CONTRIBUTIONS TO RAMANUJAN'S THETA-FUNCTIONS AND MODULAR EQUATIONS

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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OCTOBER, 2005

Dedicated to my beloved parents

Sri. Mohendra Bora

&

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ABSTRACT

In this thesis, we deal with Ramanujan's modular equations and modular relations for three sets of functions analogous to the famous Rogers-Ramanujan functions. B. C. Berndt (Ramanujan's Notebooks, part III(1991), 280-282 & 352-358; Ramanujan's Notebooks, Part V(1998), 370) proved several of Ramanujan's modular equationsof degrees 5 and 9 by using a method of parameterizations, which requires prior knowledge of the equations. In this thesis, we find alternative proofs of these modular equations by using theta-function identities. In the process, we also find more direct proofs of some of the associated theta-function identities. By employing the methods of Rogers (*Proc. London Math. Soc.* 19(1921), 387-397), Watson (*J. Indian Math. Soc.* **20**(1993), 57-69), and Bressoud (Ph.D Thesis, Temple University, 1977), Huang (*J. Number Theory* **68**(1998), 178-216) and Chen and Huang (*J. Number Theory* **93**(2002), 58-75) found 21 modular relations involving the Göllnitz-Gordon functions, which are analogous to the well known forty identities for the Rogers-Ramanujan functions. In this thesis, we find alternative proofs of these 21 modular relations as well as several new relations by using Schröter's formulas and Ramanujan's theta-function identities. We also establish many modular equations satisfied by the nonic analogues and one more set of functions analogous to the Rogers-Ramanujan functions. By the notion of colored partition, several interesting partition theoretic interpretations are derived.

DECLARATION

I, Jonali Bora, hereby declare that the subject matter in this thesis is the record of work done by me during my Ph.D. course and that the contents of this thesis have not been submitted before for any degree whatsoever by me.

The thesis is being submitted to Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

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Certificate of the Supervisor

This is to certify that the thesis entitled **"CONTRIBUTIONS TO RAMANUJAN'S THETA-FUNCTIONS AND MODULAR EQUATIONS"** submitted to Tezpur University in the **Department of Mathematical Sciences** under the **School of Science and Technology** in partial fulfillment for the award of the degree of Doctor of Philosophy in **Mathematical Sciences** is a record of research work carried out by **Ms. Jonali Bora** under my personal supervision and guidance.

All helps received by him/ her from various sources have been duly acknowledged.

No part of this thesis has been reproduced elsewhere for award of any other degree.

Signature of the Supervisor

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ACKNOWLEDGEMENT

It gives me immense pleasure to express my deep sense of gratitude to my supervisor Dr. Nayandeep Deka Baruah, Reader, Department of Mathematical Sciences, Tezpur University, for his invaluable guidance, constant encouragement and positive criticism throughout the course of study.

I would like to thank Dr. Budhin Gogoi and Mr. Pranab Gogoi, Pricipal and Vice-Principal, respectively, of Margherita College for their inspiration.

I would also like to thank the faculty members of the Department of Mathematical Sciences, Tezpur University, for their encouragement.

My sincere thanks are due to my friends Runjun, Surshree, Momi, Nipen, Naba and Abhijit for their help.

At last but not the least, I pay my special thanks to my parents and other family members for their constant inspiration.

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Contents

Chapter 1

Introduction

1.1 Introduction

Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \tag{1.1.1}$$

where $|ab| < 1$.

If we set $a = q^{2iz}$, $b = q^{-2iz}$, and $q = e^{\pi i \tau}$, where z is complex and $\mathrm{Im}(\tau) > 0$, then $f(a,b) = \vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta-functions in its standard notation [37, p. 464]. Some basic properties satisfied by $f(a,b)$ are stated in the following theorem.

Theorem 1.1.1. *[11, p. 34, Entry 18]*

$$f(a,b) = f(b,a), \tag{1.1.2}$$
$$f(1,a) = 2f(a,a^3), \tag{1.1.3}$$
$$f(-1,a) = 0, \tag{1.1.4}$$
$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}), \tag{1.1.5}$$

where n is an integer.

Jacobi's famous triple product identity [11, p. 35, Entry 19] can be put in the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{1.1.6}$$

where as usual, for any complex number a, we define

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}), \quad (a;q)_\infty := \prod_{k=1}^{\infty}(1-aq^{k-1}), \tag{1.1.7}$$

where it is also assumed here and throughout the sequel that $|q| < 1$.

Three special cases of $f(a,b)$ are

$$\phi(q) := f(q,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{1.1.8}$$

$$\psi(q) := f(q,q^3) = \frac{1}{2}f(1,q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.1.9}$$

$$f(-q) := f(-q,-q^2) = \sum_{k=0}^{\infty}(-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)/2} = (q;q)_\infty. \tag{1.1.10}$$

If we write $q = e^{2\pi i\tau}$ with $\text{Im}(\tau) > 0$, then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ is the classical Dedekind eta-function. Throughout the thesis, we shall use (1.1.2) - (1.1.10) several times without comments.

We also define

$$\chi(q) := (-q;q^2)_\infty = \prod_{k=0}^{\infty}(1+q^{2k+1}). \tag{1.1.11}$$

Next, we give the definition of a modular equation as employed by Ramanujan. The complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 sin^2\phi}} = \frac{\pi}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \tag{1.1.12}$$

where $0 < k < 1$ and where the series representation is found by expanding the integrand in a binomial series and integrating termwise. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively, where $0 < k, l < 1$. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.1.13}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is implied by (1.1.13). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α. The multiplier m is defined by $m = K/L$.

We also need to define Ramanujan's "mixed "modular equation or a modular equation of composite degree. We recall from Chapter 20 [11, p. 325]. Let K, K', L_1, L_1', L_2, L_2', L_3, L_3', denote complete elliptic integrals of first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2, and n_3, be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1\frac{K'}{K} = \frac{L_1'}{L_1}, \quad n_2\frac{K'}{K} = \frac{L_2'}{L_2}, \quad n_3\frac{K'}{K} = \frac{L_3'}{L_3} \tag{1.1.14}$$

hold. Then a "mixed "modular equation is relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, $\sqrt{\delta}$, that is induced by (1.1.14). In such an instance, we say that β, γ, and δ are of degrees n_1, n_2, and n_3 respectively. Recalling from [11, p. 101] that $z_r = \phi^2(q^r)$, we define the multipliers m and m' by $m = z_1/z_{n_1}$ and $m' = z_{n_2}/z_{n_3}$.

Ramanujan recorded many modular equations of prime degrees as well as of composite degrees in his notebooks [28] and the lost notebook [29, pp. 50 and 56]. All of Ramanujan's modular equations were proved by Berndt (see [11, Chapter 19-20], [12, Chapter 25], [13, Chapter 36], and [14, pp. 55-74]). As Ramanujan did not provide any proofs for his results, one can only speculate his proofs. It is clear from Chapter 17 of Berndt's book [11] that modular equations can be expressed as identities involving the theta-functions ϕ, ψ and f. Provably Ramanujan first derived a theta-function identity and then transcribed it into an equivalent modular equation by using his catalogue of theta-functions [11, pp. 122-124, Entries 10-12]. Therefore, often one first tries to derive a theta-function identity and then transcribes it into an equivalent modular equation. But, proofs of some of Ramanujan's modular equations given by Berndt are quite unlike this method. He sometimes reversed the process. Berndt also used a method of parameterizations in proving some of the modular equations. This method also requires prior knowledge of the equations.

In Chapter 2 of our thesis, we present alternative proofs of Ramanujan's modular equations of prime degree 5 by using theta-function identities. First we find alternative proofs of some of the associated theta-function identities and then transcribe these and their different combinations to arrive at

Ramanujan's modular equations. In the process, we also derive several new theta-function identities.

In Chapter 3, we present the proofs of all of Ramanujan's modular equations of degree 9, by using theta-function identities. In the process, we also find new proofs of some of Ramanujan's theta-function identities. The contents of this chapter are almost identical to [5].

Next, for $|q| < 1$, the well-known Rogers-Ramanujan functions are defined by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1.1.15}$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.1.16}$$

$G(q)$ and $H(q)$ are called Rogers-Ramanujan identities. In a manuscript published with the lost notebook [29], forty modular relations for $G(q)$ and $H(q)$ were recorded by Ramanujan. These are now known as Ramanujan's forty identities. Among the forty identities, the simplest and most beautiful one is

$$G(q^{11})H(q) - q^2 G(q) H(q^{11}) = 1.$$

In 1921, Darling [23] proved this identity in the Proceedings of London Mathematical Society. In the same issue of the Proceedings, Rogers [32] proved ten of the forty identities including the one proved by Darling. In 1933, Watson [36] established eight of the forty identities, two of which had been previously proved by Rogers. In 1977, Bressoud ([19] & [20]) generalized Rogers' results to prove fifteen additional identities. In 1989, Biagioli [17] established 8 identities of the remaining 9 unproved identities by using the theory of modular forms. The remaining one identity can also be proved by Biagioli's method. The primary disadvantage of Biagioli's method is that the desired identities must be known in advance, and the proofs are perhaps more properly called verifications. On the other hand, Rogers, Watson and Bressoud all employed the same bare hands approach by viewing the sum as taken over quadratic forms with variables taken from restricted residue classes. Indeed, Rogers and Bressoud also derived general formulas that were powerful for proving some of the forty identities. Recently, Berndt et al. [16] have found proofs of 35 of the 40 identities in the spirit of Ramanujan's mathematics. For each of the remaining 5 identities, they also offered heuristic arguments showing that both sides of the identity have the same asymptotic expansions as $q \to 1^-$.

Another two well known functions analogous to the Rogers-Ramanujan functions are the so called Göllnitz-Gordon Functions, defined as

$$S(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty}, \tag{1.1.17}$$

$$T(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2+2n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}. \tag{1.1.18}$$

Motivated by the similarity between the Rogers-Ramanujan and Göllnitz-Gordon functions, S.-S. Huang [26] and Chen and Huang [22] derived 21 modular relations involving $S(q)$ and $T(q)$, one new relations for $G(q)$ and $H(q)$, and 9 relations involving both the pairs $G(q)$, $H(q)$ and $S(q)$ and $T(q)$. They used the methods of Rogers [32], Watson [36], and Bressoud [19]. In Chapter 4 of this thesis, we find proofs of the modular relations involving only $S(q)$ and $T(q)$ by employing Schröter's formulas and theta functions identities. We also derive several new modular relations. The contents of this chapter are almost identical to our paper [8]

In [24] & [25], H. Hahn defined the septic analogues of the Rogers-Ramanujan functions as

$$A(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7;q^7)_\infty (q^3;q^7)_\infty (q^4;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{1.1.19}$$

$$B(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7;q^7)_\infty (q^2;q^7)_\infty (q^5;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{1.1.20}$$

$$C(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}} = \frac{(q^7;q^7)_\infty (q;q^7)_\infty (q^6;q^7)_\infty}{(q^2;q^2)_\infty}. \tag{1.1.21}$$

She derived several analogues of Ramanujan's forty identities involving $A(q)$, $B(q)$, and $C(q)$. Some of them are connected with the Rogers-Ramanujan functions and the Göllnitz-Gordon Functions. She also found partition theoretic results from some of her identities. In Chapter 5 of this thesis, we define the nonic analogues of the Rogers-Ramanujan functions as

$$D(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n} q^{3n^2}}{(q^3;q^3)_n(q^3;q^3)_{2n}} = \frac{(q^5;q^9)_\infty (q^4,q^9)_\infty (q^9;q^9)_\infty}{(q^3,q^3)_\infty}, \tag{1.1.22}$$

$$E(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n}(1-q^{3n+2})q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}} = \frac{(q^7;q^9)_\infty(q^2;q^9)_\infty(q^9;q^9)_\infty}{(q^3;q^3)_\infty}, \tag{1.1.23}$$

$$F(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n+1}q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}} = \frac{(q^8;q^9)_\infty(q;q^9)_\infty(q^9;q^9)_\infty}{(q^3;q^3)_\infty}. \tag{1.1.24}$$

We used a variety of methods to establish many modular relations involving only $D(q)$, $E(q)$, and $F(q)$ as well as several others involving other analogous functions. By the notion of colored partitions, we able to find several partition theoretic results from some of our relations. The contents of this chapter are almost identical to our paper [6].

In Chapter 6, we define another couple of functions analogues to the Rogers-Ramanujan functions. These are

$$X(q) := \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n(1-q^{n+1})q^{n(n+2)}}{(q;q)_{2n+2}} = \frac{(q;q^{12})_\infty(q^{11};q^{12})_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty}, \tag{1.1.25}$$

$$Y(q) := 1 + \sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1}(1+q^n)q^{n^2}}{(q;q)_{2n}} = \frac{(q^5;q^{12})_\infty(q^7;q^{12})_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty}. \tag{1.1.26}$$

We establish several modular relations involving only $X(q)$ and $Y(q)$ as well as several others involving other analogous functions. We also extract partition theoretic results from some of the relations. The contents of this chapter are almost identical to our paper [7].

Chapter 2

Ramanujan's Modular Equations of Degree 5 and Associated Theta-function Identities

2.1 Introduction

This chapter is devoted to proving modular equations of degree 5. Ramanujan recorded 27 modular equations of degree 5 on Chapter 19 of his second notebook [28]. Two of these modular equations were also recorded by Ramanujan in a fragment published with the lost notebook [29, p. 351]. B. C. Berndt proved [11, Entry 13, pp. 280-282] all of these modular equations. He proved most of these modular equations by a method of parameterizations. But, as we have already mentioned in the previous chapter that, Ramanujan might have first derived a theta-function identity and then transcribed it into an equivalent modular equation. Baruah and Bhattacharyya [4] found alternative proofs of three of Ramanujan's theta-function identities associated with modular equations of degree 5 and used those to derive some theorems on explicit evaluations of Ramanujan's theta-functions. Earlier those identities were proved by Berndt by using modular equations and a method of parameterizations. In this chapter, we present alternative proofs of Ramanujan's modular equations by using theta-function identities. In the meantime, we also find new proofs of some of the associated theta-function identities by using other theta-function identities of Ramanujan. The theta-function identities which we prove are different from those proved by Baruah and Bhattacharyya [4]. We also note that Berndt used modular equations and a method of parameterizations to deduce these theta-function identities.

