$

Thus, by (3.4.25) and (3.4.29), we deduce the values of λ_{15} and $\lambda_{5/3}$. The values of $\lambda_{1/15}$ and $\lambda_{3/5}$ then follow from Theorem 2.3.2. ☐

3.5 Ramanujan-type series for $\frac{1}{\pi}$

The new values of the parameters λ_n and μ_n evaluated above are connected to Ramanujan's cubic theory of elliptic functions and lead to some new Ramanujan-type of series for $1/\pi$.

In his famous paper [53], "Modular equations and approximation to π," Ramanujan offered 17 beautiful series representation for $1/\pi$. He then remarked that two of these series

$$\frac{27}{4\pi} = \sum_{m=0}^{\infty}(2+15m)\frac{(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3}\left(\frac{2}{17}\right)^m \tag{3.5.1}$$

and

$$\frac{15\sqrt{3}}{2\pi} = \sum_{m=0}^{\infty}(4+33m)\frac{(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3}\left(\frac{4}{125}\right)^m \tag{3.5.2}$$

"belongs to the theory of q_2," where

$$q_2 = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-k^2)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;k^2)}\right).$$

Ramanujan did not provide details of his proofs of (3.5.1) and (3.5.2).

Ramanujan's formulas for (3.5.1) and (3.5.2) were first proved by J. M. and P. B. Borwein [33, p. 186] by establishing a general theorem. The following version of that theorem is due to Chan and Liaw [40].

Theorem 3.5.1. *(Chan and Liaw [33, p. 186]).Let*

$$K(x) := {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right), \quad \text{and} \quad \dot{K}(x) := \frac{dK(x)}{dx}.$$

For a positive rational number n, define the cubic singular moduli to be the unique number α_n satisfying

$$\frac{K(1-\alpha_n)}{K(\alpha_n)} = \sqrt{n}. \tag{3.5.3}$$

Set

$$\varepsilon(n) = \frac{3\sqrt{3}}{8\pi}(K(a_n))^{-2} - \sqrt{n}\left(\frac{3}{2}\,\alpha_n(1-\alpha_n)\frac{\dot{K}(\alpha_n)}{K(\alpha_n)} - \alpha_n\right), \tag{3.5.4}$$

$$a_n := \frac{8\sqrt{3}}{9}\left(\varepsilon(n) - \sqrt{n}\alpha_n\right), \tag{3.5.5}$$

and

$$b_n := \frac{2\sqrt{3n}}{3}\sqrt{1-H_n}, \tag{3.5.6}$$

where

$$H_n = 4\alpha_n(1 - \alpha_n) \tag{3.5.7}$$

Then

$$\frac{1}{\pi} = \sum_{m=0}^{\infty}(a_n + b_n m)\frac{(\frac{1}{2})_m(\frac{1}{3})(\frac{2}{3})_m}{H_n^m} \tag{3.5.8}$$

The above theorem indicates that for each positive rational number n, we can easily derive a series for $1/\pi$ belonging to the "theory of q_2" if the values of α_n and $\varepsilon(n)$ (the rest of the constants can be computed from these) are known. The computation of these constants for any given n is far from trivial Using cubic Russel-type modular equations (see [39]) and Kronecker's Limit Formula, Chan and Liaw [40] discovered new series for $1/\pi$ belonging to the " theory of q_2 " They also established some new formulas satisfied by $\varepsilon(n)$ which lead to the calculation of the constant a_n in (3 5 8) They established the following theorem for the calculation of a_n

Theorem 3.5.2. *([39 p 225, Corollary 2 7]) With a_n and H_n defined in Theorem 3 5 1 we have*

$$a_n = \frac{H_n}{2\sqrt{3}}\frac{dm}{d\alpha}(1 - \alpha_n, \alpha_n),$$

where α_n is related to Ramanathan's parameter μ_n, defined as in Theorem 2 3 1, by

$$\frac{1}{\alpha_n} = \mu_n^2 + 1 \tag{3.5.9}$$

If p and q are positive integers and $n = pq$, then the constant $\frac{dm}{d\alpha}(1 - \alpha_n\ \alpha_n)$ can be calculated by employing (3 5 10) below, which is also due to Chan and Liaw [39 p 226], provided we have modular equations of degrees p and q and the singular moduli α_{pq} and $\alpha_{q/p}$

$$\frac{dm_{pq}}{d\alpha}(1 - \alpha_{pq}\ \alpha_{pq}) = m_p(\alpha_{q/p}, \alpha_{pq})\frac{dm_q}{d\alpha}(1 - \alpha_{pq}\ \alpha_{q/p}) + m_q(1 - \alpha_{pq}\ \alpha_{q/p})\frac{d\beta}{d\alpha}\ \frac{dm_p}{d\beta}(\alpha_{q\ p}\ \alpha_{pq}) \tag{3.5.10}$$

Chan and Liaw [39] calculated the constants a_n b_n and H_n for n =2 5 7 10 11 14 19 26 31 34 59 35 55, 70 91 110, 115, 119, 154 and 455 We note that the numbers which are multiples of 3 are missing above This is probably due to the non availability of cubic modular equations of degree 3 and the corresponding values of α_n. In the next section, we establish two new cubic modular equations of degree 3, which then combined with some other cubic modular equations of prime degrees and the values of μ_n, can be applied to find some new values of the constants a_n, b_n, and H_n. These values and (3.5.8) will lead to some new series for $1/\pi$.

3.6 Cubic modular equations of degree 3

We recall from Chapter 1, the cubic theta-functions

$$b(q) = \frac{f^3(-q)}{f(-q^3)}, \qquad c(q) = \frac{3q^{1/3}f^3(-q^3)}{f(-q)}, \tag{3.6.1}$$

and

$$a(q) = \left\{\frac{f^{12}(-q) + 27qf^{12}(-q^3)}{f^3(-q)f^3(-q^3)}\right\}^{1/3}. \tag{3.6.2}$$

Also, the transformation formulas [17, p. 101-103] for the above three cubic theta-functions are

$$a(q) = z, \; b(q) = (1-\alpha)^{1/3}z, \quad \text{and } c(q) = \alpha^{1/3}z, \tag{3.6.3}$$

where $z = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha\right)$.

Theorem 3.6.1. *We have*

(i) $m = \dfrac{3\beta^{1/3}}{1-(1-\alpha)^{1/3}}$,

(ii) $3\sqrt{3}\left\{\left(\dfrac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^{1/4} + \left(\dfrac{\alpha\beta}{(1-\alpha)(1-\beta)}\right)^{1/4}\right\} = \left(\dfrac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{1/2} - 9.$

Proof. (i) By Entry 1(iv)[15, p. 346], we have

$$1 + 9q\frac{f^3(-q^9)}{f^3(-q)} = \left(1 + 27q\frac{f^{12}(-q^3)}{f^{12}(-q)}\right)^{1/3}. \tag{3.6.4}$$

Cubing both sides of (3.6.4) and then employing (3.6.1) and (3.6.2), we obtain

$$b(q) + 3c(q^3) = a(q). \tag{3.6.5}$$

Transcribing (3.6.5) with the help of (3.6.3), we find that

$$(1-\alpha)^{1/3}z_1 + 3\beta^{1/3}z_3 = z_1. \tag{3.6.6}$$

Setting $m = z_1/z_3$ and simplifying (3.6.6), we finish the proof.

(ii) We rewrite the identitty in Theorem 2.2.2 as

$$(LM)^3 + \left(\frac{3}{LM}\right)^3 = \left(\frac{M}{L}\right)^6 - 9, \tag{3.6.7}$$

where

$$L = \frac{f(-q)}{q^{1/12}f(-q^3)} \qquad \textit{and} \qquad M = \frac{f(-q^3)}{q^{1/4}f(-q^9)}.$$

Employing (3.6.1) in (3.6.7), we find that

$$L^4 = \frac{3b(q)}{c(q)} \qquad \text{and} \qquad M^4 = \frac{3b(q^3)}{c(q^3)}. \tag{3.6.8}$$

Using (3.6.3) in (3.6.8), and then simplifying for $(LM)^3$ and $(M/L)^6$, we obtain

$$(LM)^3 = 3^{3/2}\left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^{1/4} \qquad \text{and} \qquad \left(\frac{M}{L}\right)^6 = \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{1/2}. \tag{3.6.9}$$

Combining (3.6.7) and (3.6.9), we complete the proof of (ii). □

Remark 3.6.1. From (3.6.8) and (3.6.9), it is clear that if we have eta-function identities of the type :

$$g(P,Q) = 0,$$

where

$$P = \frac{f(-q)}{q^{1/12}f(-q^3)} \qquad \textit{and} \qquad Q = \frac{f(-q^n)}{q^{n/12}f(-q^{3n})}$$

then we always have a cubic modular equation of degree n. Similarly, we can obtain cubic " mixed " modular equations from the eta-function identities with eight arguments of the type:

$$g(P,Q) = 0,$$

where

$$P = \frac{f(-q)f(-q^p)}{q^{1/2}f(-q^3)f(-q^{3p})} \quad \textit{and} \quad Q = \frac{f(-q^n)f(-q^{pn})}{q^{n/2}f(-q^{3n})f(-q^{3pn})},$$

where p and n are positive integers. For examples, Theorems 3.2.7-3.2.10 give cubic " mixed " modular equations for the sets of degrees {1, 2, 5, 10}, {1, 2, 7, 14}, {1, 2, 11, 22}, and { 1, 2, 13, 26 }, respectively.

By adopting the method of Chan and Liaw [39], employing the cubic modular equations and the corresponding new values of μ_n, we can obtain the new values of the constants a_n, b_n, and H_n for $n = 3$, 6, 15, and 22. For example, we obtain

$$a_3 = \frac{r\left\{7\cdot 2^{7/3} + 2^{13/3}\sqrt{3} - 10m^2 - 6\sqrt{3}m^2 + k\right\}}{m^6(2^{2/3} - 2m)^2},$$

where $r = 2\sqrt{3}(3+2\sqrt{3})$, $m = (5+3\sqrt{3})^{1/3}$, and $k = (104+60\sqrt{3})^{2/3}$

$$b_3 = \frac{2(4+3\sqrt{3})}{(5+3\sqrt{3})}, \quad \text{and} \quad H_3 = \frac{9+6\sqrt{3}}{(5+3\sqrt{3})^2}.$$

Chapter 4

Explicit Evaluations of Ramanujan-Selberg Continued Fraction

4.1 Introduction

Let $\phi(q)$ and $\psi(q)$ be defined as in (1.1.3) and (1.1.4). For $|q| < 1$, Ramanujan-Selberg continued fraction $Z(q)$ is defined by

$$Z(q) := \frac{q^{1/8}\psi(q)}{\phi(q)} = \frac{q^{1/8}}{1}\,\frac{q}{+1+q}\,\frac{q^2}{+1+q^2}\,\frac{q^3}{+1+q^3}\,{}_{+\cdots}\quad |q| < 1. \tag{4.1.1}$$

This continued fraction was recorded by Ramanujan at the beginning of Chapter 19 of his second notebook [15, p. 221]. The equality in (4.1.1) was proved by Ramanathan [48]. Closely related to $Z(q)$ is the continued fraction $H(q)$ [59, p. 82], defined by

$$H(q) := \frac{f(-q)}{q^{1/8}f(-q^4)} = q^{1/8} - \frac{q^{7/8}}{1-q}\,\frac{q^2}{+1+q^2}\,\frac{q^3}{-1-q^3}\,\frac{q^4}{+1+q^4}\,{}_{-\cdots}. \tag{4.1.2}$$

By [15, p. 115, Entry 8(xii)] and (4.1.2), we find that

$$H(q) = \frac{\phi(-q^2)}{q^{1/8}\psi(q)}. \tag{4.1.3}$$

Also, employing (1.1.4) and [15. p. 37, (22.4)], we have

$$H(q) = \frac{(q;q^2)_\infty}{q^{1/8}(-q^2;q^2)_\infty}. \tag{4.1.4}$$

Again, for $|q| < 1$, define

$$N(q) := 1 + \frac{q}{1+}\,\frac{q+q^2}{1}\,\frac{q^3}{+1+}\,\frac{q^2+q^4}{1}\,{}_{+\cdots}. \tag{4.1.5}$$

In his notebook [54, p. 290], Ramanujan asserted that

$$N(q) = \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}. \tag{4.1.6}$$

This formula was first proved in print by A. Selberg [58].

In his lost notebook, Ramanujan [56, p. 44] also stated that, if $|q| < 1$ and

$$L(q) = \frac{1+q}{1} \underset{+}{} \frac{q^2}{1} \underset{+}{} \frac{q+q^3}{1} \underset{+}{} \frac{q^4}{1} \underset{+}{} \tag{4.1.7}$$

then

$$L(q) = \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}. \tag{4.1.8}$$

From (4.1.1) and (4.1.5) - (4.1.8), we easily see that

$$Z(q) = \frac{q^{1/8}}{N(q)} = \frac{q^{1/8}}{L(q)} = \frac{q^{1/8}(-q^2;q^2)_\infty}{(-q;q^2)_\infty}. \tag{4.1.9}$$

By setting

$$T(q) := \frac{q^{1/8}}{1} \underset{+}{} \frac{-q}{1} \underset{+}{} \frac{-q+q^2}{1} \underset{+}{} \frac{-q^3}{1} \underset{+}{}, \tag{4.1.10}$$

we also note that

$$T(q) = \frac{q^{1/8}}{N(-q)} = \frac{q^{1/8}}{L(-q)} = \frac{q^{1/8}(-q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{4.1.11}$$

In Sections 4.3-4.5 of this chapter, we find several modular relations connecting the above continued fractions in different arguments.

We observe that Vasuki and Shivashankar [59] had found explicit values of $H(e^{-\pi\sqrt{n}})$ for $n = 3, 1/3, 5, 1/5, 7, 1/7, 13$ and $1/13$ by using eta-function identities and transformation formulas. In this chapter, we also find several new explicit values of $H(e^{-\pi\sqrt{n}})$ by using the parameter J_n, defined by

$$J_n = \frac{f(-q)}{\sqrt{2}q^{1/8}f(-q^4)}; \quad q := e^{-\pi\sqrt{n}}, \tag{4.1.12}$$

where n is any positive real number We to note that the parameter J_n is equivalent to Yi's parameter $r_{4,n}$ defined in Chapter 1. In Sections 4.6 and 4.7, we evaluate several explicit values of the parameter J_n and the continued fraction $H(e^{-\pi\sqrt{n}})$, respectively. In Section 4.8, we establish general formulas for explicit evaluations of $Z(e^{-\pi\sqrt{n}})$ and $Z(e^{-\pi/\sqrt{n}})$ in terms of the parameter $r_{k,n}$. We also give some particular examples. Previously, Zhang [70, p. 11, Theorems 2.1 and 2.2], established general formulas for explicit evaluations of $Z(e^{-\pi\sqrt{n}})$ and $T(e^{-\pi\sqrt{n}})$ in terms of Ramanujan's singular moduli. In fact, he proved that

$$Z(q) = \frac{\alpha_n^{1/8}}{\sqrt{2}}, \tag{4.1.13}$$

and

$$T(q) = \frac{1}{\sqrt{2}}\left(\frac{\alpha_n}{1-\alpha_n}\right)^{1/8}, \tag{4.1.14}$$

where $q = e^{-\pi\sqrt{n}}$ and the singular modulus α_n is that unique positive number between 0 and 1 satisfying

$$\sqrt{n} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha_n\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha_n\right)}.$$

Remark 4.1.1. In [70], Ramanujan-Selberg continued fraction was denoted by $S_1(q)$. In this thesis, we use the notation $Z(q)$ for $S_1(q)$.

4.2 Some eta-function identities and modular equations

In this section, we record some eta-function identities and modular equations which will be used in the subsequent sections of this chapter.

Theorem 4.2.1. *(Yi, [66, p. 36, Theorem 3.5.1]) If*

$$P = \frac{f(-q)}{q^{1/8}f(-q^4)} \quad \textit{and} \quad Q = \frac{f(-q^2)}{q^{1/4}f(-q^8)},$$

then

$$(PQ)^4 + \frac{4}{PQ}^4 = \left(\frac{Q}{P}\right)^{12} - 16\left(\frac{Q}{P}\right)^4 - 16\left(\frac{P}{Q}\right)^4. \tag{4.2.1}$$

Theorem 4.2.2. *(Yi, [66, p. 37, Theorem 3.5.2]) If*

$$P = \frac{f(-q)}{q^{1/8}f(-q^4)} \quad \textit{and} \quad Q = \frac{f(-q^3)}{q^{3/8}f(-q^{12})},$$

then

$$PQ + \frac{4}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2. \tag{4.2.2}$$

Theorem 4.2.3. *(Yi, [66, p. 38, Theorem 3.5.3]) If*

$$P = \frac{f(-q)}{q^{1/8}f(-q^4)} \quad \text{and} \quad Q = \frac{f(-q^5)}{q^{5/8}f(-q^{20})},$$

then

$$(PQ)^2 + \left(\frac{4}{PQ}\right)^2 = \left(\frac{Q}{P}\right)^3 - 5\left(\frac{Q}{P} + \frac{P}{Q}\right) + \left(\frac{P}{Q}\right)^3. \tag{4.2.3}$$

Theorem 4.2.4. *(Berndt, [15, p. 230, Entry 5(ii)]) If β has degree 3 over α, then*

$$(\alpha\beta)^{1/4} + ((1-\alpha)(1-\beta))^{1/4} = 1. \tag{4.2.4}$$

Theorem 4.2.5. *(Berndt, [15, p. 282, Entry 13(xv)]) If β has degree 5 over α then*

$$\left(Q - \frac{1}{Q}\right)^3 + 8\left(Q - \frac{1}{Q}\right) = 4\left(P - \frac{1}{P}\right), \tag{4.2.5}$$

where $P = (\alpha\beta)^{1/4}$ and $Q = (\beta/\alpha)^{1/8}$.

Theorem 4.2.6. *(Berndt, [15, p. 314, Entry 19(i)]) If β has degree 7 over α, then*

$$(\alpha\beta)^{1/8} + ((1-\alpha)(1-\beta))^{1/8} = 1. \tag{4.2.6}$$

Theorem 4.2.7. *(Berndt, [15, p. 363, Entry 7(i)]) If β has degree 11 over α, then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1. \tag{4.2.7}$$

Theorem 4.2.8. *(Berndt, [17, p. 387, Entry 62]) Let P, Q, and R be as defined by*

$$P = 1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)},$$

$$Q = 64\left(\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right),$$

and

$$R = 32\sqrt{\alpha\beta(1-\alpha)(1-\beta)},$$

respectively. Then, if β has degree 13 over α,

$$\sqrt{P}(P^3 + 8R) - \sqrt{R}(11P^2 + Q) = 0. \tag{4.2.8}$$

Theorem 4.2.9. *(Berndt, [17, p. 385, Entry 53]) If*

$$P = 1 + (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8},$$

$$Q = 4\left((\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\right),$$

and

$$R = 4\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}.$$

Then, if β has degree 15 over α,

$$P(P^2 - Q) + R = 0. \tag{4.2.9}$$

Theorem 4.2.10. *(Berndt, [17, p. 387. Entry 62]) Let P, Q, and R be as defined in Theorem 4.2.8, then, if β has degree 17 over α,*

$$P^3 - R^{1/3}(10P^2 + Q) + 13R^{2/3}P + 12R = 0. \tag{4.2.10}$$

Theorem 4.2.11. *(Berndt, [17, p. 386, Entry 58]) Let,*

$$P = 1 - (\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4},$$

$$Q = 16\left((\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}\right),$$

and

$$R = 16\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}.$$

Then, if β has degree 19 over α,

$$P^5 - 7P^2R - QR = 0. \tag{4.2.11}$$

Theorem 4.2.12. *(Berndt, [15, p. 411, Entry 15(i)]) If β has degree 23 over α, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1. \tag{4.2.12}$$

Theorem 4.2.13. *(Berndt [17, p. 385, Entry 54]) Let P, Q, and R are as defined in Theorem 4.2.9. If β has degree 31 over α. Then*

$$P^2 - Q = \sqrt{PR}. \tag{4.2.13}$$

4.3 Relations between $H(q)$ and $H(q^n)$

In this section, we state and prove some relations between $H(q)$ and $H(q^n)$.

Theorem 4.3.1. *We have*

$$\text{(i)}\quad \alpha = \frac{16}{16 + H^8(q)} \quad \textit{and} \quad \text{(ii)}\quad \beta = \frac{16}{16 + H^8(q^n)},$$

where β has degree n over *α.*

Proof. We apply Entry 12(ii) and (iv) [15, p. 124] in the definition of $H(q)$ in (4.1.2) to complete the proof. □

Theorem 4.3.2. *We have*

$$\text{(i)}\quad \alpha = \frac{-16}{H^8(-q)} \quad \textit{and} \quad \text{(ii)}\quad \beta = \frac{-16}{H^8(-q^n)},$$

where β has degree n over α.

Proof. We replace q by $-q$ in the definition of $H(q)$ and then employ Entry 12(i) and (iv) [15, p. 124] to at arrive at the desired result. □

Remark 4.3.1. By Theorem 4.3.1 and for any given modular equation of degree n, we can obtain a relation between $H(q)$ and $H(q^n)$. In the following theorem, we illustrate this with n =3, 5, and 7 in (iii), (iv), and (v) respectively.

Theorem 4.3.3. *Let* $a = H(q)$, $b = H(-q)$, $c = H(q^2)$, $u = H(q^3)$, $v = H(q^5)$, *and* $w = H(q^7)$. *Then we have*

(i) $a^8 + b^8 + 16 = 0$,

(ii) $256a^8 + 16a^{16} + 16a^8c^8 + a^{16}c^8 - c^{16} = 0$,

(iii) $a^4 - 4au - a^3u^3 + u^4 = 0$,

(iv) $a^6 - 16av - 5a^4v^2 - 5a^2v^4 - a^5v^5 + v^6 = 0$,

(v) $a^8 - 64aw - 112a^2w^2 - 112a^3w^3 - 70a^4w^4 - 28a^5w^5 - 7a^6w^6 + a^7w^7 + w^8 = 0$.

Proof. From Theorem 4.3.1(i) and Theorem 4.3.2, we easily arrive at (i). To prove (iii)-(v) we employ Theorem 4.3.1 in Theorems 4.2.4, 4.2.5, and 4.2.6, respectively. We note that (ii)-(iv) can also be proved by employing Theorems 4.2.1-4.2.3. □

4.4 Relations between $Z(q)$ and $Z(q^n)$

Theorem 4.4.1. *We have*

$$\text{(i)}\quad \alpha = 16Z^8(q), \quad \text{(ii)}\quad \beta = 16Z^8(q^n), \quad \textit{and} \quad \text{(iii)}\quad \alpha = \frac{16T^8(q)}{1+16T^8(q)}.$$

where β *has degree* n *over* α.

Proof. To prove (i) and (ii), we employ Entry 10(i) and Entry 11(i) [15, p. 122-123] in the definition of $Z(q)$ in (4.1.1). Proof of (iii) follows easily from (4.1.14). □

Remark 4.4.1. For any given modular equation of degree n, we can easily obtain the relations connecting $Z(q)$ and $Z(q^n)$ by using Theorem 4.4.1. We give some examples in the following theorem.

Theorem 4.4.2. *Let* $U = Z(q)$, $V = Z(q^3)$, $W = Z(q^5)$, *and* $X = Z(q^7)$. *Then, we have*

(i) $U^4 - UV + 4U^3V^3 - V^4 = 0$,

(ii) $U^6 - UW + 5U^4W^2 - 5U^2W^4 + 16U^5W^5 - W^6 = 0$,

(iii) $U^8 + X^8 - UX + 7U^2X^2 - 28U^3X^3 + 70U^4X^4 - 112U^5X^5 + 112U^6X^6 - 64U^7X^7 = 0$.

Proof. Employing Theorem 4.4.1 in Theorems 4.2.4 - 4.2.6, we readily deduce (i)-(iii), respectively. □

4.5 Relations connecting $H(\pm q)$, $Z(q)$ and $T(q)$

Theorem 4.5.1. *Let* $u = H(q)$, $x = H(-q)$, $U = Z(q)$, *and* $y = T(q)$. *We have*

(i) $u^8U^8 + 16U^8 - 1 = 0$,

(ii) $x^8u^8 + 1 = 0$,

(iii) $u = \dfrac{1}{y}$,

(iv) $x^8y^8 + 16y^8 + 1 = 0$.

Proof. (i) follows from Theorem 4.3.1(i) and Theorem 4.4.1(i). To prove (ii), we use Theorem 4.3.2(i) and Theorem 4.4.1(i). To prove (iii), we employ Theorem 4.3.1(i) and Theorem 4.4.1(iii). Finally, employing Theorem 4.3.2(i) and Theorem 4.4.1(iii), we easily arrive at (iv). □

4.6 Theorems on J_n and explicit values

This section is devoted to establishing some general theorems for the explicit evaluations of J_n and find some of its explicit values.

Theorem 4.6.1. *If* J_n *is defined as in* (4.1.12), *then we have*

$$J_1 = 1 \quad \textit{and} \quad J_{1/n} = \frac{1}{J_n}.$$

Proof. Follows directly from Theorem 1.1.3 and the definition of J_n. □

Theorem 4.6.2. *We have*

(i) $16\left((J_nJ_{4n})^4 + \dfrac{1}{(J_nJ_{4n})^4}\right) = \left(\dfrac{J_n}{J_{4n}}\right)^{12} - 16\left(\dfrac{J_{4n}}{J_n}\right)^4 - 16\left(\dfrac{J_n}{J_{4n}}\right)^4$,

(ii) $2\left(J_nJ_{9n} + \dfrac{1}{J_nJ_{9n}}\right) = \left(\dfrac{J_{9n}}{J_n}\right)^2 + \left(\dfrac{J_n}{J_{9n}}\right)^2$,

(iii) $4\left((J_nJ_{25n})^2 + \dfrac{1}{(J_nJ_{25n})^2}\right) = \left(\dfrac{J_{25n}}{J_n}\right)^3 - 5\left(\dfrac{J_{25n}}{J_n}\right) - 5\left(\dfrac{J_n}{J_{25n}}\right) + \left(\dfrac{J_n}{J_{25n}}\right)^3$,

(iv) $(1 + J_nJ_{49n})^8 - \left(1 + J_n^8\right)\left(1 + J_{49n}^8\right) = 0$.

Proof. Employing the definition J_n in Theorems 4.2.1-4.2.3, and 4.2.6, we complete the proof of (i)-(iv), respectively. □

Theorem 4.6.3. *We have*

(i) $J_2 = 2^{1/8}\left(1+\sqrt{2}\right)^{1/8}$,

(ii) $J_3 = \left(2+\sqrt{3}\right)^{1/4}$,

(iii) $J_4 = 2^{5/16}\left(1+\sqrt{2}\right)^{1/4}$,

(iv) $J_5 = \dfrac{1}{\sqrt{2}}\left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)^{1/2}$

(v) $J_7 = \left(8+3\sqrt{7}\right)^{1/4}$,

(vi) $J_9 = \dfrac{1}{2}+\dfrac{3^{1/4}}{\sqrt{2}}+\dfrac{\sqrt{3}}{2}$,

(vii) $J_{25} = \dfrac{1}{2}\left(3+\sqrt[4]{5}+\sqrt{5}+\sqrt[4]{5^3}\right)$,

(viii) $J_{49} = \dfrac{1}{4}\left(\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}+\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}}\right)^2$,

(ix) $J_8 = 2^{1/4}\left(1+\sqrt{2}\right)^{3/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8}$.

Proof. First we set we set $n = 1/2$, $1/3$, 1, $1/5$, $1/7$, 1, 1, and 1 in Theorem 4.6.2(i), Theorem 4.6.2(ii), Theorem 4.6.2(i), Theorem 4.6.2(iii), Theorem 4.6.2(iv), Theorem 4.6.2(ii), Theorem 4.6.2(iii), and Theorem 4.6.2(iv), respectively, and then simplify by using Theorem 4.6.1. Solve the resulting polynomial equation equations, we readily arrive at (i)-(viii).

Setting $n = 2$ in Theorem 4.8.3(i), employing the value of J_2 in (i) and solving the resulting equation, we deduce (ix). □

Remark 4.6.1. From Theorem 4.6.1 and the above theorem, the values of J_n for n =1/2, 1/3, 1/4, 1/5, 1/7, 1/9, 1/25, 1/49, and 1/8 also follow immediately.

Theorem 4.6.4. *We have*

(i) $J_6 = r_{4,6} = \left(1+\sqrt{2}\right)^{3/8}\left(2(1+\sqrt{2}+\sqrt{6})\right)^{1/8}$,

(ii) $J_{10} = \dfrac{\left(1+\sqrt{5}\right)^{9/8}\left(2+3\sqrt{2}+\sqrt{5}\right)^{1/8}}{2}$,

(iii) $J_{16} = 2^{3/8}\left(1+\sqrt{2}\right)^{1/2}\left(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4}\right)^{1/8}$,

(iv) $J_{18} = 2^{1/8}\left(\sqrt{3}+\sqrt{2}\right)\left(1+35\sqrt{2}-28\sqrt{3}\right)^{1/8}$,

(v) $J_{36} = \dfrac{\left(\sqrt{3}+1\right)^{2/3}\left(-\sqrt{2}+4+2\sqrt{3}+3^{3/4}\left(\sqrt{3}+1\right)\right)^{1/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{1/4}\right)^{1/3}}{2^{35/48}\left(\sqrt{3}-\sqrt{2}\right)^{1/3}\left(\sqrt{2}-1\right)^{5/12}}$.

Proof. First we recall from [66, p. 14, Corollary 2.1.5(i)] that

$$r_{k^2,n} = r_{k,nk} r_{k,n/k}. \tag{4.6.1}$$

Setting $k = 2$ and $n = 6$ in (4.6.1), we obtain

$$r_{4,6} = r_{2,12} \cdot r_{2,3}. \tag{4.6.2}$$

Now, from Section 1.3, we recall that

$$r_{2,3} = \left(1 + \sqrt{2}\right)^{1/6} \quad \text{and} \quad r_{2,12} = \left(1 + \sqrt{2}\right)^{5/24} \left(2(1 + \sqrt{2} + \sqrt{6})\right)^{1/8}.$$

Substituting these in (4.6.2), we complete the proof of (i).

The proofs of (ii)-(v) can be given in a similar fashion. □

Remark 4.6.2. By using Theorem 4.6.1 and the above theorem, we can easily evaluate $J_{1/n}$ for n =6, 10, 16, 18, and 36.

Theorem 4.6.5. *We have*

(i) $J_{11} = \left(\dfrac{1 + \sqrt{1 - 4a^{12}}}{2a^6}\right)^{1/4}$,

where $a = -\dfrac{2^{1/3}}{3} + \dfrac{1}{6}\left(38 - 6\sqrt{33}\right)^{1/3} + \dfrac{\left(19 + 3\sqrt{33}\right)^{1/3}}{3 \cdot 2^{2/3}}$,

(ii) $J_{13} = \left(18 + 5\sqrt{13} + 6\sqrt{18 + 5\sqrt{13}}\right)^{1/4}$,

(iii) $J_{15} = \left(\dfrac{16 + \sqrt{3}(7 + \sqrt{5})}{7 - 3\sqrt{5}}\right)^{1/4}$,

(iv) $J_{17} = \left(\dfrac{2 + \sqrt{4 - 4\left(20 + 5\sqrt{17} - 2\sqrt{206 + 50\sqrt{17}}\right)^2}}{40 + 10\sqrt{17} - 4\sqrt{206 + 50\sqrt{17}}}\right)^{1/4}$

(v) $J_{19} = \left(\dfrac{1 + \sqrt{1 - 4k^4}}{2k^2}\right)^{1/4}$,

where $k = \dfrac{1}{24}\left(-20 + \left(2944 - 384\sqrt{57}\right)^{1/3} + 4\left(46 + 6\sqrt{57}\right)^{1/3}\right)$,

(vi) $J_{23} = \left(\dfrac{1 + \sqrt{1 - 4n^{24}}}{2n^{12}}\right)^{1/4}$,

where $n = \dfrac{-1}{3 \cdot 2^{1/3}} + \dfrac{1}{6}\left(50 - 6\sqrt{69}\right)^{1/3} + \dfrac{\left(25 + 3\sqrt{69}\right)^{1/3}}{3 \cdot 2^{2/3}}$,

(vii) $J_{31} = \left(\frac{1+\sqrt{1-4d^8}}{2d^4}\right)^{1/4}$,

where $d = \frac{1}{2} + \frac{1}{6}\left(\frac{-27+3\sqrt{93}}{2}\right)^{1/3} - \frac{1}{2^{2/3}\left(-27+3\sqrt{93}\right)^{1/3}}$.

Proof of (i): Using the definition of J_n in Theorem 4.3.1, we find that

$$\alpha = \frac{1}{1+J_n^8} \quad \text{and} \quad \beta = \frac{1}{1+J_{121n}^8}, \tag{4.6.3}$$

where β has degree 11 over α.

Setting $n = 1/11$ in (4.6.3) and simplifying by using the Theorem 4.6.1, we find that

$$\alpha = J_{11}^8\beta, \quad \beta = \frac{1}{1+J_{11}^8}, \quad 1-\alpha = \beta, \quad \text{and} \quad \alpha\beta = J_{11}^8\beta^2 \tag{4.6.4}$$

Substituting (4.6.4) in Theorem 4.2.7 and simplifying, we obtain

$$2\left(J_{11}^4\beta\right)^{1/2} + 2^{4/3}\left(J_{11}^4\beta\right)^{1/3} - 1 = 0, \tag{4.6.5}$$

Solving the above polynomial equation for real positive $a := \left(J_{11}^4\beta\right)^{1/6}$, we obtain

$$a = -\frac{2^{1/3}}{3} + \frac{1}{6}\left(38 - 6\sqrt{33}\right)^{1/3} + \frac{\left(19+3\sqrt{33}\right)^{1/3}}{3\,2^{2/3}}. \tag{4.6.6}$$

Then, from (4.6.4) and (4.6.6), we arrive at

$$a^6 J_{11}^8 - J_{11}^4 + a^6 = 0. \tag{4.6.7}$$

Solving (4.6.7) for J_{11}, we complete the proof of (i).

Similarly, we can prove (ii)-(vii) by using the definition of J_n in Theorem 4.3.1, setting n =1/13, 1/15, 1/17, 1/19, 1/23, and 1/31, in turn, and then appealing to the Theorems 4.2.8-4.2.13, respectively.

Remark 4.6.3. By Theorem 4.6.1 and the above theorem, the values of $J_{1/n}$ for n =11, 13, 15. 17, 19, 23, and 31 can also be found easily.

4.7 Explicit values of $H(q)$

In this section, we establish a general formula for the explicit evaluation of $H(e^{-\pi\sqrt{n}})$ and find some explicit values by using the particular values of J_n evaluated in the above section.

Theorem 4.7.1. *We have*

$$H(e^{-\pi\sqrt{n}}) = \sqrt{2}\, J_n.$$

Proof. The proof follows directly from the definitions of $H(q)$ and J_n. □

Theorem 4.7.2. *We have*

(i) $H(e^{-\pi}) = \sqrt{2}$,

(ii) $H(e^{-\pi\sqrt{2}}) = 2^{5/8}\left(1+\sqrt{2}\right)^{1/8}$,

(iii) $H(e^{-\pi\sqrt{3}}) = \sqrt{2}\left(2+\sqrt{3}\right)^{1/4}$,

(iv) $H(e^{-2\pi}) = 2^{13/16}\left(1+\sqrt{2}\right)^{1/4}$,

(v) $H(e^{-\pi\sqrt{5}}) = \left(1+\sqrt{5}+\sqrt{2}\sqrt{1+\sqrt{5}}\right)^{1/2}$,

(vi) $H(e^{-\pi\sqrt{7}}) = \sqrt{2}\left(8+3\sqrt{7}\right)^{1/4}$,

(vii) $H(e^{-3\pi}) = \dfrac{1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}}{\sqrt{2}}$,

(viii) $H(e^{-5\pi}) = \dfrac{3+\sqrt[4]{5}+\sqrt{5}+\sqrt[4]{5^3}}{\sqrt{2}}$,

(ix) $H(e^{-7\pi}) = \dfrac{1}{2\sqrt{2}}\left(\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}+\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}}\right)^2$,

(x) $H(e^{-2\sqrt{2}\pi}) = 2^{3/4}\left(1+\sqrt{2}\right)^{3/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8}$.

Proof. Employing the value that $J_1 = 1$ in Theorem 4.7.1 we arrive at (i). To prove (ii)-(x), we employ the values of J_n from Theorem 4.6.3 in Theorem 4.7.1. □

Remark 4.7.1. From Theorems 4.6.1 and 4.7.1, it is obvious that

$$H(e^{-\pi/\sqrt{n}}) = \sqrt{2}J_{1/n} = \frac{\sqrt{2}}{J_n}. \tag{4.7.1}$$

So by employing the values of J_n from Theorem 4.6.3 in (4.7.1), we can easily evaluate $H(e^{-\pi/\sqrt{n}})$ for n =2, 3, 4, 5, 7, 9, 25, 49, and 8. For examples,

$$H(e^{-\pi/2}) = 2^{3/16}\left(\sqrt{2}-1\right)^{1/4}, \quad H(e^{-\pi/\sqrt{5}}) = \left(1+\sqrt{5}-\sqrt{2}\sqrt{1+\sqrt{5}}\right)^{1/2}$$

and

$$H(e^{-\pi/7}) = \frac{1}{2\sqrt{2}}\left(\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}-\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}}\right)^2.$$

Theorem 4.7.3. *We have*

(i) $H(e^{-\pi\sqrt{6}}) = \sqrt{2}(1+\sqrt{2})^{3/8}\left(2(1+\sqrt{2}+\sqrt{6})\right)^{1/8}$,

(ii) $H(e^{-\pi\sqrt{10}}) = \dfrac{\left(1+\sqrt{5}\right)^{9/8}\left(2+3\sqrt{2}+\sqrt{5}\right)^{1/8}}{\sqrt{2}}$,

(iii) $H(e^{-4\pi}) = 2^{7/8}\left(\sqrt{2}+1\right)^{1/2}\left(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4}\right)^{1/8}$,

(iv) $H(e^{-3\sqrt{2}\pi}) = 2^{5/8}\left(\sqrt{3}+\sqrt{2}\right)\left(1+35\sqrt{2}-28\sqrt{3}\right)^{1/8}$,

(v) $H(e^{-6\pi}) = \dfrac{\left(\sqrt{3}+1\right)^{2/3}\left(-\sqrt{2}+4+2\sqrt{3}+3^{3/4}\left(\sqrt{3}+1\right)\right)^{1/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}{2^{11/48}\left(\sqrt{3}-\sqrt{2}\right)^{1/3}\left(\sqrt{2}-1\right)^{5/12}}$.