In Section 2.2, we state some preliminary results.

In Section 2.3, we state the theta-function identities and present new proofs of some of the identities.

In the final section, we prove the modular equations by using results from the previous two sections.

2.2 Preliminary Results

In this section, we state some results which will be used to derive our theta-function identities.

Lemma 2.2.1. *[11, p. 39, Entry 24] We have*

$$\frac{\psi(q)}{\psi(-q)} = \sqrt{\frac{\phi(q)}{\phi(-q)}}, \tag{2.2.1}$$

$$f^3(-q) = \phi^2(-q)\psi(q), \tag{2.2.2}$$

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \sqrt[3]{\frac{\phi(q)}{\psi(-q)}} = \frac{\phi(q)}{f(q)} = \frac{f(-q^2)}{\psi(-q)}, \tag{2.2.3}$$

$$f^3(-q^2) = \phi(-q)\psi^2(q), \qquad \chi(q)\chi(-q) = \chi(-q^2). \tag{2.2.4}$$

Lemma 2.2.2. *[11, p. 40, Entry 25] We have*

$$\phi(q)\phi(-q) = \phi^2(-q^2), \tag{2.2.5}$$

$$\psi(q)\psi(-q) = \psi(q^2)\phi(-q^2), \tag{2.2.6}$$

$$\phi(q)\psi(q^2) = \psi^2(q), \tag{2.2.7}$$

$$\phi^4(q) - \phi^4(-q) = 16q\psi^4(q^2). \tag{2.2.8}$$

Lemma 2.2.3. *[11, p. 45, Entry 29] If $ab = cd$, then*

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc), \tag{2.2.9}$$

and

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d)$$

$$= 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right) f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \tag{2.2.10}$$

Lemma 2.2.4. *[11, pp. 122-124, Entries 10-12] If*

$$z = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \quad \textit{and} \quad y = \pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)},$$

then

$$\phi(e^{-y}) = \sqrt{z}, \tag{2.2.11}$$
$$\phi(-e^{-y}) = \sqrt{z}(1-x)^{1/4}, \tag{2.2.12}$$
$$\phi(-e^{-2y}) = \sqrt{z}(1-x)^{1/8}, \tag{2.2.13}$$
$$\phi(e^{-y/4}) = \sqrt{z}(1+x^{1/4}), \tag{2.2.14}$$
$$\phi(-e^{-y/4}) = \sqrt{z}(1-x^{1/4}), \tag{2.2.15}$$
$$\psi(e^{-y}) = \sqrt{z/2}(xe^{y})^{1/8}, \tag{2.2.16}$$
$$\psi(-e^{-y}) = \sqrt{z/2}\{x(1-x)e^{y}\}^{1/8}, \tag{2.2.17}$$
$$\psi(e^{-2y}) = \frac{1}{2}\sqrt{z}(xe^{y})^{1/4}, \tag{2.2.18}$$
$$\psi(e^{-y/2}) = \sqrt{z}\{(1+\sqrt{x})/2\}^{1/4}(xe^{y})^{1/16}, \tag{2.2.19}$$
$$\psi(-e^{-y/2}) = \sqrt{z}\{(1-\sqrt{x})/2\}^{1/4}(xe^{y})^{1/16}, \tag{2.2.20}$$
$$f(-e^{-2y}) = \sqrt{z}2^{-1/3}\{x(1-x)e^{y}\}^{1/12}, \tag{2.2.21}$$
$$f(e^{-y}) = \sqrt{z}2^{-1/6}\{x(1-x)e^{y}\}^{1/24}, \tag{2.2.22}$$
$$f(-e^{-4y}) = \sqrt{z}4^{-1/3}(1-x)^{1/24}(xe^{y})^{1/6}, \tag{2.2.23}$$
$$f(-e^{-y}) = \sqrt{z}2^{-1/6}(1-x)^{1/24}(xe^{y})^{1/24}, \tag{2.2.24}$$
$$\chi(e^{-y}) = 2^{1/6}\{x(1-x)e^{y}\}^{-1/24}, \tag{2.2.25}$$
$$\chi(-e^{-y}) = 2^{1/6}(1-x)^{1/12}(xe^{y})^{-1/24}, \tag{2.2.26}$$
$$\chi(-e^{-2y}) = 2^{1/3}(1-x)^{1/24}(xe^{y})^{-1/12}. \tag{2.2.27}$$

2.3 Theta-function identities

In this section, we state and prove some theta-function identities recorded by Ramanujan. We think that the proofs presented here are more transparent than those found by Berndt [11]-[13]. We also mention that the identities in [12] were likely unknown to the author when [11] was written.

Theorem 2.3.1.

$$\phi^2(q) - \phi^2(q^5) = 4q\frac{\chi(q)f^2(-q^{10})}{\chi(q^5)}. \tag{2.3.1}$$

Proof: Setting $a = q$, $b = -q^4$, $c = -q^2$, and $d = q^3$, in (2.2.9) and (2.2.10), we find that

$$f(q^4, q^6)f(-q^3, -q^7) = \frac{1}{2}\{f(q, -q^4)f(-q^2, q^3) + f(-q, q^4)f(q^2, -q^3)\}, \tag{2.3.2}$$

$$qf(q^2, q^8)f(-q, -q^9) = \frac{1}{2}\{f(q, -q^4)f(-q^2, q^3) - f(-q, q^4)f(q^2, -q^3)\}. \tag{2.3.3}$$

We note that [11, Entry 9(vii), p. 258],

$$f(q, q^4)f(q^2, q^3) = \frac{\phi(-q^5)f(-q^5)}{\chi(-q)}. \tag{2.3.4}$$

$$f(-q, -q^4)f(-q^2, -q^3) = f(-q)f(-q^5). \tag{2.3.5}$$

Now, multiplying (2.3.2) and (2.3.3) and using (2.3.4), (2.3.5), and (2.2.3), we find that

$$4qf(q^2, q^8)f(q^4, q^6)f(-q, -q^9)f(-q^3, -q^7) = f^2(q)f^2(q^5)\left\{1 - \frac{\phi^2(q^5)}{\phi^2(q)}\right\}. \tag{2.3.6}$$

Again, using Jacobi's Triple product identity from (1.1.6), we obtain

$$\begin{aligned} f(-q, -q^9)f(-q^3, -q^7) &= (q; q^{10})_\infty(q^3; q^{10})_\infty(q^7; q^{10})_\infty(q^9; q^{10})_\infty(q^{10}; q^{10})^2_\infty \\ &= \frac{(q; q^2)_\infty(q^{10}; q^{10})^2_\infty}{(q^5; q^{10})_\infty} \\ &= \frac{\chi(-q)f^2(-q^{10})}{\chi(-q^5)}. \end{aligned} \tag{2.3.7}$$

Using (2.2.3) and (2.3.7) in (2.3.6), we arrive at the required identity.

Theorem 2.3.2. *[11, p. 276, (12.32)] We have*

$$\frac{\phi^2(-q^{10})}{\phi^2(-q^2)} + q\left(\frac{\psi^2(q^5)}{\psi^2(q)} - \frac{\psi^2(-q^5)}{\psi^2(-q)}\right) = 1. \tag{2.3.8}$$

Proof : We have [11, p. 278], [3, (2.3)],

$$\phi(-q^5)\phi(q) - \phi(q^5)\phi(-q) = 4qf(-q^4)f(-q^{20}). \tag{2.3.9}$$

Using (2.2.7) in (2.3.9), we obtain

$$\psi^2(q)\psi^2(-q^5) - \psi^2(q^5)\psi^2(-q) = 4q\frac{f^3(-q^4)f^3(-q^{20})}{f(-q^2)f(-q^{10})}. \tag{2.3.10}$$

Dividing both sides of (2.3.10) by $\psi^2(q)\psi^2(-q)$, and then using (2.2.6), we find that

$$q\left(\frac{\psi^2(-q^5)}{\psi^2(-q)} - \frac{\psi^2(q^5)}{\psi^2(q)}\right) = 4q^2\frac{f^3(-q^4)f^3(-q^{20})}{f(-q^{10})f(-q^2)\psi^2(q^2)\phi^2(-q^2)}. \tag{2.3.11}$$

Using (2.2.3) in (2.3.11), we deduce that

$$q\left(\frac{\psi^2(-q^5)}{\psi^2(-q)} - \frac{\psi^2(q^5)}{\psi^2(q)}\right) = -4q^2\frac{\chi(-q^2)f^3(-q^{20})}{f(-q^{10})\phi^2(-q^2)}. \tag{2.3.12}$$

Now, replacing q by $-q$ in (2.3.1), we note that

$$\phi^2(-q) - \phi^2(-q^5) = -4q\frac{\chi(-q)f^2(-q^{10})}{\chi(-q^5)}. \tag{2.3.13}$$

Dividing both sides of (2.3.13) by $\phi^2(-q)$ and then replacing q by q^2, we obtain

$$1 - \frac{\phi^2(-q^{10})}{\phi^2(-q^2)} = -4q^2\frac{\chi(-q^2)f^3(-q^{20})}{f(-q^{10})\phi^2(-q^2)}. \tag{2.3.14}$$

Thus, we complete the theorem with the help of (2 3.11) and (2.3.14).

Theorem 2.3.3. *[11, p. 285]*

$$\phi^2(q)\phi^2(q^5) - \phi^2(-q)\phi^2(-q^5) - 16q^3\psi^2(q^2)\psi^2(q^{10}) = 8qf^2(-q^2)f^2(-q^{10}). \tag{2.3.15}$$

Proof: We have [11, Entry 9(vi), p. 258], [11, Entry 10(v), p. 262],

$$\psi^2(q) - q\psi^2(q^5) = \frac{\phi(-q^5)f(-q^5)}{\chi(-q)}. \tag{2.3.16}$$

Replacing q by q^2, we obtain

$$\psi^2(q^2) - q^2\psi^2(q^{10}) = \frac{\phi(-q^{10})f(-q^{10})}{\chi(-q^2)}. \tag{2.3.17}$$

Employing (2.2.3), we rewrite the above identity as

$$\psi^2(q^2) - q^2\psi^2(q^{10}) = \sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}\sqrt{\frac{\psi(q^2)}{\psi(q^{10})}}. \tag{2.3.18}$$

Now, using (2.2.3) in (2.3.1), we obtain

$$\phi^2(q) - \phi^2(q^5) = 4q\frac{\phi(q)f^2(-q^{10})f(q^5)}{f(q)\phi(q^5)}. \tag{2.3.19}$$

Employing (2.2.3) again in (2.3.19), we deduce that

$$\phi^2(q) - \phi^2(q^5) = 4q\sqrt{\frac{\phi(q)}{\phi(q^5)}}\sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}. \tag{2.3.20}$$

Replacing q by $-q$ in (2.3.20), we obtain

$$\phi^2(-q) - \phi^2(-q^5) = -4q\sqrt{\frac{\phi(-q)}{\phi(-q^5)}}\sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}. \tag{2.3.21}$$

Now replacing q by q^5 in (2.2.8), we find that

$$\phi^4(q^5) - \phi^4(-q^5) = 16q^5\psi^4(q^{10}). \tag{2.3.22}$$

From (2.2.8) and (2.3.22), we deduce that

$$\begin{aligned}&\phi^2(q)\phi^2(q^5) - \phi^2(-q)\phi^2(-q^5) - 16q^3\psi^2(q^2)\psi^2(q^{10})\\ &= \frac{1}{2}\{16q(\psi^2(q^2) - q^2\psi^2(q^{10}))^2 + (\phi^2(-q) - \phi^2(-q^5))^2\\ &- (\phi^2(q) - \phi^2(q^5))^2\}.\end{aligned} \tag{2.3.23}$$

Using (2.3.18), (2.3.20), and (2.3.21) in (2.3.23), we find that

$$\begin{aligned}&\phi^2(q)\phi^2(q^5) - \phi^2(-q)\phi^2(-q^5) - 16q^3\psi^2(q^2)\psi^2(q^{10})\\ &= 8q\frac{f^5(-q^{10})}{f(-q^2)}\left\{\frac{\psi(q^2)}{\psi(q^{10})} + q\left(\frac{\phi(-q)}{\phi(-q^5)} - \frac{\phi(q)}{\phi(q^5)}\right)\right\}\end{aligned} \tag{2.3.24}$$

Now, employing (2.2.5), (2.2.6), and (2.2.7) in (2.3.8), we obtain

$$\frac{\phi(q^5)\phi(-q^5)}{\phi(q)\phi(-q)} + q\left(\frac{\phi(q^5)\psi(q^{10})}{\phi(q)\psi(q^2)} - \frac{\phi(-q^5)\psi(q^{10})}{\phi(-q)\psi(q^2)}\right) = 1. \tag{2.3.25}$$

Multiplying both sides in (2.3.25) by $(\phi(q)\phi(-q)\psi(q^2))/(\phi(q^5)\phi(-q^5)\psi(q^{10}))$, we deduce that

$$\frac{\psi(q^2)}{\psi(q^{10})} + q\left(\frac{\phi(-q)}{\phi(-q^5)} - \frac{\phi(q)}{\phi(q^5)}\right) = \frac{\phi(q)\phi(-q)\psi(q^2)}{\phi(q^5)\phi(-q^5)\psi(q^{10})}. \tag{2.3.26}$$

Using (2.2.7) and (2.2.4) in (2.3.26), we obtain

$$\frac{\psi(q^2)}{\psi(q^{10})} + q\left(\frac{\phi(-q)}{\phi(-q^5)} - \frac{\phi(q)}{\phi(q^5)}\right) = \frac{f^3(-q^2)}{f^3(-q^{10})}. \tag{2.3.27}$$

With the help of (2.3.24) and (2.3.27) we finish the proof.

Theorem 2.3.4. *[11, p. 259]*

$$4\frac{f^5(q)}{f(q^5)} = 5\phi^3(q)\phi(q^5) - \frac{\phi^5(q)}{\phi(q^5)}. \tag{2.3.28}$$

Proof: We have [12, p. 202], [4, p. 2152, Theorem 2.2],

$$\frac{\chi^5(q)}{\chi(q^5)} = 1 + 5q\frac{\psi^2(-q^5)}{\psi^2(-q)}. \tag{2.3.29}$$

Again, replacing q by $-q$ in (2.3.16), we obtain

$$\psi^2(-q) + q\psi^2(-q^5) = \frac{\phi(q^5)f(q^5)}{\chi(q)}. \tag{2.3.30}$$

Dividing both sides of the above identity by $\psi^2(-q)$, we deduce that

$$\frac{q\psi^2(-q^5)}{\psi^2(-q)} = -1 + \frac{\phi(q^5)f(q^5)}{\chi(q)\psi^2(-q)}. \tag{2.3.31}$$

Employing (2.2.3) and (2.3.31) in (2.3.29) and then dividing both sides by $f(q^5)/f^5(q)$, we arrive at

$$4\frac{f^5(q)}{f(q^5)} = 5\frac{\phi(q^5)f^5(q)}{\chi(q)\psi^2(-q)} - \frac{\phi^5(q)}{\phi(q^5)}. \tag{2.3.32}$$

Using (2.2.2) and (2.2.3) in (2.3.32) we easily arrive at the proffered identity.

Theorem 2.3.5. *[13, p. 364, Entry 16]*

$$\frac{\psi^5(q)}{\psi(q^5)}+\frac{\psi^5(-q)}{\psi(-q^5)}+2\frac{f^5(-q^4)}{f(-q^{20})}=4\frac{\psi^5(q^2)}{\psi(q^{10})}. \tag{2.3.33}$$

Proof: Multiplying (2.3.18) by $16q\psi^2(q^2)$, we obtain

$$16q\psi^4(q^2)-16q^3\psi^2(q^2)\psi^2(q^{10})=16q\sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}\sqrt{\frac{\psi^5(q^2)}{\psi(q^{10})}}. \tag{2.3.34}$$

Again, multiplying (2.3.20) and (2.3.21) by $\phi^2(q)$ and $\phi^2(-q)$, respectively, and then subtracting the resulting identities, we deduce that

$$\begin{aligned}&\phi^4(q)-\phi^4(-q)-\phi^2(q)\phi^2(q^5)+\phi^2(-q)\phi^2(-q^5)\\&=4q\sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}\left\{\sqrt{\frac{\phi^5(q)}{\phi(q^5)}}+\sqrt{\frac{\phi^5(-q)}{\phi(-q^5)}}\right\}.\end{aligned} \tag{2.3.35}$$

Subtracting (2.3.35) from (2.3.34), and then using (2.2.8), we arrive at

$$\begin{aligned}&\phi^2(q)\phi^2(q^5)-\phi^2(-q)\phi^2(-q^5)-16q^3\psi^2(q^2)\psi^2(q^{10})\\&=4q\sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}\left\{4\sqrt{\frac{\psi^5(q^2)}{\psi(q^{10})}}-\sqrt{\frac{\phi^5(q)}{\phi(q^5)}}-\sqrt{\frac{\phi^5(-q)}{\phi(-q^5)}}\right\}.\end{aligned} \tag{2.3.36}$$

From (2.3.15) and (2.3.36), we find that

$$4\sqrt{\frac{\psi^5(q^2)}{\psi(q^{10})}}-\sqrt{\frac{\phi^5(q)}{\phi(q^5)}}-\sqrt{\frac{\phi^5(-q)}{\phi(-q^5)}}=2\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}. \tag{2.3.37}$$

Multiplying (2.3.37) by $\sqrt{\phi^5(q)/\phi(q^5)}$ and using also (2.2.3), (2.2.5), and (2.2.7) we arrive at the required identity.

Theorem 2.3.6.

$$5\phi^2(q^5)-\phi^2(q)=4\frac{\chi(q^5)\phi^2(q)}{\chi^5(q)}. \tag{2.3.38}$$

Proof: Dividing both sides of (2.3.28) by $\phi^5(q)/\phi(q^5)$, we find that

$$1+4\frac{f^5(q)\phi(q^5)}{f(q^5)\phi^5(q)}=5\frac{\phi^2(q^5)}{\phi^2(q)}. \tag{2.3.39}$$

Multiplying both sides of (2.3.39) by $\phi^2(q)$, and then employing (2.2.3), we deduce (2.3.38) to finish the proof.

Theorem 2.3.7. *[13, p. 364, Entry 15]*

$$\frac{\psi^5(-q^5)}{\psi(-q)} - \frac{\psi^5(q^5)}{\psi(q)} = 2q\frac{f^5(-q^{20})}{f(-q^4)} + 4q^3\frac{\psi^5(q^{10})}{\psi(q^2)}. \tag{2.3.40}$$

Proof: Using (2.2.3) in (2.3.38), we find that

$$\phi^2(q) - 5\phi^2(q^5) = -4\sqrt{\frac{\phi(q^5)}{\phi(q)}}\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}. \tag{2.3.41}$$

Replacing q by $-q$, we obtain

$$\phi^2(-q) - 5\phi^2(-q^5) = -4\sqrt{\frac{\phi(-q^5)}{\phi(-q)}}\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}. \tag{2.3.42}$$

Multiplying (2.3.41) and (2.3.42) by $\phi^2(q^5)$ and $\phi^2(-q^5)$, respectively, and then subtracting the resulting identities, we deduce that

$$\begin{aligned}\phi^2(q)\phi^2(q^5) &- \phi^2(-q)\phi^2(-q^5) - 80q^5\psi^4(q^{10}) \\ &= 4\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}\left\{\sqrt{\frac{\phi^5(-q^5)}{\phi(-q)}} - \sqrt{\frac{\phi^5(q^5)}{\phi(q)}}\right\}.\end{aligned} \tag{2.3.43}$$

Now, replacing q by $-q^2$ in Theorem 2.1 [4], we find that

$$\psi^2(q^2) - 5q^2\psi^2(q^{10}) = \frac{\phi^2(-q^2)}{\chi(-q^2)\chi(-q^{10})}. \tag{2.3.44}$$

With the help of (2.2.3), we rewrite (2.3.44) as

$$\psi^2(q^2) - 5q^2\psi^2(q^{10}) = \sqrt{\frac{\psi(q^{10})}{\psi(q^2)}}\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}. \tag{2.3.45}$$

Multiplying (2.3.45) by $16q^3\psi^2(q^{10})$, we find that

$$16q^3\psi^2(q^2)\psi^2(q^{10}) - 80q^5\psi^4(q^{10}) = 16q^3\sqrt{\frac{\psi^5(q^{10})}{\psi(q^2)}}\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}. \tag{2.3.46}$$

From (2.3.43) and (2.3.46), we deduce that

$$\begin{aligned}&-\phi^2(q)\phi^2(q^5)+\phi^2(-q)\phi^2(-q^5)+16q^3\psi^2(q^2)\psi^2(q^{10})\\&=4\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}\left\{4q^3\sqrt{\frac{\psi^5(q^{10})}{\psi(q^2)}}+\sqrt{\frac{\phi^5(q^5)}{\phi(q)}}-\sqrt{\frac{\phi^5(-q^5)}{\phi(-q)}}\right\}.\end{aligned} \tag{2.3.47}$$

From (2.3.15) and (2.3.47), we arrive at

$$4q^3\sqrt{\frac{\psi^5(q^{10})}{\psi(q^2)}}+\sqrt{\frac{\phi^5(q^5)}{\phi(q)}}-\sqrt{\frac{\phi^5(-q^5)}{\phi(-q)}}=-2q\sqrt{\frac{f(-q^{10})}{f(-q^2)}}. \tag{2.3.48}$$

Multiplying both sides of (2.3.48) by $\sqrt{\phi^5(q^5)/\phi(q)}$ and using (2.2.3), we find that

$$4q^3\frac{\psi^5(q^5)}{\psi(q)}+\frac{\phi^5(q^5)}{\phi(q)}-\frac{\phi^5(-q^{10})}{\phi(-q^2)}=-2q\frac{f^5(q)}{f(q^5)}. \tag{2.3.49}$$

Multiplying both sides of (2.3.49) by $(\psi^5(q^5)\phi(q))/(\phi^5(q^5)\psi(q))$ and rearranging the results by employing some identities in Lemma 2.2.1 and Lemma 2.2.2, we deduce (2.3.40) to finish the proof.

Theorem 2.3.8. *[13, p. 363, Entry 14]*

$$5\frac{\phi^2(q)}{\phi^2(q^5)}=\frac{\dfrac{\phi^5(q)}{\phi(q^5)}+4\dfrac{\psi^5(q)}{\psi(q^5)}}{\phi(q)\phi^3(q^5)+4q^2\psi(q)\psi^3(q^5)}. \tag{2.3.50}$$

Proof: With the help of (2.3.38) and (2.3.44), we obtain

$$\frac{\phi^2(q)-5\phi^2(q^5)}{\psi^2(q^2)-5q^2\psi^2(q^{10})}=-4\frac{\chi(q^5)\chi(-q^{10})}{\chi(q)\chi(-q^2)}. \tag{2.3.51}$$

Now, from (2.2.3) and (2.2.4), we notice that

$$\chi(-q^2)\chi(q)=\frac{\phi(q)}{\psi(q)}. \tag{2.3.52}$$

Using (2.3.52) in (2.3.51), we deduce that

$$\phi^2(q)-5\phi^2(q^5)=-4\frac{\psi^2(q^2)\psi(q)\phi(q^5)}{\phi(q)\psi(q^5)}+20q^2\frac{\psi^2(q^{10})\phi(q^5)\psi(q)}{\phi(q)\psi(q^5)}. \tag{2.3.53}$$

Employing (2.2.7) in (2.3.53), we arrive at

$$\phi^2(q) - 5\phi^2(q^5) = -4\frac{\psi^5(q)\phi(q^5)}{\phi^3(q)\psi(q^5)} + 20q^2\frac{\psi^3(q^5)\psi(q)}{\phi(q)\phi(q^5)}. \qquad (2.3.54)$$

Multiplying both sides of (2.3.54) by $\phi^3(q)/\phi(q^5)$, we find that

$$\frac{\phi^5(q)}{\phi(q^5)} + 4\frac{\psi^5(q)}{\psi(q^5)} = 5\phi^3(q)\phi(q^5) + 20q^2\psi(q)\psi^3(q^5)\frac{\phi^2(q)}{\phi^2(q^5)}, \qquad (2.3.55)$$

which is equivalent to (2.3.50).

2.4 Modular Equations

In this section, we find new proofs of Ramanujan's modular equations of degree 5 by using the theta-function identities established in the previous section. Berndt proved these equations by a method of parameterizations. The proofs given here are seemed to be closer to the provable proofs of Ramanujan. Throughout this section, we suppose that β has degree 5 over α and $m = z_1/z_5$ is the corresponding multiplier.

Theorem 2.4.1. *[11, p. 280, Entry 13(i)] We have*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \qquad (2.4.1)$$

Proof: Transcribing (2.3.15), by using (2.2 11), (2.2.12), (2.2.18), and (2.2.21), we easily deduce (2.4.1).

Baruah [3] has also found a different proof based on the identity (2.3.9).

Theorem 2.4.2. *[11, p. 280, Entry 13(ii)] We have*

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8} = 1 + 2^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}. \qquad (2.4.2)$$

Proof: Transcribing (2.3.33), by using (2.2.16) - (2.2.18), and (2.2.23), we easily deduce (2.4.2).

Theorem 2.4.3. *[11, p. 280, Entry 13(iii)] We have*

$$\left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^5}{\alpha}\right)^{1/8} = 1 + 2^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}. \qquad (2.4.3)$$

Proof: Transcribing (2.3.40) by means of (2.2.16) - (2.2.18), and (2.2.23), we readily deduce (2.4.3).

Theorem 2.4.4. *[11, p. 280, Entry 13(iv)] We have*

$$m = 1 + 2^{4/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}, \tag{2.4.4}$$

and

$$\frac{5}{m} = 1 + 2^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}. \tag{2.4.5}$$

Proof: Transcribing (2.3.1) with the help of (2.2.21) and (2.2.25), we arrive at (2.4.4).

Transcribing (2.3.28) by employing (2.2.11) and (2.2.22), we easily deduce (2.4.5).

Note that (2.4.5) is also the reciprocal of (2.4.4) and vice-versa.

Theorem 2.4.5. *[11, p. 280, Entry 13(v)] We have*

$$m = \frac{1 + \left((1-\beta)^5/(1-\alpha)\right)^{1/8}}{1 + \{(1-\alpha)^3(1-\beta)\}^{1/8}} = \frac{1 - \left(\beta^5/\alpha\right)^{1/8}}{1 - (\alpha^3\beta)^{1/8}}. \tag{2.4.6}$$

Proofs of the above modular equations have already been given by Berndt [11, pp. 282-283] with the help of theta-function identities.

Theorem 2.4.6. *[11, p. 280, Entry 13(vi)] We have*

$$\frac{5}{m} = \frac{1 + \left(\alpha^5/\beta\right)^{1/8}}{1 + (\alpha\beta^3)^{1/8}} = \frac{1 - \left((1-\alpha)^5/(1-\beta)\right)^{1/8}}{1 - \{(1-\alpha)(1-\beta)^3\}^{1/8}}. \tag{2.4.7}$$

These modular equations are the reciprocals of the respective modular equations in the previous theorem. Here we also offer an alternative proof.

Proof: Transcribing (2.3.50) by employing (2.2.11) and (2.2.16), we easily deduce the first equality of (2.4.7).

Replacing q by $-q$, in (2.3.38), we find that

$$\phi^2(-q) - 5\phi^2(-q^5) = -4\frac{\chi(-q^5)\phi^2(-q)}{\chi^5(-q)}. \tag{2.4.8}$$

From (2 3 38) and (2 4.8), we obtain

$$\frac{\phi^2(-q) - 5\phi^2(-q^5)}{\phi^2(q) - 5\phi^2(q^5)} = \frac{\chi(-q^5)\chi(q)}{\chi(-q)\chi(q^5)}. \tag{2.4.9}$$

Transcribing the above identity by employing (2.2 11), (2.2.12), (2.2.25), and (2 2.26), we find that

$$\frac{(1-\alpha)^{1/2} - 5/m(1-\beta)^{1/2}}{1-5/m} = \left(\frac{1-\beta}{1-\alpha}\right)^{1/8}. \tag{2.4.10}$$

Dividing both sides of the above identity by $((1-\beta)/(1-\alpha))^{1/8}$, we complete the proof of the second equality of (2 4.7) also.