Proof. We use the values of J_n from Theorem 4.6.4 in Theorem 4.7.1 to complete the proof. □

The values of $H(e^{-\pi/\sqrt{n}})$ for n =6, 10, 18, and 36 also follow from Theorem 4.6.4 and (4.7.1).

Theorem 4.7.4. *We have*

(i) $H(e^{-\pi\sqrt{11}}) = \sqrt{2}\left(\dfrac{1+\sqrt{1-4a^{12}}}{2a^6}\right)^{1/4}$,

where $a = -\dfrac{2^{1/3}}{3}+\dfrac{1}{6}\left(38-6\sqrt{33}\right)^{1/3}+\dfrac{\left(19+3\sqrt{33}\right)^{1/3}}{3\cdot 2^{2/3}}$,

(ii) $H(e^{-\pi\sqrt{13}}) = \sqrt{2}\left(18+5\sqrt{13}+6\sqrt{18+5\sqrt{13}}\right)^{1/4}$,

(iii) $H(e^{-\pi\sqrt{15}}) = \sqrt{2}\left(\dfrac{16+\sqrt{3\left(54+14\sqrt{5}\right)}}{7-3\sqrt{5}}\right)^{1/4}$,

(iv) $H(e^{-\pi\sqrt{17}}) = \sqrt{2}\left(\dfrac{2+\sqrt{4-4\left(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}\right)^2}}{40+10\sqrt{17}-4\sqrt{206+50\sqrt{17}}}\right)^{1/4}$,

(v) $H(e^{-\pi\sqrt{19}}) = \sqrt{2}\left(\dfrac{1+\sqrt{1-4k^4}}{2k^2}\right)^{1/4}$,

where $k = \dfrac{1}{24}\left(-20+\left(2944-384\sqrt{57}\right)^{1/3}+4\left(46+6\sqrt{57}\right)^{1/3}\right)$,

(vi) $H(e^{-\pi\sqrt{23}}) = \sqrt{2}\left(\dfrac{1+\sqrt{1-4n^{12}}}{2n^6}\right)^{1/4}$,

where $n = \dfrac{-1}{3\cdot 2^{1/3}}+\dfrac{1}{6}\left(50-6\sqrt{69}\right)^{1/3}+\dfrac{\left(25+3\sqrt{69}\right)^{1/3}}{3\cdot 2^{2/3}}$,

(vii) $H(e^{-\pi\sqrt{31}}) = \sqrt{2}\left(\dfrac{1+\sqrt{1+4d^8}}{2d^4}\right)^{1/4}$,

$$\text{where } d = \frac{1}{2} + \frac{1}{6}\left(\frac{-27+3\sqrt{93}}{2}\right)^{1/3} - \frac{1}{2^{2/3}(-27+3\sqrt{93})^{1/3}}.$$

Proof. Proof of the theorem follows directly from Theorem 4.6.5 and Theorem 4.7.1. □

Remark 4.7.2. Values of $H(e^{-\pi/\sqrt{n}})$ for n =11, 13, 15, 17, 19, 23, and 31 also follow readily from Theorem 4.6.5 and (4.7.1).

4.8 Explicit formulas for $Z(q)$ and explicit values

Recall the definitions of Weber-Ramanujan class invariants G_n and g_n from Chapter 1 as

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \quad \text{and} \quad g_n := 2^{-1/4}q^{-1/24}\chi(-q), \tag{4.8.1}$$

where $q := e^{-\pi\sqrt{n}}$. The two class invariants satisfy the properties (see [17, p. 187, Entry 2.1], [66, p. 18, Corollaries 2.2.4(i), (ii)])

$$g_{4n} = 2^{1/4}g_nG_n, \quad g_n^{-1} = g_{4/n}, \quad \text{and} \quad G_{1/n} = G_n. \tag{4.8.2}$$

We also note from [66, p. 13, Lemma 2.1.3(i)] and [66, p. 18, Theorem 2.2.3] that

$$r_{k,n/m} = r_{mk,n}r_{nk,m}^{-1}, \tag{4.8.3}$$

$$g_n = r_{2,n/2}, \quad \text{and} \quad G_n = \frac{r_{2,2n}}{2^{1/4}r_{2,\frac{n}{2}}}, \tag{4.8.4}$$

respectively, where $r_{k,n}$ is the as defined in (1.1.9) and k and n are positive real numbers.

Now, we state and prove two general formulas for the explicit evaluations of $Z(q)$ and then calculate some specific values.

Theorem 4.8.1. *We have*

$$Z(e^{-\pi\sqrt{n}}) = \frac{1}{2^{3/4}G_n^2g_n} = \frac{r_{2,n/2}}{2^{1/4}r_{2,2n}^2} = \frac{r_{4,n}}{2^{1/4}r_{2,2n}^3},$$

where G_n and g_n are Ramanujan's class invariants as defined in (4.8.1)

Proof. By [15, p. 39, Entry 24(iii)], we have

$$\psi(q) = \frac{f^2(-q^2)}{f(-q)} \quad \text{and} \quad \phi(q) = \frac{f^2(q)}{f(-q^2)}. \tag{4.8.5}$$

Substituting (4.8.5) in (4.1.1), we obtain

$$Z(q) = \frac{f^2(-q^2)}{2^{-1/2}q^{-1/12}f^2(q)} \times \frac{f(-q^2)}{2^{-1/4}q^{-1/24}f(-q)}. \tag{4.8.6}$$

From [15, p. 39, Entry 24(iii)], we also note that

$$\chi(q) = \frac{f(q)}{f(-q^2)}. \tag{4.8.7}$$

Now, setting $q := e^{-\pi\sqrt{n}}$ and then applying (4.8.6), (4.8.7), and (4.8.1), we complete the proof of the the first equality. Employing (4.8.4) to the first equality, we arrive at the second equality. To prove the third equality, we employ (4.8.3) to the second equality. □

Corollary 4.8.2. *We have*

(i) $Z(e^{-\pi}) = 2^{-5/8}$,

(ii) $Z(e^{-\pi\sqrt{2}}) = 2^{-1/2}\left(1+\sqrt{2}\right)^{-1/2}$,

(iii) $Z(e^{-\pi\sqrt{3}}) = 2^{-17/24}\left(1+\sqrt{3}\right)^{-1/4}$,

(iv) $Z(e^{-2\pi}) = 2^{-3/8}\left(1+\sqrt{2}\right)^{-1/4}$,

(v) $Z(e^{-\pi\sqrt{5}}) = \left(1+\sqrt{5}\right)^{-1/2}\left(\sqrt{\sqrt{5}+1}+\sqrt{2}\right)^{-1/4}$,

(vi) $Z(e^{-\pi\sqrt{6}}) = 2^{-1/2}\left(1+\sqrt{2}\right)^{-1/4}\left(1+\sqrt{2}+\sqrt{6}\right)^{-1/4}$,

(vii) $Z(e^{-\pi\sqrt{7}}) = 2^{-7/8}\left(3+\sqrt{7}\right)^{-1/4}$,

(viii) $Z(e^{-2\pi\sqrt{2}}) = 2^{-3/8}\left(1+\sqrt{2}\right)^{-3/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{-1/4}$,

(ix) $Z(e^{-3\pi}) = \dfrac{2^{1/8}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{1/4}\right)^{1/3}}{\left(1+\sqrt{3}\right)^{2/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{2/3}}$,

(x) $Z(e^{-\pi\sqrt{10}}) = 2^{1/4}\left(1+\sqrt{5}\right)^{-3/4}\left(2+3\sqrt{2}+\sqrt{5}\right)^{-1/4}$,

(xi) $Z(e^{-4\pi}) = 2^{-7/16}\left(1+\sqrt{2}\right)^{-1/4}\left(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4}\right)^{-1/4}$,

(xii) $Z(e^{-3\pi\sqrt{2}}) = 2^{-1/2}\left(\sqrt{3}+\sqrt{2}\right)^{-1}\left(1+35\sqrt{2}-28\sqrt{3}\right)^{-1/4}$,

(xiii) $Z(e^{-5\pi}) = 2^{-17/8}\left(\sqrt{5}-1\right)\left(5^{1/4}-1\right)$,

(xiv) $Z(e^{-6\pi}) = \dfrac{2^{1/6}\left(\sqrt{2}-1\right)^{5/6}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}{\left(\sqrt{3}+1\right)^{1/3}\left(\sqrt{2}+\sqrt{3}\right)^{2/3}\left(1+\sqrt{2}-\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\sqrt{6}\right)^{2/3}}$.

Proof. The parts (i)-(vi) and (viii)-(xiv) easily follow from Theorem 4.8.1 with the help of the values of $r_{k,n}$ in Section 1.3. To prove (vii), we use the values of G_7 and $g_7 = r_{2,7/2}$ from [17] and Section 1.3, respectively. □

Theorem 4.8.3. *We have*

$$Z(e^{-\pi/\sqrt{n}}) = \frac{g_n}{2^{1/2}G_n} = \frac{r_{2,n/2}^2}{2^{1/4}r_{2,2n}} = \frac{r_{4,n}^2}{2^{1/4}r_{2,2n}^3}.$$

Proof. Replacing n by $1/n$ in Theorem 4.8.1 and then simplifying by using (4.8.2), we arrive at the first equality. To prove the second equality, we employ (4.8.4) to the first. Using (4.8.3) to the second equality, we finish the proof of the third one. □

Corollary 4.8.4. *We have*

(i) $Z(e^{-\pi/\sqrt{2}}) = 2^{-3/8}\left(\sqrt{2}-1\right)^{1/8}$,

(ii) $Z(e^{-\pi/\sqrt{3}}) = 2^{-7/8}\left(\sqrt{3}+1\right)^{1/4}$,

(iii) $Z(e^{-\pi/2}) = 2^{-3/16}\left(\sqrt{2}-1\right)^{1/4}$,

(iv) $Z(e^{-\pi/2\sqrt{2}}) = 2^{-1/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{-1/8}$,

(v) $Z(e^{-\pi/5}) = 2^{-17/8}\left(\sqrt{5}-1\right)\left(5^{1/4}+1\right)$,

(vi) $Z(e^{-\pi/3}) = \dfrac{\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{1/4}\right)}{2^{7/8}\left(1+\sqrt{3}\right)^{1/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}$,

(vii) $Z(e^{-\pi/\sqrt{6}}) = 2^{-3/8}\left(\dfrac{\sqrt{2}+1}{1+\sqrt{2}+\sqrt{6}}\right)^{1/8}$,

(viii) $Z(e^{-\pi/4}) = \dfrac{2^{3/16}\left(\sqrt{2}+1\right)^{1/4}}{\left(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4}\right)^{1/8}}$,

(ix) $Z(e^{-\pi/3\sqrt{2}}) = 2^{-3/8}\left(1+35\sqrt{2}-28\sqrt{3}\right)^{-1/8}$,

(x) $Z(e^{-\pi/6}) = \dfrac{\left(\sqrt{3}+1\right)^{1/3}\left(1+\sqrt{3}+3^{3/4}\sqrt{2}\right)^{2/3}}{2^{1/4}\left(\sqrt{2}+\sqrt{3}\right)^{1/3}\left(1+\sqrt{2}-\sqrt{3}+3^{3/4}\sqrt{12}\right)^{1/3}}$,

(xi) $Z(e^{-\pi/\sqrt{10}}) = \dfrac{\left(\sqrt{5}+1\right)^{3/8}}{2^{3/4}\left(2+3\sqrt{2}+\sqrt{5}\right)^{1/8}}$.

Proof. With the help of Theorem 4.8.3 and the values of $r_{k,n}$ listed in Section 1.3, we readily complete the proof. □

Remark 4.8.1. From the last equalities of Theorems 4.8.1 and 4.8.3, we have the transformation formula for $Z(q)$: $Z(e^{-\pi/\sqrt{n}}) = r_{4,n}\, Z(e^{-\pi\sqrt{n}})$.

Chapter 5

Explicit Evaluations of Ramanujan-Göllnitz-Gordon Continued Fraction

5.1 Introduction

Let Ramanujan-Göllnitz-Gordon continued fraction $K(q)$ be defined by

$$K(q) := \frac{q^{1/2}}{1+q_+}\frac{q^2}{1+q^3{}_+}\frac{q^4}{1+q^5{}_{+\cdots}}, \quad |q|<1. \tag{5.1.1}$$

On page 299 of his second notebook [54], Ramanujan recorded a product representation of $K(q)$, namely

$$K(q) := q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}, \tag{5.1.2}$$

where $(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$.

Without the knowledge of Ramanujan's work, Göllnitz [43] and Gordon [44], independently, rediscovered and proved (5.1.2). Shortly thereafter, Andrews [4] proved (5.1.2) as a corollary of a more general result. Ramanathan [49] also found an alternative proof of (5.1.2). In addition to (5.1.2), Ramanujan offered two other identities [54, p. 299] for $K(q)$, namely,

$$\frac{1}{K(q)} - K(q) = \frac{\phi(q^2)}{q^{1/2}\psi(q^4)} \tag{5.1.3}$$

and

$$\frac{1}{K(q)} + K(q) = \frac{\phi(q)}{q^{1/2}\psi(q^4)}. \tag{5.1.4}$$

Proofs of (5.1.3) and (5.1.4) can be found in Berndt's book [15, p. 221].

Chan and Huang [37] found many identities involving the continued fraction $K(q)$. They derived several relations connecting $K(q)$ and $K(q^n)$ by using modular equations. They also evaluated explicitly $K(e^{-\pi\sqrt{n}/2})$ for several positive integers n by using Weber-Ramanujan class invariants G_n and g_n. We record the following identity established by Chan and Huang [37, p. 78, (2.5)] for our future use:

$$K^2(q) = \frac{1 - \dfrac{\phi(q^2)}{\phi(q)}}{1 + \dfrac{\phi(q^2)}{\phi(q)}}. \tag{5.1.5}$$

Recently, Vasuki and Srivatsa Kumar [60] derived three new relations connecting the continued fractions $K(q)$ and the three continued fractions $K(q^5)$, $K(q^7)$, and $K(q^{11})$ by establishing some new theta-function identities. They also gave a new approach to the relation between $K(q)$ and $K(q^3)$ established by Chan and Huang [37].

In this chapter, we present some general theorems for the explicit evaluations of $K(q)$ by parameterizations and also give some examples. We end this introduction by recalling the parameters

$$h_{2,n} = \frac{\phi(e^{-\pi\sqrt{n/2}})}{2^{1/4}\phi(e^{-\pi\sqrt{2n}})} \quad \text{and} \quad s_{k,n} = \frac{f(q)}{k^{1/4}q^{(k-1)/24}f(-(-1)^k q^k)}, \tag{5.1.6}$$

where $q = e^{-\pi\sqrt{n/k}}$ with k and n being positive real numbers. We have already mentioned in Chapter 1, the parameter $s_{k,n}$ is due to Berndt [18, p. 9, (4.7)] and $h_{2,n}$ is the particular case $k = 2$ of the parameter $h_{k,n}$ defined in (1.1.21). Employing Theorem 1.1.1 in the definition of $h_{2,n}$, it can be easily seen that

$$h_{2,1} = 1 \quad \text{and} \quad h_{2,1/n} = 1/h_{2,n}. \tag{5.1.7}$$

5.2 Values of G_n and g_n

In this section, we state and prove a formula for evaluation of the class invariant g_n in terms of parameter J_n and find some new g_n. We also record some known values of G_n and g_n which will be used in the subsequent sections.

In the following lemma, we recall some values of J_n from the previous chapter.

Lemma 5.2.1. *We have*

(i) $J_3 = \left(2+\sqrt{3}\right)^{1/4}$,

(ii) $J_5 = \frac{1}{\sqrt{2}}\left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)^{1/2}$,

(iii) $J_7 = \left(8+3\sqrt{7}\right)^{1/4}$,

(iv) $J_{13} = \left(18+5\sqrt{13}+6\sqrt{18+5\sqrt{13}}\right)^{1/4}$,

(v) $J_{15} = \left(\frac{16+\sqrt{3\left(54+14\sqrt{5}\right)}}{7-3\sqrt{5}}\right)^{1/4}$,

(vi) $J_{17} = \left(\frac{1+\sqrt{1-\left(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}\right)^2}}{20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}}\right)^{1/4}$,

(vii) $J_{25} = \frac{1}{2}\left(3+\sqrt[4]{5}+\sqrt{5}+\sqrt[4]{5^3}\right)$,

(viii) $J_{49} = \frac{1}{4}\left(\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}+\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}}\right)^2$.

For proof see Theorem 4.6.3 and Theorem 4.6.5.

Lemma 5.2.2. *We have*

(i) $G_5 = \left(\frac{1+\sqrt{5}}{2}\right)^{1/4}$,

(ii) $G_7 = 2^{1/4}$,

(iii) $G_{10} = \frac{\left\{(\sqrt{2}+1)(4+\sqrt{2+10\sqrt{2}})\right\}^{1/8}}{2^{1/4}}$,

(iv) $G_{16} = \frac{\left(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4}\right)}{2^{1/4}}$,

(v) $G_{13} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4}$,

(vi) $G_{15} = 2^{1/4}\left(\frac{1+\sqrt{5}}{2}\right)^{1/3}$,

(vii) $G_{17} = \sqrt{\frac{5+\sqrt{17}}{8}}+\sqrt{\frac{\sqrt{17}-3}{8}}$,

(viii) $G_{25} = \dfrac{1+\sqrt{5}}{2}$,

(ix) $G_{27} = 2^{1/12}\left(\sqrt[3]{2}-1\right)^{-1/3}$,

(x) $G_{36} = \dfrac{\left(2-3\sqrt{2}+3^{5/4}+3^{3/4}\right)^{1/3}}{2^{11/48}\left(\sqrt{2}-1\right)^{1/12}\left(3^{3/4}\cdot\sqrt{2}-\sqrt{3}-1\right)^{1/3}}$,

(xi) $G_{49} = \dfrac{7^{1/4}+\sqrt{4+\sqrt{7}}}{2}$.

Proofs of (iii), (iv), and (x) can be found in [66, p. 114-115, Theorem 6.2.2]. For the proofs of (i), (ii), (v)-(x), we refer to [17].

In the following theorem, we establish a relation among g_n, G_n and J_n, where J_n is defined in (4.1.12).

Theorem 5.2.3. *For any positive rational number n, we have*

$$g_n = \frac{1}{2^{1/8}}\left(\frac{J_n}{G_n}\right)^{1/2}.$$

Proof. We note from (4.1.12) that, J_n can be expressed as

$$J_n = 2^{-1/4}q^{-1/24}\frac{f(-q)}{f(-q^2)} \times 2^{-1/4}q^{-1/12}\frac{f(-q^2)}{f(-q^4)}, \tag{5.2.1}$$

where $q := e^{-\pi\sqrt{n}}$.
Applying the definition of g_n, we obtain

$$J_n = g_n g_{4n}. \tag{5.2.2}$$

Now, from [17, p. 187, Entry 2.1], we note that

$$g_{4n} = 2^{1/4} g_n G_n. \tag{5.2.3}$$

Applying (5.2.3) in (5.2.2), we arrive at the desired result. □

In the next lemma, we state some values of g_n which will be used in the last section of the chapter.

Lemma 5.2.4. *For any positive rational number n, we have*

(i) $g_5 = \dfrac{\left(1+\sqrt{5}+\sqrt{2\sqrt{1+\sqrt{5}}}\right)^{1/4}}{2^{1/4}\left(1+\sqrt{5}\right)^{1/8}}$,

(ii) $g_7 = \dfrac{\left(1+\sqrt{3}\right)^{1/4}}{2^{7/24}}$,

(iii) $g_{10} = \left(\frac{1+\sqrt{5}}{2}\right)^{1/2}$,

(iv) $g_{13} = \left(\frac{18+5\sqrt{13}+6\sqrt{18+5\sqrt{13}}}{3+\sqrt{13}}\right)^{1/8}$,

(v) $g_{15} = \frac{\left(16+\sqrt{3\left(54+14\sqrt{5}\right)}\right)^{1/8}}{2^{1/12}\left(7-3\sqrt{5}\right)^{1/8}\left(1+\sqrt{5}\right)^{1/5}}$,

(vi) $g_{16} = 2^{3/16}\left(1+\sqrt{2}\right)^{1/4}$,

(vii) $g_{17} = \frac{2^{1/4}}{\left(\sqrt{5+\sqrt{17}}+\sqrt{\sqrt{17}-3}\right)^{1/2}}\left(\frac{1+\sqrt{1-(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}})^2}}{20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}}\right)^{1/8}$;

(viii) $g_{25} = \frac{5^{1/4}+1}{2^{5/8}}$,

(ix) $g_{27} = \frac{\left(1+\sqrt{3}\right)^{1/12}\left(1-\sqrt{3}+2^{2/3}\cdot\sqrt{3}\right)^{1/3}}{2^{3/8}\left(2^{1/3}-1\right)^{1/3}}$,

(x) $g_{36} = \frac{\left(1+\sqrt{3}\right)^{1/3}\left(1+\sqrt{3}+3^{3/4}\cdot\sqrt{2}\right)^{1/3}}{2^{11/24}}$,

(xi) $g_{49} = \frac{\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}+\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}}}{2^{5/8}\left(7^{1/4}+\sqrt{4+\sqrt{7}}\right)^{1/2}}$.

Proof. To prove (i), (iv), (v), (vii), and (xi), we set n=5, 13, 15, 17, and 49 in Theorem 5.2.3 and use the values of J_n and G_n from Lemma 5.2.1 and Theorem 5.2.2, respectively. The other values are established in [17] and [66] □

5.3 Explicit values of $K(q)$ by using $h_{2,n}$

Employing the definition of the parameter $h_{2,n}$ in (5.1.5), we immediately deduce the following useful theorem.

Theorem 5.3.1. *For any positive real number n, we have*

$$K^2(e^{-\pi\sqrt{n}/2}) = \frac{2^{1/4}h_{2,n}-1}{2^{1/4}h_{2,n}+1}.$$

Remark 5.3.1. From the above theorem, it is obvious that if we know the values of $h_{2,n}$ for any positive real number n, then the values of $K(e^{-\pi\sqrt{n}/2})$ can be easily evaluated.

We have already noted in (5.1.7) that $h_{2,1} = 1$. In the next lemma, we recall some more values of $h_{2,n}$ from [66, p. 142, Theorem 9.1.6] or [69, p. 13, Theorem 4.7].

Lemma 5.3.2. *We have*

(i) $h_{2,2} = \sqrt{2\sqrt{2} - 2}$,

(ii) $h_{2,1/2} = \sqrt{\dfrac{\sqrt{2}+1}{2}}$,

(iii) $h_{2,4} = \sqrt{2} + 1 - \sqrt{\sqrt{2}+1}$,

(iv) $h_{2,1/4} = \dfrac{1+\sqrt{\sqrt{2}-1}}{\sqrt{2}}$,

(v) $h_{2,8} = \dfrac{\sqrt{2+\sqrt{2}}}{\sqrt[4]{2}+1}$,

(vi) $h_{2,1/8} = \dfrac{\sqrt{2+\sqrt{2}}}{\sqrt[4]{2}-1}$.

Next, we prove the following new theta-function identity from which we calculate some more values of $h_{2,n}$.

Theorem 5.3.3. *If*

$$P = \frac{\phi(q)}{\phi(q^2)} \qquad \text{and} \qquad Q = \frac{\phi(q^3)}{\phi(q^6)},$$

then

$$\left(\frac{P}{Q}\right)^2 - 6\left(\frac{P}{Q} + \frac{Q}{P}\right) + 4\left(\frac{2}{PQ} + PQ\right) - \left(\frac{Q}{P}\right)^2 = 0. \tag{5.3.1}$$

Proof. Transcribing P and Q using Entry 10(i) and (iv)[15, p. 122] and simplifying, we find that

$$\sqrt{1-\alpha} = \frac{2}{P^2} - 1 \qquad \text{and} \qquad \sqrt{1-\beta} = \frac{2}{Q^2} - 1, \tag{5.3.2}$$

where β has degree 3 over α.

Now, From Entry 5(x) [15, p. 231], we note that

$$m(1-\alpha)^{1/2} + (1-\beta)^{1/2} = \frac{3}{m}(1-\beta)^{1/2} - (1-\alpha)^{1/2} = 2\left\{(1-\alpha)(1-\beta)\right\}^{1/8}. \tag{5.3.3}$$

Setting $k = 2\left\{(1-\alpha)(1-\beta)\right\}^{1/8}$ in (5.3.3), we find that

$$m\sqrt{1-\alpha} + \sqrt{1-\beta} = k \tag{5.3.4}$$

and

$$\frac{3}{m}\sqrt{1-\beta} = k + \sqrt{1-\alpha}. \tag{5.3.5}$$

Multiplying (5.3.4) and (5.3.5), and then simplifying, we obtain

$$k^2 - b = -ka, \tag{5.3.6}$$

where $a = \sqrt{1-\alpha} - \sqrt{1-\beta}$ and $b = 4\sqrt{(1-\alpha)(1-\beta)}$. (5.3.7)

Squaring (5.3.6) and substituting $k^4 = 4b$, we obtain

$$4b + b^2 = k^2(a^2 + 2b). \tag{5.3.8}$$

Squaring (5.3.8) and substituting $k^4 = 4b$ once again, we arrive at

$$16b - 4(2b^2 + a^4 + 4a^2 b) + b^3 = 0, \tag{5.3.9}$$

Now, from (5.3.7) and (5.3.2), we note that

$$a = \left(\frac{2}{P^2} - \frac{2}{Q^2}\right) \qquad \text{and} \qquad b = 4\left(\frac{2}{P^2} - 1\right)\left(\frac{2}{Q^2} - 1\right). \tag{5.3.10}$$

Invoking (5.3.10) in (5.3.9), and then factoring, we find that

$$\begin{aligned}&\left(P^4 - 8PQ + 6P^3Q + 6PQ^3 - 4P^3Q^3 - Q^4\right)\\ &\qquad \times \left(P^4 + 8PQ - 6P^3Q - 6PQ^3 + 4P^3Q^3 - Q^4\right) = 0.\end{aligned} \tag{5.3.11}$$

By examining the behavior near the origin, it can be shown that the first factor of (5.3.11) is non-zero in a neighborhood of the origin. Thus, the second factor vanishes in that neighborhood. Hence, by the identity theorem, this factor vanishes identically, i.e.,

$$P^4 + 8PQ - 6P^3Q - 6PQ^3 + 4P^3Q^3 - Q^4 = 0. \tag{5.3.12}$$

Dividing the above equation by P^2Q^2 and then rearranging, we complete the proof. □

Theorem 5.3.4. *For any positive real number n, we have*

$$\left(\frac{h_{2,n}}{h_{2,9n}}\right)^2 - 6\left(\frac{h_{2,n}}{h_{2,9n}} + \frac{h_{2,9n}}{h_{2,n}}\right) + 4\sqrt{2}\left(\frac{1}{h_{2,n}h_{2,9n}} + h_{2,n}h_{2,9n}\right) - \left(\frac{h_{2,9n}}{h_{2,n}}\right)^2 = 0.$$

Proof. The theorem follows directly from Theorem 5.3.3 and the definition of $h_{2,n}$ □

Theorem 5.3.5. *We have*

(i) $h_{2,3} = \left(1+\sqrt{2}\right)^{1/2}\left(\sqrt{3}-\sqrt{2}\right)^{1/2}$,

(ii) $h_{2,1/3} = \left(\sqrt{2}-1\right)^{1/2}\left(\sqrt{3}+\sqrt{2}\right)^{1/2}$;

(iii) $h_{2,9} = \left(\sqrt{3}+\sqrt{2}\right)\left(2-\sqrt{3}\right)$,

(iv) $h_{2,1/9} = \left(\sqrt{3}-\sqrt{2}\right)\left(2+\sqrt{3}\right)$.

Proof. Setting $n = 1/3$ in Theorem 5.3.4 and then simplifying using (5.1.7), we get

$$x^4 - \frac{1}{x^4} + 6\left(x^2 + \frac{1}{x^2}\right) - 8\sqrt{2} = 0, \tag{5.3.13}$$

where $x = h_{2,3}$.
Solving the above polynomial equation, we complete the proof of (i). Now, (ii) follows immediately from (i) and the fact that $h_{2,1/3} = 1/h_{2,3}$.

To prove (iii), we set $n = 1$ in Theorem 5.3.4 and then simplifying using the result that $h_{k,1} = 1$, we find that

$$x^4 + (6 - 4\sqrt{2})(x^3 + x) - 1 = 0, \tag{5.3.14}$$

where $x = h_{2,9}$.
Solving the above polynomial equation, we arrive at (iii). Employing the result $h_{2,1/9} = 1/h_{2,9}$ in (iii), we readily arrive at (iv). □

Theorem 5.3.6. *We have*

(i) $h_{2,5} = \sqrt{\dfrac{1+\sqrt{2}+\sqrt{5}}{1+\sqrt{5}+\sqrt{10}}}$

and

(ii) $h_{2,1/5} = \sqrt{\dfrac{1+\sqrt{5}+\sqrt{10}}{1+\sqrt{2}+\sqrt{5}}}$.

To prove the above theorem, we also used some results from our next chapter.

Proof. From Theorem 6.4.9 (iii), we have

$$g'_{5,2} = \left(\frac{\sqrt{5}+1}{2}\right)^{3/2}.$$

Setting $n = 2$ in Theorem 6.4.3 (vi), using the above value of $g'_{5,2}$, and then solving the polynomial equation for $g_{5,2}$, we get

$$g_{5,2} = g_{2,5} = \sqrt{1+\sqrt{2}},$$

where the first equality is due to Theorem 6.3.1(iii).

Now, setting $q = e^{-\pi\sqrt{n/5}}$ and employing the definitions of $g_{k,n}$ and $h_{k,n}$ in Theorem 6.2.5, we obtain

$$h_{2,5}^2\left(1 + \sqrt{5}g_{2,5}^2\right) - \left(\sqrt{5} + g_{2,5}^2\right) = 0. \tag{5.3.15}$$

Solving (5.3.15), we arrive at (i). From (i) and (5.1.7), we easily deduce (ii). □

5.4 Explicit values of $K(q)$ by using $s_{4,n}$

In this section, we use the parameter $s_{4,n}$ to find explicit values of $K(e^{-\pi\sqrt{n}/4})$, where $s_{4,n}$, for $k=4$ is a particular case of the parameter $s_{k,n}$ defined in (5.1.6).

Theorem 5.4.1. *For* $q := e^{-\pi\sqrt{n}/2}$, *recall from* (5.1.6) *that*

$$s_{4,n} = \frac{f(q)}{\sqrt{2}q^{1/8}f(-q^4)}. \tag{5.4.1}$$

Then

$$K(e^{-\pi\sqrt{n}/4}) = -s_{4,n}^2 + \sqrt{s_{4,n}^4 + 1}. \tag{5.4.2}$$

Proof. Replacing q by $q^{1/2}$ in (5.1.3), we find that

$$\frac{1}{K(q^{1/2})} - K(q^{1/2}) = \frac{\phi(q)}{q^{1/4}\psi(q^2)}. \tag{5.4.3}$$

Employing (4.8.5) in (5.4.3), we deduce that

$$\frac{1}{K(q^{1/2})} - K(q^{1/2}) = \frac{f^2(q)}{q^{1/4}f^2(-q^4)}. \tag{5.4.4}$$

Using the definition of $s_{4,n}$ in (5.4.4), we arrive at

$$\frac{1}{K(e^{-\pi\sqrt{n}/4})} - K(e^{-\pi\sqrt{n}/4}) = 2s_{4,n}^2. \tag{5.4.5}$$

Now (5.4.2) is apparent from the above. □

From the above theorem, we need only to find the values of the parameter $s_{4,n}$ to get the explicit values of $K(e^{-\pi\sqrt{n}/4})$. The remaining part of this chapter is devoted to the explicit evaluations of $s_{4,n}$.

In the next theorem, we establish a relation connecting $s_{4,n}$ with $r_{k,n}$ and g_n.

Theorem 5.4.2. *We have*

$$s_{4,n} = \frac{r_{2,n/2}^2}{2^{1/4}r_{2,n/8}} = \frac{g_n^2}{2^{1/4}g_{n/4}}.$$

Proof. We can rewrite $s_{4,n}$ as

$$s_{4,n} = 2^{-1/4}q^{-1/24}\frac{f(q)}{f(-q^2)} \times 2^{-1/4}q^{-1/12}\frac{f(-q^2)}{f(-q^4)}, \tag{5.4.6}$$

where $q := e^{-\pi\sqrt{n}/2}$.

Applying the definitions of the class invariants G_n and g_n in (5.4.6), we find that

$$s_{4,n} = G_{n/4} g_n. \tag{5.4.7}$$

Now, from (4.8.4), we recall that

$$g_n = r_{2,n/2} \quad \text{and} \quad G_n = \frac{r_{2,2n}}{2^{1/4} r_{2,n/2}}. \tag{5.4.8}$$

Combining (5.4.7) and (5.4.8), we complete the proof. □

Corollary 5.4.3. *We have*

(i) $s_{4,1} = \dfrac{\left(1+\sqrt{2}\right)^{1/4}}{2^{5/16}}$,

(ii) $s_{4,2} = \dfrac{\left(1+\sqrt{2}\right)^{1/8}}{2^{1/8}}$,

(iii) $s_{4,4} = 2^{1/8}$,

(iv) $s_{4,8} = \left(1+\sqrt{2}\right)^{1/4}$,

(v) $s_{4,9} = \dfrac{\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{2/3}\left(-1-\sqrt{2}+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}-\sqrt{6}\right)^{1/3}}{2^{23/16}\left(\sqrt{2}-1\right)^{5/12}\left(\sqrt{3}-1\right)^{1/3}\left(\sqrt{3}-\sqrt{2}\right)^{4/3}}$,

(vi) $s_{4,10} = \dfrac{\left(1+\sqrt{5}\right)^{17/8}\left(2+3\sqrt{2}+\sqrt{5}\right)^{1/8}}{2^{7/4}\left(\left(3+\sqrt{5}\right)\left(1+\sqrt{2}\right)\right)^{1/2}}$,

(vii) $s_{4,12} = 2^{1/8}\left(1+\sqrt{2}\right)^{1/4}$,

(viii) $s_{4,16} = \left(1+\sqrt{3}\right)^{1/2}$,

(ix) $s_{4,18} = \dfrac{\left(\sqrt{2}+\sqrt{3}\right)}{2^{1/8}\left(-1+35\sqrt{2}+28\sqrt{3}\right)^{1/8}}$,

(x) $s_{4,24} = \left(1+\sqrt{2}\right)^{1/4}\left(1+\sqrt{2}+\sqrt{6}\right)^{1/4}$,

(xi) $s_{4,32} = 2^{-1/8}\left(1+\sqrt{2}\right)^{3/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/4}$,

(xii) $s_{4,36} = 2^{-5/8}\left(1+\sqrt{3}\right)^{2/3}\left(1+\sqrt{3}+3^{3/4}\sqrt{2}\right)^{1/3}$,

(xiii) $s_{4,72} = \left(\sqrt{2}+\sqrt{3}\right)\left(1+35\sqrt{2}-28\sqrt{3}\right)^{1/4}$.

Proof. Easily follow from Theorem 5.4.2 and the values of $r_{k,n}$ stated in Section 1.3. □

In the next, we present a relation connecting $s_{4,4n}$ with G_n and g_n.

Theorem 5.4.4. *For any positive real number n, we have*

$$s_{4,4n} = 2^{1/4} g_n G_n^2.$$

Proof. We replace n by $4n$ in Theorem 5.4.2 and simplify by using (5.2.3) to complete the proof. □

Corollary 5.4.5. *We have*

(i) $s_{4,20} = 2^{-1/2}\left(1+\sqrt{5}\right)^{3/8}\left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)^{1/4}$,

(ii) $s_{4,28} = 2^{3/8}\left(3+\sqrt{7}\right)^{1/4}$,

(iii) $s_{4,40} = 2^{3/4}\left(1+\sqrt{5}\right)^{3/4}\left(2+3\sqrt{2}+\sqrt{5}\right)^{1/4}$,

(iv) $s_{4,64} = 2^{-1/16}\left(1+\sqrt{2}\right)^{1/2}\left(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4}\right)^{2}$,

(v) $s_{4,52} = 2^{-1/4}\left(18+5\sqrt{13}+6\sqrt{18+5\sqrt{13}}\right)^{1/8}\left(3+\sqrt{13}\right)^{3/8}$,

(vi) $s_{4,60} = \left(1+\sqrt{5}\right)^{1/2}\left(\dfrac{16+\sqrt{3\left(54+14\sqrt{5}\right)}}{7-3\sqrt{5}}\right)^{1/8}$,

(vii) $s_{4,68} = 2^{-9/4}\left(\sqrt{5+\sqrt{17}}+\sqrt{\sqrt{17}-3}\right)^{3/2}$,

$$\times\left(\frac{1+\sqrt{1-\left(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}\right)^2}}{20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}}\right)^{1/8};$$

(viii) $s_{4,100} = 2^{-19/8}\left(1+\sqrt{5}\right)^{2}\left(5^{1/4}+1\right)$;

(ix) $s_{4,108} = \dfrac{2^{1/24}\left(1+\sqrt{3}\right)^{1/12}\left(1-\sqrt{3}+2^{2/3}\cdot\sqrt{3}\right)^{1/3}}{\left(2^{1/3}-1\right)}$,

(x) $s_{4,144} = \dfrac{2^{-2/3}\left(1+\sqrt{3}\right)^{1/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}\left(2-3\sqrt{2}+3^{5/4}+3^{3/4}\right)^{2/3}}{\left(\sqrt{2}-1\right)^{1/6}\left(\sqrt{2}\cdot 3^{3/4}-\sqrt{3}-1\right)^{2/3}}$;

(xi) $s_{4,196} = 2^{-19/8}\left(7^{1/4}+\sqrt{4+\sqrt{7}}\right)^{3/2}$

$$\times\left(\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}+\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}}\right).$$

Proof. We employ the values of G_n and g_n from Lemma 5.2.2 and Lemma 5.2.4 in Theorem 5.4.4. □

Our next theorem is almost similar to the Theorem 5.4.2.

Theorem 5.4.6. *For any positive real number n, we have*

$$s_{4,4n} = \frac{r_{2,2n}^3}{2^{1/4} r_{4,n}} = \frac{r_{2,2n}^3}{2^{1/4} J_n},$$

where J_n *is as defined in* (4.1.12).

Proof. Replacing n by $4n$ in Theorem 5.4.2, we obtain

$$s_{4,4n} = \frac{r_{2,2n}^2}{2^{1/4} r_{2,n/2}}. \tag{5.4.9}$$

Applying the result $r_{k,n/m} = r_{km,n} r_{nk,m}^{-1}$ [66, p. 13,Lemma 2.1.3(i)], in (5.4.9), we prove the first equality.

Second equality follows immediately from the first equality and the result $J_n = r_{4,n}$. □

Corollary 5.4.7. *We have*

(i) $s_{4,2/3} = \left(\dfrac{1+\sqrt{2}+\sqrt{6}}{2+2\sqrt{2}}\right)^{1/8}$,

(ii) $s_{4,1/2} = 2^{-3/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8}$,

(iii) $s_{4,4/7} = \dfrac{2^{7/8}\left(127+48\sqrt{7}\right)^{1/8}}{\left(3+\sqrt{7}\right)^{3/4}}$,

(iv) $s_{4,4/5} = \dfrac{\left(1+\sqrt{5}\right)^{3/8}}{\left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)^{1/4}}$,

(v) $s_{4,4/9} = 2^{3/8}$,

(vi) $s_{4,4/25} = \dfrac{2^{13/8}}{\left(5^{1/4}-1\right)\left(5^{1/4}+1\right)^2}$,

(vii) $s_{4,4/49} = \dfrac{\left(7^{1/4}+\sqrt{4+\sqrt{7}}\right)^{3/8}}{2^{3/8}\left(\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}+\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}}\right)}$,

(viii) $s_{4,2/5} = \dfrac{2^{1/4}\left(2+3\sqrt{2}+\sqrt{5}\right)^{1/8}}{\left(1+\sqrt{5}\right)^{3/8}}$.

Proof. (i) Setting $n = 1/6$ in Theorem 5.4.6, we find that

$$s_{4,2/3} = \frac{r_{4,6}}{2^{1/4} r_{2,3}^3}, \tag{5.4.10}$$

where we also used the result $r_{k,1/n} = 1/r_{k,n}$.
From [66, p. 14, Corollary 2.1.5(i)], for any positive real number k and n, we have

$$r_{k^2,n} = r_{k,nk}r_{k,n/k}. \tag{5.4.11}$$

Setting $k = 2$ and $n = 6$ in (5.4.11), we find that

$$r_{4,6} = r_{2,12}r_{2,3}. \tag{5.4.12}$$

Combining (5.4.10) and (5.4.12), we obtain

$$s_{4,2/3} = \frac{r_{2,12}}{2^{1/4}r_{2,3}^2}. \tag{5.4.13}$$

Substituting the values of $r_{2,3}$ and $r_{2,12}$ from Section 1.3, we complete the proof.
The proofs of the (ii)-(viii) are identical to the proof of (i). □

Theorem 5.4.8. *If J_n and $s_{4,n}$ are as defined in* (4.1.12) *and* (5.4.1), *respectively, then*

$$s_{4,n} = \left(\frac{2 + J_n^8 + 2\sqrt{1 + J_n^8}}{4\sqrt{1 + J_n^8}}\right)^{1/8}.$$

Proof. From Theorem 4.2.1, we find that

$$(LM)^8 + 4^4 = \frac{M^{16}}{L^8} - 16M^8 - 16L^8, \tag{5.4.14}$$

where

$$L = \frac{f(-q)}{q^{1/8}f(-q^4)} \quad \text{and} \quad M = \frac{f(-q^2)}{q^{1/4}f(-q^8)}.$$

Replacing q by $-q$, we note that $(LM)^8$, L^8, and M^4 transform to $-(RM)^8$, $-R^8$, and $-M^4$, respectively, where

$$R = \frac{f(q)}{q^{1/8}f(-q^4)}.$$

Thus, we deduce that

$$R^8\left\{(RM)^8 - 4^4\right\} = M^{16} + 16(RM)^8 - 16R^{16}. \tag{5.4.15}$$

Setting $q = e^{-\pi\sqrt{n}/2}$ and employing the definitions of $s_{4,n}$ and J_n, we find that

$$R = \sqrt{2}s_{4,n} \quad \text{and} \quad M = \sqrt{2}J_n. \tag{5.4.16}$$

Invoking (5.4.16) in (5.4.15) and then simplifying, we obtain

$$J_n^{16} + 16\,s_{4,n}^8 + 16\,J_n^8\,s_{4,n}^8 - 16\,s_{4,n}^{16} - 16\,J_n^8\,s_{4,n}^{16} = 0. \tag{5.4.17}$$

Solving the above polynomial equation for $s_{4,n}$ and considering the real positive root only, we complete the proof. □

Remark 5.4.1. From Theorem 5.4.8 it is obvious that if we know the values of the parameter J_n we can easily evaluate the values of $s_{4,n}$. Many explicit values of J_n are evaluated in Chapter 4. Also since $J_{1/n} = 1/J_n$, $s_{4,1/n}$ also follows immediately. In Corollary 5.4.9, we give few examples.

Corollary 5.4.9. *We have*

(i) $s_{4,3} = \dfrac{\left(9+4\sqrt{3}+4\sqrt{2+\sqrt{3}}\right)^{1/8}}{2^{3/8}\left(2+\sqrt{3}\right)^{1/16}}$,

(ii) $s_{4,1/3} = \dfrac{\left(9-4\sqrt{3}+4\sqrt{2-\sqrt{3}}\right)^{1/8}}{2^{3/8}\left(2-\sqrt{3}\right)^{1/16}}$,

(iii) $s_{4,5} = \dfrac{\left(1+k+2\sqrt{k}\right)^{1/8}}{2^{1/4}k^{1/16}}$, *where* $k = 1+\dfrac{1}{16}\left(1+\sqrt{5}+\sqrt{2+2\sqrt{5}}\right)^4$,

(iv) $s_{4,1/5} = \dfrac{\left(1+k_1+2\sqrt{k_1}\right)^{1/8}}{2^{1/4}k_1^{1/16}}$, *where* $k_1 = 1+\dfrac{1}{16}\left(1+\sqrt{5}-\sqrt{2+2\sqrt{5}}\right)^4$,

(v) $s_{4,7} = \dfrac{\left(129+48\sqrt{7}+\sqrt{128+48\sqrt{7}}\right)^{1/8}}{2^{1/4}\left(128+48\sqrt{7}\right)^{1/16}}$,

(vi) $s_{4,1/7} = \dfrac{\left(129-48\sqrt{7}+\sqrt{128-48\sqrt{7}}\right)^{1/8}}{2^{1/4}\left(128-48\sqrt{7}\right)^{1/16}}$,

(vii) $s_{4,25} = \dfrac{(1+m+2\sqrt{m})^{1/8}}{2^{1/4}m^{1/16}}$, *where* $m = 1+\dfrac{1}{256}\left(3+\sqrt{5}+\sqrt{10+6\sqrt{5}}\right)^8$,

(viii) $s_{4,1/25} = \dfrac{(1+m_1+2\sqrt{m_1})^{1/8}}{2^{1/4}m_1^{1/16}}$, *where* $m_1 = 1+\dfrac{1}{256}\left(3+\sqrt{5}-\sqrt{10+6\sqrt{5}}\right)^8$,

(ix) $s_{4,13} = \dfrac{(1+n+2\sqrt{n})^{1/8}}{2^{1/4}n^{1/16}}$, *where* $n = 1+\left(18+5\sqrt{13}+6\sqrt{18+5\sqrt{13}}\right)^2$,

(x) $s_{4,1/13} = \dfrac{(1+n_1+2\sqrt{n_1})^{1/8}}{2^{1/4}n_1^{1/16}}$, *where* $n_1 = 1+\left(18+5\sqrt{13}-6\sqrt{18+5\sqrt{13}}\right)^2$,

(xi) $s_{4,15} = \dfrac{\left(303-21\sqrt{5}+16r+28\sqrt{32+2r}-12\sqrt{5(32+2r)}\right)^{1/8}}{2^{7/16}\left(7-3\sqrt{5}\right)^{1/8}(16+r)^{1/6}}$, *where* $r = \sqrt{162+42\sqrt{5}}$.

Proof. We only give the proofs of (i) and (ii) only. Proofs of the remaining values follow similarly. To prove (i) and (ii), we employ the value of J_3 from Lemma 5.2.1(i) and $J_{1/3} = 1/J_3$ in Theorem 5.4.8. □

Chapter 6

Two Parameters for Ramanujan's Theta-functions and Their Explicit Values

6.1 Introduction

In his first notebook [54, Vol. I, p. 248] Ramanujan recorded many elementary values of $\phi(q)$ and $\psi(q)$. Particularly, he recorded $\psi(e^{-n\pi})$ for n=1, 2, 4, 8, 1/2, and 1/4 and $\phi(e^{-n\pi})$ and $\phi(-e^{-n\pi})$ for n=1, 2, 4, 8, 1/2, and 1/4. All these values were proved by Berndt [17, p. 325]. Ramanujan also recorded non-elementary values of $\phi(e^{-n\pi})$ for n= 3, 5, 9, 7, and 45. Berndt and Chan [20] found proofs for these. They also found new explicit values of $\phi(e^{-n\pi})$ for n= 13, 27, and 63. Recently, Yi [66, 69], evaluated many new values of $\phi(q)$ and $f(q)$ by using modular identities, transformation formulas for theta-functions and the parameters $r_{k,n}$, $r'_{k,n}$, $h_{k,n}$, and $h'_{k,n}$ which we also recall from Chapter 1 that

$$r_{k,n} := \frac{f(-q)}{k^{1/4}q^{(k-1)/24}f(-q^k)}; \quad q = e^{-2\pi\sqrt{n/k}}, \tag{6.1.1}$$

$$r'_{k,n} := \frac{f(q)}{k^{1/4}q^{(k-1)/24}f(q^k)}; \quad q = e^{-\pi\sqrt{n/k}}, \tag{6.1.2}$$

$$h_{k,n} := \frac{\phi(q)}{k^{1/4}\phi(q^k)}; \quad q = e^{-\pi\sqrt{n/k}}, \tag{6.1.3}$$

and

$$h'_{k,n} = \frac{\phi(-q)}{k^{1/4}\phi(-q^k)}, \quad q = e^{-2\pi\sqrt{n/k}}. \tag{6.1.4}$$

In particular, she evaluated $\phi(e^{-n\pi})$ for n=1, 2, 3, 4, 5, and 6 and $\phi(-e^{-n\pi})$ for n=1, 2, 4, 6, 8, 10, and 12. Motivated by Yi's work, we define, for any positive real numbers k and n, the two parameters $g_{k,n}$ and $g'_{k,n}$ of the theta-function $\psi(q)$, by

$$g_{k,n} := \frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)} \qquad q = e^{-\pi\sqrt{n/k}}, \tag{6.1.5}$$

and

$$g'_{k,n} := \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)}, \qquad q = e^{-\pi\sqrt{n/k}}. \tag{6.1.6}$$

In this chapter, we establish many general properties of these parameters, which are analogous to those of $h_{k,n}$ and $h'_{k,n}$. We also find several general theorems for the explicit evaluations of these parameters by using theta-function identities. In particular, we obtain several new explicit values of the theta-function $\psi(q)$ and quotients of $\psi(q)$ and of $\phi(q)$. We will use the explicit values of $r_{k,n}$, and $r'_{k,n}$ from [66] and listed in Section 1.3 for finding some of the explicit values of $g_{k,n}$ and $g'_{k,n}$. In addition, we will establish some theorems for the explicit evaluations of Rogers-Ramanujan continued fraction and Ramanujan cubic continued fraction using the parameters $g_{k,n}$, $g'_{k,n}$, $h_{k,n}$, and $h'_{k,n}$.

6.2 Theta-function identities

In this section we state and prove some theta-function identities which we will use in the subsequent section. Proofs of the new identities are also given.

Theorem 6.2.1. *(Ramanujan [54, p. 327]; Berndt [16, p. 233, Entry 66])) If*

$$P = \frac{\psi(q)}{q^{1/2}\psi(q^5)} \quad and \quad Q = \frac{\psi(q^3)}{q^{3/2}\psi(q^{15})},$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 - \left(\frac{P}{Q}\right)^2 + 3\left(\frac{Q}{P} + \frac{P}{Q}\right). \tag{6.2.1}$$

Theorem 6.2.2. *If*

$$P = \frac{\psi(-q)}{q\psi(-q^9)} \quad and \quad Q = \frac{\psi(-q^3)}{q^3\psi(-q^{27})},$$

then

$$\left(\frac{3}{Q} + Q + 3\right)\left(\frac{3}{P} + P + 3\right) = \left(\frac{Q}{P}\right)^2. \tag{6.2.2}$$

Proof. The proof of the theorem follows directly from [15, Entry 1 (ii), p. 345]. □

Theorem 6.2.3. *(Adiga et al. [3, p. 10, Theorem 5.1]; Baruah & Bhattacharyya [11, p. 2157])) Let*

$$P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \textit{and} \quad Q = \frac{\phi(q)}{\phi(q^3)},$$

then

$$Q^4 + P^4Q^4 = 9 + P^4. \tag{6.2.3}$$

Theorem 6.2.4. *(Adiga et al. [3, p. 10. Theorem 5.2]; Baruah & Bhattacharyya [11, p 2156])) Let*

$$P = \frac{\psi(-q)}{q\psi(-q^9)} \quad \textit{and} \quad Q = \frac{\phi(q)}{\phi(q^9)},$$

then

$$Q + PQ = 3 + P. \tag{6.2.4}$$

Theorem 6.2.5. *(Adiga et al. [3, p. 10, Theorem 5.3]; Baruah & Bhattacharyya [11, p. 2156])) Let*

$$P = \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \textit{and} \quad Q = \frac{\phi(q)}{q\phi(q^5)},$$

then

$$Q^2 + P^2Q^2 = 5 + P^2. \tag{6.2.5}$$

Theorem 6.2.6. *If*

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \textit{and} \quad Q = \frac{\psi(q^2)}{q^{1/2}\psi(q^6)},$$

then

$$\left(\frac{P}{Q}\right)^2 + \frac{3}{P^2} - P^2 + \left(\frac{Q}{P}\right)^2 = 0. \tag{6.2.6}$$

Proof. Using

$$\psi(q) = \frac{f^2(-q^2)}{f(-q)},$$

we find that

$$P = \frac{f^2(-q^2)f(-q^3)}{q^{1/4}f(-q)f^2(-q^6)} \quad \text{and} \quad Q = \frac{f^2(-q^4)f(-q^6)}{q^{1/2}f(-q^2)f^2(-q^{12})} \tag{6.2.7}$$

We set

$$L_1 := \frac{f(-q)}{q^{1/12}f(-q^2)} \quad \text{and} \quad L_2 := \frac{f^2(-q^2)}{q^{1/3}f^2(-q^6)}, \tag{6.2.8}$$

$$M_1 := \frac{f(-q^2)}{q^{1/6}f(-q^6)} \quad \text{and} \quad M_2 := \frac{f^2(-q^4)}{q^{2/3}f^2(-q^{12})}. \tag{6.2.9}$$

Then, from (6.2.7), (6.2.8), and (6.2.9), we have

$$P = \frac{L_2}{L_1}, \quad Q = \frac{M_2}{M_1}, \quad L_2M_1 = M_1^3. \tag{6.2.10}$$

Now from (6.2.8), (6.2.9), and [16, p.204, Entry 51], we have

$$(L_1M_1)^2 + \left(\frac{3}{L_1M_1}\right)^2 = \left(\frac{M_1}{L_1}\right)^6 + \left(\frac{L_1}{M_1}\right)^6, \tag{6.2.11}$$

$$L_2M_2 + \left(\frac{9}{L_2M_2}\right) = \left(\frac{M_2}{L_2}\right)^3 + \left(\frac{L_2}{M_2}\right)^3, \tag{6.2.12}$$

Replacing L_1 and M_2 in (6.2.11) and (6.2.12), respectively by using (6.2.10), we find that

$$M_1^{12} = \frac{P^{12} - 9P^8}{P^4 - 1}, \tag{6.2.13}$$

and

$$M_1^6 = \frac{Q^6 - 9Q^2}{Q^4 - 1}, \tag{6.2.14}$$

respectively. Thus, from (6.2.13) and (6.2.14), we have

$$\frac{P^{12} - 9P^8}{P^4 - 1} = \left(\frac{Q^6 - 9Q^2}{Q^4 - 1}\right)^2. \tag{6.2.15}$$

Simplifying the above equation (6.2.15), we obtain

$$(P^4 - 3Q^2 + P^4Q^2 + Q^4)(-P^4 - 3Q^2 + P^4Q^2 - Q^4)(9 - P^4 - Q^4 + P^4Q^4) = 0. \tag{6.2.16}$$

By examining the behavior of the first and the last factors of the left hand side of (6.2.16) near $q = 0$, it can be seen that there is a neighborhood about the origin, where these factors are not zero. Then the second factor is zero in this neighborhood. By the identity theorem this factor is identically zero. Thus, we have

$$P^4 + 3Q^2 - P^4Q^2 + Q^4 = 0. \tag{6.2.17}$$

Dividing the above equation by P^2Q^2, we complete the proof. □

Theorem 6.2.7. *If*

$$P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \textit{and} \quad Q = \frac{\psi(q^2)}{q^{1/2}\psi(q^6)},$$

then

$$\left(\frac{P}{Q}\right)^2 + \frac{3}{P^2} + P^2 - \left(\frac{Q}{P}\right)^2 = 0. \tag{6.2.18}$$

Proof. Replacing q by $-q$ in Theorem 6.2.6, we complete the proof. □

Theorem 6.2.8. *If*

$$P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \textit{and} \quad Q = \frac{\psi(q)}{q^{1/4}\psi(q^3)}$$

then

$$\left(\frac{P}{Q}\right)^4 + \left(\frac{Q}{P}\right)^4 + \left(\left(\frac{P}{Q}\right)^2 - \left(\frac{Q}{P}\right)^2\right)\left(\left(\frac{3}{PQ}\right)^2 - (PQ)^2\right) - 10 = 0. \tag{6.2.19}$$

Proof. Using

$$\psi(q)=\frac{f^2(-q^2)}{f(-q)},$$

we find that

$$P=\frac{f^2(-q^2)f(q^3)}{q^{1/4}f(q)f^2(-q^6)} \quad \text{and} \quad Q=\frac{f^2(-q^2)f(-q^3)}{q^{1/2}f(-q)f^2(-q^6)} \tag{6.2.20}$$

We set

$$L_1:=\frac{f(q)}{q^{1/12}f(q^3)} \quad \text{and} \quad L_2:=\frac{f^2(-q^2)}{q^{1/3}f^2(-q^6)}, \tag{6.2.21}$$

$$M_1:=\frac{f(-q)}{q^{1/6}f(-q^3)} \quad \text{and} \quad M_2:=\frac{f^2(-q^2)}{q^{1/3}f^2(-q^6)}. \tag{6.2.22}$$

Then, we have

$$P=\frac{L_2}{L_1}, \quad Q=\frac{M_2}{M_1}, \quad \text{and} \quad L_2=M_2. \tag{6.2.23}$$

Now by applying (6.2.21) in [16, p. 204, (51.3)], we obtain

$$L_1^8M_2^4-9L_1^4M_2^2=M_2^6-L_1^{12}. \tag{6.2.24}$$

Replacing L_1 in the above equation using (6.2.23), and simplifying using the result $L_2=M_2$, we find that

$$M_2^6=\frac{P^{12}+9P^8}{P^4+1}. \tag{6.2.25}$$

Again,from (6.2.22) in [16, p. 204, Entry 51], we obtain

$$M_1^2M_2+\frac{9}{M_1^2M_2}=\frac{M_1^3}{M_1^6}+\frac{M_2^6}{M_2^3}. \tag{6.2.26}$$

Replacing L_1 in the above equation using (6.2.23), and simplifying using the result $L_2=M_2$, we find that

$$M_2^6=\frac{Q^6(Q^6-9Q^2)}{Q^4-1}. \tag{6.2.27}$$

From (6.2.25) and (6.2.27), we have

$$\frac{P^{12}+9P^8}{P^4+1}=\frac{Q^6(Q^6-9Q^2)}{Q^4-1}. \tag{6.2.28}$$

Simplifying, we get

$$9P^4+P^8-9Q^4-10P^4Q^4-P^8Q^4+Q^4+P^4Q^8=0. \tag{6.2.29}$$

Dividing the above equation P^4Q^4 and rearranging the terms, we complete the proof. □

Theorem 6.2.9. *If*

$$P=\frac{\psi(q)}{q^{1/2}\psi(q^5)} \quad \textit{and} \quad Q=\frac{\psi(q^2)}{q\psi(q^{10})},$$

then

$$\left(\frac{P}{Q}\right)^2-\frac{5}{P^2}-P^2+\left(\frac{Q}{P}\right)^2+4=0. \tag{6.2.30}$$

Proof. We employ [16, p. 206, Entry 53 and (53.2)] and proceed as in the proof of Theorem 6.2.6. □

Theorem 6.2.10. *If*

$$P = \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \textit{and} \quad Q = \frac{\psi(q^2)}{q\psi(q^{10})},$$

then

$$\left(\frac{P}{Q}\right)^2 - \frac{5}{P^2} - P^2 + \left(\frac{Q}{P}\right)^2 - 4 = 0. \tag{6.2.31}$$

Proof. We replace q by $-q$ in Theorem 6.2.9. □

Theorem 6.2.11. *If*

$$P = \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \textit{and} \quad Q = \frac{\psi(q)}{q\psi(q^5)},$$

then

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q} - \frac{Q}{P}\right)\left(\frac{5}{PQ} - PQ\right) - 6 = 0. \tag{6.2.32}$$

Proof. We use [16, p. 206, Entry 53] and proceed as in the proof of Theorem 6.2.8. □

Theorem 6.2.12. *If*

$$P = \frac{\psi(q)}{q^{1/8}\psi(q^2)} \quad \textit{and} \quad Q = \frac{\psi(q^2)}{q^{1/4}\psi(q^4)},$$

then

$$P^2 - \left(\frac{2}{PQ}\right)^2 - \left(\frac{Q}{P}\right)^2 = 0. \tag{6.2.33}$$

Proof. From [66, p. 21, Theorem 3.2.2], we note that

$$(L_1M_1)^4 + \left(\frac{2}{L_1M_1}\right)^4 = \left(\frac{M_1}{L_1}\right)^{12}, \tag{6.2.34}$$

where

$$L_1 := \frac{f(-q)}{q^{1/24}f(-q^2)} \quad \text{and} \quad M_1 := \frac{f(-q^2)}{q^{1/12}f(-q^4)}. \tag{6.2.35}$$

Let

$$L_2 := \frac{f^2(-q^2)}{q^{1/6}f^2(-q^4)} \quad \text{and} \quad M_2 := \frac{f^2(-q^4)}{q^{1/3}f^2(-q^8)}. \tag{6.2.36}$$

Now, we proceed as in the proof of Theorem 6.2.6 with applications of (6.2.34) instead of [16, p. 204, Entry 51] to complete the proof. □

Theorem 6.2.13. *If*

$$P = \frac{\phi(q)}{\phi(q^5)} \quad \textit{and} \quad Q = \frac{\phi(-q)}{\phi(-q^5)},$$

then

$$PQ + \frac{5}{PQ} - 4 = \frac{Q}{P} + \frac{P}{Q}. \tag{6.2.37}$$

Proof. From [15, p. 39, Entry 24(iii)], we note that

$$\phi(q) = \frac{f^2(q)}{f(-q^2)}. \tag{6.2.38}$$

Thus, P and Q can be written as

$$P = \frac{f^2(q)f(-q^{10})}{f(-q^2)f^2(q^5)} \quad \text{and} \quad Q = \frac{f^2(-q)f(-q^{10})}{f(-q^2)f^2(-q^5)}. \tag{6.2.39}$$

Setting

$$L_1 := \frac{f(-q^2)}{q^{1/3}f(-q^{10})} \quad \text{and} \quad L_2 := \frac{f^2(q)}{q^{1/3}f^2(q^5)}, \tag{6.2.40}$$

$$M_1 := \frac{f(-q^2)}{q^{1/3}f(-q^{10})} \quad \text{and} \quad M_2 := \frac{f^2(-q)}{q^{1/3}f^2(-q^5)},. \tag{6.2.41}$$

we find that

$$P = \frac{L_2}{L_1}, \quad Q = \frac{M_2}{M_1}, \quad \text{and} \quad M_1 = L_1. \tag{6.2.42}$$

Now, from (6.2.40) and [16, p. 207, (53.3)], we have

$$L_1^4L_2^2 - 5L_1^2L_2 = L_1^6 - L_2^3. \tag{6.2.43}$$

From (6.2.42) and (6.2.43), we obtain

$$L_1^3 = \frac{5P - P^3}{P^2 - 1}. \tag{6.2.44}$$

Again, from (6.2.41) and [16, p. 206, Entry 53]

$$M_1^4M_2^2 + 5M_1^2M_2 = M_1^6 + M_2^3. \tag{6.2.45}$$

From (6.2.42) and (6.2.45), we find that

$$M_1^3 = \frac{Q^3 - 5Q}{Q^2 - 1}. \tag{6.2.46}$$

Since $L_1 = M_1$, so from (6.2.44) and (6.2.46), we deduce that

$$\frac{5P - P^3}{P^2 - 1} = \frac{Q^3 - 5Q}{Q^2 - 1}. \tag{6.2.47}$$

Simplifying (6.2.47), we arrive at

$$(P + Q)(5 - P^2 - 4PQ - Q^2 + P^2Q^2) = 0. \tag{6.2.48}$$

Since the first factor is non-zero in a neighborhood of the origin, we deduce that

$$5 - P^2 - 4PQ - Q^2 + P^2Q^2 = 0. \tag{6.2.49}$$

Dividing the above equation by PQ, we complete the proof. □

Theorem 6.2.14. *If*

$$P = \frac{\phi(-q)}{\phi(-q^5)} \quad \textit{and} \quad Q = \frac{\phi(-q^2)}{\phi(-q^{10})},$$

then

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 - Q^2 - \frac{5}{Q^2} + 4 = 0. \tag{6.2.50}$$

Proof. Employing (6.2.38), we note that

$$P = \frac{f^2(-q)f(-q^{10})}{f(-q^2)f^2(-q^5)} \quad \text{and} \quad Q = \frac{f^2(-q^2)f^2(-q^{20})}{f(-q^4)f^2(-q^{10})}. \tag{6.2.51}$$

Setting

$$L_1 := \frac{f(-q^2)}{q^{1/3}f(-q^{10})} \quad \text{and} \quad L_2 := \frac{f^2(-q)}{q^{1/3}f^2(-q^5)}, \tag{6.2.52}$$

$$M_1 := \frac{f(-q^4)}{q^{2/3}f(-q^{20})} \quad \text{and} \quad M_2 := \frac{f^2(-q^2)}{q^{2/3}f^2(-q^{10})}, \tag{6.2.53}$$

we deduce that

$$P = \frac{L_2}{L_1}, \quad Q = \frac{M_2}{M_1}, \quad \text{and} \quad M_2 = L_1^2. \tag{6.2.54}$$

Now, from (6.2.52) and [16, p.206, Entry 53], we deduce that

$$L_1M_1 + \frac{5}{L_1M_1} = \left(\frac{M_1}{L_1}\right)^3 + \left(\frac{L_1}{M_1}\right)^3. \tag{6.2.55}$$

Applying the results in (6.2.54) and simplifying, we find that

$$L_1^6 = \frac{Q^6 - 9Q^4}{Q^2 - 1}. \tag{6.2.56}$$

Similarly, from (6.2.53) and [16, p. 206, (53.3)], we obtain

$$L_1^3 = \frac{P^3 - 5P}{P^2 - 1}. \tag{6.2.57}$$

From (6.2.56) and (6.2.57), we find that

$$\left(\frac{P^3 - 5P}{P^2 - 1}\right)^2 = \frac{Q^6 - 9Q^4}{Q^2 - 1}. \tag{6.2.58}$$

Simplifying the above equation, we obtain

$$(5 - P^2 - Q^2 + P^2Q^2)(-5P^2 + P^4 + 4P^2Q^2 + Q^4 - P^2Q^4) = 0. \tag{6.2.59}$$

Now, proceeding as in Theorem 6.2.6, it can be shown that the first factor of (6.2.59) is non-zero in a neighborhood of zero. Thus, we have

$$5P^2 - P^4 - 4P^2Q^2 - Q^4 + P^2Q^4 = 0. \tag{6.2.60}$$

Dividing the above equation by P^2Q^2, we complete the proof. □

Theorem 6.2.15. *If*

$$P = \frac{\phi(q)}{\phi(q^5)} \quad \text{and} \quad Q = \frac{\phi(-q^2)}{\phi(-q^{10})},$$

then

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 - Q^2 - \frac{5}{Q^2} + 4 = 0. \tag{6.2 61}$$

Proof Replacing q by $-q$ in Theorem 6.2.14, we readily complete the proof. □

Theorem 6.2.16. *(Berndt [15, p. 306, Entry 18(vi)]) If*

$$\mu = \frac{f^4(-q)}{qf^4(-q^7)} \quad \text{and} \quad \nu = \frac{f(-q^{1/7})}{q^{2/7}f(-q^7)},$$

then

$$2\mu = 7(\nu^3 + 5\nu^2 + 7\nu) + (\nu^2 + 7\nu + 7)(4\nu^3 + 21\nu^2 + 28\nu)^{1/2}. \tag{6.2.62}$$

6.3 Properties of $g_{k,n}$ and $g'_{k,n}$

Theorem 6.3.1. *For all positive real numbers k and n, we have*

(i) $g_{k,1} = 1$,

(ii) $g_{k,\frac{1}{n}} = g_{k,n}^{-1}$,

(iii) $g_{k,n} = g_{n,k}$.

Remark 6.3.1.. By using the definitions of $\psi(q)$ and $g_{k,n}$, it can be seen that $g_{k,n}$ increases as n increases when $k > 1$. Thus, by Theorem 6.3.1(i), $g_{k,n} > 1$ for all $n > 1$ if $k > 1$.

Proof Using the definition of $g_{k,n}$ and Theorem 1.1.5, we easily arrive at $g_{k,1} = 1$. Replacing n by $1/n$ in $g_{k,n}$ and using Theorem 1.1.5, we find that $g_{k,n}\, g_{k,1/n} = 1$. Interchanging n and k in $g_{k,n}$, we complete the proof of (iii). □

Theorem 6.3.2. *For all positive real numbers k, m, and n*

$$g_{k,\frac{n}{m}} = g_{mk,n}\, g_{nk,m}^{-1}$$

Proof By the definition of $g_{k,n}$, we find that

$$g_{mk,n}\, g_{nk\,m}^{-1} = g_{\frac{m}{n},\frac{1}{k}}$$

Employing Theorem 6 3 1(ii) and (iii), we complete the proof □

Theorem 6.3.3. *For all positive real numbers a, b, c, and d, we have*

$$g_{\frac{a}{b},\frac{c}{d}} = \frac{g_{ad,bc}}{g_{ac,bd}} \tag{6 3 1}$$

Proof Applying Theorem 6 3 1(iii) in Theorem 6 3 2, we deduce that, for all positive real numbers a, b, and n

$$g_{\frac{a}{b},n} = g_{a,bn}\, g^{-1}_{b,an} \tag{6 3 2}$$

Again employing Theorem 6 3 2 and Theorem 6 3 1(iii) in (6 3 2), we arrive at (6 3 1) □

Theorem 6.3.4. *For all positive real numbers k and n, we have*

$$g_{k^2,n} = g_{k,nk}\, g_{k,\frac{n}{k}}$$

Proof Setting $a = k$, $b = 1/k$, $c = n$, and $d = 1$ in Theorem 6 3 3, we deduce that

$$g_{k^2,n} = \frac{g_{k,\frac{n}{k}}}{g_{\frac{1}{k},nk}}$$

Employing Theorem 6 3 1(ii) and (iii), we readily complete the proof □

Theorem 6.3.5. *For all positive real numbers a and b, we have*

(i) $g_{\frac{a}{b},\frac{a}{b}} = g_{b,b}\, g_{a,\frac{a}{b^2}}$,

(ii) $g_{a,a}\, g_{a,\frac{b^2}{a}} = g_{b,b}\, g_{b,\frac{a^2}{b}}$,

(iii) $g_{a,a}\, g_{b,a^2b} = g_{b,b}\, g_{a,ab^2}$

Proof Let a and b be any positive real numbers By using Theorem 6 3 3 and Theorem 6 3 1(ii), we find that

$$g_{\frac{a}{b},\frac{a}{b}} = g_{b,b}\, g_{a,\frac{a}{b^2}} \tag{6 3 3}$$

So we complete the proof of (i) Similarly, we find that

$$g_{\frac{b}{a},\frac{b}{a}} = g_{a,a}\, g^{-1}_{b,\frac{a^2}{b}} \tag{6 3 4}$$

From (6 3 3) and (6 3 4), we derive (ii) By using Theorem 6 3 1(ii) and Theorem 6 3 2, we find that

$$g_{\frac{a}{b},\frac{a}{b}} = g_{b,b}\, g_{ab^2,a}\, g^{-1}_{a^2,b^2} \tag{6 3 5}$$

Similarly, we find that

$$g_{\frac{b}{a},\frac{b}{a}} = g_{a,a}\, g_{a^2b,b}\, g^{-1}_{b^2,a^2} \tag{6 3 6}$$

From (6 3 5), (6 3 6), and Theorem 6 3 1(ii) and (iii), we complete the proof of (iii) □

Theorem 6.3.6. *For all positive real numbers k, a, b, c, and d with $ab = cd$, we have*

$$g_{a,b}\, g_{kc,kd} = g_{ka,kb}\, g_{c,d}$$

Proof From the definition of $g_{k,n}$ and using $ab = cd$, we derive that for all positive numbers k, a, b, c, and d,

$$g_{ka,kb}\, g^{-1}_{a,b} = g_{kc,kd}\, g^{-1}_{c,d}$$

Rearranging the terms, we complete the proof □

Theorem 6.3.7. *For all positive real numbers n and p, we have*

$$g_{np,np} = g_{n,np^2}\, g_{p,p}.$$

Proof. The result follows immediately from Theorem 6.3.1(i) and (iii), and Theorem 6.3.6 with $a = p^2$, $b = 1$, $c = d = p$, and $k = n$. □

Now, we give relations between the parameters $g_{k,n}$, $g'_{k,n}$, $r_{k,n}$, and $r'_{k,n}$ and then use these relations to determine the values of $g_{k,n}$ and $g'_{k,n}$ by using known values of $r_{k,n}$ and $r'_{k,n}$, where $r_{k,n}$ and $r'_{k,n}$ are given by (6.1.1) and (6.1.2).