Theorem 2.4.7. *[11, p. 280, Entry 13(vii)] We have*

$$\begin{aligned}(\alpha\beta^3)^{1/8} + \{(1-\alpha)(1-\beta)^3\}^{1/8} &= 1 - 2^{1/3}\left(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right)^{1/24} \\ &= (\alpha^3\beta)^{1/8} + \{(1-\alpha)^3(1-\beta)\}^{1/8} \\ &= \left(\frac{1+(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right)^{1/2}.\end{aligned} \tag{2.4.11}$$

Proof: Replacing q by $q^{1/4}$, in (2.3.9), we find that

$$\phi(q^{5/4})\phi(-q^{1/4}) - \phi(-q^{5/4})\phi(q^{1/4}) = -4q^{1/4}f(-q)f(-q^5). \tag{2.4.12}$$

Transcribing this equation by using (2.2.11), (2 2 12), and (2.2.24), we obtain

$$\alpha^{1/4} - \beta^{1/4} = 2^{2/3}(\alpha\beta)^{1/24}((1-\alpha)(1-\beta))^{1/6}. \tag{2.4.13}$$

The reciprocal of the above equation is given by

$$(1-\beta)^{1/4} - (1-\alpha)^{1/4} = 2^{2/3}(\alpha\beta)^{1/6}((1-\alpha)(1-\beta))^{1/24}. \tag{2.4.14}$$

Berndt et al. [16, lemma 9.1, p. 20] have also given a proof of (2.4.14) by using a method of parameterizations. They used this modular equation to prove some results on Ramanujan's famous forty identities for the Rogers-Ramanujan functions.

Multiplying both sides of (2.4.14) by $((1-\alpha)(1-\beta))^{1/8}$, we find that

$$((1-\beta)^3(1-\alpha))^{1/8}-((1-\alpha)^3(1-\beta))^{1/8} = 2^{2/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/6}. \quad (2.4.15)$$

The reciprocal of this equation is given by

$$(\alpha^3\beta)^{1/8} - (\beta^3\alpha)^{1/8} = 2^{2/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/6}. \quad (2.4.16)$$

From (2.4.15) and (2.4.16), we deduce that

$$((1-\beta)^3(1-\alpha))^{1/8} - ((1-\alpha)^3(1-\beta))^{1/8} = (\alpha^3\beta)^{1/8} - (\beta^3\alpha)^{1/8}. \quad (2.4.17)$$

We rewrite the above identity as

$$(\alpha\beta^3)^{1/8} + \{(1-\alpha)(1-\beta)^3\}^{1/8} = (\alpha^3\beta)^{1/8} + \{(1-\alpha)^3(1-\beta)\}^{1/8}. \quad (2.4.18)$$

Now, dividing both sides of (2.4.12) by $(\alpha\beta)^{1/24}$, we obtain

$$\left(\frac{\alpha^5}{\beta}\right)^{1/24} - \left(\frac{\beta^5}{\alpha}\right)^{1/24} = 2^{2/3}((1-\alpha)(1-\beta))^{1/6}. \quad (2.4.19)$$

Multiplying both sides of the above identity by $2^{1/3}\left((1-\alpha)^5/(1-\beta)\right)^{1/24}$, we find that

$$2^{1/3}\left(\frac{(\alpha(1-\alpha))^5}{\beta(1-\beta)}\right)^{1/24} - 2^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/24} = 2((1-\alpha)^3(1-\beta))^{1/8}. \quad (2.4.20)$$

Again, multiplying both sides of (2.4.18) by $2^{1/3}\left((1-\beta)^5/(1-\alpha)\right)^{1/24}$, we arrive at

$$2^{1/3}\left(\frac{(\alpha(1-\beta))^5}{\beta(1-\alpha)}\right)^{1/24} - 2^{1/3}\left(\frac{(\beta(1-\beta))^5}{\alpha(1-\alpha)}\right)^{1/24} = 2\left((1-\beta)^3(1-\alpha)\right)^{1/8}. \quad (2.4.21)$$

The reciprocal of (2.4.21) is given by

$$2^{1/3}\left(\frac{(\alpha(1-\beta))^5}{\beta(1-\alpha)}\right)^{1/24} - 2^{1/3}\left(\frac{(\alpha(1-\alpha))^5}{\beta(1-\beta)}\right)^{1/24} = 2(\alpha^3\beta)^{1/8}. \quad (2.4.22)$$

Adding (2.4.20) and (2.4.22), we deduce that

$$2^{1/3}\left(\frac{(\alpha(1-\beta))^5}{\beta(1-\alpha)}\right)^{1/24} - 2^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/24} = 2\{(\alpha^3\beta)^{1/8} + ((1-\alpha)^3(1-\beta))^{1/8}\}. \quad (2.4.23)$$

Dividing both sides of the first part of (2.4.49) by $2^{-1/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/24}$, we obtain

$$2^{1/3}\left(\frac{(\alpha(1-\beta))^5}{\beta(1-\alpha)}\right)^{1/24} = 2 - 2^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/24}. \qquad (2.4.24)$$

Employing (2.4.24) in (2.4.23), we find that

$$(\alpha^3\beta)^{1/8} + \{(1-\alpha)^3(1-\beta)\}^{1/8} = 1 - 2^{1/3}\left(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right)^{1/24}. \qquad (2.4.25)$$

Now, multiplying both sides of (2.4.24) by $4^{1/3}\left((\beta(1-\alpha))^5/\alpha(1-\beta)\right)^{1/24}$, we obtain

$$4^{1/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/6}+4^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/12} = 4^{2/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/24}. \qquad (2.4.26)$$

The above identity can also be written as

$$\begin{aligned}1-(16\alpha\beta(1-\alpha)(1-\beta))^{1/6} &= 1 + 4^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/12} \\ &\quad - 2.2^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/24} \\ &= \left(1 - 2^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/24}\right)^2. \qquad (2.4.27)\end{aligned}$$

Now, we recast (2.4.1) as

$$1-(16\alpha\beta(1-\alpha)(1-\beta))^{1/6} = \left(\frac{1+(\alpha\beta)^{1/2}+((1-\alpha)(1-\beta))^{1/2}}{2}\right)^{1/2}. \qquad (2.4.28)$$

From (2.4.27) and (2 4.28), we arrive at

$$1-2^{1/3}\left(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right)^{1/24} = \left(\frac{1+(\alpha\beta)^{1/2}+((1-\alpha)(1-\beta))^{1/2}}{2}\right)^{1/2}. \qquad (2.4.29)$$

Thus, from (2.4.18), (2.4.25), and (2.4.29), we obtain (2.4.11) to finish the proof.

Theorem 2.4.8. *[13, p. 366, Entry 20] We have*

$$(\alpha\beta^3)^{1/8} + \{(1-\alpha)(1-\beta)^3\}^{1/8} = \sqrt{1-(\alpha\beta(1-\alpha)(1-\beta))^{1/6}}. \quad (2.4.30)$$

Proof: This identity follows readily from (2 4.11) and (2.4.28).

Theorem 2.4.9. *[11, p. 280, Entry 13(viii)] If a and b are arbitrary complex numbers, then*

$$m = \frac{a + 2^{4/3}(a-b)\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24} + 4^{1/3}b\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}}{a - b\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}} \quad (2.4.31)$$

and

$$m = \frac{1 - 2^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24} - 4^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}}{(1-3\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} + \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/3})^{1/2}}. \quad (2.4.32)$$

Proof: Subtracting (2.4.43) from (2.4.41), we find that

$$2^{4/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24} - 2^{2/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12} = \frac{m}{2}\{1-(\alpha\beta)^{1/2} - ((1-\alpha)(1-\beta))^{1/2}\}. \quad (2.4\ 33)$$

Now, from (2.4.1) we obtain

$$1-(\alpha\beta)^{1/2} - ((1-\alpha)(1-\beta))^{1/2} = 2(16\alpha\beta(1-\alpha)(1-\beta))^{1/6}. \quad (2.4.34)$$

Using (2.4.33) in (2.4.34), we deduce that

$$m = \frac{2^{4/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24} - 2^{2/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}}{\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}}. \quad (2.4.35)$$

Multiplying (2.4.4) by a, we obtain

$$ma = a\left(1 + 2^{4/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}\right). \quad (2.4.36)$$

Again, multiplying the numerator and denominator of the right side of (2.4.35) by b, we find that

$$m = \frac{b\{2^{4/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24} - 2^{2/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}\}}{b\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}}. \tag{2.4.37}$$

From (2.4.36) and (2.4.37), we readily deduce (2.4.31).

Now, from [27, Corollary 2.3, p. 95], we have

$$(\phi^2(q) - \phi^2(q^5))(5\phi^2(q^5) - \phi^2(q)) = 16qf^2(-q^2)f^2(-q^{10}). \tag{2.4.38}$$

Transcribing this equation by employing (2.2.11). (2.2.21) we find that

$$\frac{1}{4m}\{6m - m^2 - 5\} = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}. \tag{2.4.39}$$

Now,

$$\begin{aligned}(1 - 3\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} &+ \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/3}) \\ &= 1 - 3\frac{1}{4m}\{6m - m^2 - 5\} + \left(\frac{1}{4m}\{6m - m^2 - 5\}\right)^2 \\ &= \left(\frac{m^2-5}{4m}\right)^2.\end{aligned} \tag{2.4.40}$$

Again, from (2.4.4), we find that

$$\frac{m-1}{2} = 2^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}. \tag{2.4.41}$$

Thus,

$$1 - 2^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24} - 4^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12} = \frac{m^2-5}{4}. \tag{2.4.42}$$

Equating (2.4.40) and (2.4.42), we arrive at (2.4.32).

Theorem 2.4.10. *[11, p. 280, Entry 13(xi)] We have*

$$1 + 4^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12} = \frac{1}{2}m\left(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}\right) \tag{2.4.43}$$

and

$$1+4^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/12}=\frac{5}{2m}\left(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}\right). \tag{2.4.44}$$

Proof: We have (2.4.63)

$$m+\frac{5}{m}=2\left(2+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}\right). \tag{2.4.45}$$

$$m^2+5=2m\left(2+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}\right). \tag{2.4.46}$$

Again, from (2 4 41), we obtain

$$m-1=2^{4/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}. \tag{2.4.47}$$

Squaring both side in the above equation, we find that

$$m^2+1=2m+2^{8/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}. \tag{2 4.48}$$

Using (2.4.48), in (2.4.46), we obtain (2.4.43).

Taking reciprocal of (2.4.43), we arrive at (2.4 44).

Theorem 2.4.11. *[11, p. 280, Entry 13(x)] We have*

$$\begin{aligned}\{\alpha(1-\beta)\}^{1/4}+\{\beta(1-\alpha)\}^{1/4}&=4^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}\\&=m\{\alpha(1-\alpha)\}^{1/4}+\{\beta(1-\beta)\}^{1/4}\\&=\{\alpha(1-\alpha)\}^{1/4}+\frac{5}{m}\{\beta(1-\beta)\}^{1/4}.\end{aligned} \tag{2.4.49}$$

Proof: We rewrite (2.4.60) as

$$m(\alpha(1-\alpha))^{1/4}+(\beta(1-\beta))^{1/4}=(\beta(1-\alpha))^{1/4}+(\alpha(1-\beta))^{1/4}. \tag{2.4.50}$$

Again, from (2.4.61), we obtain

$$\frac{5}{m}(\beta(1-\beta))^{1/4}+(\alpha(1-\alpha))^{1/4}=(\alpha(1-\beta))^{1/4}+(\beta(1-\alpha))^{1/4}. \tag{2.4.51}$$

Now, replacing q by $-q$, in (2.3.16), we find that

$$\psi^2(-q) + q\psi^2(-q^5) = \frac{\phi(q^5)f(q^5)}{\chi(q)}. \tag{2.4.52}$$

Transcribing this with (2.2.11), (2.2.17), (2.2.22), and (2.2.25), we deduce that

$$4^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = m\{\alpha(1-\alpha)\}^{1/4} + \{\beta(1-\beta)\}^{1/4}. \tag{2.4.53}$$

Equating (2.4.50), (2.4.51), and (2.4.53), we obtain (2.4.49).

Theorem 2.4.12. *[11, p. 280, Entry 13(xi)] We have*

$$\left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8} + \left(\frac{\beta^5}{\alpha}\right)^{1/8} = m\left(\frac{1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right)^{1/2} \tag{2.4 54}$$

and

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} + \left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8} = \frac{5}{m}\left(\frac{1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right)^{1/2}. \tag{2.4.55}$$

Proof: We have (2.4.6)

$$m = \frac{1+((1-\beta)^5/(1-\alpha))^{1/8}}{1+\{(1-\alpha)^3(1-\beta)\}^{1/8}} = \frac{1-(\beta^5/\alpha)^{1/8}}{1-(\alpha^3\beta)^{1/8}}. \tag{2.4.56}$$

Thus,

$$m\{1+((1-\alpha)^3(1-\beta))^{1/8}\} = 1 + \left(\frac{(1-\beta)^5}{(1-\alpha)}\right)^{1/8} \tag{2.4.57}$$

and

$$m\{1-(\alpha^3\beta)^{1/8}\} = 1 - \left(\frac{\beta^5}{\alpha}\right)^{1/8}. \tag{2.4.58}$$

Subtracting (2.4.58) from (2.4.57), we find that

$$\left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8} + \left(\frac{\beta^5}{\alpha}\right)^{1/8} = m\{(\alpha^3\beta)^{1/8} + \{(1-\alpha)^3(1-\beta)\}^{1/8}\}. \tag{2.4.59}$$

Using (2.4.11) in (2.4.59), we deduce (2.4.54).

The modular equation (2.4.55) is the reciprocal of (2.4.54).

Theorem 2.4.13. *[11, p. 280, Entry 13(xii)] We have*

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}. \qquad (2.4.60)$$

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}. \qquad (2.4.61)$$

Proof: Transcribing (2.3.8), by employing (2.2 13), (2.2.16), and (2.2.17), we easily deduce (2.4.60).

The identity (2.4.61) is the reciprocal of (2.4.60).

Ramanujan recorded this two modular equation in a fragment published with the lost notebook [29, p. 351].