Theorem 6.3.8. *Let k and n be any positive real numbers. Then*

(i) $g_{k,n} = \dfrac{r^2_{k,n}}{r'_{k,n}}$

(ii) $g'_{k,n} = \dfrac{r_{2,\frac{nk}{2}}}{r_{2,\frac{n}{2k}}}\, r_{k,n}.$

Proof. (i) Let $q = e^{-\pi\sqrt{n/k}}$. From [15, p.39, Entry 24(iii)]

$$\psi(q) = \frac{f^2(-q^2)}{f(-q)}. \tag{6.3.7}$$

Replacing q by $-q$ in (6.3.7) and using the definitions of $g_{k,n}$ and $r_{k,n}$, we find that

$$g_{k,n} = \frac{G_{nk}}{G_{\frac{n}{k}}} r_{k,n}, \tag{6.3.8}$$

where the class invariant G_n is given by

$$G_n = 2^{-1/4} q^{-1/24} \chi(q),$$

where $q := e^{-\pi\sqrt{n}}$, n is a positive real number, and $\chi(q) = (-q;q^2)_\infty$.
By [66, p. 17, Theorem 2.2.1], we note that

$$\frac{G_{\frac{n}{k}}}{G_{nk}} = \frac{r'_{k,n}}{r_{k,n}}. \tag{6.3.9}$$

Using (6.3.9) in (6.3.8), we complete the proof of (i).
(ii) Let $q := e^{-\pi\sqrt{n/k}}$. Employing (6.3.7) and the definitions of $g'_{k,n}$ and $r_{k,n}$, we find that

$$g'_{k,n} = \frac{g_{nk}}{g_{\frac{n}{k}}} r_{k,n}, \tag{6.3.10}$$

where the class invariant g_n is given by

$$g_n = 2^{-1/4} q^{-1/24} \chi(-q),$$

where $q := e^{-\pi\sqrt{n}}$, n is a positive real number and $\chi(q) = (-q; q^2)_\infty$.
Also, by [66, p. 18, Theorem 2.3.3(i)], we have

$$g_n = r_{2,\frac{n}{2}}. \tag{6.3.11}$$

Using (6.3.11) in (6.3.10), we complete the proof of (ii). □

Theorem 6.3.9. *For every positive real number n, we have*

$$g'_{n,1} = r_{4,n}. \tag{6.3.12}$$

Proof. From [66, p. 13, Lemma 2.1.3(i)], we note that

$$r_{k,n/m} = \frac{r_{mk,n}}{r_{nk,m}}. \tag{6.3.13}$$

Employing (6.3.13), Theorem 6.3.8(ii) and Theorem 6.3.1(i), we complete the proof. □

Theorem 6.3.10. *For all positive real numbers k and n, we have*

(i) $g_{k,n} = \dfrac{G_{nk}}{G_{\frac{n}{k}}} r_{k,n}$

(ii) $g'_{k,n} = \dfrac{g_{nk}}{g_{\frac{n}{k}}} r_{k,n}$.

Proof. These are (6.3.8) and (6.3.10), respectively. □

Theorem 6.3.11. *For every positive real number n, we have*

(i) $g_{n,n} = G_{n^2}\, r_{n,n}$

(ii) $g'_{n,n} = 2^{1/8}\, r_{2,\frac{n^2}{2}}\, r_{n,n} = 2^{1/8} g_{n^2}\, r_{n,n}$.

Proof. (i) With $k = n$ in Theorem 6.3.8 (i) and then using [66, p. 17, Corollary 2.2.2], we complete the proof.
(ii) Setting $k = n$ in Theorem 6.3.8(ii) and using the value $r_{2,2} = 2^{1/8}$ from [66, p. 42, Theorem 4.1.2(i)], we complete the proof of (ii). □

6.4 General theorems for explicit evaluations of $g_{k,n}$ and $g'_{k,n}$

In this section, we find some general theorems on $g_{k,n}$ and $g'_{k,n}$ and then use these theorems to find some explicit values of $g_{k,n}$ and $g'_{k,n}$.

Theorem 6.4.1. *We have*

(i) $\left(1+\sqrt{3}g_{3,n}g_{3,9n}\right)^3 = \left(1+3g_{3,9n}^4\right)$,

(ii) $\left(\sqrt{5}g_{5,n}g_{5,9n} + \dfrac{\sqrt{5}}{g_{5,n}g_{5,9n}}\right) = \left(\dfrac{g_{5,9n}}{g_{5,n}}\right)^2 - 3\left(\dfrac{g_{5,9n}}{g_{5,n}}\right) - 3\left(\dfrac{g_{5,n}}{g_{5,9n}}\right) - \left(\dfrac{g_{5,n}}{g_{5,9n}}\right)^2$,

(iii) $3\left(g_{3,n}g_{3,25n}\right)^2 + \dfrac{3}{\left(g_{3,n}g_{3,25n}\right)^2} + 5\left(\dfrac{g_{3,25n}}{g_{3,n}}\right)^2 + 5\left(\dfrac{g_{3,n}}{g_{3,25n}}\right)^2$

$= \left(\dfrac{g_{3,25n}}{g_{3,n}}\right)^3 - \left(\dfrac{g_{3,n}}{g_{3,25n}}\right)^3 + 5\left(\dfrac{g_{3,25n}}{g_{3,n}} - \dfrac{g_{3,n}}{g_{3,25n}}\right)$,

(iv) $k_1\left(\sqrt{3}g_{3,n}g_{3,49n}\right)^3 + k_2\left(\sqrt{3}g_{3,n}g_{3,49n}\right) = k_3\left(\sqrt{3}g_{3,n}g_{3,49n}\right)^2 + k_4\left(\dfrac{g_{3,n}}{g_{3,49n}}\right)^2 - k_5$,

where $k_1 = \left(\dfrac{g_{3,n}}{g_{3,49n}}\right)^8 - 1$, $k_2 = -42g_{3,n}^4\left(\left(\dfrac{g_{3,n}}{g_{3,49n}}\right)^4 - 1\right)$, $k_3 = -3g_{3,n}^4(7+3g_{3,n}^4)$,

$k_4 = 63g_{3,n}^4(g_{3,n}^4+1)$, *and* $k_5 = 27\left(\dfrac{g_{3,n}}{g_{3,49n}}\right)^4 - 63g_{3,n}^4\left(1+\left(\dfrac{g_{3,n}}{g_{3,49n}}\right)^4 - g_{3,n}^4\right)$,

(v) $\left(\dfrac{\sqrt{3}}{g_{9,9n}} + \sqrt{3}g_{9,9n} + 3\right)\left(\dfrac{\sqrt{3}}{g_{9,n}} + \sqrt{3}g_{9,n} + 3\right) = \left(\dfrac{g_{9,9n}}{g_{9,n}}\right)^2$.

Proof. Proof of (i) follows from [15, p. 345, Entry 1(ii)] and the definition of $g_{k,n}$. (ii)-(v) follow from Theorems 6.2.1, 2.2.11,2.2.12, and 6.2.2, respectively, and the definition of $g_{k,n}$. □

Theorem 6.4.2. *For any positive real number n, we have*

(i) $\left(1-\dfrac{\sqrt{3}}{g'_{3,n}g'_{3,9n}}\right)^3 = \left(1-\dfrac{3}{g'^4_{3,9n}}\right)$,

(ii) $\sqrt{5}g'_{5,n}g'_{5,9n} + \dfrac{\sqrt{5}}{g'_{5,n}g'_{5,9n}} = \left(\dfrac{g'_{5,9n}}{g'_{5,n}}\right)^2 + 3\left(\dfrac{g'_{5,9n}}{g'_{5,n}}\right) + 3\left(\dfrac{g'_{5,n}}{g'_{5,9n}}\right) - \left(\dfrac{g'_{5,n}}{g'_{5,9n}}\right)^2$,

(iii) $3\left(g'_{3,n}g'_{3,25n}\right)^2 + \dfrac{3}{\left(g'_{3,n}g'_{3,25n}\right)^2} + 5\left(\dfrac{g'_{3,25n}}{g'_{3,n}}\right)^2 - 5\left(\dfrac{g'_{3,n}}{g'_{3,25n}}\right)^2 = \left(\dfrac{g'_{3,25n}}{g'_{3,n}}\right)^3$

$-\left(\dfrac{g'_{3,n}}{g'_{3,25n}}\right)^3 - 5\left(\dfrac{g'_{3,25n}}{g'_{3,n}} - \dfrac{g'_{3,n}}{g'_{3,25n}}\right)$,

(iv) $k_1\left(\sqrt{3}g'_{3,n}g'_{3,49n}\right)^3 + k_2\left(\sqrt{3}g'_{3,n}g'_{3,49n}\right) = k_3\left(\sqrt{3}g'_{3,n}g'_{3,49n}\right)^2 + k_4\left(\dfrac{g'_{3,n}}{g'_{3,49n}}\right)^2 - k_5$,

where $k_1 = \left(\dfrac{g'_{3,n}}{g'_{3,49n}}\right)^8 - 1$, $k_2 = 42g'^4_{3,n}\left(\left(\dfrac{g'_{3,n}}{g'_{3,49n}}\right)^4 - 1\right)$, $k_3 = 3g'^4_{3,n}(7-3g'^4_{3,n})$,

$k_4 = 63g'^4_{3,n}(g'^4_{3,n}-1)$, *and* $k_5 = 27\left(\dfrac{g'_{3,n}}{g'_{3,49n}}\right)^4 + 63g'^4_{3,n}\left(1+\left(\dfrac{g'_{3,n}}{g'_{3,49n}}\right)^4 + g'^4_{3,n}\right)$,

(v) $\left(\dfrac{\sqrt{3}}{g'_{9,9n}} + \sqrt{3}g'_{9,9n} - 3\right)\left(\dfrac{\sqrt{3}}{g'_{9,n}} + \sqrt{3}g'_{9,n} - 3\right) = \left(\dfrac{g'_{9,9n}}{g'_{9,n}}\right)^2.$

Proof. *Proof of (i) follows easily from [15, p. 345, Entry 1(i)] and the definition of* $g'_{k,n}$. *Proofs* of (ii) - (v) follow from Theorems 6.2.1, 2.2.11,2.2.12, and 6.2.2, respectively, and the definition of $g'_{k,n}$. □

Theorem 6.4.3. *We have*

(i) $\left(\dfrac{g'_{3,n}}{g'_{3,4n}}\right)^2 + \sqrt{3}\left(\dfrac{1}{(g'_{3,n})^2} - (g'_{3,n})^2\right) + \left(\dfrac{g'_{3,4n}}{g'_{3,n}}\right)^2 = 0,$

(ii) $\left(\dfrac{g_{3,n}}{g'_{3,4n}}\right)^2 + \sqrt{3}\left(\dfrac{1}{(g_{3,n})^2} + (g_{3,n})^2\right) - \left(\dfrac{g'_{3,4n}}{g_{3,n}}\right)^2 = 0,$

(iii) $\left(\dfrac{g_{3,n}}{g'_{3,n}}\right)^4 + \left(\dfrac{g'_{3,n}}{g_{3,n}}\right)^4 + 3\left\{\left(\dfrac{g_{3,n}}{g'_{3,n}}\right)^2 - \left(\dfrac{g'_{3,n}}{g_{3,n}}\right)^2\right\}\left\{\left(\dfrac{1}{g_{3,n}g'_{3,n}}\right)^2 - (g'_{3,n}g_{3,n})^2\right\} - 10 = 0,$

(iv) $\left(\dfrac{g'_{5,n}}{g'_{5,4n}}\right)^2 - \sqrt{5}\left(\dfrac{1}{(g'_{5,n})^2} + (g'_{5,n})^2\right) + \left(\dfrac{g'_{5,4n}}{g'_{5,n}}\right)^2 + 4 = 0,$

(v) $\left(\dfrac{g_{5,n}}{g'_{5,4n}}\right)^2 - \sqrt{5}\left(\dfrac{1}{(g_{5,n})^2} + (g_{5,n})^2\right) + \left(\dfrac{g'_{5,4n}}{g_{5,n}}\right)^2 - 4 = 0,$

(vi) $\left(\dfrac{g_{5,n}}{g'_{5,n}}\right)^2 + \left(\dfrac{g'_{5,n}}{g_{5,n}}\right)^2 + \sqrt{5}\left(\dfrac{g_{5,n}}{g'_{5,n}} - \dfrac{g'_{5,n}}{g_{5,n}}\right)\left(\dfrac{1}{g_{5,n}g'_{5,n}} - g'_{5,n}g_{5,n}\right) - 6 = 0,$

(vii) $\sqrt{2}\left((g'_{2,n})^2 - \dfrac{\sqrt{2}}{(g'_{2,n}g'_{2,4n})^2}\right) - \left(\dfrac{g'_{2,4n}}{g'_{2,n}}\right)^2 = 0.$

Proof. Proofs of (i) - (vii) follow from Theorems 6.2.6 - 6.2.12, respectively, and the definitions of $g_{k,n}$ and $g'_{k,n}$. □

Theorem 6.4.4. *We have*

(i) $g_{3,3} = \dfrac{\psi(-e^{-\pi})}{3^{1/4}e^{-\pi/4}\psi(-e^{-3\pi})} = \left(3 + 2\sqrt{3}\right)^{1/4}$

and

(iii) $g_{3,9} = \dfrac{\psi(-e^{-\pi/\sqrt{3}})}{3^{1/4}e^{-\pi\sqrt{3}/4}\psi\left(-e^{-3\sqrt{3}\pi}\right)} = \dfrac{(1 + 2^{1/3})^2}{\sqrt{3}}.$

Proof. Setting $n = 1/3$ in Theorem 6.4.1(i) and employing Theorem 6.3.1(ii), we obtain

$$(1 + \sqrt{3})^3 = (1 + 3g_{3,3}^4),$$

which readily gives (i). Again, setting $n = 1$ in Theorem 6.4.1(i) and recalling the value $g_{k,1} = 1$ from Theorem 6.3.1(i), we find that

$$(1 + \sqrt{3}g_{3,9})^3 = (1 + 3g_{3,9}^4). \tag{6.4.1}$$

Solving (6.4.1) and using the remark given after Theorem 6.3.1, we prove (ii). □

Theorem 6.4.5. *We have*

(i) $g_{5,9} = \frac{1}{2}\left(3 + \sqrt{3} + \sqrt{5} + \sqrt{15}\right)$

and

(ii) $g_{5,3} = (17\sqrt{5} + 38)^{1/6}$.

Proof. Setting $n = 1$ in Theorem 6.4.1(ii) and recalling that $g_{k,1} = 1$ from Theorem 6.3.1(i), we find that

$$\sqrt{5}\left(g_{5,9} + \frac{1}{g_{5,9}}\right) = (g_{5,9})^2 - 3\left(g_{5,9} + \frac{1}{g_{5,9}}\right) - \left(\frac{1}{g_{5,9}}\right)^2. \tag{6.4.2}$$

Solving (6.4.2) and using the fact that $g_{k,n} > 1$ from the remark after Theorem 6.3.1, we prove (i). Again, setting $n = 1/3$ in Theorem 6.4.1(ii) and recalling $g_{k,1/n} = 1/g_{k,n}$ from Theorem 6.3.1(ii), we find that,

$$\left(g_{5,3}^4 - \frac{1}{g_{5,3}^4}\right) - 3\left(g_{5,3}^2 + \frac{1}{g_{5,3}^2}\right) = 2\sqrt{5}. \tag{6.4.3}$$

Solving (6.4.3) and employing $g_{k,n} > 1$ again, we prove (ii). □

Theorem 6.4.6. *We have*

(i) $$g_{3,25} = \frac{\left(1 + \sqrt[3]{10} + \sqrt{5 + 2\sqrt[3]{10} + \sqrt[3]{10^2}}\right)^2}{2(1+\sqrt{5})},$$

(ii) $$g_{3,7} = \frac{\left(\sqrt{3} + \sqrt{7}\right)^{3/4}}{2^{3/4}\left(2 - \sqrt{3}\right)^{1/4}},$$

(iii) $$g_{13,3} = \frac{\left(\sqrt{11 + \sqrt{13}} + \sqrt{3 + \sqrt{13}}\right)^2}{2\sqrt{2}\left(\sqrt{5 + \sqrt{13}} + \sqrt{\sqrt{13} - 3}\right)},$$

(iv) $$g_{3,49} = \frac{\left(3 + \sqrt[3]{4}\cdot\sqrt[3]{7} + \sqrt[3]{2}\cdot\sqrt[3]{49} + \sqrt{49 + 13\cdot\sqrt[3]{4}\cdot\sqrt[3]{7} + 8\cdot\sqrt[3]{2}\cdot\sqrt[3]{49}}\right)^2}{6\left(\sqrt{3} + \sqrt{7}\right)},$$

(v) $$g_{25,9} = \frac{(\sqrt[4]{60} + 2 - \sqrt{3} + \sqrt{5})^2(\sqrt[4]{60} - 2 - \sqrt{3} + \sqrt{5})}{(\sqrt[4]{60} - 2 + \sqrt{3} - \sqrt{5})^2(\sqrt[4]{60} + 2 + \sqrt{3} - \sqrt{5})},$$

(vi) $$g_{13,9} = \frac{\left((\sqrt{3}+1)(\sqrt{3}+\sqrt{13}) + 2\sqrt{(3+2\sqrt{3})(4+\sqrt{13})}\right)^2}{4\left((\sqrt{3}+1)(\sqrt{13}-\sqrt{3}) + 2\sqrt{(3+2\sqrt{3})(4-\sqrt{13})}\right)},$$

(vii) $$g_{25,7} = \frac{\left(4\sqrt{5} + a + b + \sqrt{(4\sqrt{5}+a+b)^2 - 36}\right)^2}{6\left(2\sqrt{5} + c + d + \sqrt{(2\sqrt{5}+c+d)^2 - 36}\right)},$$

where $a = \left(\frac{1}{2}(2251\sqrt{5} + 9\sqrt{105})\right)^{1/3}$, $b = \left(\frac{1}{2}(2251\sqrt{5} - 9\sqrt{105})\right)^{1/3}$,

$c = \left(\frac{1}{2}(17\sqrt{5} + 3\sqrt{105})\right)^{1/3}$, *and* $d = \left(\frac{1}{2}(17\sqrt{5} - 3\sqrt{105})\right)^{1/3}$,

(viii) $$g_{25,49} = \frac{(a' + 2\sqrt{5b'} + \sqrt{(a' + 2\sqrt{5b'})^2 - 64})^2}{8(c' + 2\sqrt{5d'} + \sqrt{(c' + 2\sqrt{5d'})^2 - 64})},$$

where $a' = 1497 + 651\sqrt{5} + 565\sqrt{7} + 247\sqrt{35}$, $b' = 437430 + 195566\sqrt{5} + 165333\sqrt{7} + 73917\sqrt{35}$, $c' = 1497 - 651\sqrt{5} + 565\sqrt{7} - 247\sqrt{35}$, *and* $d' = 437430 - 195566\sqrt{5} + 165333\sqrt{7} - 73917\sqrt{35}$.

Proof. The proof of the theorem follows from Theorem 6.3.8(i) and the corresponding values of $r_{k,n}$ and $r'_{k,n}$ from Sections 1.3 and 1.4, respectively. □

Theorem 6.4.7. *We have*

(i) $g'_{1,1} = 1$,

(ii) $g'_{2,1} = 2^{1/8}\left(1 + \sqrt{2}\right)^{1/8}$,

(iii) $g'_{3,1} = \sqrt{\frac{\sqrt{3}+1}{\sqrt{2}}}$,

(iv) $g'_{4,1} = 2^{5/16}\left(1 + \sqrt{2}\right)^{1/4}$,

(v) $g'_{5,1} = \left(\frac{1 + \sqrt{5} + \sqrt{2}\sqrt{1+\sqrt{5}}}{2}\right)^{1/2}$,

(vi) $g'_{8,1} = 2^{1/4}\left(1 + \sqrt{2}\right)^{3/8}\left(4 + \sqrt{2 + 10\sqrt{2}}\right)^{1/8}$,

(vii) $g'_{9,1} = \frac{1}{2}\left(1 + \sqrt{2}\sqrt[4]{3} + \sqrt{3}\right)$,

(viii) $g'_{25,1} = \frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}$.

Proof. The proof of the theorem follows from Theorem 6.3.9 and the corresponding values of $r_{4,n}$ from Section 1.3. □

Theorem 6.4.8. *We have*

(i) $g'_{2,4} = \left(\sqrt{\sqrt{2}-1} + \sqrt{\sqrt{2}+1}\right)^{1/2}$,

(ii) $g'_{2,8} = \sqrt{2 + \sqrt{2}}$,

(iii) $g'_{3,4} = \frac{\sqrt{3}+1}{\sqrt{2}}$,

(iv) $g'_{3,16} = \dfrac{(1+\sqrt{3})(\sqrt{2}+1)}{\sqrt{2}}$,

(v) $g'_{3,64} = \left(102 + 72\sqrt{2} + 59\sqrt{3} + 42\sqrt{6} + \sqrt{41680 + 29472\sqrt{2} + 24064\sqrt{3} + 17016\sqrt{6}}\right)^{1/2}$,

(vi) $g'_{3,12} = \left(3 + 2\sqrt{3} + \sqrt{24 + 14\sqrt{3}}\right)^{1/2}$,

(vii) $g'_{3,36} = \left(13\sqrt{3} + 10\cdot 2^{1/3}\sqrt{3} + 8\cdot 2^{2/3}\cdot\sqrt{3} + 2\sqrt{373 + 296\cdot 2^{1/3} + 235\cdot 2^{2/3}}\right)^{1/2}$,

(viii) $g'_{3,20} = \dfrac{1}{\sqrt{2}}\left(\sqrt{3} + \sqrt{3}\left(38+17\sqrt{5}\right)^{2/3} + \sqrt{3 + 10\left(38+17\sqrt{5}\right)^{2/3} + 3\left(38+17\sqrt{5}\right)^{4/3}}\right)^{1/2}$,

(ix) $g'_{3,7} = \left(\dfrac{12208 + 7048\sqrt{3} + 4614\sqrt{7} + 2664\sqrt{21} - \sqrt{6k}}{582 + 333\sqrt{3} + 218\sqrt{7} + 127\sqrt{21}}\right)^{1/4}$,

where $k = 9623566 + 55561688\sqrt{3} + 36373663\sqrt{7} + 21000344\sqrt{21}$

(x) $g'_{3,9} = \dfrac{746 + 592\sqrt{3} + 470\sqrt[3]{4} + \sqrt{1641279 + 1302684\sqrt{3} + 1033941\sqrt[3]{4}}}{19 + 15\sqrt[3]{2} + 12\sqrt[3]{4}}$,

(xi) $g'_{5,4} = \left(\dfrac{1}{2}\left(4 + 2\sqrt{5} + \sqrt{2(1+\sqrt{5})} + \sqrt{10(1+\sqrt{5})}\right)\right)^{1/2}$,

(xii) $g'_{5,3} = \left(\dfrac{\sqrt{5} - 6\left(38+17\sqrt{5}\right)^{1/3} + \sqrt{5}\left(38+17\sqrt{5}\right)^{1/3} + \sqrt{r}}{-2 + 2\sqrt{5}\left(38+17\sqrt{5}\right)^{1/3}}\right)^{1/2}$,

where $r = -675 - 304\sqrt{5} + 19\left(38+17\sqrt{5}\right)^{1/3} + 77\sqrt{5}\left(38+17\sqrt{5}\right)^{1/3} + 22\left(38+17\sqrt{5}\right)^{1/3}$,

(xiii) $g'_{5,9} = \left(\dfrac{132 + 76\sqrt{3} + 59\sqrt{5} + 34\sqrt{15} + 2\sqrt{16406 + 9472\sqrt{3} + 7337\sqrt{5} + 4236\sqrt{15}}}{8 + 5\sqrt{3} + 4\sqrt{5} + 2\sqrt{15}}\right)^{1/2}$.

Proof. To prove (i) and (ii), we set $n = 1$ in Theorem 6.4.3(vii) and use the value of $g'_{2,1}$ from Theorem 6.4.7(ii) and the value of $g'_{2,2} = 2^{3/8}$ from Theorem 6.5.7(ii), respectively.

To prove (iii), we set $n = 1$ in Theorem 6.4.3(i) and use the value of $g'_{3,1}$ from Theorem 6.4.7(iii).

To prove (iv) and (v), we set $n = 4$ and 16, respectively in Theorem 6.4.3(i) and successively use the values of $g'_{3,4}$ and $g'_{3,16}$ from the same theorem.

To prove (vi)-(viii), we set $n = 3, 9$, and 5 in Theorem 6.4.3(ii) and use the value of $g_{3,3}$, $g_{3,9}$, and $g_{3,5}$ *from Theorem 6.4.4(i), (iii), Theorem 6.4.5(iii), and Theorem 6.3.1(iii), respectively.*

We set n =7 and 9 in Theorem 6.4.3(iii) and use the values of $g_{3,7}$ and $g_{3,9}$ from Theorem 6.4.6(iii) and Theorem 6.4.4(iii), respectively, to complete the proof of (ix) and (x).

We set $n = 1$ in Theorem 6.4.3(iv) and use the value of $g'_{5,1}$ from Theorem 6.4.7(v) to prove (xi).

To prove (xii) and (xiii), we set $n = 3$, and 9 in Theorem 6.4.3(vi) and use the values of $g_{5,3}$ and $g_{5,9}$ from Theorem 6.4.5(iii) and (i), respectively. □

Theorem 6.4.9. *We have*

(i) $g'_{3,2} = (1+\sqrt{2})^{1/2}$,

(ii) $g'_{4,2} = 2^{3/8}(1+\sqrt{2})^{3/4}$,

(iii) $g'_{5,2} = \left(\frac{\sqrt{5}+1}{2}\right)^{3/2}$,

(iv) $g'_{2,8} = 2^{1/4}\left(1+\sqrt{2}\right)^{1/2}$,

(v) $g'_{9,2} = \sqrt{2}+\sqrt{3}$,

(vi) $g'_{4,8} = 2^{3/8}\left(1+\sqrt{2}\right)^{5/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8}$,

(vii) $g'_{2,16} = 2^{3/8}\left(1+\sqrt{2}\right)^{3/4}(4+\sqrt{2+10\sqrt{2}})^{3/8}$,

(viii) $g'_{2,32} = 2^{9/16}\left(1+\sqrt{2}\right)^{3/4}\left(16+15\sqrt[4]{2}+12\sqrt{2}+9\cdot 2^{3/4}\right)^{3/4}$.

Proof. The proof of the theorem follows directly from Theorem 6.3.8(ii) and the values of $r_{k,n}$ from Section 1.3. □

Theorem 6.4.10. *We have*

(i) $g_{7,7} = \frac{1}{2}\left(7^{1/4}+\sqrt{4+\sqrt{7}}\right)\left(\frac{35}{2}+7\sqrt{7}+\frac{7}{2}\sqrt{21+8\sqrt{7}}+\sqrt{147+56\sqrt{7}}\right)^{1/4}$,

(ii) $g'_{7,7} = 2^{-1/8}\, g_{49}\left(35+14\sqrt{7}+2\sqrt{147+56\sqrt{7}}+\sqrt{7(147+56\sqrt{7})}\right)^{1/4}$,

where $g_{49} = \left(\frac{G_{49}^{12}+\sqrt{G_{49}^{24}-1}}{2G_{49}^{4}}\right)^{1/8}$ *and* $G_{49} = \frac{7^{1/4}+\sqrt{4+\sqrt{7}}}{2}$.

Proof. First we find the explicit values of $r_{7,7}$ and $r'_{7,7}$ in the following Lemma. □

Lemma 6.4.11. *We have*

(i) $r_{7,7} = \left(\frac{1}{2}\left(35+14\sqrt{7}+2\sqrt{147+56\sqrt{7}}+\sqrt{7(147+56\sqrt{7})}\right)\right)^{1/4}$,

(ii) $r'_{7,7} = \dfrac{2^{3/4}\left(35+14\sqrt{7}+7\sqrt{21+8\sqrt{7}}+2\sqrt{147+56\sqrt{7}}\right)^{1/4}}{7^{1/4}+\sqrt{4+\sqrt{7}}}$.

Proof of the lemma. We set $q := e^{-2\pi}$ in Theorem 6.2.16 and then apply Theorem 1.1.3, to obtain

$$\nu = \frac{f(-e^{-2\pi/7})}{e^{-4\pi/7}f(-e^{-14\pi})} = \sqrt{7} \tag{6.4.4}$$

and

$$\mu = \frac{f(-e^{-2\pi})}{e^{-4\pi}f(-e^{-14\pi})} = 7\, r_{7,7}^4. \qquad (6.4.5)$$

Using (6.4.4) and (6.4.5) in (6.2.62), we obtain

$$r_{7,7} = \left(\frac{1}{2}\left(35 + 14\sqrt{7} + 2\sqrt{147 + 56\sqrt{7}} + \sqrt{7(147 + 56\sqrt{7})}\right)\right)^{1/4}, \qquad (6.4.6)$$

to complete the proof of (i).

From [66, p. 17, Corollary 2.2.2], we have

$$r_{n,n} = G_{n^2} r'_{n,} \qquad (6.4.7)$$

Setting $n = 7$ and using the value of G_{49} [17, p. 191] and (6.4.6) in ((6.4.7), we complete the proof of (ii).

Proof of Theorem 6.4.10. Using Theorem 6.3.11 and the above lemma, we easily complete the proof.

6.5 Explicit values for $\psi(\pm q)$

In this section, we find explicit formulae for the theta functions $\psi(e^{-n\pi})$, $\psi(-e^{-n\pi})$, $\psi(e^{-\pi/n})$, and $\psi(-e^{-\pi/n})$ for any positive real number n and give some examples.

Lemma 6.5.1. *Let $a = \pi^{1/4}/\Gamma(\frac{3}{4})$. Then*

(i) $\psi(e^{-\pi}) = a2^{-5/8}e^{\pi/8}$

(ii) $\psi(-e^{-\pi}) = a2^{-3/4}e^{\pi/8}$.

Proof. See [15, p. 123, Entry 11(i) and (ii)]. □

Theorem 6.5.2. *For every positive real number n, we have*

(i) $\psi(-e^{-n\pi}) = \dfrac{a2^{-3/4}e^{n\pi/8}}{n^{1/4}g_{n,n}} = \dfrac{a2^{-3/4}e^{n\pi/8}}{n^{1/4}G_{n^2}r_{n,n}}$

(ii) $\psi(e^{-n\pi}) = \dfrac{a2^{-5/8}e^{n\pi/8}}{n^{1/4}g'_{n,n}} = \dfrac{a2^{-3/4}e^{n\pi/8}}{n^{1/4}r_{2,\frac{n^2}{2}}r_{n,n}}$.

Proof. Using the definitions of $g_{n,n}$, $g'_{n,n}$, Lemma 6.5.1, and Theorem 6.3.11, we complete the proofs of (i) and (ii). □

Theorem 6.5.3. *For every positive number n, we have*

(i) $\psi(-e^{-\pi/n}) = \dfrac{an^{1/4}2^{-3/4}e^{\pi/8n}}{g_{n,n}} = \dfrac{an^{1/4}2^{-3/4}e^{\pi/8n}}{G_{n^2}r_{n,n}}$

(ii) $\psi(e^{-\pi/n}) = \dfrac{an^{1/4}2^{-5/8}e^{\pi/8n}}{g'_{\frac{1}{n},\frac{1}{n}}} = \dfrac{an^{1/4}2^{-3/4}r_{2,2n^2}e^{\pi/8n}}{r_{n,n}}.$

Proof. Replacing n by $1/n$ in Theorem 6.5.2(i) and (ii), and using the fact that $g_{1/n,1/n} = g_{n,n}$ and $r_{k,1/n} = r_{k,n}^{-1}$ [66, p. 12, Theorem 2.1.2], we complete the proof of (i) and (ii). □

In Theorem 6.4.4(i) and Theorem 6.4.10(i) and (ii), we have evaluated $g_{3,3}$, $g_{7,7}$, and $g'_{7,7}$, respectively. Now, we give some more explicit values of $g_{n,n}$ and $g'_{n,n}$ and then use these values to determine some values of theta-function $\psi(q)$.

Theorem 6.5.4. *We have*

(i) $g_{1,1} = 1$,

(ii) $g_{2,2} = 2^{-1/16}(\sqrt{2}+1)^{1/4}$,

(iii) $g_{4,4} = 2^{1/16}(1+\sqrt{2})^{1/2}(9\cdot 2^{1/4}+4\sqrt{2}-3\cdot 2^{3/4})^{1/8}$,

(iv) $g_{5,5} = \dfrac{(5+\sqrt{5})^{3/2}}{2^{3/2}\sqrt{5}}$,

(v) $g_{6,6} = \dfrac{3^{1/4}(\sqrt{3}+1)^{5/6}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{2/3}(2-3\sqrt{2}+3\cdot 3^{1/4}+3^{3/4})^{1/3}}{2^{85/48}}$,

(vi) $g_{9,9} = 2+\sqrt{3}+\dfrac{1}{3}\left(1269+729\sqrt{3}-27\sqrt{156+90\sqrt{3}}\right)^{1/3}+\left(47+27\sqrt{3}+\sqrt{156+90\sqrt{3}}\right)^{1/3}.$

Proof. The value in (i) readily follows from Theorem 6.3.1. The proofs of (ii) - (v) follow from Theorem 6.3.8 (i) and the values of $r_{k,n}$ and $r'_{k,n}$ given Sections 1.3 and 1.4, respectively. Next, we set $n = 1$ in Theorem 6.4.1(v) and use the value $g_{k,1} = 1$, to obtain

$$\left(\sqrt{3}\left(g_{9,9}+\frac{1}{g_{9,9}}\right)+3\right)\left(2\sqrt{3}+3\right) = g_{9,9}^2. \tag{6.5.1}$$

Solving equation (6.5.1), we easily arrive at (vii).

Adiga et al. [3] also found the value of $g_{9,9}$. The same value is also evaluated in Chapter 2 of this thesis. □

Theorem 6.5.5. *We have*

(i) $\psi(-e^{-\pi}) = a2^{-3/4}e^{\pi/8}$,

(ii) $\psi(-e^{-2\pi}) = a2^{-15/16}(\sqrt{2}-1)^{1/4}e^{\pi/4}$,

(iii) $\psi(-e^{-3\pi}) = \dfrac{a2^{-3/4}e^{3\pi/8}}{3^{1/4}(3+2\sqrt{3})^{1/4}}$,

(iv) $\psi(-e^{-4\pi}) = \dfrac{a2^{-21/16}(\sqrt{2}-1)^{1/2}e^{\pi/2}}{(9\sqrt[4]{2}+4\sqrt{2}-3\cdot 2^{3/4})^{1/8}}$,

(v) $\psi(-e^{-5\pi}) = \dfrac{ae^{5\pi/8}(5-\sqrt{5})^{3/2}}{2^{9/4}5^{5/4}}$,

(vi) $\psi(-e^{-6\pi}) = \dfrac{ae^{3\pi/4}2^{37/48}}{\sqrt{3}\left(\sqrt{3}+1\right)^{5/6}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{2/3}\left(2-3\sqrt{2}+3^{5/4}+3^{3/4}\right)^{1/3}}$,

(vii) $\psi(-e^{-7\pi}) = \dfrac{a2^{1/2}e^{7\pi/8}}{7^{1/4}\left(7^{1/4}+\sqrt{4+\sqrt{7}}\right)\left(35+14\sqrt{7}+7\sqrt{21+8\sqrt{7}}+2\sqrt{147+56\sqrt{7}}\right)^{1/4}}$,

(viii) $\psi(-e^{-9\pi}) = \dfrac{a2^{-3/4}e^{9\pi/8}}{\sqrt{3}g_{9,9}}$, *where* $g_{9,9}$ *is as given in Theorem* 6.5.4(*vi*).

Proof. The proof of the theorem follows from Theorem 6.5.2 (i) and the values of $g_{n,n}$ from Theorem 6.4.4(i), Theorem 6.4.10(i) and Theorem 6.5.4. □

Theorem 6.5.5(iii), (v) and (viii) were also proved by Baruah and Bhattacharyya [11].

Theorem 6.5.6. *We have*

(i) $\psi(-e^{-\pi/2}) = a\,2^{-7/16}e^{\pi/16}(\sqrt{2}-1)^{1/4}$,

(ii) $\psi(-e^{-\pi/3}) = \dfrac{a\,3^{1/4}2^{-3/4}e^{-\pi/24}}{(3+2\sqrt{3})^{1/4}}$,

(iii) $\psi(-e^{-\pi/4}) = \dfrac{a\,2^{-5/16}e^{\pi/32}(\sqrt{2}-1)^{1/2}}{(9\cdot 2^{1/4}+4\sqrt{2}-3\cdot 2^{3/4})^{1/8}}$,

(iv) $\psi(-e^{-\pi/5}) = \dfrac{a\,2^{3/4}5^{3/4}e^{\pi/40}}{(\sqrt{5}+5)^{3/2}}$,

(v) $\psi(-e^{-\pi/6}) = \dfrac{a\,2^{61/48}e^{\pi/48}}{(\sqrt{3}+1)^{5/6}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{2/3}(2-3\sqrt{2}+3^{5/4}+3^{3/4})^{1/3}}$,

(vi) $\psi(-e^{-\pi/7}) = \dfrac{a\,2^{1/2}\,7^{1/4}e^{\pi/56}}{\left(7^{1/4}+\sqrt{4+\sqrt{7}}\right)\left(35+14\sqrt{7}+7\sqrt{21+8\sqrt{7}}+2\sqrt{147+56\sqrt{7}}\right)^{1/4}}$,

(vii) $\psi(-e^{-\pi/9}) = \dfrac{\sqrt{3}\,a\,2^{-3/4}e^{\pi/72}}{g_{9,9}}$,

where $g_{9,9}$ *is as given in Theorem 6.5.4(vi).*

Proof The proofs follow from Theorem 6.5.3(i) and the values of $g_{n,n}$ from Theorem 6.4 4(i), and Theorem 6.4.10(i), and Theorem 6.5.4. □

Theorem 6.5.7. *We have*

(i) $g'_{1,1} = 1,$

(ii) $g'_{2,2} = 2^{3/8},$

(iii) $g'_{3,3} = \dfrac{3^{1/3}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3}(1+\sqrt{3})^{1/6}}{\sqrt{2}},$

(iv) $g'_{4,4} = 2^{3/8}(1+\sqrt{2})^{1/2},$

(v) $g'_{5,5} = \dfrac{(5+\sqrt{5})^{1/2}(5^{1/4}+1)}{2},$

(vi) $g'_{6,6} = \dfrac{3^{1/8}(1+\sqrt{3})^{5/6}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{2/3}}{2^{29/24}},$

(vii) $g'_{9,9} = \dfrac{a+(2(b-2c))^{1/3}+(2(b+2c))^{1/3}}{2},$

where $a = 2+\sqrt{2}\cdot 3^{1/4}+2\sqrt{3}+\sqrt{2}\cdot 3^{3/4}$, $b = 82+45\sqrt{2}+48\sqrt{3}+25\sqrt{2}\cdot 3^{3/4}$,

and $c = \sqrt{3\left(88+47\sqrt{2}\cdot 3^{1/4}+50\sqrt{3}+27\sqrt{2}\cdot 3^{3/4}\right)}.$

Proof. The proof of (i)-(vi) follow from Theorem 6.3.11 (ii) and the values of $r_{k,n}$ listed in Section 1.3.
Next, we set $n = 1$ in Theorem 6.4.2(v), to obtain

$$\left(\frac{\sqrt{3}}{g'_{9,9}}+\sqrt{3}g'_{9,9}-3\right)\left(\frac{\sqrt{3}}{g'_{9,1}}+\sqrt{3}g'_{9,1}-3\right) = \left(\frac{g'_{9,9}}{g'_{9,1}}\right)^2. \tag{6.5.2}$$

Substituting the value of $g'_{9,1}$ from Theorem 6.4.7(vii) in (6.5.2) and solving the resulting polynomial equation, we complete the proof of (vii). □

Theorem 6.5.8. *We have*

(i) $\psi(e^{-\pi}) = a\,2^{-5/8}e^{\pi/8},$

(ii) $\psi(e^{-2\pi}) = a\,2^{-5/4}e^{\pi/4},$

(iii) $\psi(e^{-3\pi}) = \dfrac{a\,2^{-1/8}e^{3\pi/8}}{3^{1/3}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3}(1+\sqrt{3})^{1/6}},$

(iv) $\psi(e^{-4\pi}) = a\,2^{-2}(2-\sqrt{2})^{1/2},$

(v) $\psi(e^{-5\pi}) = \dfrac{a\,2^{3/8}e^{5\pi/8}}{5^{1/4}(5+\sqrt{5})^{1/2}(1+5^{1/4})},$

(vi) $\psi(e^{-6\pi}) = \dfrac{a\,2^{1/4}e^{3\pi/4}}{3^{3/8}(1+\sqrt{3})^{5/6}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{2/3}},$

(vii) $\psi(e^{-7\pi}) = \dfrac{a\,7^{-1/4}2^{-1/2}e^{7\pi/8}}{g'_{7,7}},$

(viii) $\psi(e^{-9\pi}) = \dfrac{a2^{-5/8}e^{9\pi/8}}{\sqrt{3}g'_{9,9}}$,

where $g'_{7,7}$ and $g'_{9,9}$ are as given in Theorems 6.4.10(ii) and 6.5.7, respectively.

Proof. The proof of the theorem follows from Theorem 6.4.2(ii) and the values of $g'_{n,n}$ from Theorem 6.4.10(ii) and Theorem 6.5.7. □

Theorem 6.5.8(i) and (ii) were also proved by Berndt [17, p. 325].

Theorem 6.5.9. *We have*

(i) $\psi(e^{-\pi/2}) = a\,2^{-7/16}(\sqrt{2}+1)^{1/4}e^{\pi/16}$,

(ii) $\psi(e^{-\pi/3}) = a\,2^{-27/24}3^{-1/8}(\sqrt{3}+1)^{1/6}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3}e^{\pi/24}$,

(iii) $\psi(e^{-\pi/4}) = a\,2^{-7/8}(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4})^{1/8}$,

(iv) $\psi(e^{-\pi/5}) = \dfrac{a\,2^{3/8}e^{\pi/40}}{(5+\sqrt{5})^{1/2}(5^{1/4}+1)}$,

(v) $\psi(e^{-\pi/6}) = \dfrac{a\,2^{-11/12}e^{\pi/48}\left((\sqrt{2}+\sqrt{3})(\sqrt{3}+1)\left(1+\sqrt{2}-\sqrt{3}+2\cdot 3^{5/4}\right)\right)^{1/3}}{3^{1/8}(1+\sqrt{3})^{1/2}(1+\sqrt{3}+\sqrt{2}\,3^{3/4})^{1/3}(\sqrt{2}-1)^{5/12}}$.

Proof. The proof of the theorem follows from Theorem 6.5.3 (ii) and the values of $r_{k,n}$ listed in Section 1.3. □

Theorem 6.5.9(i) and (iii) were also proved by Berndt [17, p. 325].

Remark 6.5.1. Many non-elementary quotients of theta-function $\psi(q)$ can be evaluated by employing the values of J_n from Theorems 4.6.3-4.6.5 in Theorem 6.3.9.

6.6 Explicit values of quotients of the theta-function $\phi(q)$

In this section, we give theorems for the explicit evaluations of quotients of the theta-function $\phi(q)$ in terms of the parameter $g'_{k,n}$ and then use these theorems to find some new explicit values.

Theorem 6.6.1. *For any positive real number n, we have*

(i) $\dfrac{\phi(-e^{-\pi\sqrt{n/3}})}{\phi(-e^{-\pi\sqrt{3n}})} = \left(\dfrac{9-3g'^4_{3,n}}{1-3g'^4_{3,n}}\right)^{1/4}$,

(ii) $\dfrac{\phi(-e^{-\pi\sqrt{n/9}})}{\phi(-e^{-\pi\sqrt{9n}})} = \left(\dfrac{3-\sqrt{3}g'_{9,n}}{1-\sqrt{3}g'_{9,n}}\right)$,

(iii) $\dfrac{\phi(-e^{-\pi\sqrt{n/5}})}{\phi(-e^{-\pi\sqrt{5n}})} = \left(\dfrac{5-\sqrt{5}g'^2_{5,n}}{1-\sqrt{5}g'^2_{5,n}}\right)^{1/2}$.

Proof. We set $q = -e^{-\pi\sqrt{n/3}}$, $-e^{-\pi\sqrt{n/9}}$, and $-e^{-\pi\sqrt{n/5}}$ in Theorems 6.2.3 - 6.2.5, respectively and use the definition of $g'_{k,n}$ to complete the proofs. □

Theorem 6.6.2. *We have*

(i) $$\frac{\phi(-e^{-\pi/\sqrt{3}})}{\phi(-e^{-\pi\sqrt{3}})} = \left(\frac{9-3(2+\sqrt{3})}{1-3(2+\sqrt{3})}\right)^{1/4},$$

(ii) $$\frac{\phi(-e^{-\pi})}{\phi(-e^{-3\pi})} = \left(\frac{36-3^{9/2}\left(1+\sqrt{3}\right)^{2/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{4/3}}{4-3^{9/2}\left(1+\sqrt{3}\right)^{2/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{4/3}}\right)^{1/4},$$

(iii) $$\frac{\phi(-e^{-\pi\sqrt{2/3}})}{\phi(-e^{-\pi\sqrt{6}})} = \left(\frac{3\sqrt{2}}{4+3\sqrt{2}}\right)^{1/4}$$

(iv) $$\frac{\phi(-e^{-\pi/\sqrt{9}})}{\phi(-e^{-\pi\sqrt{9}})} = \frac{\sqrt{3}+\sqrt{6}\sqrt[4]{3}-3}{\sqrt{3}+\sqrt{6}\sqrt[4]{3}+1},$$

(v) $$\frac{\phi(-e^{-\pi})}{\phi(-e^{-9\pi})} = \left(\frac{3-b_2}{1-b_2}\right),$$

where $b_2 = \sqrt{3}g'_{9,9}$ *and* $g'_{9,9}$ *is given by Theorem* 6.5.7(*vii*),

(vi) $$\frac{\phi(-e^{-\pi\sqrt{2/9}})}{\phi(-e^{-3\pi\sqrt{2}})} = 3-\sqrt{6},$$

(vii) $$\frac{\phi(-e^{-\pi/\sqrt{5}})}{\phi(-e^{-\pi\sqrt{5}})} = \left(\frac{\sqrt{5}+\sqrt{5}\sqrt{2(1+\sqrt{5})}-5}{\sqrt{5}+\sqrt{5}\sqrt{2(1+\sqrt{5})}+3}\right)^{1/2},$$

(viii) $$\frac{\phi(-e^{-\pi})}{\phi(-e^{-5\pi})} = \left(\frac{5\left(1+5^{1/4}+\sqrt{5}+5^{3/4}\right)}{13+5\cdot 5^{1/4}+5\sqrt{5}+5\cdot 5^{3/4}}\right)^{1/2},$$

(ix) $$\frac{\phi(-e^{-\pi\sqrt{2/5}})}{\phi(-e^{-\pi\sqrt{10}})} = \left(\frac{\sqrt{5}}{2+\sqrt{5}}\right)^{1/2}.$$

Proof. Proofs of (i)-(iii) directly follow from Theorem 6.6.1(i) and the values of $g'_{3,1}$ from Theorem 6.4.7(iii), $g'_{3,3}$ from Theorem 6.5.7(iii), and $g'_{3,2}$ from Theorem 6.4.9(i), respectively.

Similarly, proofs of (iv)-(vi) follow from Theorem 6.6.1(ii) and the values of $g'_{9,1}$ from Theorem 6.4.7(vii), $g'_{9,9}$ from Theorem 6.5.7(viii), and $g'_{9,2}$ from Theorem 6.4.9(v), respectively and proofs of (vii)-(ix) follow from Theorem 6.6.1(iii) and the values of $g'_{5,1}$ from Theorem 6.4.7(v), $g'_{5,5}$ from Theorem 6.5.7(v), and $g'_{5,2}$ from Theorem 6.4.9(iii), respectively. □

Several other quotients of $\psi(q)$ and $\phi(q)$ are also evaluated in Chapter 2 of this thesis.

Theorem 6.6.3. *For any positive real number n, we have*

(i) $$\sqrt{5}\left(h_{5,n}h'_{5,n/4}+\frac{1}{h_{5,n}h'_{5,n/4}}\right)-4 = \frac{h'_{5,n/4}}{h_{5,n}}+\frac{h_{5,n}}{h'_{5,n/4}},$$

(ii) $\left(\frac{h'_{5,n}}{h'_{5,4n}}\right)^2 + \left(\frac{h'_{5,4n}}{h'_{5,n}}\right)^2 - \sqrt{5}\left(h'^2_{5,4n} + \frac{1}{h'^2_{5,4n}}\right) + 4 = 0,$

(iii) $\left(\frac{h_{5,n}}{h'_{5,n}}\right)^2 + \left(\frac{h'_{5,n}}{h_{5,n}}\right)^2 - \sqrt{5}\left(h'^2_{5,n} + \frac{1}{h'^2_{5,n}}\right) + 4 = 0.$

Proof. The proof follows from Theorems 6.2.13-6.2.15 and the definitions of $h_{k,n}$ and $h'_{k,n}$ from (6.1.3) and (6.1.4), respectively. □

Theorem 6.6.4. *We have*

(i) $h_{5,1} = 1,$

(ii) $h_{5,3} = \frac{\sqrt{5\sqrt{5}-1}}{\sqrt{2}},$

(iii) $h_{5,1/3} = \frac{\sqrt{5\sqrt{5}+1}}{\sqrt{2}},$

(iv) $h_{5,9} = \frac{\sqrt{3}+1}{\sqrt{3}+\sqrt{5}},$

(iv) $h_{5,1/9} = \frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}+1}.$

For proofs see [66, p. 134, 146, 148].

Theorem 6.6.5. *We have*

(i) $h'_{5,1/4} = \frac{2+\sqrt{2\sqrt{5}-2}}{\sqrt{5}-1},$

(ii) $h'_{5,1} = \left(\frac{2-\sqrt{2\sqrt{5}-2}}{\sqrt{5}-1}\right)^{1/2},$

(iii) $h'_{5,4} = \left(\frac{2(-2+2\sqrt{5}-\sqrt{10(-1+\sqrt{5})}+\sqrt{-2+2\sqrt{5}}-2\sqrt{1+\sqrt{5}+2\sqrt{-2+2\sqrt{5}}})}{4+\sqrt{10(1+\sqrt{5})}-5\sqrt{-2+2\sqrt{5}}}\right)^{1/2},$

(iv) $h'_{5,3} = \left(\frac{-2+2\sqrt{5}+\sqrt{6(3-\sqrt{5})}}{3-\sqrt{5}}\right)^{1/2},$

(v) $h'_{5,1/3} = \left(\frac{2+2\sqrt{5}+\sqrt{6(3+\sqrt{5})}}{3+\sqrt{5}}\right)^{1/2},$

(vi) $h'_{5,9} = \left(\frac{8+4\sqrt{3}+3\sqrt{5}+2\sqrt{15}+2\sqrt{46+32\sqrt{3}+25\sqrt{5}+12\sqrt{15}}}{5\sqrt{3}+4\sqrt{5}-2\sqrt{15}-8}\right)^{1/2};$

(vii) $h'_{5,1/9} = \left(\frac{8+4\sqrt{3}+3\sqrt{5}+2\sqrt{15}+2\sqrt{46+32\sqrt{3}+25\sqrt{5}+12\sqrt{15}}}{4+3\sqrt{3}+4\sqrt{5}+2\sqrt{15}}\right)^{1/2}.$

Proof. (i) Setting $n = 1$ in Theorem 6.6.3(i) and then using Theorem 6.6.4(i), we find that

$$(\sqrt{5}-1)\left(x+\frac{1}{x}\right)-4=0. \tag{6.6.1}$$

Solving the above polynomial equation (6.6.1) for x, we complete the proof.

(ii) Setting $n = 1$ in Theorem 6.6.3(iii) and then using Theorem 6.6.4(i), we deduce that

$$(1-\sqrt{5})\left(x^2+\frac{1}{x^2}\right)+4=0. \tag{6.6.2}$$

Solving the above polynomial equation (6.6.2), we complete the value of (ii).

(iii) Setting n =1 in Theorem 6.6.3(ii), substituting the value of $h'_{5,1}$ from (ii) and solving the resulting polynomial equation for $h'_{5,1}$ we readily complete the proof.

(iv) - (vii) Setting n =3, 1/3, 9, and 1/9 in Theorem 6.6.3(iii) and employing the values of $h_{5,3}$, $h_{5,1/3}$, $h_{5,9}$, and $h_{5,1/9}$ from Theorem 6.6.4, respectively, and then solving the corresponding polynomial equations, we complete the proofs. □

Remark 6.6.1. Yi [66, 69] also found the value of $h'_{5,1}$.

6.7 Explicit evaluations of the Rogers-Ramanujan continued fraction

In this section, we discuss about the applications of the parameters $h_{k,n}$, $h'_{k,n}$, $g_{k,n}$, and $g'_{k,n}$ to the explicit evaluations of the famous Rogers-Ramanujan continued fraction $R(q)$ defined in (1.1.6).

Theorem 6.7.1. *[11, p. 2157, (3.42)] We have,*

$$\frac{f^6(q)}{qf^6(q^5)}=\frac{\psi^2(-q)}{q\psi^2(-q^5)}\times\frac{\phi^4(q)}{\phi^4(q^5)} \tag{6.7.1}$$

and

$$\frac{f^6(-q^2)}{q^2f^6(-q^{10})}=\frac{\phi^2(q)}{\phi^2(q^5)}\times\frac{\psi^4(-q)}{q^2\phi^4(-q^5)}. \tag{6.7.2}$$

Now, we recall from (1.1.8) that

$$\frac{1}{R^5(q)}-11-R^5(q)=\frac{f^6(-q)}{qf^6(-q^5)}. \tag{6.7.3}$$

Replacing q by q^2 and $-q$, in succession, we find that

$$\frac{1}{R^5(q^2)}-11-R^5(q^2)=\frac{f^6(-q^2)}{q^2f^6(-q^{10})} \tag{6.7.4}$$

and

$$\frac{1}{S^5(q)} + 11 - S^5(q) = \frac{f^6(q)}{qf^6(q^5)}, \tag{6.7.5}$$

where $S(q) = -R(-q)$.

Employing (6.7.1) - (6.7.5) and the definitions of $h_{k,n}$, $h'_{k,n}$, $g_{k,n}$, and $g'_{k,n}$, we easily find the following theorem.

Theorem 6.7.2. *We have*

(i) $$\frac{1}{R^5\left(e^{-\pi\sqrt{n/5}}\right)} - 11 - R^5\left(e^{-\pi\sqrt{n/5}}\right) = 5\sqrt{5}g'^2_{5,n}h'^4_{5,n/4}; \tag{6.7.6}$$

(ii) $$\frac{1}{R^5\left(e^{-2\pi\sqrt{n/5}}\right)} - 11 - R^5\left(e^{-2\pi\sqrt{n/5}}\right) = 5\sqrt{5}g^4_{5,n}h^2_{5,n}; \tag{6.7.7}$$

(iii) $$\frac{1}{S^5\left(e^{-\pi\sqrt{n/5}}\right)} + 11 - S^5\left(e^{-\pi\sqrt{n/5}}\right) = 5\sqrt{5}g^2_{5,n}h^4_{5,n}. \tag{6.7.8}$$

From the above theorem, it is clear that we can find explicit values of $R\left(e^{-\pi\sqrt{n/5}}\right)$, $R\left(e^{-2\pi\sqrt{n/5}}\right)$ and $S\left(e^{-\pi\sqrt{n/5}}\right)$ by using the known values of $h_{k,n}$, $h'_{k,n}$, $g_{k,n}$, and $g'_{k,n}$. For example, setting $n = 4$ in Theorem 6.7.2(i) and using Theorem 6.4.8(xi) and Theorem 6.6.5(ii), or setting $n = 1$ in Theorem 6.7.2(ii) and using Theorem 6.3.1(i) and Theorem 6.6.4(i), we find that

$$\frac{1}{R^5\left(e^{-2\pi/\sqrt{5}}\right)} - 11 - R^5\left(e^{-2\pi/\sqrt{5}}\right) = 5\sqrt{5}. \tag{6.7.9}$$

Solving (6.7.9) for $R^5\left(e^{-2\pi/\sqrt{5}}\right)$, we conclude that

$$R^5\left(e^{-2\pi/\sqrt{5}}\right) = \frac{1}{2}\left\{\sqrt{10\left(25+11\sqrt{5}\right)} - \left(5\sqrt{5}+11\right)\right\}.$$

This was first evaluated by Yi [67, Corollary 4.3].

Similarly, setting setting $n = 1$ in Theorem 6.7.2(iii) and using Theorem 6.3.1(i) and Theorem 6.6.4(i), we obtain

$$\frac{1}{S^5\left(e^{-\pi/\sqrt{5}}\right)} + 11 - S^5\left(e^{-\pi/\sqrt{5}}\right) = 5\sqrt{5}. \tag{6.7.10}$$

Solving (6.7.10) for $S^5\left(e^{-\pi/\sqrt{5}}\right)$, we deduce that

$$S^5\left(e^{-\pi/\sqrt{5}}\right)=\frac{1}{2}\left\{\sqrt{10\left(25-11\sqrt{5}\right)}-\left(5\sqrt{5}-11\right)\right\}.$$

This was recorded by Ramanujan [56, p. 210] and the first proof was given by Berndt, Chan and Zhang [26]. Kang [45] and Yi [67] also established this value.

6.8 Explicit evaluations of Ramanujan's cubic continued fraction

In this section, we discuss about the applications of the parameters $h_{k,n}$, $h'_{k,n}$, $g_{k,n}$, and $g'_{k,n}$ to the explicit evaluations of Ramanujan's cubic continued fraction $G(q)$ defined in (1.1.11).

From Theorem 2.3.21 and the definition of $g_{k,n}$ and $g'_{k,n}$ the following theorem is apparent.

Theorem 6.8.1. *We have*

(i) $$G^3\left(-e^{-\pi\sqrt{n/3}}\right)=\frac{-1}{1+3g_{3,n}^4}; \tag{6.8.1}$$

(ii) $$G^3\left(e^{-\pi\sqrt{n/3}}\right)=\frac{1}{3g'^4_{3,n}-1}. \tag{6.8.2}$$

Employing the values of $g_{3,n}$ for $n=1$, 3, 1/3, 9, 1/9, 5, 1/5, 25, 1/25, 7, 1/7, 13, 1/13, 49, and 1/49 from Theorems 6.4.4 - 6.4.6 in Theorem 6.8.1(i), the values of $G\left(-e^{-\pi\sqrt{n/3}}\right)$ can be found by solving a cubic equation.

Yi [66] and Adiga et al. [1] also found the values of $G\left(-e^{-\pi\sqrt{n/3}}\right)$ for $n=1$, 3, 1/3, 9, 1/9, 5, 1/5, 25, 1/25, 7, and 1/7 . The same values are also evaluated in Chapter 2 of this thesis.

Employing the values of $g'_{3,n}$ for $n=1$, 2, 3, 4, 7, 9, 12, 16, 20, 36, and 64 from Theorems 6.4.7 - 6.4.9 and Theorem 6.5.7 in Theorem 6.8.1(ii), the values of $G\left(e^{-\pi\sqrt{n/3}}\right)$ can be found by solving a cubic equation.

Ramanathan [47] and Yi [66] also evaluated $G\left(e^{-\pi\sqrt{n/3}}\right)$ for $n=1$, 2, 3, 4, 9, and 36

Remark 6.8.1. Theorem 6.4.3(i) - (iii) imply that if we know $g_{3,n}$, then $g'_{3,n}$, and hence $g'_{3,4n}$ can be evaluated. Thus, by Theorem 6.8.1, if we know $G\left(-e^{-\pi\sqrt{n/3}}\right)$, then $G\left(e^{-\pi\sqrt{n/3}}\right)$ and $G\left(e^{-2\pi\sqrt{n/3}}\right)$ can also be evaluated.

The next theorem follows easily from Entry 1(i) [15, p. 345] and the definitions of $g'_{k,n}$ and $h_{k,n}$.

Theorem 6.8.2. *We have*

$$\text{(i)} \quad G\left(e^{-\pi\sqrt{n}}\right) = \frac{1}{\sqrt{3}g'_{9,n} - 1}; \tag{6.8.3}$$

$$\text{(ii)} \quad G\left(-e^{-\pi\sqrt{n}}\right) = \frac{1 - \sqrt{3}h_{9,n}}{2}. \tag{6.8.4}$$

Chapter 7

Some New Weber-Ramanujan Class-Invariants G_n and g_n

7.1 Introduction

Let Weber-Ramanujan class invariants G_n and g_n be as defined in (4.8.1). Since from [15, p. 124], $\chi(q) = 2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}$ and $\chi(-q) = 2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}$, it follows from (4.8.1) that

$$G_n = \{4\alpha(1-\alpha)\}^{-1/24} \quad \text{and} \quad g_n = 2^{-1/12}(1-\alpha)^{1/12}\alpha^{-1/24}. \tag{7.1.1}$$

Also, if β has degree r over α, then

$$G_{r^2n} = \{4\beta(1-\beta)\}^{-1/24} \quad \text{and} \quad g_{r^2n} = 2^{-1/12}(1-\beta)^{1/12}\beta^{-1/24}. \tag{7.1.2}$$

In his notebooks [54] and paper [53], Ramanujan recorded a total of 116 class invariants. An account of Ramanujan's class invariants can be found in Chapter 34 of Berndt's book [17]. The table at the end of Weber's book [64, p. 721-726] contains the values of 107 class invariants.

In 2001, Yi [66, p. 120-124] evaluated several class invariants g_n by using her parameter $r_{k,n}$ defined in (1.1.9). In particular, she established the result

$$g_n = r_{2,n/2}. \tag{7.1.3}$$

For our future use, we also note from [66] that

$$r_{k,1} = 1, \qquad r_{k,1/n} = 1/r_{k,n} \qquad \text{and} \qquad r_{k,n} = r_{n,k}. \tag{7 1.4}$$

Adiga et al. [2] also evaluated some values of g_n.

Again, on pages 86 and 88 of Notebook I [54, Vol. I], Ramanujan recorded 11 Schläfli-type "mixed" modular equations or modular equations of composite degrees, which were not recorded in Notebook II [54, Vol. II]. One of these 11 equations follows from a modular equation recorded by Ramanujan in Chapter 20 of Notebook II. This was observed by K. G. Ramanathan [52, pp. 419-420]. But the corresponding modular equation was proved by B. C. Berndt [15, p. 423, Entry 18(v)] by using the theory of modular forms. Berndt [17, p. 382-384] also proved the other 10 equations by invoking to the theory of modular forms. Baruah [6, 10] proved nine of these equations by employing some theta-function identities and modular equations. In the process, he also found three new Schläfli-type "mixed " modular equations of the same nature. Baruah [8] also used Schläfli-type modular equations of composite degrees combined with the prime degrees to prove some values of Ramanujan's class invariants G_n.

Motivated by the above work, in this chapter, we present alternative proofs of some of the class invariants by using Ramanujan's Schläfli-type modular equations. In the process, we also find some new class invariants.

In Section 7.2, we record the Schläfli-type modular equations which will be used in the subsequent sections of this chapter.

In Sections 7.3 and 7.4, we find the values of g_n and G_n, respectively.

We end this introduction by recalling from [15, p. 124, Entry 12(i), (iii)], that

$$f(q) = \sqrt{z}2^{-1/6}(\alpha(1-\alpha)q)^{1/24} \quad \text{and} \quad f(-q^2) = \sqrt{z}2^{-1/3}(\alpha(1-\alpha)q)^{1/12}, \tag{7.1.5}$$

where $f(-q)$ is as defined in (1.1.5).

7.2 Schläfli-type modular equations

This section is devoted to recording some Schläfli-type modular equations.

In the following four lemmas, we set

$$L := 2^{1/6}\left(\alpha\beta(1-\alpha)(1-\beta)\right)^{1/24} \quad \text{and} \quad S := \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/24}. \tag{7.2.1}$$

Lemma 7.2.1. *(Berndt [17, p. 378. Entry 41]) If β has degree 11 over α, then*

$$S^6 + \frac{1}{S^6} - 2\sqrt{2}\left(\frac{2}{L^5} - \frac{11}{L^3} + \frac{22}{L} - 22L + 11L^3 - 2L^5\right) = 0. \tag{7.2.2}$$

Lemma 7.2.2. *(Berndt [17, p. 378. Entry 41]) If β has degree 13 over α, then*

$$S^7 + \frac{1}{S^7} + 13\left(S^5 + \frac{1}{S^5}\right) + 52\left(S^3 + \frac{1}{S^3}\right) + 78\left(S + \frac{1}{S}\right) - 8\left(L^6 - \frac{1}{L^6}\right) = 0. \tag{7.2.3}$$

Lemma 7.2.3. *(Berndt [17, p. 378, Entry 41]) If β has degree 17 over α, then*

$$S^9 + \frac{1}{S^9} - 34\left(S^6 + \frac{1}{S^6}\right) + 17\left(S^3 + \frac{1}{S^3}\right)\left(\frac{4}{L^4} + 7 + 4L^4\right)$$
$$- \left(\frac{16}{L^8} - \frac{136}{L^4} - 340 - 136L^4 + 16L^8\right) = 0. \tag{7.2.4}$$

Lemma 7.2.4. *(Berndt [17, p. 378, Entry 41]) If β has degree 19 over α, then*

$$S^{10} + \frac{1}{S^{10}} + 114\left(S^6 + \frac{1}{S^6}\right) - 190\sqrt{2}\left(S^4 + \frac{1}{S^4}\right)\left(\frac{1}{L^3} - L^3\right)$$
$$+19\left(S^2 + \frac{1}{S^2}\right)\left(\frac{8}{L^6} - 5 + 8L^6\right) - 4\sqrt{2}\left(\frac{4}{L^9} + \frac{19}{L^3} - 19L^3 - 4L^9\right) = 0. \tag{7.2.5}$$

In the remaining lemmas of this section, we set

$$P := (256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/48}, \tag{7.2.6}$$

$$Q := \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/48}, \tag{7.2.7}$$

$$R := \left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/48}, \tag{7.2.8}$$

and

$$T := \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/48}. \tag{7.2.9}$$

Lemma 7.2.5. *(Berndt [17, p. 381, Entry 50]; Baruah [10, p. 274, Theorem 6]) If α, β, γ, and δ have degrees 1, 5, 7, and 35, respectively, then*

$$R^4 + \frac{1}{R^4} - \left(Q^6 + \frac{1}{Q^6}\right) + 5\left(Q^4 + \frac{1}{Q^4}\right) - 10\left(Q^2 + \frac{1}{Q^2}\right) + 15 = 0. \tag{7.2.10}$$

Lemma 7.2.6. *(Berndt [17, p. 381, Entry 48]; Baruah [10, p. 274, Theorem 4]) If α, β, γ, and δ have degrees 5, 1, 7, and 35, respectively, then*

$$Q^6 + \frac{1}{Q^6} + 5\sqrt{2}\left(Q^3 + \frac{1}{Q^3}\right)\left(P + \frac{1}{P}\right) - 4\left(P^4 + \frac{1}{P^4}\right) + 10 = 0. \tag{7.2.11}$$

Lemma 7.2.7. *(Berndt [17, p. 380, Entry 43]) If α, β, γ, and δ have degrees 3, 1, 5, and 15, respectively, then*

$$Q^4 + \frac{1}{Q^4} - 2\left(P^2 + \frac{1}{P^2}\right) + 3 = 0. \tag{7.2.12}$$

Lemma 7.2.8. *(Berndt [17, p. 381, Entry 51]) If α, β, γ, and δ have degrees 1, 13, 3, and 39, respectively, then*

$$Q^4 + \frac{1}{Q^4} - 3\left(Q^2 + \frac{1}{Q^2}\right) - \left(T^2 + \frac{1}{T^2}\right) + 3 = 0. \tag{7.2.13}$$

Lemma 7.2.9. *(Berndt [17, p. 380, Entry 47]) If α, β, γ, and δ have degrees 3, 1, 11, and 33, respectively, then*

$$Q^4 + \frac{1}{Q^4} + 3\left(Q^2 + \frac{1}{Q^2}\right) - 2\left(P^2 + \frac{1}{P^2}\right) = 0. \tag{7.2.14}$$

Lemma 7.2.10. *(Berndt [17, p. 380, Entry 44]; Baruah [10, p. 273, Theorem 1]) If α, β, γ, and δ have degrees 5, 1, 3, and 15, respectively, then*

$$Q^6 + \frac{1}{Q^6} - 4\left(P^4 + \frac{1}{P^4}\right) + 10\left(P^2 + \frac{1}{P^2} - 1\right) = 0. \tag{7.2.15}$$

Lemma 7.2.11. *(Baruah [10, p. 277, Lemma 3.1]) If α, β, γ, and δ have degrees 1, 3, 7, and 21, respectively, then*

$$R^2 + \frac{1}{R^2} = Q^4 + \frac{1}{Q^4} - 3. \tag{7.2.16}$$

Lemma 7.2.12. *(Baruah [10, p. 283, Theorem 4.1]) If α, β, γ, and δ have degrees 1, 3, 7, and 21, respectively, then*

$$T^{12} + \frac{1}{T^{12}} - 18\left(T^6 + \frac{1}{T^6}\right) + 18\sqrt{2}\left(T^3 + \frac{1}{T^3}\right)\left(P^3 + \frac{1}{P^3}\right) - 8\left(P^6 + \frac{1}{P^6}\right) - 54 = 0. \tag{7.2.17}$$

7.3 Evaluations of g_n

In this section, we find some values of g_n by using the Schläfli-type modular equations recorded in the previous section.

Theorem 7.3.1. *We have*

$$g_{22} = \left(19601 + 13860\sqrt{2}\right)^{1/24} \quad \textit{and} \quad g_{2/11} = \left(19601 - 13860\sqrt{2}\right)^{1/24}.$$

Proof. We set

$$A := \frac{f(q)}{q^{1/24}f(-q^2)} \qquad \text{and} \qquad B := \frac{f(q^{11})}{q^{11/24}f(-q^{22})}. \tag{7.3.1}$$

so that, by (7.1.5), we have

$$A = \frac{2^{1/6}}{(\alpha(1-\alpha))^{1/24}} \qquad \text{and} \qquad B = \frac{2^{1/6}}{(\beta(1-\beta))^{1/24}}, \tag{7.3.2}$$

where β has degree 11 over α.

Now, from (7.2.1) and (7.3.2), we find that

$$L = \frac{2^{1/2}}{(AB)} \qquad \text{and} \qquad S = \frac{A}{B}, \tag{7.3.3}$$

where L and S are related by Lemma 7.2.1.
Replacing q by $-q$ in the definition of A and B, we observe from (7.3.3) that L^2 and S^{12} are transformed into $-L_1^2$ and $-S_1^{12}$, respectively, where

$$L_1 = \frac{2^{1/2}}{(A_1B_1)} \qquad \text{and} \qquad S_1 = \frac{A_1}{B_1}, \tag{7.3.4}$$

where

$$A_1 = \frac{f(-q)}{q^{1/24}f(-q^2)}, \qquad \text{and} \qquad B_1 = \frac{f(-q^{11})}{q^{11/24}f(-q^{22})}. \tag{7.3.5}$$

Squaring (7.2.2) and substituting $-L_1^2$ and $-S_1^{12}$ for L^2 and S^{12}, respectively, we obtain

$$9746 + \frac{32}{L_1^{10}} + \frac{352}{L_1^8} + \frac{1672}{L_1^6} + \frac{4576}{L_1^4} + \frac{8096}{L_1^2} + 8096L_1^2 + 4576L_1^4$$

$$+1672L_1^6 + 352L_1^8 + 32L_1^{12} - \frac{1}{S_1^{12}} - S_1^{12} = 0. \tag{7.3.6}$$

Now, setting $q = e^{-2\pi\sqrt{n/2}}$ and applying the definition of $r_{k,n}$ in (7.3.4), we obtain

$$L_1 = \frac{\sqrt{2}}{\sqrt{2}r_{2,n}r_{2,121n}} \qquad \text{and} \qquad S_1 = \frac{r_{2,n}}{r_{2,121n}}. \tag{7.3.7}$$

Setting $n = 1/11$ in (7.3.7) and using (7.1.4), we find that

$$L_1 = 1 \qquad \text{and} \qquad S_1 = \frac{1}{r_{2,11}^2}. \tag{7.3.8}$$

So, invoking (7.3.8) in (7.3.6), we find that

$$r_{2,11}^{24} + \frac{1}{r_{2,11}^{24}} - 39202 = 0. \tag{7.3.9}$$

Solving (7.3.9) for $r_{2,11}$, we deduce that

$$r_{2,11} = \left(19601 + 13860\sqrt{2}\right)^{1/24}. \tag{7.3.10}$$

Using (7.1.3) and (7.1.4), we complete the proof. □

Theorem 7.3.2. *We have*

$$g_{34} = \left(9 + 2\sqrt{17} + 2\sqrt{37 + 9\sqrt{17}}\right)^{1/6} \quad \textit{and} \quad g_{2/17} = \left(9 + 2\sqrt{17} - 2\sqrt{37 + 9\sqrt{17}}\right)^{1/6}$$

Proof. We set,

$$A := \frac{f(q)}{q^{1/24} f(-q^2)}, \qquad \text{and} \qquad B := \frac{f(q^{17})}{q^{17/24} f(-q^{34})}. \tag{7.3.11}$$

Transcribing (7.3.11) by using (7.1.5), we find that

$$A = \frac{2^{1/6}}{(\alpha(1-\alpha))^{1/24}} \qquad \text{and} \qquad B = \frac{2^{1/6}}{(\beta(1-\beta))^{1/24}}, \tag{7.3.12}$$

where β has degree 17 over α.
From (7.2.1) and (7.3.12), we find that

$$L = \frac{2^{1/2}}{(AB)} \qquad \text{and} \qquad S = \frac{A}{B}, \tag{7.3.13}$$

where L and S are defined in (7.2.1) and are related by Lemma 7.2.3.
Replacing q by $-q$ in the definition of A and B, we observe from (7.3.13) that L^4 and S^3 are transformed to $-L_1^4$ and S_1^3, respectively, where L_1 and S_1 are given by

$$L_1 = \frac{2^{1/2}}{(A_1 B_1)} \qquad \text{and} \qquad S_1 = \frac{A_1}{B_1}, \tag{7.3.14}$$

where $$A_1 = \frac{f(-q)}{q^{1/24} f(-q^2)}, \qquad \text{and} \qquad B_1 = \frac{f(-q^{17})}{q^{17/24} f(-q^{34})}. \tag{7.3.15}$$

Replacing $-L_1^4$ and S_1^3 for L^4 and S^3, respectively, in (7.2.4), we obtain

$$S_1^9 + \frac{1}{S_1^9} - 34\left(S_1^6 + \frac{1}{S_1^6}\right) - 17\left(S_1^3 + \frac{1}{S_1^3}\right)\left(4L_1^4 + \frac{4}{L_1^4} - 7\right)$$

$$-\left(\frac{16}{L_1^8}+\frac{136}{L_1^4}-340+136L_1^4+16L_1^8\right)=0. \tag{7.3.16}$$

Now, setting $q=e^{-2\pi\sqrt{n/2}}$ and applying the definition of $r_{k,n}$ in (7.3.14) and (7.3.15), we find that

$$L_1=\frac{\sqrt{2}}{\sqrt{2}r_{2,n}r_{2,289n}} \qquad \text{and} \qquad S_1=\frac{r_{2,n}}{r_{2,289n}}. \tag{7.3.17}$$

Setting $n=1/17$ in (7.3.17) and using (7.1.4), we deduce that

$$L_1=1 \qquad \text{and} \qquad S_1=\frac{1}{r_{2,17}^2}. \tag{7.3.18}$$

So, invoking (7.3.18) in (7.3.16), we arrive at

$$r_{2,17}^{18}+\frac{1}{r_{2,17}^{18}}-17\left(r_{2,17}^6+\frac{1}{r_{2,17}^6}\right)-34\left(r_{2,17}^{12}+\frac{1}{r_{2,17}^{12}}\right)+36=0. \tag{7.3.19}$$

Solving the above equation for real positive $r_{2,17}$, we obtain

$$r_{2,17}=\left(9+2\sqrt{17}+\sqrt{37+9\sqrt{17}}\right)^{1/6}. \tag{7.3.20}$$

Employing (7.1.3) and (7.1.4), we easily complete the proof. □

Remark 7.3.1. By setting $n=1$ in (7.3.17) and noting $r_{2,1}=1$ from (7.1.4), and then proceeding similarly as in the above proof, we can also evaluate the values of g_{578} and $g_{2/289}$.