Theorem 2.4.14. *[11, p. 280, Entry 13(xiii)] We have*

$$m - \frac{5}{m} = \frac{4((\alpha\beta)^{1/2} - \{(1-\alpha)(1-\beta)\}^{1/2})}{((1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})/2)^{1/2}} \qquad (2.4.62)$$

and

$$m + \frac{5}{m} = 2\left(2 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}\right). \qquad (2.4.63)$$

Proof: Cubing both sides of the equation (2.4.19), we obtain

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} - \left(\frac{\beta^5}{\alpha}\right)^{1/8} - 3.2^{2/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/6} = 4((1-\alpha)(1-\beta))^{1/2}. \qquad (2.4.64)$$

Taking reciprocal of this equation, we find that

$$\left(\frac{(1-\beta)^5}{(1-\alpha)}\right)^{1/8} - \left(\frac{(1-\alpha)^5}{(1-\beta)}\right)^{1/8} - 3.2^{2/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/6} = 4(\alpha\beta)^{1/2}. \qquad (2.4.65)$$

Subtracting (2.4.65) from (2.4.64), we obtain

$$\left(\frac{(1-\beta)^5}{(1-\alpha)}\right)^{1/8} + \left(\frac{\beta^5}{\alpha}\right)^{1/8} - \left(\left(\frac{(1-\alpha)^5}{(1-\beta)}\right)^{1/8} + \left(\frac{\alpha^5}{\beta}\right)^{1/8}\right)$$
$$= 4\{(\alpha\beta)^{1/2} - ((1-\alpha)(1-\beta))^{1/2}\}. \qquad (2.4.66)$$

Subtracting (2.4.55) from (2.4.54), we find that

$$\left(\frac{(1-\beta)^5}{(1-\alpha)}\right)^{1/8}+\left(\frac{\beta^5}{\alpha}\right)^{1/8}-\left(\left(\frac{(1-\alpha)^5}{(1-\beta)}\right)^{1/8}+\left(\frac{\alpha^5}{\beta}\right)^{1/8}\right)$$
$$=\left(m-\frac{5}{m}\right)\left(\frac{1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right)^{1/2}. \qquad (2.4.67)$$

Equating (2.4.66) and (2.4.67), we find that

$$m-\frac{5}{m}=\frac{4((\alpha\beta)^{1/2}-\{(1-\alpha)(1-\beta)\}^{1/2})}{((1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2})/2)^{1/2}}. \qquad (2.4.68)$$

Again, transcribing (2.4.38) by employing (2.2.11) and (2.2.21), we obtain

$$m+\frac{5}{m}=2\{2(1-\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6})\}+2. \qquad (2.4.69)$$

Employing (2.4.1) in the above identity, we deduce (2.4.63).

Theorem 2.4.15. *[11, p. 280, Entry 13(xiv)] If*

$$P=\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \quad \textit{and} \quad Q=(\beta(1-\beta)/\alpha(1-\alpha))^{1/8}, \qquad (2.4.70)$$

then

$$Q+\frac{1}{Q}+2\left(P-\frac{1}{P}\right)=0. \qquad (2.4.71)$$

Proof: We note that

$$PQ=2^{1/3}\left(\frac{(\beta(1-\beta))^5}{\alpha(1-\alpha)}\right)^{1/24}, \qquad (2.4.72)$$

$$\frac{P}{Q}=2^{1/3}\left(\frac{(\alpha(1-\alpha))^5}{\beta(1-\beta)}\right)^{1/24}. \qquad (2.4.73)$$

Now, Adding (2.4.73) and (2.4.72), and using (2.4.4) and (2.4.5), we find that

$$PQ+\frac{P}{Q}=\frac{m-1}{2}+\frac{5/m-1}{2}. \qquad (2.4.74)$$

$$PQ + \frac{P}{Q} = \frac{m + 5/m - 2}{2}. \tag{2.4.75}$$

Using (2.4.63), in (2.4 75), we obtain

$$PQ + \frac{P}{Q} = \frac{-2 + 2((\alpha\beta)^{1/2} + 2 + ((1-\alpha)(1-\beta))^{1/2})}{2}. \tag{2 4 76}$$

Now, using (2.4.1) in (2.4.76) and simplifying, we obtain that

$$Q + \frac{1}{Q} + 2\left(P - \frac{1}{P}\right) = 0. \tag{2.4 77}$$

Theorem 2.4.16. *[11, p. 280, Entry 13(xv)] If*

$$P = (\alpha\beta)^{1/4} \quad \textit{and} \quad Q = (\beta/\alpha)^{1/8}. \tag{2.4.78}$$

then

$$\left(Q - \frac{1}{Q}\right)^3 + 8\left(Q - \frac{1}{Q}\right) = 4\left(P - \frac{1}{P}\right). \tag{2.4.79}$$

Proof: We note that

$$PQ = (\beta^3\alpha)^{1/8}. \tag{2.4.80}$$

$$\frac{P}{Q} = (\alpha^3\beta)^{1/8}. \tag{2.4.81}$$

Subtracting (2.4.81), from (2.4.80), we find that

$$P\left(Q - \frac{1}{Q}\right) = (\beta^3\alpha)^{1/8} - (\alpha^3\beta)^{1/8}. \tag{2.4 82}$$

Using (2 4 16), in (2.4.82), we obtain

$$-P\left(Q - \frac{1}{Q}\right) = 2^{2/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/6}. \tag{2.4.83}$$

Using (2 4.78) in (2.4.83), we obtain

$$Q - \frac{1}{Q} = -2^{2/3}\frac{((1-\alpha)(1-\beta))^{1/6}}{(\alpha\beta)^{1/12}}. \tag{2.4.84}$$

Cubbing both side in the above equation, we find that

$$\left(Q - \frac{1}{Q}\right)^3 = -4\frac{((1-\alpha)(1-\beta))^{1/2}}{(\alpha\beta)^{1/4}}. \tag{2.4.85}$$

From (2.4.84) and (2.4.85) we obtain,

$$\left(Q-\frac{1}{Q}\right)^3+8\left(Q-\frac{1}{Q}\right)$$
$$=-\frac{4}{(\alpha\beta)^{1/4}}\{((1-\alpha)(1-\beta))^{1/2}+2^{5/3}((1-\alpha)(1-\beta)\alpha\beta)^{1/6}\}. \tag{2.4.86}$$

Now, using (2.4.1), in (2.4.86), we find that

$$\left(Q-\frac{1}{Q}\right)^3+8\left(Q-\frac{1}{Q}\right)=-\frac{4}{(\alpha\beta)^{1/4}}\{1-(\alpha\beta)^{1/2}\}. \tag{2.4.87}$$

Using (2.4.78) in (2.4.87), we obtain (2.4.71).

Chapter 3

New Proofs of Ramanujan's Modular Equations of degree 9

3.1 Introduction

In this chapter, we find proofs of Ramanujan's modular equations of degree 9 by using theta function identities. Ramanujan recorded 14 modular equations of degrees 1, 3, 9 in Chapter 20 of his second notebook [28]. He also recorded two more equations on pages 286 and 296 of his first notebook [28], but the second equation is incorrect as shown by Berndt [13, p. 370, Entry 28]. All of Ramanujan's modular equations of degrees 1, 3, 9 have been proved by Berndt (See [11, pp. 352-358, Entry 3] and [13, p. 370, Entry 27]). As we have already mentioned in Chapter 1, modular equations can be expressed as identities involving the theta-functions ϕ, ψ and f. Therefore, often one first tries to derive a theta-function identity and then transcribes it into an equivalent modular equation. But, proofs of some modular equations of composite degree 9 given by Berndt are quite unlike this method. He sometimes reversed the process. In this chapter 3, we find new proofs of these modular equations by using theta-function identities. In the meantime, we also find new proofs of some of the theta-function identities. Earlier these identities were proved by Berndt by using modular equations and a method of parameterizations.

In Section 3.2, we state some preliminary results.

In Section 3.3, we state the theta-function identities and present new proofs of some of the identities.

In the final section, we prove the modular equations by using results from the previous two sections.

3.2 Preliminary Results

In this section, we state some results which will be used to derive our theta-function identities.

Lemma 3.2.1. *[11, p. 51, Example (v)] We have*

$$f(q, q^5) = \psi(-q^3)\chi(q). \tag{3.2.1}$$

Lemma 3.2.2. *[11, p. 350, (2.3)] We have*

$$f(q, q^2) = \frac{\phi(-q^3)}{\chi(-q)}. \tag{3.2.2}$$

Lemma 3.2.3. *[11, p. 49, Entry 31 {Corollary (i) and (ii)}] We have*

$$\phi(q) = \phi(q^9) + 2qf(q^3, q^{15}), \tag{3.2.3}$$

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{3.2.4}$$

3.3 Theta-function identities

In this section, we state and prove some theta-function identities recorded by Ramanujan. We think that the proofs presented here are more transparent than those found by Berndt [11]-[13]. We also mention that the identities in [12] were likely unknown to the author when [11] was written.

Theorem 3.3.1. *[11, p. 345, Entry 1(i)]*

$$1 + \frac{\chi^3(-q^3)}{q^{1/3}\chi(-q)} = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}, \tag{3.3.1}$$

and

$$1 + \frac{\chi^9(-q^3)}{q\chi^3(-q)} = \frac{\psi^4(q)}{q\psi^4(q^3)}. \tag{3.3.2}$$

Here we present a proof of (3.3.1) slightly different from Berndt [11, p. 345].

Proof: By (3.2.4) and (3.2.2), we find that

$$\psi(q) - q\psi(q^9) = \frac{\phi(-q^9)}{\chi(-q^3)}. \tag{3.3.3}$$

Dividing both sides by $q\psi(q^9)$ and then using (2.2.3), we obtain

$$\frac{\psi(q)}{q\psi(q^9)} - 1 = \frac{\chi^3(-q^9)}{q\chi(-q^3)}. \tag{3.3.4}$$

Replacing q by $q^{1/3}$, we easily arrive at (3.3.1).

For a proof of (3.3.2) see [11, p. 346, Entry 1(i)].

Theorem 3.3.2. *[11, p. 345, Entry 1(ii)]*

$$1 + \frac{\psi(-q^{1/3})}{q^{1/3}\psi(-q^3)} = \left(1 + \frac{\psi^4(-q)}{q\psi^4(-q^3)}\right)^{1/3}. \tag{3.3.5}$$

Berndt [11, p. 347] proved that this theorem follows from (3.3.1) and (3.3.2).

Theorem 3.3.3. *[11, p. 345, Entry 1(ii)]*

$$\frac{\phi(q^{1/3})}{\phi(q^3)} = 1 + \left(\frac{\phi^4(q)}{\phi^4(q^3)} - 1\right)^{1/3}. \tag{3.3.6}$$

Berndt [11, p. 218 and p. 347] offered two proofs for this theorem. Here we offer an alternative proof.

Proof: Berndt [12, Entry 8, pp. 144-146] proved the following beautiful theta-function identity due to Ramanujan.

$$\{f(a,b) - f(a^6b^3, a^3b^6)\}^3 = \frac{f(a^3,b^3)}{f(a^6b^3,a^3b^6)} f^3(a^2b, ab^2) - f^3(a^6b^3, a^3b^6). \tag{3.3.7}$$

Putting $a = b = q$ in (3.3.7), we find that

$$\{\phi(q) - \phi(q^9)\}^3 = \frac{\phi(q^3)}{\phi(q^9)}\phi^3(q^3) - \phi^3(q^9). \tag{3.3.8}$$

Simplifying (3.3.8), we obtain

$$\frac{\phi^3(q)}{\phi^3(q^9)} - 3\frac{\phi^2(q)}{\phi^2(q^9)} + 3\frac{\phi(q)}{\phi(q^9)} = \frac{\phi^4(q^3)}{\phi^4(q^9)}. \tag{3.3.9}$$

Replacing q by $q^{1/3}$ in (3.3.9), we can easily arrive at (3.3.6).

Theorem 3.3.4. *[11, p. 349, Entry 2(i)]*

$$\phi(q)\phi(q^9) - \phi^2(q^3) = 2q\phi(-q^2)\psi(q^9)\chi(q^3). \tag{3.3.10}$$

Proof: At first, we prove the following lemma.

Lemma 3.3.5.

$$\phi^2(q) - \phi^2(q^3) = 4q\chi(q)\chi(-q^2)\psi(q^3)\psi(q^6). \tag{3.3.11}$$

Transcribing by using (2.2.11), (2.2.16), (2.2.18), (2.2.25), and (2.2.27) it can be seen that the above theta-function identity is equivalent to a modular equation of degree 3 [11, Entry 5(iii), first equation, p. 230]. Here we present a more direct proof of this theta-function identity.

Proof: Putting $a = q^2$, $b = q^4$, $c = q$, and $d = q^5$ in (2.2.9), and then using (3.2.1), (3.2.2), (1.1.9), (1.1.10), and (2.2.4), we find that

$$\frac{\phi(-q^6)\psi(-q^3)}{\chi(-q)} + f(-q^2)\chi(-q)\psi(q^3) = 2\psi(q^3)f(q^5,\ q^7). \tag{3.3.12}$$

Using (2.2.6) and (2.2.3), we can rewrite the above identity as

$$\frac{\psi^2(-q^3)}{\psi(q^6)\chi(-q)} + f(-q) = 2f(q^5, q^7). \tag{3.3.13}$$

Replacing q by $-q$ in (3.3.13) and then employing (2.2.7), we obtain

$$\frac{\phi(q^3)}{\chi(q)} + f(q) = 2f(-q^5, -q^7). \tag{3.3.14}$$

Similarly, putting $a = q^2$, $b = q^4$, $c = q$, and $d = q^5$ in (2 2.10), and then proceeding as above, we find that

$$-\frac{\phi(q^3)}{\chi(q)} + f(q) = 2qf(-q, -q^{11}). \tag{3.3.15}$$

Multiplying (3.3.14) and (3.3.15), and then using (2.2.3), we obtain

$$\phi^2(q) - \phi^2(q^3) = 4q\chi^2(q)f(-q^5, -q^7)f(-q, -q^{11}). \tag{3.3.16}$$

Now, using the Jacobi triple product identity (1 1.6), we find that

$$\begin{aligned} &f(-q^5, -q^7)f(-q, -q^{11}) \\ &= (q;\, q^{12})_\infty (q^5;\, q^{12})_\infty (q^7;\, q^{12})_\infty (q^{11};\, q^{12})_\infty (q^{12};\, q^{12})_\infty^2 \\ &= \frac{(q;\, q^2)_\infty (q^{12};\, q^{12})_\infty^3}{(q^3;\, q^{12})_\infty (q^9;\, q^{12})_\infty (q^{12};\, q^{12})_\infty}. \end{aligned} \tag{3.3.17}$$

Using (1.1.11), (1.1.9), and then (1.1.10), we obtain

$$f(-q^5, -q^7)f(-q, -q^{11}) = \frac{\chi(-q)f^3(-q^{12})}{\psi(-q^3)}. \tag{3.3.18}$$

Now, from (2.2.3), we note that

$$f(-q^2) = \chi(-q)\psi(q) = \sqrt[3]{\phi(-q)\psi^2(q)}. \tag{3.3.19}$$

Replacing q by q^6 in (3.3.19), we deuce that

$$f^3(-q^{12}) = \phi(-q^6)\psi^2(q^6). \tag{3.3.20}$$

Using (2 2.6), we find that

$$f^3(-q^{12}) = \psi(q^3)\psi(-q^3)\psi(q^6). \tag{3.3.21}$$

Thus, (3.3.18) can be written as

$$f(-q^5, -q^7)f(-q, -q^{11}) = \chi(-q)\psi(q^3)\psi(-q^3)\psi(q^6). \tag{3.3.22}$$

Employing (3.3.22) in (3.3.16), and then using (2.2.4), we arrive at (3.3.11), which completes the proof of the lemma.