Theorem 7.3.3. *We have*

$$g_{26}=\left(\frac{1}{6}\left(m+\sqrt{-36+m^2}\right)\right)^{1/4} \quad \textit{and} \quad g_{2/13}=\left(\frac{1}{6}\left(m-\sqrt{-36+m^2}\right)\right)^{1/4}$$

where $\quad m=8+\left(359-12\sqrt{78}\right)^{1/3}+\left(359+12\sqrt{78}\right)^{1/3}.$

Proof. We set

$$A:=\frac{f(q)}{q^{1/24}f(-q^2)} \qquad \text{and} \qquad B:=\frac{f(q^{13})}{q^{17/24}f(-q^{26})}. \tag{7.3.21}$$

Transcribing this by using (7.1.5), we find that

$$A=\frac{2^{1/6}}{(\alpha(1-\alpha))^{1/24}} \qquad \text{and} \qquad B=\frac{2^{1/6}}{(\beta(1-\beta))^{1/24}}, \tag{7.3.22}$$

where β has degree 13 over α.

From (7.2.1) and (7.3.22), we obtain

$$L=\frac{2^{1/2}}{(AB)} \qquad \text{and} \qquad S=\frac{A}{B}, \tag{7.3.23}$$

where L and S are defined in (7.2.1) and are related by Lemma 7.2.2.

Replacing q by $-q$ in the definition of A and B, we observe from (7.3.23) that L^{12} and S^2 are transformed into $-L_1^{12}$ and $-S_1^2$, respectively, where L_1 and S_1 are given by

$$L_1 = \frac{2^{1/2}}{(A_1 B_1)} \qquad \text{and} \qquad S_1 = \frac{A_1}{B_1}, \tag{7.3.24}$$

where

$$A_1 = \frac{f(-q)}{q^{1/24} f(-q^2)}, \qquad \text{and} \qquad B_1 = \frac{f(-q^{13})}{q^{17/24} f(-q^{26})}. \tag{7 3 25}$$

Squaring (7.2.3) and substituting $-L_1^{12}$ and $-S_1^2$ for L^{12} and S^2, respectively, we obtain

$$18044 + \frac{64}{L_1^{12}} + 64L_1^{12} - \frac{1}{S_1^{14}} + \frac{26}{S_1^{12}} - \frac{273}{S_1^{10}} + \frac{1508}{S_1^8} - \frac{4888}{S_1^6} + \frac{10244}{S_1^4} - \frac{15574}{S_1^2}$$

$$+10244S_1^4 - 4888S_1^6 + 1508S_1^8 - 273S_1^{10} + 26S_1^{12} - S_1^{14} = 0. \tag{7.3.26}$$

, Now, setting $q = e^{-2\pi\sqrt{n/2}}$ in (7.3.24), we find that

$$L_1 = \frac{\sqrt{2}}{\sqrt{2} r_{2,n} r_{2,169n}} \qquad \text{and} \qquad S_1 = \frac{r_{2,n}}{r_{2,169n}}. \tag{7.3.27}$$

Taking $n = 1/13$ in (7.3.27) and using (7.1.4), we find that

$$L_1 = 1 \qquad \text{and} \qquad S_1 = \frac{1}{r_{2,13}^2}. \tag{7.3.28}$$

Employing (7.3.28) in (7.3.26), we deduce that

$$\left(1 - 9x^4 + 20r_{2,13}^8 - 9x^{12} + x^{16}\right)^2 \left(1 - 8x^4 + 8x^8 - 18x^{12} + 8x^{16} - 8x^{20} + x^{24}\right) = 0, \tag{7.3.29}$$

where $x = r_{2,13}$. Since the first two equal factors have no real root for $r_{2,13}$, we arrive at

$$1 - 8r_{2,13}^4 + 8r_{2,13}^8 - 18r_{2,13}^{12} + 8r_{2,13}^{16} - 8r_{2,13}^{20} + r_{2,13}^{24} = 0. \tag{7.3.30}$$

Setting $z = r_{2,13}^4 + r_{2,13}^{-4}$ in the above equation, we find that

$$z^3 - 8z^2 + 5z - 2 = 0. \tag{7.3.31}$$

Solving the above equation for real positive z, we have

$$z = \frac{1}{3}\left(8 + (359 - 12\sqrt{78})^{1/3} + (359 + 12\sqrt{78})^{1/3}\right) \tag{7.3.32}$$

Thus,

$$r_{2,13}^4 = \frac{1}{6}\left(m + \sqrt{-36 + m^2}\right). \tag{7 3 33}$$

where $m = 8 + \left(359 - 12\sqrt{78}\right)^{1/3} + \left(359 + 12\sqrt{78}\right)^{1/3}$ Using (7 1.3) and (7.1.4), we complete the proof □

Remark 7.3.2. The values of g_{26} and $g_{2/13}$ can also be obtained by using the eta-function identity Entry 57 [16, p. 211] instead of Lemma 7.2.2.

Theorem 7.3.4. *We have*

$$g_{10/7} = \left(\frac{47-21\sqrt{5}}{2}\right)^{1/8}\left(99+70\sqrt{2}\right)^{1/12},$$

$$g_{14/5} = \left(\frac{47+21\sqrt{5}}{2}\right)^{1/8}\left(99-70\sqrt{2}\right)^{1/12},$$

$$g_{70} = \left(\frac{47+21\sqrt{5}}{2}\right)^{1/8}\left(99+70\sqrt{2}\right)^{1/12},$$

and

$$g_{2/35} = \left(\frac{47-21\sqrt{5}}{2}\right)^{1/8}\left(99-70\sqrt{2}\right)^{1/12}.$$

Proof. We define

$$A := \frac{f(q)}{q^{1/24}f(-q^2)}, B := \frac{f(q^5)}{q^{5/24}f(-q^{10})}, C := \frac{f(q^7)}{q^{7/24}f(-q^{14})}, \text{ and } D := \frac{f(q^{35})}{q^{35/24}f(-q^{70})}. \quad (7.3.34)$$

With the help of (7.1.5), the above expressions can be written as

$$A = \frac{2^{1/6}}{(\alpha(1-\alpha))^{1/24}}, \quad B = \frac{2^{1/6}}{(\beta(1-\beta))^{1/24}}, \quad C = \frac{2^{1/6}}{(\gamma(1-\gamma))^{1/24}},$$

and

$$D = \frac{2^{1/6}}{(\delta(1-\delta))^{1/24}}, \quad (7.3.35)$$

where α, β, γ, and δ have degrees 1, 5, 7, and 35, respectively. Thus, from (7.2.7), and (7.2.8), we find that

$$Q^2 = \frac{BC}{AD} \quad \text{and} \quad R^2 = \frac{AB}{CD}, \quad (7.3.36)$$

where Q and R are related by Lemma 7.2.5.

Replacing q by $-q$, we observe that Q^2 and R^4 transforms to $-Q_1^2$ and $-R_1^4$, respectively, with

$$Q_1^2 = \frac{B_1C_1}{A_1D_1} \quad \text{and} \quad R_1^2 = \frac{A_1B_1}{C_1D_1}, \quad (7.3.37)$$

where

$$A_1 = \frac{f(-q)}{q^{1/24}f(-q^2)}, B_1 = \frac{f(-q^5)}{q^{5/24}f(-q^{10})}, C_1 = \frac{f(-q^7)}{q^{7/24}f(-q^{14})}, \text{ and } D_1 = \frac{f(-q^{35})}{q^{35/24}f(-q^{70})}. \quad (7.3.38)$$

Replacing Q^2 and R^4 by $-Q_1^2$ and $-R_1^4$, respectively, in Lemma 7.2.5, we obtain

$$R_1^4 + \frac{1}{R_1^4} - \left(Q_1^6 + \frac{1}{Q_1^6}\right) - 5\left(Q_1^4 + \frac{1}{Q_1^4}\right) - 10\left(Q_1^2 + \frac{1}{Q_1^2}\right) - 15 = 0. \quad (7.3.39)$$

Setting $q = e^{-2\pi\sqrt{n/2}}$ and applying the definition of $r_{k,n}$ in (7.3.37) and (7.3.38), we find that

$$Q_1^2 = \frac{r_{2,25n}r_{2,49n}}{r_{2,n}r_{2,1225n}} \qquad \text{and} \qquad R_1^2 = \frac{r_{2,n}r_{2,25n}}{r_{2,49n}r_{2,1225n}}. \tag{7.3.40}$$

Setting $n = 1/35$ in (7.3.40) and using (7.1.4), we obtain

$$Q_1^2 = 1 \qquad \text{and} \qquad R_1^2 = \left(\frac{r_{2,5/7}}{r_{2,35}}\right)^2. \tag{7.3.41}$$

Invoking (7.3.41) in (7.3.39), we deduce that

$$\left(\frac{r_{2,5/7}}{r_{2,35}}\right)^4 + \left(\frac{r_{2,5/7}}{r_{2,35}}\right)^{-4} - 47 = 0. \tag{7.3.42}$$

Solving the above equation for positive real $r_{2,5/7}/r_{2,35}$, we obtain

$$\frac{r_{2,5/7}}{r_{2,35}} = \left(\frac{47 - 21\sqrt{5}}{2}\right)^{1/4}. \tag{7.3.43}$$

Again, if α, β, γ, and δ are of degrees 5, 1, 7, and 35, respectively, then from (7.3.35), (7.2.6), and (7.2.7), we find that

$$P^2 = \frac{2}{ABCD} \qquad \text{and} \qquad Q^2 = \frac{BC}{AD}, \tag{7.3.44}$$

where P and Q are related by Lemma 7.2.6.

Replacing q by $-q$, we observe from (7.3.44) and the definition of A, B, C, and D in (7.3.34) that P^2 and Q^6 are converted to P_1^2 and Q_2^6, respectively, where P_1^2 and Q_2^2 are defined by

$$P_1^2 = \frac{2}{A_2B_2C_1D_1} \qquad \text{and} \qquad Q_2^2 = \frac{B_2C_1}{A_2D_1}, \tag{7.3.45}$$

where C_1, and D_1 are given in (7.3.38) and A_2 and B_2 are defined as

$$A_2 := \frac{f(-q^5)}{q^{5/24}f(-q^{10})} \quad \text{and} \quad B_2 := \frac{f(-q)}{q^{1/24}f(-q^2)}. \tag{7.3.46}$$

Now, squaring (7.2.11) and substituting P_1^2 and Q_2^6 for P^2 and Q^6, respectively, we obtain

$$\left\{Q_2^6 + \frac{1}{Q_2^6} - 4\left(P_1^4 + \frac{1}{P_1^4}\right) + 10\right\}^2 = 50\left(Q_2^6 + \frac{1}{Q_2^6} + 2\right)\left(P_1^2 + \frac{1}{P_1^2} + 2\right). \tag{7.3.47}$$

Setting $q = e^{-2\pi\sqrt{n/2}}$ and employing the definition of $r_{k,n}$ in (7.3.45), we find that

$$P_1^2 = \frac{2}{2r_{2,n}r_{2,25n}r_{2,49n}r_{2,1225n}} \qquad \text{and} \qquad Q_2^2 = \frac{r_{2,n}r_{49n}}{r_{2,25n}r_{2,1225n}}. \tag{7.3.48}$$

Setting $n = 1/35$ and invoking to (7.1.4), we obtain

$$P_1^2 = 1 \qquad \text{and} \qquad Q_2^2 = \frac{1}{\left(r_{2,5/7}r_{2,35}\right)^2}. \tag{7.3.49}$$

Applying (7.3.49) in (7.3.47), we have

$$\left(r_{2,5/7}r_{2,35}\right)^6 + \left(r_{2,5/7}r_{2,35}\right)^{-6} - 198 = 0. \tag{7.3.50}$$

Solving the above equation for positive real $r_{2,5/7}r_{2,35}$, we obtain

$$r_{2,5/7}r_{2,35} = \left(99 + 70\sqrt{2}\right)^{1/6}. \tag{7.3.51}$$

With the help of (7.3.43), (7.3.51), (7.1.3) and (7.1.4), the values of $g_{10/7}, g_{14/5}, g_{70}$, and $g_{2/35}$ readily follow. □

Theorem 7.3.5. *We have*

$$g_{10/3} = \left(\frac{7-3\sqrt{5}}{2}\right)^{1/8}\left(19+6\sqrt{10}\right)^{1/12}, \qquad g_{6/5} = \left(\frac{7+3\sqrt{5}}{2}\right)^{1/8}\left(19-6\sqrt{10}\right)^{1/12},$$

$$g_{30} = \left(\frac{7+3\sqrt{5}}{2.}\right)^{1/8}\left(19+6\sqrt{10}\right)^{1/12}, \quad \textit{and} \quad g_{2/15} = \left(\frac{7-3\sqrt{5}}{2}\right)^{1/8}\left(19-6\sqrt{10}\right)^{1/12}.$$

Proof. Set

$$E := \frac{f(q^3)}{q^{1/8}f(-q^6)}, A := \frac{f(q)}{q^{1/24}f(-q^2)}, B := \frac{f(q^5)}{q^{5/24}f(-q^{10})}, \text{ and } H := \frac{f(q^{15})}{q^{5/8}f(-q^{30})}, \tag{7.3.52}$$

so that, by (7.1.5), we have

$$E = \frac{2^{1/6}}{(\alpha(1-\alpha))^{1/24}}, A = \frac{2^{1/6}}{(\beta(1-\beta))^{1/24}}, B = \frac{2^{1/6}}{(\gamma(1-\gamma))^{1/24}}, \text{ and } H = \frac{2^{1/6}}{(\delta(1-\delta))^{1/24}}, \tag{7.3.53}$$

where α, β, γ, and δ have degrees 3, 1, 5, and 15, respectively.

From (7.3.54), (7.2.6), and (7.2.7), we find that

$$P^2 = \frac{2}{ABEH} \qquad \text{and} \qquad Q^2 = \frac{AB}{EH}, \tag{7.3.54}$$

where P and Q are related as in Lemma 7.2.7.

Replacing q by $-q$ in the definition of E, A, B, and H, we observe from (7.3.52) and (7.3.54) that P^2 and Q^4 are transformed to $-P_2^2$ and $-Q_3^4$, respectively, where P_2^2 and Q_3^2 are given by

$$P_2^2 = \frac{2}{A_1B_1E_1H_1} \qquad \text{and} \qquad Q_3^2 = \frac{A_1B_1}{E_1H_1}, \tag{7.3.55}$$

where A_1 and B_1 are defined in (7.3.38), and E_1 and H_1 are defined as

$$E_1 := \frac{f(-q^3)}{q^{1/8}f(-q^6)} \qquad \text{and} \qquad H_1 := \frac{f(-q^{15})}{q^{5/8}f(-q^{30})}. \tag{7.3.56}$$

So, replacing P^2 and Q^4 by $-P_2^2$ and $-Q_3^4$, respectively, in Lemma 7.2.7, we have

$$Q_3^4 + \frac{1}{Q_3^4} - 2\left(P_2^2 + \frac{1}{P_2^2}\right) - 3 = 0. \tag{7.3.57}$$

Now, we set $q = e^{-2\pi\sqrt{n/2}}$ and apply the definition of $r_{k,n}$ in (7.3.55), to obtain

$$P_2^2 = \frac{2}{2r_{2,n}r_{2,9n}r_{2,25n}r_{2,225n}} \qquad \text{and} \qquad Q_3^2 = \frac{r_{2,n}r_{25n}}{r_{2,9n}r_{2,225n}}. \tag{7.3.58}$$

Setting $n = 1/15$ in (7.3.58) and using (7.1.4), we deduce that

$$P_2^2 = 1 \qquad \text{and} \qquad Q_3^2 = \left(\frac{r_{2,5/3}}{r_{2,15}}\right)^2. \tag{7.3.59}$$

Invoking (7.3.59) in (7.3.57), we arrive at

$$\left(r_{2,5/3}/r_{2,15}\right)^4 + \left(r_{2,5/3}/r_{2,15}\right)^{-4} - 7 = 0. \tag{7.3.60}$$

Solving the above equation for positive real $r_{2,5/3}/r_{2,15}$, we obtain

$$r_{2,5/3}/r_{2,15} = \left(\frac{7-3\sqrt{5}}{2}\right)^{1/4}. \tag{7.3.61}$$

Now, if we consider α, β, γ, and δ of degrees 5, 1, 3, and 15, respectively, then (7.3.53), (7.2.6), and (7.2.7), implies that

$$P^2 = \frac{2}{ABEH} \qquad \text{and} \qquad Q^2 = \frac{AB}{EH}, \tag{7.3.62}$$

where P and Q are related by Lemma 7.2.10.
Replacing q by $-q$, we observe from (7.3.62) and the definitions of E, A, B, and H in (7.3.52) that P^2 and Q^6 are converted to $-P_1^2$ and Q_4^6, respectively, where P_2^2 and Q_4^2 are defined by

$$P_2^2 = \frac{2}{A_1B_3E_2H_1} \qquad \text{and} \qquad Q_4^2 = \frac{A_1B_3}{E_2H_1}, \tag{7.3.63}$$

where $E_2 = B_1$, $B_3 = E_1$, and A_1, and H_1 are given in (7.3.38) and (7.3.56), respectively.
Substituting P_2^2 and Q_4^6 for P^2 and Q^6, respectively, in Lemma 7.2.10, we obtain

$$Q_4^6 + \frac{1}{Q_4^6} - 4\left(P_2^4 + \frac{1}{P_2^4}\right) - 10\left(P_2^2 + \frac{1}{P_2^2} + 1\right) = 0. \tag{7.3.64}$$

Setting $q = e^{-2\pi\sqrt{n/2}}$ and employing the definition of $r_{k,n}$ in (7.3.63), we obtain

$$P_2^2 = \frac{2}{2r_{2,n}r_{2,9n}r_{2,25n}r_{2,225n}} \qquad \text{and} \qquad Q_4^2 = \frac{r_{2,n}r_{9n}}{r_{2,25n}r_{2,225n}}. \tag{7.3.65}$$

Setting $n = 1/15$ above and employing (7.1.4) in (7.3.64), we obtain

$$P_2^2 = 1 \qquad \text{and} \qquad Q_4^2 = \frac{1}{\left(r_{2,5/3}r_{2,15}\right)^2}. \tag{7.3.66}$$

Applying (7.3.66) in (7.3.64), we find that

$$\left(r_{2,5/3}r_{2,15}\right)^6 + \left(r_{2,5/3}r_{2,15}\right)^{-6} - 38 = 0. \tag{7.3.67}$$

Solving (7.3.67) for positive real $r_{2,5/3}r_{2,15}$, we obtain

$$r_{2,5/3}r_{2,15} = \left(19 + 6\sqrt{10}\right)^{1/6}. \tag{7.3.68}$$

Employing (7.1.3) and (7.1.4), the values of $g_{10/3}, g_{6/5}, g_{30}$, and $g_{2/15}$ follow from (7.3.61) and (7.3.68). □

Theorem 7.3.6. *We have*

$$g_{6/7} = \left(\frac{5-\sqrt{21}}{2}\right)^{1/4}\left(15+4\sqrt{14}\right)^{1/12}, \quad g_{14/3} = \left(\frac{5+\sqrt{21}}{2}\right)^{1/4}\left(15-4\sqrt{14}\right)^{1/12},$$

$$g_{42} = \left(\frac{5+\sqrt{21}}{2}\right)^{1/4}\left(15+4\sqrt{14}\right)^{1/12}, \quad \text{and} \quad g_{2/21} = \left(\frac{5-\sqrt{21}}{2}\right)^{1/4}\left(15-4\sqrt{14}\right)^{1/12}.$$

Proof. We define

$$A := \frac{f(q)}{q^{1/24}f(-q^2)}, \quad E := \frac{f(q^3)}{q^{1/8}f(-q^6)}, \quad C := \frac{f(q^7)}{q^{7/24}f(-q^{14})}, \quad \text{and } G := \frac{f(q^{21})}{q^{7/8}f(-q^{42})}, \tag{7.3.69}$$

so that, by (7.1.5),

$$A = \frac{2^{1/6}}{(\alpha(1-\alpha))^{1/24}}, \quad E = \frac{2^{1/6}}{(\beta(1-\beta))^{1/24}}, \quad C = \frac{2^{1/6}}{(\gamma(1-\gamma))^{1/24}}, \quad \text{and } G = \frac{2^{1/6}}{(\delta(1-\delta))^{1/24}}, \tag{7.3.70}$$

where α, β, γ, and δ have degrees 1, 3, 7 and 21.

From (7.3.70), (7.2.7), and (7.2.8), we find that

$$Q^2 = \frac{CE}{AG} \qquad \text{and} \qquad R^2 = \frac{AE}{CG}, \tag{7.3.71}$$

where Q and R are related by Lemma 7.2.11.

Replacing q by $-q$ in the definitions of A, E, C, and G, we observe from (7.3.71) that R^2 and Q^4 are transformed to $-R_2^2$ and $-Q_5^4$, with

$$Q_5^2 = \frac{C_1E_1}{A_1G_1} \qquad \text{and} \qquad R_2^2 = \frac{A_1E_1}{C_1G_1} \tag{7.3.72}$$

where A_1 and C_1 are defined in (7.3.38), E_1 is defined in (7.3.56), and G_1 is given by

$$G_1 = \frac{f(-q^{21})}{q^{7/8}f(-q^{42})}.$$

Replacing R^2 and Q^4 by $-R_2^2$ and $-Q_5^4$ in Lemma 7.2.11, we obtain

$$\frac{1}{R_2^2} + R_2^2 = Q_5^4 + \frac{1}{Q_5^4} + 3. \tag{7.3.73}$$

Now setting $q = e^{-2\pi\sqrt{n/2}}$ and applying the definition of $r_{2,n/2}$ in (7.3.72), we find that

$$R_2^2 = \frac{r_{2,n}r_{2,9n}}{r_{2,49n}r_{2,441n}} \qquad \text{and} \qquad Q_5^2 = \frac{r_{2,9n}r_{2,49n}}{r_{2,n}r_{2,441n}}. \tag{7.3.74}$$

Setting $n = 1/21$ and using (7.1.4) in (7.3.74), we obtain

$$R_2^2 = \left(r_{2,3/7}/r_{2,21}\right)^2 \qquad \text{and} \qquad Q_5^2 = 1. \tag{7.3.75}$$

Invoking (7.3.75) in (7.3.73), we deduce that

$$\left(r_{2,3/7}/r_{2,21}\right)^2 + \left(r_{2,3/7}/r_{2,21}\right)^{-2} - 5 = 0. \tag{7.3.76}$$

Solving the above equation for real positive $r_{2,3/7}/r_{2,21}$, we obtain

$$r_{2,3/7}/r_{2,21} = \left(\frac{5-\sqrt{21}}{2}\right)^{1/2}. \tag{7.3.77}$$

Again, considering α, β, γ, and δ to be of degrees 1, 3, 7 and 21, from (7.2.9) and (7.3.70), we notice that

$$P^2 = \frac{2}{AECG} \qquad \text{and} \qquad T^2 = \frac{AC}{EG}. \tag{7.3.78}$$

Replacing q by $-q$ in (7.3.69) we observe from (7.3.78) that P^6 and T^6 are transformed to P_3^6 and T_1^6, where P and T are related by Lemma 7.2.12 and

$$P_3^2 = \frac{2}{A_1E_1C_1G_1} \qquad \text{and} \qquad T_1^2 = \frac{A_1C_1}{E_1G_1}. \tag{7.3.79}$$

Squaring (7.2.17) and replacing P_3^6 and T_1^6 by P^6 and T^6, respectively, we obtain

$$\left\{T_1^{12} + \frac{1}{T_1^{12}} - 18\left(T_1^6 + \frac{1}{T_1^6}\right) - 8\left(P_3^6 + \frac{1}{P_3^6}\right) - 54\right\}^2 = 648\left(T_1^6 + \frac{1}{T_1^6} + 2\right)\left(P_3^6 + \frac{1}{P_3^6} + 2\right). \tag{7.3.80}$$

Setting again $q = e^{-2\pi\sqrt{n/2}}$ and applying the definition of $r_{2,n/2}$ in (7.3.79), we find that

$$P_3^2 = \frac{1}{r_{2,n}r_{2,9n}r_{2,49n}r_{2,441n}} \qquad \text{and} \qquad T_1^2 = \frac{r_{2,n}r_{2,49n}}{r_{2,9n}r_{2,441n}}. \tag{7.3.81}$$

We set $n = 1/21$ in the above equation and apply (7.1.4) to arrive at

$$P_3^2 = 1 \qquad \text{and} \qquad T_1^2 = \left(r_{2,3/7}r_{2,21}\right)^{-2}. \tag{7.3.82}$$

Invoking (7.3.82) in (7.3.80), we obtain

$$\left\{x^{12} + \frac{1}{x^{12}} - 18\left(x^6 + \frac{1}{x^6}\right) - 70\right\}^2 = 2592\left(x^6 + \frac{1}{x^6}\right) + 5148, \tag{7.3.83}$$

where $x = \left(r_{2,3/7}r_{2,21}\right)$. Solving the above equation for x and noticing that $r_{2,n} > r_{2,m}$ for $n > m$, we derive that

$$x := \left(r_{2,3/7}r_{2,21}\right) = \left(15 + 4\sqrt{14}\right)^{1/6}. \tag{7.3.84}$$

The values of $g_{6/7}, g_{14/3}, g_{42}$, and $g_{2/21}$ follow from (7.3.77), (7.3.84) and the properties (7.1.3) and (7.1.4). □

Theorem 7.3.7. *We have*

$$g_{22/3} = \frac{1}{2}\left(3 + \sqrt{33} - \sqrt{26 + 6\sqrt{33}}\right)^{1/2}\left(\sqrt{2} + \sqrt{3}\right)^{1/4}\left(7\sqrt{2} + 3\sqrt{11}\right)^{1/12} \times \left(\sqrt{\frac{7 + \sqrt{33}}{8}} + \sqrt{\frac{\sqrt{33} - 1}{8}}\right)^{1/2}$$

and

$$g_{6/11} = \frac{2^{7/4}}{\left(3 + \sqrt{33} - \sqrt{26 + 6\sqrt{33}}\right)^{1/12}\left(\sqrt{2} + \sqrt{3}\right)^{1/4}\left(7\sqrt{2} + 3\sqrt{11}\right)^{1/12}} \times \frac{1}{\left(\sqrt{7 + \sqrt{33}} + \sqrt{\sqrt{33} - 1}\right)^{1/2}}.$$

Proof. Define

$$L := \frac{f(q^3)}{q^{1/8}f(-q^6)}, M := \frac{f(q)}{q^{1/24}f(-q^2)}, N := \frac{f(q^{11})}{q^{11/24}f(-q^{22})}, \text{ and } K := \frac{f(q^{33})}{q^{11/8}f(-q^{66})}. \tag{7.3.85}$$

Transcribing these with the help of (7.1.5), we find that

$$L = \frac{2^{1/6}}{(\alpha(1-\alpha))^{1/24}}, M = \frac{2^{1/6}}{(\beta(1-\beta))^{1/24}}, N = \frac{2^{1/6}}{(\gamma(1-\gamma))^{1/24}}, \text{ and } K = \frac{2^{1/6}}{(\delta(1-\delta))^{1/24}}, \tag{7.3.86}$$

where α, β, γ, and δ have degrees 3, 1, 11, and 33, respectively.

Theorem 7.3.8. *We have*

$$g_{6/13} = \left(\frac{-3+\sqrt{13}}{2}\right)^{1/2} \left(5+\sqrt{26}\right)^{1/6}$$

and

$$g_{26/3} = \frac{2}{\left(5+\sqrt{26}\right)^{1/6}\sqrt{-6+2\sqrt{13}}}.$$

Proof. We set

$$L' := \frac{f(q)}{q^{1/24}f(-q^2)}, \quad M' := \frac{f(q^{13})}{q^{13/24}f(-q^{26})}, \quad N' := \frac{f(q^3)}{q^{1/8}f(-q^6)}, \quad \text{and } K' := \frac{f(q^{39})}{q^{13/8}f(-q^{78})}. \tag{7.3.96}$$

With the help of (7.1.5), we rewrite the above expressions as

$$L' = \frac{2^{1/6}}{(\alpha(1-\alpha))^{1/24}}, \quad M' = \frac{2^{1/6}}{(\beta(1-\beta))^{1/24}}, \quad N' = \frac{2^{1/6}}{(\gamma(1-\gamma))^{1/24}}, \quad \text{and } K' = \frac{2^{1/6}}{(\delta(1-\delta))^{1/24}}, \tag{7.3.97}$$

where α, β, γ, and δ have degrees 1, 13, 3, and 39, respectively.
Proceeding as in the case of the previous theorem, we have

$$Q^2 = \frac{M'N'}{L'K'} \qquad \text{and} \qquad T^2 = \frac{L'N'}{M'K'}, \tag{7.3.98}$$

where Q and T are related by Lemma 7.2.8.
Replacing q by $-q$, we see that Q^2 and T^2 are transformed to $-Q_1^2$ and T_1^2, where

$$Q_1^2 = \frac{M_2N_2}{L_2K_2} \qquad \text{and} \qquad T_1^2 = \frac{L_2N_2}{M_2K_2}, \tag{7.3.99}$$

with

$$L_2 := \frac{f(-q)}{q^{1/24}f(-q^2)}, \quad M_2 := \frac{f(-q^{13})}{q^{13/24}f(-q^{26})}, \quad N_2 := \frac{f(-q^3)}{q^{1/8}f(-q^6)}, \quad \text{and } K_2 := \frac{f(-q^{39})}{q^{13/8}f(-q^{78})}. \tag{7.3.100}$$

So, replacing q by $-q$ in Lemma 7.2.8 and substituting $-Q_1^2$ and T_1^2 for Q^2 and T^2, respectively, we have

$$Q_1^4 + \frac{1}{Q_1^4} + 3\left(Q_1^2 + \frac{1}{Q_1^2}\right) - \left(T_1^2 + \frac{1}{T_1^2}\right) + 3 = 0. \tag{7.3.101}$$

Setting $q = e^{-2\pi\sqrt{n/2}}$ and applying the definition of $r_{2,n}$ in (7.3.99), we find that

$$Q_1^2 = \frac{r_{2,169n}r_{2,9n}}{r_{2,n}r_{2,1521n}} \qquad \text{and} \qquad T_1^2 = \frac{r_{2,n}r_{2,9n}}{r_{2,169n}r_{2,1521n}}. \tag{7.3.102}$$

Now, setting $n = 1/39$ and applying (7.1.4) in (7.3.102), we obtain

$$Q_1^2 = 1 \qquad \text{and} \qquad T_1^2 = \left(\frac{r_{2,3/13}}{r_{2,39}}\right)^2. \tag{7.3.103}$$

Invoking (7.3.103) in (7.3.101), we deduce that

$$\left(\frac{r_{2,3/13}}{r_{2,39}}\right)^2 + \frac{1}{\left(\frac{r_{2,3/13}}{r_{2,39}}\right)^2} - 11 = 0. \tag{7.3.104}$$

Solving the above equation for $r_{2,3/13}/r_{2,39}$ and noting that $r_{2,n} > 1$ and $g_n = r_{2,n/2}$, we find that

$$\frac{r_{2,3/13}}{r_{2,39}} = \frac{g_{6/13}}{g_{78}} = \left(\frac{11 - 3\sqrt{13}}{2}\right)^{1/2}. \tag{7.3.105}$$

Now, from [17, p. 202], we recall that

$$g_{78} = \left(\frac{3 + \sqrt{13}}{2}\right)^{1/2} \left(5 + \sqrt{26}\right)^{1/6}. \tag{7.3.106}$$

Combining (7.3.105) and (7.3.106), we obtain the value of $g_{6/13}$. In a similar way, employing (7.1.3) and (7.1.4), we arrive at the value of $g_{26/3}$. □

7.4 Evaluations of G_n

In this section, we use some of Schläfli-type modular equations listed in Section 7.2 to find some class invariants G_n. We note that $G_{1/n} = 1/G_n$, which will be used throughout this section without further comment.

Theorem 7.4.1. *We have*

$$G_{11} = \frac{1}{2\sqrt{3}\,a}\left\{\sqrt{2}\,b + \sqrt{144\,a^4 + 2\,b^2}\right\}^{1/2},$$

where $a = (17 + 3\sqrt{33})^{1/6}$ *and* $b = -2 + 2a^2 + a^4$.

Proof. Applying the definition of G_n in Lemma 7.2.1, we find that

$$L = \frac{1}{G_n G_{121n}} \quad \text{and} \quad S = \frac{G_n}{G_{121n}}. \tag{7.4.1}$$

Setting $n = 1/11$ in (7.4.1), we obtain

$$L = \frac{1}{G_{11}^2} \quad \text{and} \quad S = 1. \tag{7.4.2}$$

Invoking (7.4.2) in (7.2.2), we deduce that

$$1 - \sqrt{2}\left\{\left(G_{11}^{10} - \frac{1}{G_{11}^{10}}\right) - 11\left(G_{11}^{6} - \frac{1}{G_{11}^{6}}\right) + 22\left(G_{11}^{2} - \frac{1}{G_{11}^{2}}\right)\right\} = 0. \tag{7.4.3}$$

Setting

$$u = \left(G_{11}^2 - \frac{1}{G_{11}^2}\right) \tag{7.4.4}$$

in (7.4.3), we arrive at

$$1 + \sqrt{2}u + \sqrt{2}u^3 - 2\sqrt{2}u^5 = 0. \tag{7.4.5}$$

Solving the above polynomial equation for u, we get

$$u = \frac{-2 + 2(17 + 3\sqrt{33})^{1/3} + (17 + 3\sqrt{33})^{2/3}}{3\sqrt{2}(17 + 3\sqrt{33})^{1/3}}. \tag{7.4.6}$$

Thus,

$$G_{11}^2 = \frac{1}{12a^2}\left(\sqrt{2}\, b + \sqrt{144a^4 + 2b^2}\right), \tag{7.4.7}$$

where $a = (17 + 3\sqrt{33})^{1/6}$ and $b = -2 + 2a + a^4$. Thus, we complete the proof. □

Theorem 7.4.2. *We have*

$$G_{13} = \left(18 + 5\sqrt{13}\right)^{1/12}.$$

Proof. Applying the definition of G_n in Lemma 7.2.2, we find that

$$L = \frac{1}{G_n G_{169n}} \quad \text{and} \quad S = \frac{G_n}{G_{169n}}. \tag{7.4.8}$$

Setting $n = 1/13$ in (7.4.8), we obtain

$$L = \frac{1}{G_{13}^2} \quad \text{and} \quad S = 1. \tag{7.4.9}$$

Employing the above expressions in (7.2.3), we deduce that

$$G_{13}^{12} - \frac{1}{G_{13}^{12}} + 36 = 0. \tag{7.4.10}$$

Solving the above equation and noting that $G_n > 1$, we readily finish the proof. □

With the help of Lemmas 7.2.3 and 7.2.4, the next two theorems can be proved similarly.

Theorem 7.4.3.

$$G_{17} = \left(\frac{17 + 5\sqrt{17} + \sqrt{698 + 170\sqrt{17}}}{4}\right)^{1/8}$$

Theorem 7.4.4.

$$G_{19} = 2^{-5/12}\left(\frac{a + \sqrt{288 + a^2}}{3}\right)^{1/6},$$

where $a = 38 + (20528 - 1296\sqrt{57})^{1/3} + 2(2566 + 162\sqrt{57})^{1/3}$.

Next, we use a couple of Schläfli-type "mixed" modular equations listed in Section 7.2 to find the class invariants G_{35} and $G_{7/5}$.

Theorem 7.4.5. *We have*

$$G_{35} = 2^{-1/4}\left(\frac{b+\sqrt{-144+b^2}}{a-\sqrt{-36+a^2}}\right)^{1/4}$$

and

$$G_{7/5} = \frac{1}{\sqrt{6}2^{1/4}}\left\{\left(a-\sqrt{-36+a^2}\right)\left(b+\sqrt{-144+b^2}\right)\right\}^{1/4}$$

where

$$a = 5 + (62 - 6\sqrt{105})^{1/3} + (62 + 6\sqrt{105})^{1/3}$$

and

$$b = 2\sqrt{2} + (142\sqrt{2} - 6\sqrt{210})^{1/3} + (142\sqrt{3} + 6\sqrt{210})^{1/3}.$$

Proof. Applying the definition of G_n in Lemma 7.2.5, we obtain

$$Q^2 = \frac{G_{25n}G_{49n}}{G_nG_{35^2n}} \quad \text{and} \quad R^2 = \frac{G_nG_{25n}}{G_{49n}G_{35^2n}} \tag{7.4.11}$$

Setting $n = 1/35$ in (7.4.11), we deduce that

$$Q^2 = \left(\frac{G_{5/7}}{G_{35}}\right)^2 \quad \text{and} \quad R^2 = 1. \tag{7.4.12}$$

Invoking (7.4.12) in (7.2.10), we find that

$$2 - \left(\left(\frac{G_{5/7}}{G_{35}}\right)^6 + \left(\frac{G_{5/7}}{G_{35}}\right)^{-6}\right) + 5\left(\left(\frac{G_{5/7}}{G_{35}}\right)^4 + \left(\frac{G_{5/7}}{G_{35}}\right)^{-4}\right)$$
$$-10\left(\left(\frac{G_{5/7}}{G_{35}}\right)^2 + \left(\frac{G_{5/7}}{G_{35}}\right)^{-2}\right) + 15 = 0. \tag{7.4.13}$$

Setting

$$y = \left(\frac{G_{5/7}}{G_{35}}\right)^2 + \left(\frac{G_{5/7}}{G_{35}}\right)^{-2} \tag{7.4.14}$$

in (7.4.13), we arrive at

$$y^3 - 5y^2 + 7y - 7 = 0. \tag{7.4.15}$$

Solving the above polynomial equation, we get

$$y = \frac{1}{3}\left(5 + \left(62 - 6\sqrt{105}\right)^{1/3} + \left(62 + 6\sqrt{105}\right)^{1/3}\right). \tag{7.4.16}$$

Thus, we have

$$\frac{G_{5/7}}{G_{35}} = \left(\frac{a - \sqrt{-36 + a^2}}{6}\right)^{1/2}, \tag{7.4.17}$$

where $a = 5 + \left(62 - 6\sqrt{105}\right)^{1/3} + \left(62 + 6\sqrt{105}\right)^{1/3}$.

Again, applying the definition of G_n in Lemma 7.2.6, we find that

$$P^2 = \frac{1}{G_n G_{25n} G_{49n} G_{35^2 n}} \quad \text{and} \quad Q^2 = \frac{G_n G_{49n}}{G_{25n} G_{35^2 n}}. \tag{7.4.18}$$

Setting $n = 1/35$,we obtain

$$P^2 = \frac{1}{(G_{5/7} G_{35})^2} \quad \text{and} \quad Q^2 = 1. \tag{7.4.19}$$

Invoking (7.4.19) in (7.2.11), we find that

$$1 + 5\sqrt{2}\left((G_{5/7}G_{35}) + \frac{1}{(G_{5/7}G_{35})}\right) - 2\left((G_{5/7}G_{35})^4 + \frac{1}{(G_{5/7}G_{35})^{-4}}\right) + 5 = 0. \tag{7.4.20}$$

Setting $z = G_{5/7}G_{35} + (G_{5/7}G_{35})^{-1}$ in (7.4.20), we deduce that

$$2 + 5\sqrt{2}z + 8z^2 - 2z^4 = 0. \tag{7.4.21}$$

Solving the above polynomial equation for z, we get

$$z = \frac{1}{6}\left\{2\sqrt{2} + \left(142\sqrt{2} - 6\sqrt{210}\right)^{1/3} + \left(142\sqrt{2} + 6\sqrt{210}\right)^{1/3}\right\} \tag{7.4.22}$$

Therefore,

$$G_{5/7}G_{35} = \frac{1}{12}\left(b + \sqrt{-144 + b^2}\right), \tag{7.4.23}$$

where

$$b = 2\sqrt{2} + \left(142\sqrt{2} - 6\sqrt{210}\right)^{1/3} + \left(142\sqrt{2} + 6\sqrt{210}\right)^{1/3}$$

□

Dividing (7.4.23) by (7.4.17), and then simplifying, we obtain the class invariant G_{35}. Similarly multiplying (7.4.23) and (7.4.17), and then simplifying, we derive the value of $G_{5/7}$.

In his paper [53] and also on page 294 of his second notebook [54, Vol. II], Ramanujan recorded two simple formulas relating the class invariants g_n and G_n, namely, for $n > 0$,

$$g_{4n} = 2^{1/4} g_n G_n \quad \text{and} \quad (g_n G_n)^8 \left(G_n^8 - g_n^8\right) = \frac{1}{4}.$$

Thus, if we know g_n and g_{4n} or only g_n then the corresponding G_n can be calculated by the above formulas. But, the values may not be as elegant as we expect. As for examples, in the following theorem, we list some class invariants, which we find by using this process.

Theorem 7.4.6. *We have*

$$G_{22} = \frac{\left(\sqrt{2178+1540\sqrt{2}} + \sqrt{19601+13860\sqrt{2}}\right)^{1/8}}{2^{1/8}\left(19601+13860\sqrt{2}\right)^{1/48}};$$

$$G_{2/11} = \frac{\left(\sqrt{2178-1540\sqrt{2}} + \sqrt{19601-13860\sqrt{2}}\right)^{1/8}}{2^{1/8}\left(19601-13860\sqrt{2}\right)^{1/48}};$$

$$G_{34} = \frac{\left(m + \sqrt{1+\left(9+2\sqrt{17}+2\sqrt{37+9\sqrt{17}}\right)^4}\right)^{1/8}}{2^{1/8}\left(9+2\sqrt{17}+2\sqrt{37+9\sqrt{17}}\right)^{1/12}},$$

where $m = \left(297+72\sqrt{17}+36\sqrt{37+9\sqrt{17}}+8\sqrt{629+153\sqrt{17}}\right)$;

$$G_{2/17} = \frac{\left(m_1 + \sqrt{1+\left(9+2\sqrt{17}-2\sqrt{37+9\sqrt{17}}\right)^4}\right)^{1/8}}{2^{1/8}\left(9+2\sqrt{17}-2\sqrt{37+9\sqrt{17}}\right)^{1/12}},$$

where $m_1 = \left(297+72\sqrt{17}-36\sqrt{37+9\sqrt{17}}-8\sqrt{629+153\sqrt{17}}\right)$;

$$G_{10/7} = \frac{\left(a + \sqrt{376-168\sqrt{5}+4\left(99+70\sqrt{2}\right)^2\left(2207-987\sqrt{5}\right)^2}\right)^{1/8}}{2^{1/4}\left(47-21\sqrt{5}\right)^{1/8}\left(99+70\sqrt{2}\right)^{1/24}},$$

where $a = \left(436896+308980\sqrt{2}-195426\sqrt{5}-138180\sqrt{10}\right)$;

$$G_{14/5} = \frac{\left(a_1 + \sqrt{376+168\sqrt{5}+4\left(99-70\sqrt{2}\right)^2\left(2207+987\sqrt{5}\right)^2}\right)^{1/8}}{2^{1/4}\left(47+21\sqrt{5}\right)^{1/8}\left(99-70\sqrt{2}\right)^{1/24}},$$

where $a_1 = \left(436896-308980\sqrt{2}+195426\sqrt{5}-138180\sqrt{10}\right)$;

$$G_{70} = \frac{\left(a_2 + \sqrt{376+168\sqrt{5}+4\left(99+70\sqrt{2}\right)^2\left(2207+987\sqrt{5}\right)^2}\right)^{1/8}}{2^{1/4}\left(47+21\sqrt{5}\right)^{1/8}\left(99+70\sqrt{2}\right)^{1/24}},$$

where $a_2 = \left(436896+308980\sqrt{2}+195426\sqrt{5}+138180\sqrt{10}\right)$.

Chapter 8

Explicit Evaluations of Cubic and Quartic Theta-Functions

8.1 Introduction

In his famous paper [53], [55, p. 23-39], Ramanujan offered 17 elegant series for $1/\pi$ and remarked that 14 of these series belong to the "corresponding theories" in which the base q in classical theory of elliptic functions is replaced by one or other of the functions "

$$q_r := q_r(x) = exp\left(-\pi csc(\pi/r)\frac{{}_2F_1\left(\frac{1}{r},\frac{r-1}{r},1,1-x\right)}{{}_2F_1\left(\frac{1}{r},\frac{r-1}{r},1,x\right)}\right), \tag{8.1.1}$$

where r =3, 4, and 6, where ${}_2F_1$ denotes the Gaussian hypergeometric function. In the classical theory the variable $q = q_2$. Ramanujan did not offer any proof of these 14 series for $1/\pi$ or any of his theorems in the " corresponding"or " alternative" theories. In 1987, J.M. Borwein and P.B. Borwein [33] proved the formulas for $1/\pi$. However, in his second notebook [54, Vol. II], Ramanujan recorded, without proof, some of his theorems in alternative theories which were first proved by Berndt, Bhargava and Garvan [19] in 1995. These theories are now known as the theory of signature r, where r =3, 4, and 6. In particular, the theories of signature 3 and 4 are called cubic and quartic theories, respectively. An account of this work may also be found in Berndt's book [17].

In Ramanujan's cubic theory, the theta-functions $a(q)$, $b(q)$, and $c(q)$ are defined by

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \qquad b(q) = \sum_{m,n=-\infty}^{\infty} w^{m-n}q^{m^2+mn+n^2}, \tag{8.1.2}$$

and

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}, \tag{8.1.3}$$

where $w = \exp(2\pi i/3)$. These theta-functions were first introduced by Borweins [34], who also proved that

$$a^3(q) = b^3(q) + c^3(q). \tag{8.1.4}$$

Cubic theta-functions $b(q)$ and $c(q)$ are related with the Dedekind eta-function by [17, p. 109, Lemma 5.1]

$$b(q) = \frac{f^3(-q)}{f(-q^3)} \quad \text{and} \quad c(q) = \frac{3q^{1/3}f^3(-q^3)}{f(-q)}. \tag{8.1.5}$$

The Borwein brothers [34, (2.2)] also established the following three transformation formulas:

$$a(e^{-2\pi t}) = \frac{1}{t\sqrt{3}}\, a(e^{-2\pi/3t}), \tag{8.1.6}$$

$$b(e^{-2\pi t}) = \frac{1}{t\sqrt{3}}\, c(e^{-2\pi/3t}), \tag{8.1.7}$$

and

$$c(e^{-2\pi t}) = \frac{1}{t\sqrt{3}}\, b(e^{-2\pi/3t}), \tag{8.1.8}$$

where $\mathrm{Re}(t) > 0$. Cooper [42] also found alternate proofs of (8.1.6)-(8.1.8).

In quartic theory, Berndt, Bhargava, and Garvan [19] (see also [17, p. 146, (9.7)]) established a " transfer "principle of Ramanujan by which formulas in this theory can be derived from those of the classical theory. Taking place of $a(q)$, $b(q)$, and $c(q)$ in cubic theory are the functions $A(q)$, $B(q)$, and $C(q)$ [23], defined by

$$A(q) = \phi^4(q) + 16q\psi^4(q^2), \qquad B(q) = \phi^4(q) - 16q\psi^4(q^2), \tag{8.1.9}$$

and

$$C(q) = 8\sqrt{q}\phi^2(q)\psi^2(q^2), \tag{8.1.10}$$

which also satisfy the equality

$$A^2(q) = B^2(q) + C^2(q). \tag{8.1.11}$$

Berndt, Chan, and Liaw [23] used (8.1.11) to establish the inversion formula

$$z_4 := {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1, x\right) = \sqrt{A(q)}, \tag{8.1.12}$$

where $q := q_4$ is given by (8.1.1). Therefore, they did able to prove the theorems in the quartic theory directly.

The quartic analogues of (8.1.5) is given by [23, p. 139, Theorem 3.1]

$$B(q) = \left(\frac{f^2(-q)}{f(-q^2)}\right)^4 \quad \text{and} \quad C(q) = 8\sqrt{q}\left(\frac{f^2(-q^2)}{f(-q)}\right)^4. \tag{8.1.13}$$

In this chapter of our thesis, we find explicit values of cubic and quartic theta-functions and their quotients by using some parameterizations defined in the previous chapters. In the process, we also find some transformation formulas of these theta-functions.

While proving the explicit values of $\phi(q)$ and $\psi(q)$ recorded by Ramanujan in his notebooks, Berndt and Chan [17], explicitly determined the value of cubic theta-function $a(e^{-2\pi})$ [17, p. 328, Corollary 3], namely

$$\frac{a(e^{-2\pi})}{\phi^2(e^{-\pi})} = \frac{1}{(12)^{1/8}\sqrt{\sqrt{3}-1}}, \tag{8.1.14}$$

where $\phi(e^{-\pi}) = \pi^{1/4}/\Gamma(\frac{3}{4})$ is classical [65]. Certain quotients of $A(q)$, $B(q)$ and $C(q)$ were also evaluated by Berndt et al. [23] while deriving the series for $\frac{1}{\pi}$ associated with the theory of signature 4.

In Sections 8.2 and 8.3, we deal with explicit evaluations of cubic theta-functions and their quotients.

The last two sections of this chapter are on explicit evaluations of the quartic theta-functions and their quotients.

8.2 Theorems on explicit evaluation of $a(q)$, $b(q)$ and $c(q)$

In this section, we present some general formulas for the explicit evaluations of cubic theta-functions and their quotients by parameterizations given in previous chapters. In the process, we also establish some transformation formulas of quotients of cubic theta-functions.

Theorem 8.2.1. *For any positive real number n, we have*

$$\frac{b(e^{-2\pi\sqrt{n/3}})}{c(e^{-2\pi\sqrt{n/3}})} = r_{3,n}^4 = \mu_n^{2/3},$$

where $r_{k,n}$ and μ_n are as defined in (1.1.9) and (1.1.12), respectively.

Proof. Using the definitions of $b(q)$ and $c(q)$ from (8.1.5), we have

$$\frac{3b(q)}{c(q)} = \left(\frac{f(-q)}{q^{1/12}f(-q^3)}\right)^4. \tag{8.2.1}$$

Setting $q = e^{-2\pi\sqrt{n/3}}$ and then employing the definitions of $r_{k,n}$ and μ_n, we finish the proof. □

Remark 8.2.1. Replacing n by $1/n$ in Theorem 8.2.1 and noting that $r_{3,1/n} = 1/r_{3,n}$, we also have

$$\frac{b(e^{-2\pi\sqrt{n/3}})}{c(e^{-2\pi\sqrt{n/3}})} = \frac{c(e^{-2\pi/\sqrt{3n}})}{b(e^{-2\pi/\sqrt{3n}})}. \tag{8.2.2}$$

Thus, if we know the value of one quotient of (8.2.2) then the other quotient follows readily.

From Theorem 8.2.1 and (8.1.4), the following theorem is apparent.

Theorem 8.2.2. *We have*

$$\frac{a(e^{-2\pi\sqrt{n/3}})}{c(e^{-2\pi\sqrt{n/3}})} = (r_{3,n}^{12} + 1)^{1/3}.$$

Theorem 8.2.3. *For any positive real number n, we have*

$$\frac{b(e^{-2\pi\sqrt{n}})}{c(e^{-2\pi\sqrt{n}/3})} = \frac{r_{9,n}}{\sqrt{3}}.$$

Proof. From the definitions $b(q)$ and $c(q)$ in (8.1.5), we observe that

$$\frac{b(q^3)}{c(q)} = \frac{f(-q)}{3q^{1/3}f(-q^9)}. \tag{8.2.3}$$

Setting $q = e^{-2\pi\sqrt{n}/3}$ in (8.2.3) and then employing the definition of $r_{k,n}$, we arrive at the desired result. □

Remark 8.2.2. Noting that $r_{9,1/n} = 1/r_{9,n}$ from (7.1.4) and using Theorem 8.2.3 , we find that

$$\frac{3\,b(e^{-2\pi\sqrt{n}})}{c(e^{-2\pi\sqrt{n}/3})} = \frac{c(e^{-2\pi/3\sqrt{n}})}{b(e^{-2\pi/\sqrt{n}})}. \tag{8.2.4}$$

Now, from (8.2.4) it is obvious that if we know the value of one quotient then the other quotient can easily be evaluated.

In the next theorem, we give a relation between $c(q)$ and the parameter $h_{k,n}$ as defined in (6.1.4).

Theorem 8.2.4. *For any positive real number n, we have*

$$\frac{c(e^{-8\pi\sqrt{n/3}})}{c(e^{-2\pi\sqrt{n/3}})} = \frac{1}{4}\left(1 - \sqrt{3}(h'_{3,n})^2\right).$$

Proof. From [23, p. 111, Lemma 5.5], we note that

$$1 - 4\frac{c(q^4)}{c(q)} = \left(\frac{\phi(-q)}{\phi(-q^3)}\right)^2. \tag{8.2.5}$$

Now applying the definition of $h'_{k,n}$, with $k = 3$, in (8.2.5), we complete the proof. □

The next theorem connects $a(q)$ with the parameter $r_{k,n}$ defined in (1.1.9).

Theorem 8.2.5. *For any positive real number n, we have*

$$a^{12}(e^{-2\pi\sqrt{n/3}}) = \frac{27\left(r_{3,n}^{12}+1\right)^4 e^{-2\pi\sqrt{n/3}} f^{24}(-e^{-2\pi\sqrt{n/3}})}{r_{3,n}^{36}}.$$

Proof. From [36, p. 196, (2.9)], we note that

$$27qf^{24}(-q) = a^{12}(q)(1-\alpha(q))^3\alpha(q), \tag{8.2.6}$$

where $\alpha(q) = c^3(q)/a^3(q)$.
Setting $q = e^{-2\pi\sqrt{n/3}}$ and then applying (8.2.2) in (8.2.6), we obtain

$$27e^{-2\pi\sqrt{n/3}} f^{24}(-e^{-2\pi\sqrt{n/3}}) = a^{12}(e^{-2\pi\sqrt{n/3}})\left(1 - \frac{1}{r_{3,n}^{12}+1}\right)^3\left(\frac{1}{r_{3,n}^{12}+1}\right),$$

which on simplification gives the required result. □

Theorem 8.2.6. *We have*

$$a(e^{-3n\pi}) = \frac{1}{3}\left\{a(e^{-n\pi}) + 2b(e^{-n\pi})\right\}.$$

Proof. From [17, p. 93, (2.8)], we have

$$b(q) = \frac{1}{2}\left\{3a(q^3) - a(q)\right\}. \tag{8.2.7}$$

Setting $q = e^{-n\pi}$ in (8.2.7), we readily complete the proof. □

Theorem 8.2.7. *For any positive real number n, we have*

$$\text{(i)}\quad b(e^{-n\pi})=\frac{f^3(-e^{-n\pi})}{f(-e^{-3n\pi})}$$

and

$$\text{(ii)}\quad b(-e^{-n\pi})=\frac{f^3(e^{-n\pi})}{f(e^{-3n\pi})}.$$

Proof. Setting $q=e^{-n\pi}$ and $q=-e^{-n\pi}$ in (8.1.5), we readily arrive at (i) and (ii), respectively. □

Theorem 8.2.8. *For all positive real numbers n, we have*

$$\text{(i)}\quad b(e^{-2\pi\sqrt{n/3}})=3^{1/4}e^{-\pi\sqrt{n}/6\sqrt{3}}f^2(-e^{-2\pi\sqrt{n/3}})r_{3,n}$$

and

$$\text{(ii)}\quad b(-e^{-\pi\sqrt{n/3}})=3^{1/4}e^{-\pi\sqrt{n}/12\sqrt{3}}f^2(e^{-\pi\sqrt{n/3}})r'_{3,n},$$

where the parameters $r_{3,n}$ and $r'_{3,n}$ are defined in (1.1.9) and (6.1.2), respectively.

Proof. We rewrite $b(q)$ in (8.1.5) as

$$b(q)=f^2(-q)q^{1/12}\frac{f(-q)}{q^{1/12}f(-q^3)}.\tag{8.2.8}$$

Setting $q=e^{-2\pi\sqrt{n/3}}$ and employing the definition of $r_{3,n}$, we arrive at (i). To prove (ii), we replace q by $-q$ in (8.2.8) and then use the definition of $r'_{3,n}$. □

Theorem 8.2.9. *For all positive real number n, we have*

$$\text{(i)}\quad c(e^{-n\pi})=3e^{-n\pi/3}\frac{f^3(-e^{-3n\pi})}{f(-e^{-n\pi})}$$

and

$$\text{(ii)}\quad c(-e^{-n\pi})=-3e^{-n\pi/3}\frac{f^3(e^{-3n\pi})}{f(e^{-n\pi})}.$$

Proof. Follow readily from (8.1.5) with $q=e^{-n\pi}$ and $q=-e^{-n\pi}$. □

Theorem 8.2.10. *For all positive real number n, we have*

$$c(e^{-2\pi\sqrt{n/3}})=\frac{3^{3/4}e^{-\pi\sqrt{n}/2\sqrt{3}}f^2(-e^{-2\pi\sqrt{3n}})}{r_{3,n}}.$$

Proof. We set $q=e^{-2\pi\sqrt{n/3}}$ in (8.1.5) and then employ the definition of the parameter $r_{k,n}$ to finish the proof. □

8.3 Explicit values of $a(q)$, $b(q)$ and $c(q)$

In this section, we find explicit values of cubic theta-functions and their quotients by using the results established in the previous section.

Theorem 8.3.1. *We have*

(i) $\dfrac{b(e^{-2\pi/\sqrt{3}})}{c(e^{-2\pi/\sqrt{3}})} = 1,$

(ii) $\dfrac{b(e^{-2\pi\sqrt{2/3}})}{c(e^{-2\pi\sqrt{2/3}})} = \left(1+\sqrt{2}\right)^{2/3},$

(iii) $\dfrac{b(e^{-2\pi})}{c(e^{-2\pi})} = \dfrac{3^{1/2}\left(1+\sqrt{3}\right)^{2/3}}{2^{1/3}},$

(iv) $\dfrac{b(e^{-4\pi\sqrt{3}})}{c(e^{-4\pi\sqrt{3}})} = \left(\dfrac{1+\sqrt{3}}{\sqrt{2}}\right)^{2},$

(v) $\dfrac{b(e^{-2\pi\sqrt{5/3}})}{c(e^{-2\pi\sqrt{5/3}})} = \left(\dfrac{1+\sqrt{5}}{2}\right)^{10/3},$

(vi) $\dfrac{b(e^{-2\pi\sqrt{7/3}})}{c(e^{-2\pi\sqrt{7/3}})} = \dfrac{\sqrt{3}+\sqrt{7}}{2\left(2-\sqrt{3}\right)},$

(vii) $\dfrac{b(e^{-4\pi\sqrt{2/3}})}{c(e^{-4\pi\sqrt{2/3}})} = \left(1+\sqrt{2}\right)^{4/3}\left(\sqrt{2}+\sqrt{3}\right),$

(viii) $\dfrac{b(e^{-2\pi\sqrt{3}})}{c(e^{-2\pi\sqrt{3}})} = \dfrac{3^{2/3}}{(2^{1/3}-1)^{4/3}},$

(ix) $\dfrac{b(e^{-6\pi\sqrt{2/3}})}{c(e^{-6\pi\sqrt{2/3}})} = 3^{2/3}\left(1+\sqrt{2}\right)^{10/9}\left(2+\sqrt{2}\left(1+\sqrt{2}\right)^{1/3} + \left(1+\sqrt{2}\right)^{2/3}\right)^{4/3},$

(x) $\dfrac{b(e^{-2\pi\sqrt{13/3}})}{c(e^{-2\pi\sqrt{13/3}})} = \left(\dfrac{\sqrt{11+\sqrt{13}}+\sqrt{3+\sqrt{13}}}{2\sqrt{2}}\right)^{4},$

(xi) $\dfrac{b(e^{-10\pi/\sqrt{3}})}{c(e^{-10\pi/\sqrt{3}})} = \dfrac{1}{16}\left(1+\sqrt[3]{10}+\sqrt{5+2\sqrt[3]{10}+\sqrt[3]{10^2}}\right)^{4},$

(xi) $\dfrac{b(e^{-14\pi/\sqrt{3}})}{c(e^{-14\pi/\sqrt{3}})} = \left(\dfrac{3+\sqrt[3]{2^2}\sqrt[3]{7}+\sqrt[3]{2}\sqrt[3]{7^2}+\sqrt{49+13\sqrt[3]{2^2}\sqrt[3]{7}+8\sqrt[3]{2}\sqrt[3]{7^2}}}{2\sqrt{3}}\right)^{4}.$

Proof. Follows directly from Theorem 8.2.1 and the corresponding values of $r_{3,n}$ listed in Section 1.3. ☐

More values can be calculated by employing Theorem 8.2.1 and the corresponding values of μ_n evaluated in Chapters 2 and 3 of this thesis.

Theorem 8.3.2. *We have*

(i) $\dfrac{a(e^{-2\pi/\sqrt{3}})}{c(e^{-2\pi/\sqrt{3}})} = \sqrt[3]{2},$

(ii) $\dfrac{a(e^{-2\pi\sqrt{2/3}})}{c(e^{-2\pi\sqrt{2/3}})} = 2^{1/3}\left(2+\sqrt{2}\right)^{1/3},$

(iii) $\dfrac{a(e^{-2\pi})}{c(e^{-2\pi})} = \left(\dfrac{3^{3/2}\left(1+\sqrt{3}\right)^2}{2}+1\right)^{1/3},$

(iv) $\dfrac{a(e^{-4\pi/\sqrt{3}})}{c(e^{-4\pi/\sqrt{3}})} = \left(\left(\dfrac{(1+\sqrt{3})}{\sqrt{2}}\right)^6+1\right)^{1/3},$

(v) $\dfrac{a(e^{-2\pi\sqrt{5/3}})}{c(e^{-2\pi\sqrt{5/3}})} = \left(\left(\dfrac{1+\sqrt{5}}{2}\right)^{10}+1\right)^{1/3},$

(vi) $\dfrac{a(e^{-2\pi\sqrt{7/3}})}{c(e^{-2\pi\sqrt{7/3}})} = \left(\left(\dfrac{\sqrt{3}+\sqrt{7}}{2(2-\sqrt{3})}\right)^3+1\right)^{1/3},$

(vii) $\dfrac{a(e^{-4\pi\sqrt{2/3}})}{c(e^{-4\pi\sqrt{2/3}})} = \left(\left(1+\sqrt{2}\right)^4\left(\sqrt{2}+\sqrt{3}\right)^4+1\right)^{1/3},$

(viii) $\dfrac{a(e^{-10\pi/\sqrt{3}})}{c(e^{-10\pi/\sqrt{3}})} = \left(\left(\dfrac{1+\sqrt[3]{16}+\sqrt{5+2\sqrt[3]{10}+\sqrt[3]{10^2}}}{2}\right)^{12}+1\right)^{1/3},$

(ix) $\dfrac{a(e^{-6\pi/\sqrt{3}})}{c(e^{-6\pi/\sqrt{3}})} = \left(\dfrac{9}{(2^{1/4}-1)^4}+1\right)^{1/3},$

(x) $\dfrac{a(e^{-14\pi/\sqrt{3}})}{c(e^{-14\pi/\sqrt{3}})} = \left(\left(\dfrac{3+\sqrt[3]{2}\sqrt[3]{7}+\sqrt[3]{2}\sqrt[3]{7^2}+\sqrt{49+13\sqrt[3]{2}\sqrt[3]{7}+8\sqrt[3]{2}\sqrt[3]{7^2}}}{2\sqrt{3}}\right)+1\right)^{1/3},$

(xi) $\dfrac{a(e^{-6\pi})}{c(e^{-6\pi})} = \left(3^4\left(1+\sqrt{2}\right)^{10/3}\left(2+\sqrt{2}\left(1+\sqrt{2}\right)^4+\left(1+\sqrt{2}\right)^8\right)+1\right)^{1/3}.$

Proof. Follows easily from (8.2.2) and the corresponding values of $r_{3,n}$ listed in Section 1.3. □

Theorem 8.3.3. *We have*

(i) $\dfrac{b(e^{-2\pi})}{c(e^{-2\pi/3})} = \dfrac{1}{\sqrt{3}},$

(ii) $\dfrac{b(e^{-2\pi\sqrt{2}})}{c(e^{-2\pi\sqrt{2}/3})} = \dfrac{\left(\sqrt{3}+\sqrt{2}\right)^{1/3}}{\sqrt{3}},$

(iii) $\dfrac{b(e^{-2\pi\sqrt{3}})}{c(e^{-2\pi/\sqrt{3}})} = \left(\dfrac{1}{3\left(\sqrt[3]{2}-1\right)}\right)^{1/3}$,

(iv) $\dfrac{b(e^{-4\pi})}{c(e^{-4\pi/3})} = \dfrac{1}{2\sqrt{3}}\left(1+\sqrt{2}\cdot 3^{1/4}+\sqrt{3}\right)$,

(v) $\dfrac{b(e^{-2\sqrt{5}\,\pi})}{c(e^{-2\sqrt{5}\,\pi/3})} = \dfrac{1}{\sqrt{3}}\left(104+60\sqrt{3}+45\sqrt{5}+26\sqrt{15}\right)^{1/6}$.

Proof. Follows from Theorem 8.2.3 and the corresponding values of $r_{9,n}$ in Section 1.3. □

Lemma 8.3.4. *We have*

(i) $h'_{1,1} = 1$,

(ii) $h'_{2,2} = 2^{1/16}\left(\sqrt{2}-1\right)^{1/4}$,

(iii) $h'_{3,3} = \dfrac{2^{1/3}3^{1/8}\left(\sqrt{3}-1\right)^{1/6}}{\left(1+\sqrt{3}+\sqrt{2}\sqrt[4]{3^3}\right)^{1/3}}$,

(iv) $h'_{4,4} = \dfrac{2^{1/4}}{\left(16+15\sqrt[4]{2}+12\sqrt{2}+9\sqrt[4]{2^3}\right)^{1/8}}$,

(v) $h'_{5,5} = \dfrac{1}{2}\left(\sqrt[4]{5}-1\right)\sqrt{5+\sqrt{5}}$,

(vi) $h'_{6,6} = \dfrac{2^{1/4}3^{1/8}\left(\sqrt{2}-1\right)^{1/12}\left(\sqrt{3}+1\right)^{1/6}\left(-1-\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}{\left(2-3\sqrt{2}+3^{5/4}+3^{3/4}\right)^{1/3}}$,

(vii) $h'_{3,1} = 2^{-1/4}\sqrt{\sqrt{3}-1}$.

For proofs (i)-(vi), see [69, p. 21, Theorem 5.6] or [66, p. 152 ,Theorem 9.2.6]. For proof of (vii), see [69, p. 15, Theorem 4.11] or [66, p. 145, Theorems 9.1.10] .

Theorem 8.3.5. *We have*

(i) $\dfrac{c(e^{-8\pi/\sqrt{3}})}{c(e^{-2\pi/\sqrt{3}})} = \dfrac{1}{4}\left(\dfrac{\sqrt{2}+\sqrt{3}-3}{\sqrt{2}}\right)$,

(ii) $\dfrac{c(e^{-8\pi})}{c(e^{-2\pi})} = \dfrac{1}{4}\left(1-\sqrt{3}\left(\dfrac{2^{2/3}3^{1/4}\left(\sqrt{3}-1\right)^{1/3}}{\left(1+\sqrt{3}+\sqrt{2}\sqrt[4]{3^3}\right)^{2/3}}\right)\right)$.

Proof. We set $n = 1$ and 3 in Theorem 8.2.4 and then employ the values of $h'_{3,1}$ and $h'_{3,3}$ from Lemma 8.3.4(vii) and (iii), respectively to finish the proof. □

For the remaining part of this chapter, we set $a := \phi(e^{-\pi}) = \pi^{1/4}/\Gamma(3/4)$.

Lemma 8.3.6. *We have*

(i) $f(-e^{-\pi}) = a\, 2^{-3/8} e^{\pi/24}$,

(ii) $f(e^{-\pi}) = a\, 2^{-1/4} e^{\pi/24}$,

(iii) $f(-e^{-2\pi}) = a\, 2^{-1/2} e^{\pi/12}$,

(iv) $f(-e^{-3\pi}) = \dfrac{a\, e^{\pi/4}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}{3^{3/8} 2^{17/24} (1+\sqrt{3})^{1/6}}$,

(v) $f(-e^{-4\pi}) = a\, 2^{-7/8} e^{\pi/6}$,

(vi) $f(-e^{-6\pi}) = a\, 2^{-7/12} 3^{-3/8} e^{\pi/4} \left(\sqrt{3}-1\right)^{1/4}$,

(vii) $f(-e^{-12\pi}) = \dfrac{a\, e^{\pi/2}}{2^{5/24} 3^{3/8} \sqrt{1+\sqrt{3}}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}$,

(viii) $f(-e^{-\pi/3}) = \dfrac{a\, 2^{7/24} 3^{1/8} e^{\pi/72}}{\sqrt{1+\sqrt{3}}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}$,

(ix) $f(-e^{-2\pi/3}) = a\, 2^{-7/12} 3^{1/8} e^{\pi/36} \left(\sqrt{3}-1\right)^{1/6}$,

(x) $f(e^{-2\pi}) = a\, 2^{-13/16} e^{\pi/12} \left(\sqrt{2}+1\right)^{1/4}$,

(xi) $f(e^{-3\pi}) = a\, 2^{-1/3} 3^{-3/8} e^{\pi/8} \left(\sqrt{3}+1\right)^{1/6}$,

(xii) $f(e^{-6\pi}) = \dfrac{a\, e^{\pi/4}\left(2-3\sqrt{2}+3^{5/4}+3^{3/4}\right)^{1/3}}{2^{15/16} 3^{3/8} \left(\sqrt{2}-1\right)^{1/12} \left(\sqrt{3}+1\right)^{1/6}}$.

For a proof of the lemma, we refer to [17, p. 326, Entry 6] and [66, p. 125-129].

Theorem 8.3.7. *We have*

(i) $b(e^{-\pi}) = \dfrac{a^2 3^{3/8} \left(1+\sqrt{3}\right)^{1/6}}{2^{5/12}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}$,

(ii) $b(e^{-2\pi}) = \dfrac{a^2 3^{3/8}}{2^{11/12}\left(\sqrt{3}-1\right)^{1/6}}$,

(iii) $b(e^{-4\pi}) = a^2 2^{-29/12} 3^{3/8} \left(1+\sqrt{3}\right)^{1/2} \left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}$,

(iv) $b(e^{-\pi/3}) = \dfrac{a^2 2^{5/4} 3^{3/8}}{\left(1+\sqrt{3}\right)^{3/2}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)}$,

(v) $b(e^{-2\pi/3}) = \dfrac{a^2 3^{1/8} \left(\sqrt{3}-1\right)^{1/3}}{2^{13/12}\left(\sqrt{3}+1\right)^{1/6}}$,

(vi) $$b(-e^{-\pi}) = \frac{a^2 3^{3/8}}{2^{5/12}\left(\sqrt{3}+1\right)^{1/6}},$$

(vii) $$b(-e^{-2\pi}) = \frac{a^2 3^{3/8}\left(\sqrt{2}+1\right)^{3/4}\left(\sqrt{2}-1\right)^{1/12}\left(\sqrt{3}+1\right)^{1/6}}{2^{3/2}\left(2-3\sqrt{2}+3^{5/4}+3^{3/4}\right)^{1/3}}.$$

Proof. To prove (i)-(v), we set $n = 1$, 2, 4, 1/3, and 2/3, respectively, in Theorem 8.2.7(i) and use the corresponding values of $f(\pm e^{-\pi n})$ from Lemma 8.3.6.

To prove (vi) and (vii), we set n =1 and 2, respectively in Theorem 8.2.7(ii) and then use the corresponding values $f(\pm e^{-\pi n})$ from Lemma 8.3.6. □

Theorem 8.3.8. *We have*

(i) $$c(e^{-4\pi/3}) = \frac{a^2 3^{7/8}\left(1+\sqrt{3}\right)^{1/6}}{2^{17/12}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}},$$

(ii) $$c(e^{-2\pi/3}) = a^2 2^{-13/12} 3^{7/8}\left(\sqrt{3}+1\right)^{1/6},$$

(iii) $$c(e^{-\pi/3}) = 2^{-17/12} 3^{7/8} a^2\left(1+\sqrt{3}\right)^{1/2}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3},$$

(iv) $$c(e^{-4\pi}) = \frac{a^2 3^{3/8}}{2^{1/4}\left(1+\sqrt{3}\right)^{3/2}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)},$$

(v) $$c(e^{-2\pi}) = \frac{a^2\left(\sqrt{3}-1\right)^{1/3}}{3^{3/8}2^{13/12}\left(\sqrt{3}+1\right)^{1/6}},$$

(vi) $$c(e^{-\pi}) = \frac{a^2\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)}{3^{1/8}2^{7/4}\left(1+\sqrt{3}\right)^{1/2}},$$

(vii) $$c(e^{-8\pi}) = \frac{1}{4}\left(1-\sqrt{3}\left(\frac{2^{2/3}\cdot 3^{1/4}\left(\sqrt{3}-1\right)^{1/3}}{\left(1+\sqrt{3}+\sqrt{2}+\sqrt[4]{3^2}\right)^{2/3}}\right)\right)\times\frac{a^2\left(\sqrt{3}-1\right)^{1/3}}{3^{3/8}\cdot 2^{13/12}\left(\sqrt{3}+1\right)^{1/6}}.$$

Proof. To prove (i)-(v), we set t =1/2, 1, 2, 1/6, and 1/3, respectively in (8.1.7) and then apply the corresponding values of $b(e^{-n\pi})$ from Theorem 8.3.7.

To prove (vi), we set $n = 1$ in Theorem 8.2.9 and use the corresponding values of $f(-e^{-n\pi})$ from Lemma 8.3.6. At last, (vii) follows from Theorem 8.3.8(v) and Theorem 8.3.5(ii). □

Remark 8.3.1. Setting $t = 1/2$ in (8.1.8) and then employing the value of $c(e^{-\pi})$ from Theorem 8.3.8(vi), we can also evaluate $b(e^{-4\pi/3})$.