Proof: From (3.2.3) and (3.2.1), we find that

$$\phi(q) = \phi(q^9) + 2q\psi(-q^9)\chi(q^3). \tag{3.3.23}$$

Multiplying both sides of (3.3.23) by $\phi(q^9)$, we obtain

$$\phi(q)\phi(q^9) = \phi^2(q^9) + 2q\phi(q^9)\psi(-q^9)\chi(q^3). \tag{3.3.24}$$

Now, by (2.2.6) and (2.2.7), we deduce that

$$\phi(q)\psi(-q) = \psi(q)\phi(-q^2). \tag{3.3.25}$$

Replacing q by q^9 in (3.3.25) and then using (3.3 24), we find that

$$\phi(q)\phi(q^9) = \phi^2(q^9) + 2q\psi(q^9)\phi(-q^{18})\chi(q^3). \tag{3.3.26}$$

Using (3.3.23) in (3.3.26), we obtain

$$\phi(q)\phi(q^9) = \phi^2(q^9) + 2q\psi(q^9)\chi(q^3)\{\phi(-q^2) + 2q^2\psi(q^{18})\chi(-q^6)\}. \tag{3.3.27}$$

Employing (3.3.11) in (3.3.27) we easily arrive at Theorem 3.3.1 to complete the proof. Berndt et al. [16, lemma 9.1, p. 20] have also given a proof of Theorem (3.3.1). They used this modular equation to prove some results on Ramanujan's famous forty identities for the Rogers-Ramanujan functions.

Theorem 3.3.6. *[11, p. 349, Entry 2(ii)]*

$$\psi(q) - 3q\psi(q^9) = \frac{\phi(-q)}{\chi(-q^3)}. \tag{3.3.28}$$

Proof: Replacing q by $-q$ in (3.3.23) and then using the resulting identity in (3.3.3), we easily deduce (3.3.28).

Theorem 3.3.7. *[13, p. 357, Entry 4]*

$$\{3\phi(-q^9) - \phi(-q)\}^3 = 8\psi^3(q)\frac{\phi(-q^3)}{\psi(q^3)}. \tag{3.3.29}$$

Proof: Replacing q by $-q$ in (3.3.23), we find that

$$\phi(-q^9) - \phi(-q) = 2q\psi(q^9)\chi(-q^3). \tag{3.3.30}$$

Using (2.2.3), this can be written as

$$\phi(-q^9) - \phi(-q) = 2q\psi(q^9)\sqrt[3]{\frac{\phi(-q^3)}{\psi(q^3)}}. \tag{3.3.31}$$

Now, using (2.2.3) in (3.3.3), we deduce that

$$\psi(q) - q\psi(q^9) = \phi(-q^9)\sqrt[3]{\frac{\psi(q^3)}{\phi(-q^3)}}. \tag{3.3.32}$$

Using (3.3.32) in (3.3.31), we find that

$$3\phi(-q^9) - \phi(-q) = 2\psi(q)\sqrt[3]{\frac{\phi(-q^3)}{\psi(q^3)}}. \tag{3.3.33}$$

So, we complete the proof by cubing (3.3.33).

Theorem 3.3.8. *[11, p. 349, Entry 2(iii)]*

$$\phi(q)\phi(q^9) + \phi^2(q^3) = 2\psi(q)\phi(-q^{18})\chi(q^3). \tag{3.3.34}$$

Proof: Replacing q by q^3 in (3.3.6) and then simplifying, we obtain

$$1 + 3\frac{\phi^2(q)\phi^2(q^9)}{\phi^4(q^3)} = \frac{\phi(q)\phi(q^9)}{\phi^4(q^3)}\left(\phi^2(q) + 3\phi^2(q^9)\right). \tag{3.3.35}$$

Now, (3.3.23) can be rewritten as

$$\phi(q) - \phi(q^9) = 2q\psi(-q^9)\chi(q^3). \tag{3.3.36}$$

Again, replacing q by $-q$ in (3.3.33) and then using (2.2.3), we deduce that

$$3\phi(q^9) - \phi(q) = 2\psi(-q)\chi(q^3). \tag{3.3.37}$$

Multiplying (3.3.36) and (3.3.37), we obtain

$$3\phi^2(q^9) + \phi^2(q) = 4\phi(q)\phi(q^9) - 4q\psi(-q)\psi(-q^9)\chi^2(q^3). \tag{3.3.38}$$

Using (3.3.38) in (3.3.35), we find that

$$1 - \frac{\phi^2(q)\phi^2(q^9)}{\phi^4(q^3)} = -4q\frac{\phi(q)\phi(q^9)\psi(-q)\psi(-q^9)\chi^2(q^3)}{\phi^4(q^3)}. \tag{3.3.39}$$

With the aid of (3.3.10) the above identity can be written as

$$\phi(q)\phi(q^9) + \phi^2(q^3) = 2\chi(q^3)\frac{\phi(q)\psi(-q)\phi(q^9)\psi(-q^9)}{\phi(-q^2)\psi(q^9)}. \tag{3.3.40}$$

We complete the proof by employing (3.3.25) in (3.3.40).

Theorem 3.3.9. *[11, p. 358, Entry 4(i)]*

$$\frac{\phi(-q^{18})}{\phi(-q^2)} + q\left(\frac{\psi(q^9)}{\psi(q)} - \frac{\psi(-q^9)}{\psi(-q)}\right) = 1. \tag{3.3.41}$$

Berndt [11, p. 359] proved this by using the modular equation (3.4.12). Here we give an alternative proof.
Proof: Replacing q by $-q$ in (3.3.3), we obtain

$$\psi(-q) + q\psi(-q^9) = \frac{\phi(q^9)}{\chi(q^3)}. \tag{3.3.42}$$

From (3.3.3) and (3.3 42), we obtain

$$q\left\{\frac{\psi(q^9)}{\psi(q)} - \frac{\psi(-q^9)}{\psi(-q)}\right\} = 2 - \frac{\phi(-q^9)}{\chi(-q^3)\psi(q)} - \frac{\phi(q^9)}{\chi(q^3)\psi(-q)}. \tag{3.3.43}$$

Now, adding (3.3.10) and (3.3.34), we obtain

$$\phi(q)\phi(q^9) = \psi(q)\phi(-q^{18})\chi(q^3) + q\phi(-q^2)\psi(q^9)\chi(q^3). \tag{3.3.44}$$

Using (2.2.5), we obtain

$$\sqrt{\frac{\phi(q)}{\phi(-q)}}\phi(q^9) = \psi(q)\frac{\phi(-q^{18})}{\phi(-q^2)}\chi(q^3) + q\psi(q^9)\chi(q^3). \tag{3.3.45}$$

Employing (2.2.1) in (3.3.45), we find that

$$\phi(q^9) = \frac{\chi(q^3)\phi(-q^{18})\psi(-q)}{\phi(-q^2)} + q\frac{\psi(q^9)\chi(q^3)\psi(-q)}{\psi(q)}. \tag{3.3.46}$$

Replacing q by $-q$ in (3 3.46), we obtain

$$\phi(-q^9) = \frac{\chi(-q^3)\phi(-q^{18})\psi(q)}{\phi(-q^2)} - q\frac{\psi(-q^9)\chi(-q^3)\psi(q)}{\psi(-q)}. \tag{3.3.47}$$

Using (3.3.46) and (3.3.47) in (3.3.43), we deduce (3.3.41) to complete the proof.

Lemma 3.3.10. *[11, p. 358, Entry 4(ii)]*

$$\frac{\phi(-q^2)}{\phi(-q^{18})} + \frac{1}{q}\left(\frac{\psi(q)}{\psi(q^9)} - \frac{\psi(-q)}{\psi(-q^9)}\right) = 3. \tag{3.3.48}$$

This identity was proved by Berndt [11, p. 359] by using the modular equation (3.4.13). Here, we present an alternative proof.

Proof: Replacing q by $-q$, in (3.3.28), we obtain

$$\psi(-q)+3q\psi(-q^9)=\frac{\phi(q)}{\chi(q^3)}. \tag{3.3.49}$$

From (3.3.28) and (3.3.49), we find that

$$\frac{\psi(q)}{q\psi(q^9)}-\frac{\psi(-q)}{q\psi(-q^9)}=6+\frac{\phi(-q)}{q\chi(-q^3)\psi(q^9)}-\frac{\phi(q)}{q\chi(q^3)\psi(-q^9)}. \tag{3.3.50}$$

Now, from (3.3.44), we deduce that

$$\frac{\phi(q)\phi(q^9)}{\phi(-q^{18})}=\psi(q)\chi(q^3)+q\frac{\phi(-q^2)\psi(q^9)\chi(q^3)}{\phi(-q^{18})}. \tag{3.3.51}$$

Using (2.2.5) and (2.2.1) in (3.3.51), we find that

$$\frac{\phi(q)}{q\chi(q^3)\psi(-q^9)}=\frac{\psi(q)}{q\psi(q^9)}+\frac{\phi(-q^2)}{\phi(-q^{18})}. \tag{3.3.52}$$

Replacing q by $-q$, we obtain

$$\frac{\phi(-q)}{q\chi(-q^3)\psi(q^9)}=\frac{\psi(-q)}{q\psi(-q^9)}-\frac{\phi(-q^2)}{\phi(-q^{18})}. \tag{3.3.53}$$

Using (3.3.52) and (3.3.53) in (3.3.50), we obtain (3.3.48). Thus, we complete the proof.

The theta-function identities in the following theorem were recorded by Ramanujan in the unorganized portions of his second notebook [28, p. 310]. Berndt [12, p. 185] proved this theorem by using parameterizations. Here we give alternative proofs by using other simple theta-function identities of Ramanujan.

Theorem 3.3.11. *[12, p. 185, Entry 33] For* $|q|<1$

(i)
$$\frac{\phi^3(q^{1/3})}{\phi(q)}=\frac{\phi^3(q)}{\phi(q^3)}+6q^{1/3}\frac{f^3(q^3)}{f(q)}+12q^{2/3}\frac{f^3(-q^6)}{f(-q^2)}, \tag{3.3.54}$$

(ii)
$$\frac{\psi^3(q^{1/3})}{\psi(q)}=\frac{\psi^3(q)}{\psi(q^3)}+3q^{1/3}\frac{f^3(-q^3)}{f(-q)}+3q^{2/3}\frac{f^3(-q^6)}{f(-q^2)}. \tag{3.3.55}$$

Proof of (i): Replacing q by $q^{1/3}$ in (3.3 9), we find that

$$\frac{\phi^3(q^{1/3})}{\phi^3(q^3)} - 3\frac{\phi^2(q^{1/3})}{\phi^2(q^3)} + 3\frac{\phi(q^{1/3})}{\phi(q^3)} = \frac{\phi^4(q)}{\phi^4(q^3)}. \tag{3.3.56}$$

Multiplying both sides of (3.3 56) by $\phi^3(q^3)/\phi(q)$, we obtain

$$\frac{\phi^3(q^{1/3})}{\phi(q)} = \frac{\phi^3(q)}{\phi(q^3)} + 3\frac{\phi(q^{1/3})\phi(q^3)}{\phi(q)}\left(\phi(q^{1/3}) - \phi(q^3)\right). \tag{3.3.57}$$

Now, replacing q by $q^{1/3}$ in (3.3.23), we obtain

$$\phi(q^{1/3}) = \phi(q^3) + 2q^{1/3}\chi(q)\psi(-q^3). \tag{3.3.58}$$

Employing (3.3.58), we deduce from (3 3.57) that

$$\frac{\phi^3(q^{1/3})}{\phi(q)} = \frac{\phi^3(q)}{\phi(q^3)} + 6q^{1/3}\frac{\phi^2(q^3)\chi(q)\psi(-q^3)}{\phi(q)} + 12q^{2/3}\frac{\chi^2(q)\psi^2(-q^3)\phi(q^3)}{\phi(q)}. \tag{3 3.59}$$

From (2.2.3), we note that

$$\frac{\chi(q)}{\phi(q)} = \frac{1}{f(q)} \quad \text{and} \quad \frac{\chi(q)}{f(q)} = \frac{1}{f(-q^2)}. \tag{3.3.60}$$

Using (3.3.60), (2.2.2) and (2.2.4) in (3.3.59), we arrive at

$$\frac{\phi^3(q^{1/3})}{\phi(q)} = \frac{\phi^3(q)}{\phi(q^3)} + 6q^{1/3}\frac{f^3(q^3)}{f(q)} + 12q^{2/3}\frac{f^3(-q^6)}{f(-q^2)},$$

which completes the proof of (3.3.54).

Proof of (ii): From (3 3.5), we obtain

$$\frac{\psi^3(-q^{1/3})}{q\psi^3(-q^3)} + 3\frac{\psi^2(-q^{1/3})}{q^{2/3}\psi^2(-q^3)} + 3\frac{\psi(-q^{1/3})}{q^{1/3}\psi(-q^3)} = \frac{\psi^4(-q)}{q\psi^4(-q^3)}. \tag{3.3.61}$$

Multiplying both sides of (3.3.61) by $q\psi^3(-q^3)/\psi(-q)$, we deduce that

$$\frac{\psi^3(-q^{1/3})}{\psi(-q)} = \frac{\psi^3(-q)}{\psi(-q^3)} - 3q^{1/3}\frac{\psi(-q^{1/3})\psi(-q^3)}{\psi(-q)}\left(\psi(-q^{1/3}) + q^{1/3}\psi(-q^3)\right). \tag{3.3.62}$$

Replacing q by $-q$, in (3.3.62) we find that

$$\frac{\psi^3(q^{1/3})}{\psi(q)} = \frac{\psi^3(q)}{\psi(q^3)} + 3q^{1/3}\frac{\psi(q^{1/3})\psi(q^3)}{\psi(q)}\left(\psi(q^{1/3}) - q^{1/3}\psi(q^3)\right). \qquad (3.3.63)$$

Now, by (3.2.4) and (3.2.2), we obtain

$$\psi(q) = \frac{\phi(-q^9)}{\chi(-q^3)} + q\psi(q^9). \qquad (3.3.64)$$

Replacing q by $q^{1/3}$, we rewrite (3.3.64) as

$$\psi(q^{1/3}) = \frac{\phi(-q^3)}{\chi(-q)} + q^{1/3}\psi(q^3). \qquad (3.3.65)$$

Employing (3.3.65), we obtain from (3.3.63) that

$$\frac{\psi^3(q^{1/3})}{\psi(q)} = \frac{\psi^3(q)}{\psi(q^3)} + 3q^{1/3}\frac{\psi(q^3)\phi^2(-q^3)}{\psi(q)\chi^2(-q)} + 3q^{2/3}\frac{\psi^2(q^3)\phi(-q^3)}{\psi(q)\chi(-q)}. \qquad (3.3.66)$$

From (2.2.3), we now note that

$$\psi(q)\chi(-q) = f(-q^2) \quad \text{and} \quad \chi(-q)f(-q^2) = f(-q). \qquad (3.3.67)$$

Using (3.3.67), (2.2.2) and (2.2.4), we conclude that

$$\frac{\psi^3(q^{1/3})}{\psi(q)} = \frac{\psi^3(q)}{\psi(q^3)} + 3q^{1/3}\frac{f^3(-q^3)}{f(-q)} + 3q^{2/3}\frac{f^3(-q^6)}{f(-q^2)},$$

which is (3.3.55)

3.4 Modular Equations

In this section, we find, except for two modular equations, new proofs of Ramanujan's modular equations of composite degree 9. Throughout this section, suppose β and γ are of the third and ninth degrees, respectively, with respect to α and $m = z_1/z_3$ and $m' = z_3/z_9$ are the corresponding multipliers.

Theorem 3.4.1. *[11, p. 352, Entry 3(i)] We have*

$$1+4^{1/3}\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/24}=\frac{3}{\sqrt{mm'}}. \tag{3.4.1}$$

Proof: From (3.3.29), we obtain

$$\{3\phi(-q^9)-\phi(-q)\}^3=8\psi^3(q)\frac{\phi(-q^3)}{\psi(q^3)}. \tag{3.4.2}$$

Replacing q by $-q$, we obtain,

$$3\phi(q^9)-\phi(q)=2\psi(-q)\left\{\frac{\phi(q^3)}{\psi(-q^3)}\right\}^{1/3}. \tag{3.4.3}$$

Transcribing (3.4.3) by using (2.2.11) and (2.2.17), we readily obtain (3.4.1).

Theorem 3.4.2. *[11, p. 352, Entry 3(ii)]*

$$1+4^{1/3}\left(\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}\right)^{1/24}=\sqrt{mm'}. \tag{3.4.4}$$

Proof: By using (3.2.3) and (3.2.1), we find that

$$\phi(q)=\phi(q^9)+2q\chi(q^3)\psi(-q^9). \tag{3.4.5}$$

Transcribing this by employing (2.2.11), (2.2.17), and (2.2.25), we easily deduce (3.4.4).

Theorem 3.4.3. *[11, p. 352, Entry 3(iii)] We have*

$$1-2^{4/3}\left(\frac{\alpha^3\gamma^3(1-\alpha)^3(1-\gamma)^3}{\beta^2(1-\beta)^2}\right)^{1/24}=\frac{m'}{m}. \tag{3.4.6}$$

Proof: Multiplying (3.3.10) and (3.3.34), and then transcribing the resulting identity by using (2.2.11), (2.2.13), (2.2.16), and (2.2.25), we easily arrive at (3.4.6).

Theorem 3.4.4. *[11, p. 352, Entry 3(iv)] We have*

$$1-4^{1/3}\left(\frac{\gamma^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/24}=\sqrt{\frac{m}{m'}}=4^{1/3}\left(\frac{\alpha^3(1-\gamma)^3}{\beta(1-\beta)}\right)^{1/24}-1. \tag{3.4.7}$$

For a proof, see [11, p. 355].

Theorem 3.4.5. *[11, p. 352, Entry 3(v)] We have*

$$(\alpha\gamma)^{1/2}+\{(1-\alpha)(1-\gamma)\}^{1/2}$$
$$+2\{4\beta(1-\beta)\}^{1/3}=1+8\{\beta(1-\beta)\}^{1/4}\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{1/8}. \quad (3.4.8)$$

Proof: Using (3.4.6) in (3.4.36), we obtain

$$(\alpha\gamma)^{1/2}+\{(1-\alpha)(1-\gamma)\}^{1/2}$$
$$+2\{4\beta(1-\beta)\}^{1/3}\left\{1-2^{4/3}\left(\frac{\alpha^3\gamma^3(1-\alpha)^3(1-\gamma)^3}{\beta^2(1-\beta)^2}\right)^{1/24}\right\}=1. \quad (3.4.9)$$

Simplifying this, we easily arrive at (3.4.8), to complete our proof.

Theorem 3.4.6. *[11, p. 352, Entry 3(vi)] We have*

$$\{\alpha(1-\gamma)\}^{1/8}+\{\gamma(1-\alpha)\}^{1/8}=2^{1/3}\{\beta(1-\beta)\}^{1/24}. \quad (3.4.10)$$

Proof: Adding (3.3.10) and (3.3.34), we find that

$$\phi(q)\phi(q^9)=\psi(q)\phi(-q^{18})\chi(q^3)+q\phi(-q^2)\psi(q^9)\chi(q^3). \quad (3.4.11)$$

Transcribing this via (2.2.11), (2.2.13), (2.2.16), and (2.2.25), we deduce (3.4.10) to complete the proof.

Theorem 3.4.7. *[11, p. 352, Entry 3(x)] We have*

$$\left(\frac{\gamma}{\alpha}\right)^{1/8}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}-\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8}=\sqrt{mm'}. \quad (3.4.12)$$

Proof: We transcribe (3.3.41) by (2.2.13), (2.2.16) and (2.2.17) to arrive at (3.4.12).

Theorem 3.4.8. *[11, p. 352, Entry 3(xi)]*

$$\left(\frac{\alpha}{\gamma}\right)^{1/8}+\left(\frac{1-\alpha}{1-\gamma}\right)^{1/8}-\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/8}=\frac{3}{\sqrt{mm'}}. \quad (3.4.13)$$

Proof: In this case, we transcribe (3.3.48) to deduce (3.4.13).

Theorem 3.4.9. *[11, p. 352, Entry 3(xii)] We have*

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4}+\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4}-\left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}=-3\frac{m}{m'}. \tag{3.4.14}$$

Proof: Replacing $q^{1/3}$ by $-q$ in (3.3.5) and cubing both sides, we obtain

$$\left(1-\frac{\psi(q)}{q\psi(q^9)}\right)^3=1-\frac{\psi^4(q^3)}{q^3\psi^4(q^9)}. \tag{3.4.15}$$

Simplifying this, we find that

$$\frac{\psi^4(q^3)}{q\psi^2(q)\psi^2(q^9)}-3q\frac{\psi(q^9)}{\psi(q)}+3-\frac{\psi(q)}{q\psi(q^9)}=0. \tag{3.4.16}$$

Replacing q by $-q$, in (3.4.16), we obtain

$$\frac{\psi^4(-q^3)}{q\psi^2(-q)\psi^2(-q^9)}+3q\frac{\psi(-q^9)}{\psi(-q)}+3+\frac{\psi(-q)}{q\psi(-q^9)}=0 \tag{3.4.17}$$

From (3.4.16) and (3.4.17), we deduce that

$$\frac{\psi^4(q^3)}{q\psi^2(q)\psi^2(q^9)}-\frac{\psi^4(-q^3)}{q\psi^2(-q)\psi^2(-q^9)}$$
$$=3q\left(\frac{\psi(q^9)}{\psi(q)}-\frac{\psi(-q^9)}{\psi(-q)}\right)+\frac{1}{q}\left(\frac{\psi(q)}{\psi(q^9)}-\frac{\psi(-q)}{\psi(-q^9)}\right)-6. \tag{3.4.18}$$

Employing (3.3.41) and (3.3.48) in (3.4.18), we find that

$$\frac{\psi^4(q^3)}{q\psi^2(q)\psi^2(q^9)}-\frac{\psi^4(-q^3)}{q\psi^2(-q)\psi^2(-q^9)}=-3\frac{\phi(-q^{18})}{\phi(-q^2)}-\frac{\phi(-q^2)}{\phi(-q^{18})}. \tag{3.4.19}$$

Now, replacing $q^{1/3}$ by q in (3.3.6) and then simplifying, we deduce that

$$3\frac{\phi(q^9)}{\phi(q)}+\frac{\phi(q)}{\phi(q^9)}=3+\frac{\phi^4(q^3)}{\phi^2(q)\phi^2(q^9)}. \tag{3.4.20}$$

Replacing q by $-q^2$, in (3.4.20), we obtain

$$3\frac{\phi(-q^{18})}{\phi(-q^2)}+\frac{\phi(-q^2)}{\phi(-q^{18})}=3+\frac{\phi^4(-q^6)}{\phi^2(-q^2)\phi^2(-q^{18})}. \tag{3.4.21}$$

Now, using (3.4.21) in (3.4.19), we find that

$$\frac{\psi^4(q^3)}{q\psi^2(q)\psi^2(q^9)} - \frac{\psi^4(-q^3)}{q\psi^2(-q)\psi^2(-q^9)} + 3 = -\frac{\phi^4(-q^6)}{\phi^2(-q^2)\phi^2(-q^{18})}. \qquad (3.4.22)$$

Transcribing this by using (2.2.13), (2.2.17) and (2.2.17), we easily deduce (3.4.14).

Theorem 3.4.10. *[11, p. 352, Entry 3(xiii)] We have*

$$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4} - \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4} = \frac{m'}{m}. \qquad (3.4.23)$$

Proof: Replacing q by q^3 in (3.3.5) and then simplifying, we find that

$$1 - 3q\frac{\psi^2(-q)\psi^2(-q^9)}{\psi^4(-q^3)} = \frac{\psi(-q)\psi(-q^9)}{\psi^4(-q^3)}\left(\psi^2(-q) + 3q^2\psi^2(-q^9)\right). \qquad (3.4.24)$$

Again, multiplying (3.3.42) and (3.3.49), we find that

$$\psi^2(-q) + 3q^2\psi^2(-q^9) = \frac{\phi(q)\phi(q^9)}{\chi^2(q^3)} - 4q\psi(-q)\psi(-q^9). \qquad (3.4.25)$$

From (3.4.24) and (3.4.25), we obtain

$$1 + q\frac{\psi^2(-q)\psi^2(-q^9)}{\psi^4(-q^3)} = \frac{\psi(-q)\psi(-q^9)\phi(q)\phi(q^9)}{\psi^4(-q^3)\chi^2(q^3)}. \qquad (3.4.26)$$

Multiplying (3.3.39) and (3.4.26), we find that

$$1 + q\frac{\psi^2(-q)\psi^2(-q^9)}{\psi^4(-q^3)} = \frac{\phi^2(q)\phi^2(q^9)}{\phi^4(q^3)}\left(1 - 3q\frac{\psi^2(-q)\psi^2(-q^9)}{\psi^4(-q^3)}\right). \qquad (3.4.27)$$

We transcribe this by employing (2.2.11) and (2.2.17) to arrive at

$$\frac{m'}{m} = \frac{(\alpha\gamma(1-\alpha)(1-\gamma))^{1/4}}{(\beta(1-\beta))^{1/2}} = 1 - 3\frac{m}{m'}\frac{(\alpha\gamma(1-\alpha)(1-\gamma))^{1/4}}{(\beta(1-\beta))^{1/2}}. \qquad (3.4.28)$$

Using the expression of $-3m/m'$ from (3.4.14) in (3.4.28), we readily deduce (3.4.23) to complete the proof.

Theorem 3.4.11. *[11, p. 352, Entry 3(xiv)] We have*

$$\frac{2^{1/3}\{\beta(1-\beta)\}^{1/24}}{\{\alpha(1-\gamma)\}^{1/8}-\{\gamma(1-\alpha)\}^{1/8}}=\sqrt{\frac{m}{m'}}. \tag{3.4.29}$$

Proof: Subtracting (3.3.10) from (3.3.34), dividing the result by 2, we obtain

$$\phi^2(q^3)=\psi(q)\phi(-q^{18})\chi(q^3)-q\phi(-q^2)\psi(q^9)\chi(q^3). \tag{3.4.30}$$

Transcribing (3.4.30) via (2.2.11), (2.2.13), (2.2.16), and (2.2.25), we easily arrive at (3.4.29).

Theorem 3.4.12. *[11, p. 352, Entry 3(xv)] We have*

$$\begin{aligned}(\alpha^{1/4}-\gamma^{1/4})^4&+\{(1-\gamma)^{1/4}-(1-\alpha)^{1/4}\}^4\\ &=(\{\alpha(1-\gamma)\}^{1/4}-\{\gamma(1-\alpha)\}^{1/4})^4.\end{aligned} \tag{3.4.31}$$

Proof: From [10, p. 338], we note the following general result of Ramanujan: If the modular equation of degree $n-1$ is

$$\{\alpha\beta\}^{1/n}+\{(1-\alpha)(1-\beta)\}^{1/n}=1, \tag{3.4.32}$$

then

$$\begin{aligned}\{\{\alpha(1-\beta)\}^{1/n}&-\{\beta(1-\alpha)\}^{1/n}\}^n\\ &=\{\alpha^{1/n}-\beta^{1/n}\}^n+\{(1-\beta)^{1/n}-(1-\alpha)^{1/n}\}^n,\end{aligned} \tag{3.4.33}$$

is a modular equation of degree $(n-1)^2$.