Theorem 8.3.9. *We have*

(i) $$a(e^{-2\pi}) = \frac{a^2\left(10+6\sqrt{3}\right)^{1/3}}{2\left(3+2\sqrt{3}\right)^{1/4}},$$

(ii) $$a(e^{-2\pi/3}) = \frac{a^2\sqrt{3}\left(10+6\sqrt{3}\right)^{1/3}}{2\left(3+2\sqrt{3}\right)^{1/4}},$$

(iii) $a(e^{-6\pi}) = \frac{a^2}{3}\left(3^{1/4}\left(1+\sqrt{2}\right)^{1/6} + \frac{\left(10+6\sqrt{3}\right)^{1/3}}{2\left(3+2\sqrt{3}\right)^{1/4}}\right),$

(iv) $a(e^{-2\pi/9}) = \sqrt{3}a^2\left\{\left(3^{1/4}\left(1+\sqrt{2}\right)^{1/6} + \frac{\left(10+6\sqrt{3}\right)^{1/3}}{2\left(3+2\sqrt{3}\right)^{1/4}}\right)\right\}.$

Proof. To prove (i), we set n =3 in Theorem 8.2.5 and use $f(-e^{-2\pi})$ from Lemma 8.3.6 and the values of $r_{3,3}$ from Section 1.3.

To prove (ii), we set $t = 1$ in (8.1.6) and then employ Theorem 8.3.9(i).

To prove (iii), we set $n = 2$ in Theorem 8.2.6 and then employ the values of $a(e^{-2\pi})$ and $b(e^{-2\pi})$ from Theorems 8.3.9(i) and 8.3.7(ii), respectively.

To prove (iv), we set $t = 3$ in (8.1.6) and use the value of $a(e^{-6\pi})$. □

8.4 Theorems on explicit evaluations of $A(q)$, $B(q)$, and $C(q)$

In this section, we use the parameters $r_{k,n}$, $h_{k,n}$, $g'_{k,n}$, and J_n, defined in (1.1.9), (1.1.21), (1.1.24), and (4.1.12), respectively, to establish some formulas for the explicit evaluations of quartic theta-functions and their quotients.

Theorem 8.4.1. *For any positive real number n, we have*

$$\frac{B(e^{-\pi\sqrt{2n}})}{C(e^{-\pi\sqrt{2n}})} = r_{2,n}^{12} = g_{2n}^{12}.$$

Proof. Employing the definition of $B(q)$ and $C(q)$ given in (8.1.13), we find that

$$\frac{B(q)}{C(q)} = \left(\frac{f(-q)}{2^{1/4}q^{1/24}f(-q^2)}\right)^{12}. \tag{8.4.1}$$

Setting $q = e^{-2\pi\sqrt{n/2}}$ in (8.4.1) and then using the definition of $r_{k,n}$, we arrive at the first equality. Second equality readily follows from (1.1.19) and (8.4.1). □

Remark 8.4.1. From Theorem 8.4.1 and (7.1.4), we have the following transformation formula

$$\frac{B(e^{-\pi\sqrt{2/n}})}{C(e^{-\pi\sqrt{2/n}})} = \frac{C(e^{-\pi\sqrt{2n}})}{B(e^{-\pi\sqrt{2n}})}. \tag{8.4.2}$$

Thus, if we know the value of one of the quotient of (8.4.2) then the other one follows immediately.

Theorem 8.4.2. *We have*

$$\frac{B(e^{-2\pi\sqrt{n}})}{C(e^{-\pi\sqrt{n}})} = \frac{J_n^4}{2}.$$

Proof. Theorem follows easily from (8.1.13) and the definition of J_n in (4.1.12). □

Remark 8.4.2. Using the fact that $J_{1/n} = 1/J_n$ from Theorem 4.6 1, in Theorem 8.4.2 we have the following transformation formula

$$\frac{4\,B(e^{-2\pi/\sqrt{n}})}{C(e^{-\pi/\sqrt{n}})} = \frac{C(e^{-\pi\sqrt{n}})}{B(e^{-2\pi\sqrt{n}})}. \tag{8.4.3}$$

Hence, if we know one quotient of (8.4.3) then the other quotient follows immediately.

Lemma 8.4.3. *We have*

$$\text{(i)}\quad \phi(-e^{-2n\pi}) = \frac{a}{2^{1/8}n^{1/4}h'_{n,n}} = \frac{ar_{2,2n^2}}{n^{1/4}2^{1/4}r_{n,n}},$$

$$\text{(ii)}\quad \phi(e^{-n\pi}) = \frac{a}{n^{1/4}h_{n,n}} = \frac{aG_{n^2}^2}{n^{1/4}r_{n,n}},$$

$$\text{(iii)}\quad \psi(e^{-n\pi}) = \frac{a2^{-5/8}e^{n\pi/8}}{n^{1/4}g'_{n,n}} = \frac{a2^{-3/4}e^{n\pi/8}}{n^{1/4}r_{2,n^2/2}r_{n,n}},$$

$$\text{(iv)}\quad \psi(e^{-\pi/n}) = \frac{an^{1/4}2^{-3/4}r_{2,2n^2}e^{n\pi/8}}{r_{n,n}},$$

where the parameters $r_{k,n}$, $h_{k,n}$, $h'_{k,n}$, $g'_{k,n}$, and G_n are as defined in (1.1.9), (1.1.21), (6.1.4), (1.1.24), *and* (1.1.19), *respectively.*

For proofs of (i) and (ii), we refer to [66, p. 150] or [69]. For proofs of (iii) and (iv), we refer to Theorem 6.5.2(ii) and Theorem 6.5.3(ii), respectively.

Theorem 8.4.4. *For any positive real number n, we have*

$$\text{(i)}\quad B(e^{-2\pi n}) = \frac{a^4}{\sqrt{2}\,n\,h'^4_{n,n}} = \frac{a^4r^4_{2,\,2n^2}}{2\,n\,r^4_{n,n}},$$

$$\text{(ii)}\quad B(e^{-2\pi/n}) = \frac{a^4n\,r^4_{2,\,2/n^2}}{2\,r^4_{n,n}},$$

where $h'_{k,n}$ is as defined in (6.1.4).

Proof. From [15, p. 39, Entry 24(iii)], we note that

$$\phi(-q) = \frac{f^2(-q)}{f(-q^2)}. \tag{8.4.4}$$

Employing (8.4.4) in (8.1.13), we obtain

$$B(q) = \phi^4(-q) \tag{8.4.5}$$

Setting $q = e^{-2\pi n}$ in (8.4.5) and then employing Lemma 8.4.3(i), we arrive at (i).

To prove (ii), we replace n by $1/n$ in (i) and employ the result $r_{1/n,1/n} = r_{n,n}$, which is easily derivable from (7.1.4). □

Theorem 8.4.5. *We have*

(i) $B(-e^{-n\pi}) = \dfrac{a^4}{nh_{n,n}^4} = \dfrac{a\,G_{n^2}^8}{nr_{n,n}^4}$,

(ii) $B(-e^{-\pi/n}) = \dfrac{na^4}{h_{n,n}^4} = \dfrac{a^4nG_{n^2}^8}{r_{n,n}^4}$,

where $h'_{k,n}$ *is as defined in* (6.1.4).

Proof Replacing q by $-q$ in (8.4.5) and setting $q = e^{-n\pi}$, we have

$$B(-e^{-n\pi}) = \phi^4(e^{-n\pi}), \tag{8.4.6}$$

Employing Lemma 8.4.3(ii) in (8.4.6), we finish the proof of (i).

To prove (ii), we replace n by $1/n$ in (i) and use the results $h_{n,n} = h_{1/n,1/n}$ [69] and $G_{1/n} = G_n$. □

Remark 8.4.3. The following transformation formula is apparent from Theorem 8.4.5(i) and (ii),

$$n^2B(-e^{-n\pi}) = B(-e^{-\pi/n}). \tag{8.4.7}$$

Theorem 8.4.6. *For any positive real number n, we have*

$$C(e^{-n\pi}) = \frac{\sqrt{2}a^4e^{n\pi/2}}{ng'^4_{n,n}},$$

where $g'_{k,n}$ *is as defined in* (1.1.24).

Proof. From [15, p. 39, Entry 24(iii)], we notice that

$$\psi(q) = \frac{f^2(-q^2)}{f(-q)}. \tag{8.4.8}$$

Thus, from (8.4.8) and (8.1.13), we find that

$$C(e^{-n\pi}) = 8e^{-n\pi/2}\psi^4(e^{-n\pi}), \tag{8.4.9}$$

Setting $q = e^{-n\pi}$ in (8.4.9) and employing Lemma 8.4.3(iii), we easily complete the proof. □

Theorem 8.4.7. *We have*

$$C(e^{-\pi/n}) = \frac{na^4r_{2,2n^2}^4}{r_{n,n}^4}.$$

Proof. Applying (8.4.8) in the definition of $C(q)$ given in (8.1.13) and setting $q = e^{-\pi/n}$, we find that

$$C(e^{-\pi/n}) = 8e^{-\pi/2n}\psi^4(e^{-\pi/n}), \tag{8.4.10}$$

Now, employing Lemma 8.4.3(iv) in (8.4.10), we finish the proof. □

8.5 Explicit values of quartic theta-functions

In this section, we find explicit values of the quartic theta-functions $A(q)$, $B(q)$, and $C(q)$ and also their quotients by using the results established in the previous section

Theorem 8.5.1. *We have*

(i) $\dfrac{B(e^{-\pi\sqrt{2}})}{C(e^{-\pi\sqrt{2}})} = 1,$

(ii) $\dfrac{B(e^{-2\pi})}{C(e^{-2\pi})} = 2^{3/2},$

(iii) $\dfrac{B(e^{-\pi\sqrt{6}})}{C(e^{-\pi\sqrt{6}})} = 3 + 2\sqrt{2},$

(iv) $\dfrac{B(e^{-\pi 2\sqrt{2}})}{C(e^{-\pi 2\sqrt{2}})} = 2^{3/2}\left(1+\sqrt{2}\right)^{3/2},$

(v) $\dfrac{B(e^{-\pi\sqrt{10}})}{C(e^{-\pi\sqrt{10}})} = \left(\dfrac{1+\sqrt{5}}{2}\right)^{6},$

(vi) $\dfrac{B(e^{-\pi 2\sqrt{3}})}{C(e^{-\pi 3\sqrt{3}})} = \sqrt{2}\left(\sqrt{3}+1\right)^{3},$

(vii) $\dfrac{B(e^{-\pi\sqrt{14}})}{C(e^{-\pi\sqrt{14}})} = \left(\dfrac{\sqrt{2}+1\sqrt{2\sqrt{2}-1}}{2}\right)^{6},$

(viii) $\dfrac{B(e^{-4\pi})}{C(e^{-4\pi})} = 2^{9/4}\left(1+\sqrt{2}\right)^{3},$

(ix) $\dfrac{B(e^{-\pi 3\sqrt{2}})}{C(e^{-\pi 3\sqrt{2}})} = \left(\sqrt{3}+\sqrt{2}\right)^{4},$

(x) $\dfrac{B(e^{-\pi 2\sqrt{5}})}{C(e^{-\pi 2\sqrt{5}})} = \dfrac{1}{8}\left(1+\sqrt{5}\right)^{3}\left(\sqrt{\sqrt{5}+1}+\sqrt{2}\right)^{3},$

(xi) $\dfrac{B(e^{-\pi 2\sqrt{6}})}{C(e^{-\pi 2\sqrt{6}})} = \left(1+\sqrt{2}\right)^{5/2}\left(2(1+\sqrt{2}+\sqrt{6})\right)^{3/2},$

(xii) $\dfrac{B(e^{-\pi 4\sqrt{2}})}{C(e^{-\pi 4\sqrt{2}})} = 2^{3/2}\left(1+\sqrt{2}\right)^{3}\left(4+\sqrt{2+10\sqrt{2}}\right)^{3/2},$

(xiii) $\dfrac{B(e^{-6\pi})}{C(e^{-6\pi})} = 2^{-11/2}\left(1+\sqrt{3}\right)^{4}\left(1+\sqrt{3}+\sqrt{2}\;\;3^{3/4}\right)^{4},$

(xiv) $\dfrac{B(e^{-\pi 2\sqrt{10}})}{C(e^{-\pi 2\sqrt{10}})} = 2^{-6}\left(1+\sqrt{5}\right)^{15/2}\left(2+3\sqrt{2}+\sqrt{5}\right)^{3/2},$

(xv) $\dfrac{B(e^{-5\pi})}{C(e^{-5\pi})} = \dfrac{\left(5^{1/4}+1\right)^{12}}{2^{15/2}},$

(xvi) $\dfrac{B(e^{-\pi 3\sqrt{6}})}{C(e^{-\pi 6\sqrt{6}})} = \left(1+\sqrt{2}\right)^{10/3} \left\{\sqrt{2}+\sqrt{2}(1+\sqrt{2})^{1/3}+(1+\sqrt{2})^{2/3}\right\}^4,$

(xvii) $\dfrac{B(e^{-8\pi})}{C(e^{-8\pi})} = 2^{9/4}\left(1+\sqrt{2}\right)^3\left(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4}\right)^{3/2},$

(xviii) $\dfrac{B(e^{-\pi 6\sqrt{2}})}{C(e^{-\pi 6\sqrt{2}})} = \dfrac{\left(2\left(1+35\sqrt{2}-28\sqrt{3}\right)\right)^{3/2}}{\left(\sqrt{3}-\sqrt{2}\right)^8},$

(xix) $\dfrac{B(e^{-\pi 7\sqrt{2}})}{C(e^{-\pi 7\sqrt{2}})} = \left(\dfrac{1+\sqrt{7+2\sqrt{14}}}{2\sqrt{2}} + \dfrac{\sqrt{\sqrt{14}+\sqrt{7+2\sqrt{14}}}}{2}\right)^{12},$

(xx) $\dfrac{B(e^{-10\pi})}{C(e^{-10\pi})} = \dfrac{2^{15/2}}{(5^{1/4}-1)^{12}},$

(xxi) $\dfrac{B(e^{-12\pi})}{C(e^{-12\pi})} = \dfrac{\left((\sqrt{2}+\sqrt{3})(\sqrt{3}+1)(1+\sqrt{2}-\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\sqrt{6})\right)^4}{2^{21/4}\left(\sqrt{2}-1\right)^5},$

(xxii) $\dfrac{B(e^{-3\pi})}{C(e^{-3\pi})} = \dfrac{\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{1/4}\right)^4}{2^{13/2}},$

(xxiii) $\dfrac{B(e^{-3\pi/\sqrt{2}})}{C(e^{-3\pi/\sqrt{2}})} = \dfrac{\left(-1+35\sqrt{2}+28\sqrt{3}\right)^{3/2}}{2^{3/2}\left(\sqrt{2}+\sqrt{3}\right)^4},$

(xxiv) $\dfrac{B(e^{-\pi\sqrt{3}})}{C(e^{-\pi\sqrt{3}})} = \dfrac{\left(\sqrt{3}+1\right)^3}{2^{7/2}},$

(xxv) $\dfrac{B(e^{-\pi 3\sqrt{2}})}{C(e^{-\pi 3\sqrt{2}})} = \dfrac{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}+2^{2/3}\cdot\sqrt{3}\right)^4}{2^{9/2}\left(2^{1/3}-1\right)^4},$

(xxvi) $\dfrac{B(e^{-\pi\sqrt{7}})}{C(e^{-\pi\sqrt{7}})} = \dfrac{\left(3+\sqrt{7}\right)^3}{2^{9/2}},$

(xxvii) $\dfrac{B(e^{-\pi 3\sqrt{7}})}{C(e^{-\pi 3\sqrt{7}})} = \dfrac{\left(7-2\sqrt{3}+\sqrt{21}+(3+\sqrt{3})\sqrt{3+16\sqrt{21}-27\sqrt{7}}\right)^4}{2^{13/2}(\sqrt{3}-1)^8(3-\sqrt{7})}.$

Proof. We employ the values of $r_{2,n}$ from Section 1.3 in Theorem 8.4.1 to finish the proof. □

Theorem 8.5.2. *We have*

(i) $\dfrac{B(e^{-2\pi})}{C(e^{-\pi})} = \dfrac{1}{2},$

(ii) $\dfrac{B(e^{-\pi 2\sqrt{2}})}{C(e^{-\pi\sqrt{2}})} = \left(\dfrac{\sqrt{2}+1}{2}\right)^{1/2},$

(iii) $\dfrac{B(e^{-2\pi\sqrt{3}})}{C(e^{-\pi\sqrt{3}})} = \dfrac{1}{2}\left(\sqrt{2}+\sqrt{3}\right),$

(iv) $\dfrac{B(e^{-4\pi})}{C(e^{-2\pi})} = 2^{1/4}\left(\sqrt{2}+1\right),$

(v) $\dfrac{B(e^{-2\pi\sqrt{5}})}{C(e^{-\pi\sqrt{5}})} = \dfrac{1}{8}\left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)^2,$

(vi) $\dfrac{B(e^{-2\pi\sqrt{7}})}{C(e^{-\pi\sqrt{7}})} = \dfrac{\left(127+48\sqrt{7}\right)^{1/2}}{2},$

(vii) $\dfrac{B(e^{-6\pi})}{C(e^{-3\pi})} = \dfrac{1}{2}\left(\dfrac{1}{2}+\dfrac{3^{1/4}}{\sqrt{2}}+\dfrac{\sqrt{3}}{2}\right)^4,$

(viii) $\dfrac{B(e^{-10\pi})}{C(e^{-5\pi})} = \dfrac{1}{2}\left(\dfrac{3}{2}+\dfrac{\sqrt{5}}{2}+\sqrt{\dfrac{1}{2}(5+3\sqrt{5})}\right)^4,$

(ix) $\dfrac{B(e^{-14\pi})}{C(e^{-7\pi})} = \dfrac{1}{2}\left(\dfrac{\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}+\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}}}{2}\right)^8.$

Proof. Follows easily from Theorem 8.4.2 and the values of J_n from Theorem 4.6.3. □

Theorem 8.5.3. *We have*

(i) $B(e^{-2\pi}) = 2^{-1/2}a^4,$

(ii) $B(e^{-4\pi}) = 2^{-7/4}a^4(1+\sqrt{2}),$

(iii) $B(e^{-6\pi}) = \dfrac{a^4\left(1+\sqrt{3}+\sqrt{2}\cdot\sqrt[4]{3^3}\right)^{4/3}}{2^{11/6}\cdot 3^{3/2}\left(\sqrt{3}-1\right)^{2/3}},$

(iv) $B(e^{-8\pi}) = 2^{-7/2}a^4(16+15\sqrt[4]{2}+12\sqrt{2}+9\sqrt[4]{2^3})^{1/2},$

(v) $B(e^{-10\pi}) = \dfrac{a^4 2^{7/2}}{5\left(\sqrt[4]{5}-1\right)^4\left(5+\sqrt{5}\right)^{1/2}},$

(vi) $B(e^{-12\pi}) = \dfrac{a^4\left(2-3\sqrt{2}+3^{5/4}+3^{3/4}\right)^{4/3}}{3^{3/2}2^{19/12}(\sqrt{2}-1)^{1/3}(\sqrt{3}+1)^{2/3}(-1-\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{4/3}},$

(vii) $B(e^{-3\pi}) = \dfrac{a^4\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{1/4}\right)^{4/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^4}{2^4\cdot 3^{3/2}\left(1+\sqrt{3}\right)^{2/3}},$

(viii) $B(e^{-5\pi}) = \dfrac{a^4\left(5^{1/4}+1\right)^4\left(3+2\cdot 5^{1/4}\right)}{5^2\cdot 2^2\left(1+\sqrt{5}\right)}.$

Proof. (i)-(vi) follow readily from the first equality of Theorem 8.4.4(i) and the corresponding values of $h'_{n,n}$ in Lemma 8.3.4(i)-(vi), respectively. To prove (vii) and (viii), we employ the corresponding values of $r_{k,n}$ listed in Section 1.3 to the second equality of Theorem 8.4.4(i). □

Theorem 8.5.4. *We have*

(i) $B(e^{-\pi}) = \dfrac{a^4}{2},$

(ii) $B(e^{-\pi/2}) = \dfrac{a^4}{2(1+\sqrt{2})^2},$

(iii) $B(e^{-\pi/3}) = \dfrac{2^4\sqrt{3}a^4}{(1+\sqrt{3})^{10/3}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{8/3}}$

Proof. We set n =2, 3, and 6, respectively, in Theorem 8 4 4(ii) and then use the corresponding values of $r_{k,n}$ from Section 1.3 to complete the proofs. □

Lemma 8.5.5. *We have*

(i) $h_{1,1} = 1,$

(ii) $h_{2,2} = \sqrt{2\sqrt{2}-2},$

(iii) $h_{3,3} = \left(2\sqrt{3}-3\right)^{1/4} = \dfrac{3^{1/8}\sqrt{\sqrt{3}-1}}{2^{1/4}},$

(iv) $h_{4,4} = \dfrac{2^{3/4}}{\sqrt[4]{2}+1},$

(v) $h_{5,5} = \sqrt{5-2\sqrt{5}},$

(vi) $h_{6,6} = \dfrac{2^{3/4}3^{1/8}\left(\left(\sqrt{2}-1\right)\left(\sqrt{3}-1\right)\right)^{1/6}}{\left(-4+3\sqrt{2}+3^{5/4}+2\sqrt{3}-3^{3/4}+2\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}.$

We refer to [69, p. 19, Theorem 5.4] or [66, p. 150, Theorem 9.2.4] for proofs of the above assertions.

Theorem 8.5.6. *We have*

(i) $B(-e^{-\pi}) = a^4,$

(ii) $B(-e^{-2\pi}) = \dfrac{a^4}{8(\sqrt{2}-2)^2},$

(iii) $B(-e^{-3\pi}) = \dfrac{2a^4}{3^{3/2}(\sqrt{3}-1)^2},$

(iv) $B(-e^{-4\pi}) = \dfrac{a^4(\sqrt[4]{2}+1)^4}{32},$

(v) $B(-e^{-5\pi}) = \dfrac{a^4}{5(5-2\sqrt{5})^2},$

(vi) $B(-e^{-6\pi}) = \dfrac{a^4\left(-4+3\sqrt{2}+3^{5/4}+2\sqrt{3}-3^{3/4}+2\sqrt{2}\cdot 3^{3/4}\right)^{4/3}}{2^4\cdot 3^3\left((\sqrt{2}-1)(\sqrt{3}-1)\right)^{2/3}}.$

Proof. We employ the values of $h_{n,n}$ given in the above lemma in Theorem 8.4.5(i) to finish the proof. □

Theorem 8.5.7. *We have*

(i) $B(-e^{-\pi/2}) = \dfrac{a^4}{2\left(\sqrt{2}-2\right)^2}$,

(ii) $B(-e^{-\pi/3}) = \dfrac{3a^4}{2\sqrt{3}-3}$,

(iii) $B(-e^{-\pi/4}) = \dfrac{a^4}{2}(\sqrt[4]{2}+1)^4$,

(iv) $B(-e^{-\pi/5}) = \dfrac{5a^4}{(5-2\sqrt{5})^2}$,

(v) $B(-e^{-\pi/6}) = \dfrac{\sqrt{3}\,a^4\left(-4+3\sqrt{2}+3^{5/4}+2\sqrt{3}-3^{3/4}+2\sqrt{2}\cdot 3^{3/4}\right)^{4/3}}{2^2\sqrt{3}\left((\sqrt{2}-1)(\sqrt{3}-1)\right)^{2/3}}$.

Proof. We use the values of $h_{n,n}$ from Lemma 8.5.5 in Theorem 8.4.5(ii) . □

Theorem 8.5.8. *We have*

(i) $C(e^{-\pi}) = \sqrt{2}a^4$,

(ii) $C(e^{-2\pi}) = \dfrac{a^4}{4}$,

(iii) $C(e^{-3\pi}) = \dfrac{2^{5/2}a^4}{3^{7/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{4/3}\left(1+\sqrt{3}\right)^{2/3}}$,

(iv) $C(e^{-4\pi}) = \dfrac{a^4}{2^3\left(1+\sqrt{2}\right)^2}$,

(v) $C(e^{-5\pi}) = \dfrac{2^{9/2}a^4}{5\left(5+\sqrt{5}\right)^2(5^{1/4}+1)^4}$,

(vi) $C(e^{-6\pi}) = \dfrac{2^{13/3}a^4}{3^{3/2}\left(1+\sqrt{3}\right)^{10/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{8/3}}$,

(vii) $C(e^{-9\pi}) = \dfrac{a^4\sqrt{2}}{9g'^4_{9,9}}$, *where* $g'_{9,9}$ *is given in Theorem* 6.5.7(*vii*).

Proof. The proof of the theorem follows from Theorem 8.4.6 and the values of $g'_{n,n}$ from Theorem 6.5.7. □

Theorem 8.5.9. *We have*

(i) $C(e^{-\pi/2}) = 2^{5/4}a^4\left(1+\sqrt{2}\right)$,

(ii) $C(e^{-\pi/3}) = 2^{-3/2}\sqrt{3}a^4\left(1+\sqrt{3}\right)^{2/3}\left(\sqrt{2}\cdot 3^{3/4}+\sqrt{3}+1\right)^{4/3}$,

(iii) $C(e^{-\pi/4}) = 2^{3/2}a^4 \left(16 + 15 \cdot 2^{1/4} + 12\sqrt{2} + 9 \cdot 2^{3/4}\right)^{1/2}$,

(iv) $C(e^{-\pi/5}) = \dfrac{5 \cdot 2^{9/2}a^4}{\left(5+\sqrt{5}\right)^2 \left(5^{1/4}-1\right)^4}$,

(v) $C(e^{-\pi/6}) = \dfrac{2^{17/3}3^{1/2}a^4 \left(\left(\sqrt{2}+\sqrt{3}\right)\left(1+\sqrt{3}\right)\left(1+\sqrt{2}-\sqrt{3}+\sqrt{12}\cdot 3^{3/4}\right)\right)^{4/3}}{\left(1+\sqrt{3}\right)^2 \left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{4/3} \left(\sqrt{2}-1\right)^{5/3}}$

Proof. We set n =2, 3, 4, 5, and 6 in Theorem 8.4.7 and then employ the corresponding values of $r_{k,n}$ listed in Section 1.3. □

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Bibliography

[1] **Adiga, C., Naika, M. S. M., and Vasuki, K. R.** , Some new explicit evaluations for Ramanujan's cubic continued fraction, *New Zealand J. Math.* **31**(2) (2002), 109-114.

[2] **Adiga, C., Naika, M. S. M., and Shivashankara, K.**, On some $P - Q$ eta-function identities of Ramanujan, *Indian J. Math.* **44**(3) (2002), 253-267.

[3] **Adiga, C., Kim, T., Naika, M. S. M., and Madhusudhan, H. S.**, On Ramanujan's cubic continued fraction and explicit evaluations of theta-functions, *Indian J. Pure Appl. Math.*, **35**(9) (2004), 1047-1062.

[4] **Andrews, G. E.**, On q-difference equations for certain well-poised basic hypergeometric series, *Quart. J. Math. (Oxford)* **19** (1968), 433-447.

[5] **Andrews, G. E., Berndt, B. C., Jacobsen, L., and Lamphere, R. L.**, The continued fraction found in unorganised portions of Ramanujan's notebooks, *Mem. Am. Math. Soc.*, **477**, Providence, RI, 1992.

[6] **Baruah, N. D.**, A few theta-function identities and some of Ramanujan's modular equations, *Ramanujan J.* **4**(3) (2000), 239-250.

[7] **Baruah, N. D.**, On some of Ramanujan's identities for eta-functions, *Indian J. Math.* **42** (3) (2000), 253-266.

[8] **Baruah, N. D.** , On some of class invariants of Ramanujan, *J. Indian Math. Soc.* **68**(1-4) (2001), 113-131.

[9] **Baruah, N. D.**, Modular equations for Ramanujan's cubic continued fraction, *J. Math. Anal. Appl.* **268** (2002), 244-255.

[10] **Baruah, N. D.**, On some of Ramanujan's Schläfli-type " mixed "modular equations, *J. Number Theory* **100** (2003), 270-294.

[11] **Baruah, N. D. and Bhattacharyya, P.**, Some theorems on the explicit evaluation of Ramanujan's theta-functions, *Int. J. Math. Math. Sci.* **40** (2004), 2149-2159.

[12] **Baruah, N. D. and Saikia, N.**, Some general theorems on the explicit evaluations of Ramanujan's cubic continued fraction, *Journal Comput. Appl. Math.*, **160**(1-2) (2003), 37-51.

[13] **Baruah, N. D. and Saikia, N.**, Some New Explicit Values of Ramanujan's Continued Fractions, *Indian J. Math.*, **46**(2-3) (2004), 197-222.

[14] **Baruah, N. D. and Saikia, N.**, Two parameters for Ramanujan's theta-functions and their explicit values, *Rocky Mountain J. Math.* **to appear**.

[15] **Berndt, B. C.**, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[16] **Berndt, B. C.**, *Ramanujan's Notebooks, Part IV*, Springer-Verlag, New York, 1994.

[17] **Berndt, B. C.**, *Ramanujan's Notebooks, Part V*, Springer-Verlag, New York, 1998.

[18] **Berndt, B. C.**, Flowers which we cannot yet see growing in Ramanujan's garden of hypergeometeric series, elliptic functions, and q's, in Special Functions 2000: Current Perspective and Future Directions (J. Bustoz, M. E. H. Ismail, and S. K. Suslov, eds.), Kluwer, Dordrecht, (2001), 61-85.

[19] **Berndt, B. C., Bhargava, S., and Garvan, F. G.**, Ramanujan's theories of elliptic functions to alternative bases, *Trans. Amer. Math. Soc.* **347** (1995), 4163-4244.

[20] **Berndt, B. C. and Chan, H. H.**, Ramanujan's explicit values for the classical theta-function, *Mathematika*, **42** (1995), 278-294.

[21] **Berndt, B. C. and Chan, H. H.**, Eisentein series and approximation to π, *Illinois J Math.* **45** (2001), 75-90.

[22] **Berndt, B. C. and Chan, H. H.**, Some values for the Rogers-Ramanujan continued fraction, *Canad. J. Math.* **47**(5) (1995), 897-914.

[23] **Berndt, B. C., Chan, H. H., and Liaw, W.-C.**, On Ramanujan Quartic Theory of Elliptic Functions, *J. Number Theory* **88** (2001), 129-156.

[24] **Berndt, B. C., Chan, H. H., Kang, S.-Y., and Zhang, L.-C.**, A certain quotient of eta-funtions found in Ramanujan's lost notebook, *Pacific J. Math.* **202**(2) (2002), 267-304.

[25] **Berndt, B. C., Chan, H. H., and Zhang, L.-C.**, Ramanujan's class invariants with applications to the values of q-continued fractions and theta-functions, in *Special Functions and q-Series and Related Topics* (M. Ismail, D. Masson, and M. Rahman, eds.), Fields Institute Communications Series, **14**, Amer. Math. Soc., Providence, RI, 1997, 37-53.

[26] **Berndt, B. C., Chan, H. H., Zhang, L.-C.**, Ramanujan's class invariants and cubic continued fraction, *Acta Arith.* **73** (1995), 67-85.

[27] **Berndt, B. C., Chan, H. H., Zhang, L.-C.**, Explicit evaluations of the Rogers-Ramanujan continued fraction, *J. Reine Angew. Math.* **480** (1996), 141-159.

[28] **Berndt, B. C., Chan, H. H., Zhang, L.-C.**, Ramanujan's remarkable product of theta-functions, *Proc. Edinburg Math. Soc.* **40** (1997), 583-612.

[29] **Berndt, B. C., Chan, H. H., Zhang, L.-C.**, Ramanujan's class invariants, Kronecker's limit formula, and modular equations, *Trans. Amer. Math. Soc.* **349** (1997), 2125-2173.

[30] **Berndt, B. C. and Rankin, R. A.**, *Ramanujan: Letters and Commentary*, Amer. Math Soc., Providence, RI, 1995; jointly published by the London Math. Soc. London, 1995.

[31] **Berndt, B. C. and Zhang, L.-C.**, Ramanujan's identities for eta-functions, *Math. Ann.* **292** (1992), 561-573.

[32] **Bhargava, S., Vasuki, K.R., and Sreeramamurthy, T.G.**, Some evaluations of Ramanujan's cubic continued fraction, *Indian J. Pure and Appl Math.*, **35**(8) (2004), 1003-1025.

[33] **Borwein, J. M. and Borwein, P. B.**, *Pi and the AGM*, Wiley, New York, 1987.

[34] **Borwein, J. M. and Borwein, P. B.**, A cubic counterpart of Jacobi's identity and the AGM, *Trans Amer. Math. Soc.* **323** (1991), 691-701.

[35] **Chan, H. H.**, On Ramanujan's cubic continued fraction, *Acta Arith.* **73** (1995), 343-355.

[36] **Chan, H. H.**, On Ramanujan's cubic transformation for ${}_2F_1\left(\frac{1}{2},\frac{2}{3},1;z\right)$, *Math. Proc Cambridge Philos. Soc.* **124** (1998), 193-204.

[37] **Chan, H. H. and Huang, S.-S.**, On the Ramanujan-Göllnitz-Gordon continued fraction, *Ramanujan J.* 1 (1997), 75-90.

[38] **Chan, H. H., Gee, A., and Tan, V.**, Cubic singular moduli, the Rogers-Ramanujan continued fraction, and special values of theta-functions, *Pacific J. Math* **208**(1) (2003), 23-37.

[39] **Chan, H. H. and Liaw, W.-C.**, On Russell-type modular equations, *Canad. J. Math.* **52** (2000), 31-46.

[40] **Chan, H. H. and Liaw, W.-C.**, Cubic modular equations and new Ramanujan-type series for $1/\pi$, *Pacific J. Math.* **192**(2) (2000), 219-238.

[41] **Chan, H. H., Liaw, W.-C., Tan, V.**, Ramanujan's class invariants λ_n and a new class of series for $1/\pi$, *J. London Math. Soc.* **64**(2) (2001), 93-106.

[42] **Cooper, S.**, Cubic theta-functions, *Journal Comput. Appl. Math.* **160**(1-2) (2003), 77-94.

[43] **Gollnitz, G**, Partitionen mit Differenzenbedingungen, *J Reine Angew Math* **225** (1967), 154-190

[44] **Göllnitz, H.**, Some continued fractions of Rogers-Ramanujan's type, *Duke Math* **16** (1985), 695-724

[45] **Kang, S.-Y** , Ramanujan s formulas for the explicit evaluations of the Rogers-Ramanujan continued fraction and theta-functions, *Acta Arith* **90** (1999), 49-68

[46] **Liaw, W.-C.**, *Contributions to Ramanujan's theories of modular equations, Ramanujan type series for* $1/\pi$*, and partitions*, Ph D Thesis, University of Illinois at Urbana-Champaign, 1999

[47] **Ramanathan, K. G.**, On Ramanujan s continued fraction, *Acta Arith* **43** (1984), 209-226

[48] **Ramanathan, K. G.**, Hypergeometric series and continued fraction, *Proc Indian Acad Sci (Math Sci)* **97** (1987), 227-296

[49] **Ramanathan, K. G.**, Ramanujan's continued fraction, *Indian J Pure Appl Math* **16** (1985), 695-724

[50] **Ramanathan, K. G.**, Some applications of Kronecker's limit formula , *Indian J Pure Appl Math* **52** (1987), 71-89

[51] **Ramanathan, K. G.**, On some theorems stated by Ramanujan, in *Number Theory and Related Topics*, Oxford University Press, Bombay, 1989, 151-160

[52] **Ramanathan, K. G.**, Ramanujan's Modular Equations , *Acta Arith* **53** (1990), 403-420

[53] **Ramanujan, S.**, Modular equations and approximations to π, *Quart J Math* **45** (1914), 350-372

[54] **Ramanujan, S.**, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[55] **Ramanujan, S.**, *Collected Papers*, Chelsa, NewYork, 1962.

[56] **Ramanujan, S.**, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[57] **Rogers, L. J.**, On a type of modular relation, *Proc. London Math. Soc.* 19 (1921), 387-397.

[58] **Selberg, A.**, Über einige arithmetische Identitäten, *Avh. Norske Vid.-Akad. Oslo I. Mat.-Naturv. Kl.* (1936), 2-23.

[59] **Vasuki, K. R. and Shivashankara, K.**, On Ramanujan's continued fractions, *Ganita* 53 (2002), 81-88.

[60] **Vasuki, K. R. and Srivatsa Kumar, B. R.**, Certain identities for Ramanujan-Göllnitz Gordon continued fraction, preprint.

[61] **Venkatachalienger, K.**, *Development of Elliptic Functions According to Ramanujan*, Technical Report 2, Madurai University, Madurai, 1988.

[62] **Watson, G. N.**, Theorems stated by Ramanujan (VII): Theorems on continued fractions, *J. London Math. Soc.* 4 (1929), 39-48.

[63] **Watson, G. N.**, Theorems stated by Ramanujan (IX): Two continued fractions, *J. London Math. Soc.* 4 (1929), 231-237.

[64] **Weber, H.**, *Lehrburg der Algebra II*, Chelsea, New York, 1961.

[65] **Whittaker, E. T. and Watson, G. N.**, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1966. Indian edition is published by Universal Book Stall, New Delhi, 1991.

[66] Yi, J., *Construction and application of modular equation* , Ph. D thesis, University of Illionis, 2001

[67] Yi, J., Evaluations of the Rogers-Ramanujan continued fraction $R(q)$ by modular equations, *Acta Arith* **97** (2001), 103-127

[68] Yi, J., Modular equations for the Rogers-Ramanujan continued fraction and the Dedekind eta-functions, *Ramanujan J.* **5** (2001), 377-384

[69] Yi, J., Theta-function identities and the explicit formulas for theta-function and their applications, *J Math Anal. Appl.* **292** (2004), 381-400

[70] Zhang, L.-C., Explicit evaluation of Ramanujan-Selberg continued fraction, *Proc of Amer. Soc Math* **130**(1) (2002), 9-14.