Now, we know from Entry 5 [11, p. 230] that

$$(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}=1. \tag{3.4.34}$$

where β has degree 3 over α. Thus,

$$\begin{aligned}(\alpha^{1/4}-\gamma^{1/4})^4&+\{(1-\gamma)^{1/4}-(1-\alpha)^{1/4}\}^4\\ &=(\{\alpha(1-\gamma)\}^{1/4}-\{\gamma(1-\alpha)\}^{1/4})^4,\end{aligned} \tag{3.4.35}$$

where γ has degree 9 over α. This completes the proof.

Theorem 3.4.13. *[11, p. 352, Entry 3(xvi)] We have*

$$1=(\alpha\gamma)^{1/2}+\{(1-\alpha)(1-\gamma)\}^{1/2}+2\{4\beta(1-\beta)\}^{1/3}\frac{m'}{m}. \tag{3.4.36}$$

Proof: The proof of this modular equation is somewhat different in nature from the other proofs

Setting $\mu = 5$, $\nu = 4$, $A = 1$, $B = -1$ in Schöter's formula (4.3.20), we deduce that

$$\frac{1}{2}\{\phi(Q)\phi(-q) - \phi(-Q)\phi(q)\} = -2qf(-Q^8, -Q^{12})f(-q^8, -q^{12}) - 2q^9 f(-Q^4, -Q^{16})f(-q^4, -q^{16}), \quad (3.4.37)$$

where $Q = q^9$. Now, $G(q)$ and $H(q)$ are known as the Rogers-Ramanujan functions defined in (1.1.15) and (1.1.16) . Ramanujan [29, pp. 236-237] found forty modular relations for $G(q)$ and $H(q)$, which are called Ramanujan's forty identities. The sixth of these forty identities is

$$G(Q)G(q) + q^2 H(Q)H(q) = \frac{f^2(-q^3)}{f(-q)f(-q^9)}, \quad (3\,4\,38)$$

where $f(-q)$ is as defined in (1.1.10). The first proof of (3 4.38) was given by Rogers [32]. Berndt et al. [16] and Baruah et al. [8] also found several new proofs.

Now, using the Jacobi triple product identity (1.1.6) in (1.1.15) and (1.1.16), we easily find that

$$G(q) = \frac{f(-q^2, -q^3)}{f(-q)} \quad \text{and} \quad H(q) = \frac{f(-q, -q^4)}{f(-q)}. \quad (3.4.39)$$

Employing (3.4.39) in (3.4.37), we find that

$$\frac{1}{2}\{\phi(Q)\phi(-q) - \phi(-Q)\phi(q)\} = -2q\{G(Q^4)G(q^4) + q^8 H(Q^4)H(q^4)\}f(-q^4)f(-q^{36}). \quad (3.4.40)$$

Replacing q by q^4 in (3.4.38). and then using the resultant identity in (3.4.40). we deduce that

$$\{\phi(Q)\phi(-q) - \phi(-Q)\phi(q)\} = -2qf^2(-q^{12}). \quad (3.4.41)$$

Replacing q by $q^{1/2}$, we obtain

$$\{\phi(Q^{1/2})\phi(-q^{1/2}) - \phi(-Q^{1/2})\phi(q^{1/2})\} = -2q^{1/2}f^2(-q^6). \quad (3.4.42)$$

We complete the proof by transcribing (3.4.42) by employing (2.2.14), (2.2.15) and (2.2.21).

Theorem 3.4.14. *[13, p. 370, Entry 27]*

$$\left(\frac{\alpha}{\gamma}\right)^{1/8}\left(\frac{1-\alpha}{1-\gamma}\right)^{1/4}\sqrt{mm'}+\frac{3}{mm'}=\left(\frac{\alpha}{\gamma}\right)^{1/8}+\left(\frac{1-\alpha}{1-\gamma}\right)^{1/4}. \tag{3.4.43}$$

Proof: From (3.3.3) and (3.3 28), we deduce that

$$\psi(q)\phi(-q^9)-3q\psi(q^9)\phi(-q^9)=\psi(q)\phi(-q)-q\psi(q^9)\phi(-q). \tag{3.4 44}$$

Employing (2.2.6) in (3.4.44), we obtain

$$\psi(q)\phi(-q^9)-3q\psi(q^{9/2})\psi(-q^{9/2})=\psi(q^{1/2})\psi(-q^{1/2})-q\psi(q^9)\phi(-q). \tag{3.4 45}$$

Transcribing this by employing (2.2.12), (2.2.16), (2.2.19), and (2.2.20), we easily arrive at (3.4.43).

Chapter 4

Some New Proofs of Modular Relations for the Göllnitz-Gordon Functions

4.1 Introduction

We recall from Chapter 1, that for $|q| < 1$, the Rogers-Ramanujan functions, are defined by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{4.1.1}$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{4.1.2}$$

when the later equalities are the famous Rogers-Ramanujan identities. Ramanujan recorded forty modular relations for $G(q)$ and $H(q)$ in a manuscript published with the lost notebook [29]. These are now known as Ramanujan's forty identities. Darling [23] established one of the identities in 1921 in the Proceedings of London Mathematical Society. Rogers [32] established ten of the forty identities including the one proved by Darling. Watson [36] proved eight of the forty identities, two of them from the list that Rogers proved. In 1977, Bressoud ([19], [20]) generalized Rogers' results to prove fifteen from the list of forty. In 1989, Biagioli [17] used modular forms to prove seven of the remaining nine identities. Recently, Berndt et al. [16] have found proofs of 35 of the 40 identities in the spirit of Ramanujan's mathematics. For the remaining 5 identities, they also offered heuristic arguments showing that both sides of the identity have the same asymptotic expansions as $q \to 1^-$.

Now, we recall the definitions of Göllnitz-Gordon Functions,

$$S(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty}, \tag{4.1.3}$$

$$T(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2+2n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}. \tag{4.1.4}$$

Many of the Göllnitz-Gordon identities are similar in character to the Rogers-Ramanujan's identities. For example, the quotient of $G(q)$ and $H(q)$ gives the Rogers-Ramanujan continued fraction, while the quotient of $S(q)$ and $T(q)$ gives the Ramanujan-Göllnitz-Gordon continued fraction [28, Vol. 2, p. 229]. Chan and Huang [21] succeeded in obtaining several relations involving the Ramanujan- Göllnitz-Gordon continued fraction. Motivated by the similarity between the Rogers-Ramanujan and Göllnitz-Gordon functions, S.-S. Huang [26] and Chen and Huang [22] derived 21 modular relations involving $S(q)$ and $T(q)$, one new relations for $G(q)$ and $H(q)$, and 9 relations involving both the pairs $G(q)$, $H(q)$ and $S(q)$ and $T(q)$. They used the methods of Rogers [32], Watson [36], and Bressoud [19]. In this Chapter, we find alternative proofs of the modular relations involving only $S(q)$ and $T(q)$ by employing Schröter's formulas and theta functions identities. We also find several new modular relations, and many more can be found by using the same method.

In Section 4.2, we give the list of the modular relations for the Göllnitz-Gordon functions which will be proved in this chapter.

In Section 4.3, we state some preliminary results.

In Sections 4.4-4.12, we present the proofs of the identities listed in section 4.2.

In our last section, we present the new modular relations for the Göllnitz-Gordon functions.

4.2 Modular Relations for the Göllnitz-Gordon Functions

For convenience, we denote $f(-q^n)$ by f_n.

$$S(q)S(q) + qT(q)T(q) = \left\{\frac{f_2f_2}{f_1f_4}\right\}^3, \tag{4.2.1}$$

$$S(q)S(q) - qT(q)T(q) = \frac{f_4f_4}{f_1f_2}\left\{\frac{f_4}{f_8}\right\}^2, \tag{4.2.2}$$

$$S(q^7)T(q) - q^3S(q)T(q^7) = 1, \tag{4.2.3}$$

$$S(q^3)S(q) + q^2T(q^3)T(q) = \frac{f_3f_4}{f_1f_{12}}, \tag{4.2.4}$$

$$S(q^3)T(q) - qS(q)T(q^3) = \frac{f_1f_{12}}{f_3f_4}, \tag{4.2.5}$$

$$S(q^5)S(q) + q^3T(q^5)T(q) = \frac{f_2f_{10}}{f_1f_{20}}, \tag{4.2.6}$$

$$S(q^5)T(q) - q^2S(q)T(q^5) = \frac{f_2f_{10}}{f_4f_5}, \tag{4.2.7}$$

$$S(q^4)T(q^2) - qS(q^2)T(q^4) = \frac{f_1f_{32}}{f_2f_{16}}, \tag{4.2.8}$$

$$S(q^9)T(q) - q^4S(q)T(q^9) = \frac{f_6f_6}{f_3f_{12}}, \tag{4.2.9}$$

$$S(q^9)S(q) + q^5T(q^9)T(q) = \frac{f_2f_3f_{12}f_{18}}{f_1f_4f_9f_{36}}, \tag{4.2.10}$$

$$S(q^{15})S(q) + q^8T(q^{15})T(q) = \frac{f_2f_3f_5f_{12}f_{20}f_{30}}{f_1f_4f_6f_{10}f_{15}f_{60}}, \tag{4.2.11}$$

$$S(q^5)T(q^3) - qS(q^3)T(q^5) = \frac{f_1f_4f_6f_{10}f_{15}f_{60}}{f_2f_3f_5f_{12}f_{20}f_{30}}, \tag{4.2.12}$$

$$\frac{S(q^5)S(q^3) + q^4T(q^5)T(q^3)}{S(q^{15})T(q) - q^7S(q)T(q^{15})} = 1, \tag{4.2.13}$$

$$\frac{S(q^{39})S(q) + q^{20}T(q^{39})T(q)}{S(q^{13})T(q^3) - q^5S(q^3)T(q^{13})} = \frac{f_2f_3f_{12}f_{13}f_{52}f_{78}}{f_1f_4f_6f_{26}f_{39}f_{156}}, \tag{4.2.14}$$

$$\frac{S(q^{55})S(q) + q^{28}T(q^{55})T(q)}{S(q^{11})T(q^5) - q^3S(q^5)T(q^{11})} = \frac{f_2f_5f_{11}f_{20}f_{44}f_{110}}{f_1f_4f_{10}f_{22}f_{55}f_{220}}, \tag{4.2.15}$$

$$\frac{S(q^{63})T(q) - q^{31}S(q)T(q^{63})}{S(q^9)T(q^7) - qS(q^7)T(q^9)} = \frac{f_2f_7f_9f_{28}f_{36}f_{126}}{f_1f_4f_{14}f_{18}f_{63}f_{252}}, \tag{4.2.16}$$

$$\{S(q^7)S(q^3) + q^5T(q^7)T(q^3)\}\{S(q^7)T(q^3) - q^2S(q^3)T(q^7)\}$$
$$= \frac{1}{2q}\left\{\frac{f_2f_2f_2f_{14}f_{14}f_{14}f_{21}}{f_1f_4f_7f_7f_{12}f_{28}f_{28}} - \frac{f_1f_6f_6f_6f_{42}f_{42}f_{42}}{f_3f_3f_{12}f_{12}f_{21}f_{28}f_{84}}\right\}, \tag{4.2.17}$$
$$\{S(q^{21})S(q) + q^{11}T(q^{21})T(q)\}\{S(q^{21})T(q) - q^{10}S(q)T(q^{21})\}$$
$$= \frac{1}{2q}\left\{\frac{f_2f_2f_2f_3f_{14}f_{14}f_{14}}{f_1f_1f_4f_4f_7f_{28}f_{84}} - \frac{f_6f_6f_6f_7f_{42}f_{42}f_{42}}{f_3f_4f_{12}f_{21}f_{21}f_{84}f_{84}}\right\}, \tag{4.2.18}$$
$$S(q)T(-q) + S(-q)T(q) = \frac{2f_{16}}{f_2f_8}f(-q^6, -q^{10}), \tag{4.2.19}$$
$$S(q)T(-q) - S(-q)T(q) = \frac{2qf_{16}}{f_2f_8}f(-q^2, -q^{14}), \tag{4.2.20}$$
$$S(q)S(-q) - qT(q)T(-q) = \frac{f_8^2}{f_2f_4f_{16}}f(-q, q^3). \tag{4.2.21}$$

4.3 Preliminary Results

Lemma 4.3.1. *[11, p. 40, Entry 25] We have*

$$\phi(q) + \phi(-q) = 2\phi(q^4), \tag{4.3.1}$$
$$\phi(q) - \phi(-q) = 4q\psi(q^8), \tag{4.3.2}$$
$$\phi^2(q) - \phi^2(-q) = 8q\psi^2(q^4), \tag{4.3.3}$$
$$\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2). \tag{4.3.4}$$

Lemma 4.3.2. *[11, p. 48, Entry 31 with $k = 2$] We have*

$$f(a, b) = f(a^3b, ab^3) + af(b/a, a^5b^3). \tag{4.3.5}$$

Lemma 4.3.3. *[11, p. 46] We have*

$$f(a, b) + f(-a, -b) = 2f(a^3b, ab^3). \tag{4.3.6}$$

Lemma 4.3.4. *[11, p. 46] We have*

$$f(a, b) - f(-a, -b) = 2af(b/a, a^5b^3). \tag{4.3.7}$$

Lemma 4.3.5. *[11, p. 46] We have*

$$f(a, b)f(-a, -b) = f(-a^2, -b^2)\phi(-ab). \tag{4.3.8}$$

Lemma 4.3.6. *[11, p. 51, Example (iv), with q replaced by −q] We have*

$$\phi(q)+\phi(q^2)=2\frac{f^2(-q^3,-q^5)}{\psi(-q)}, \tag{4.3.9}$$

and

$$\phi(q)-\phi(q^2)=2q\frac{f^2(-q,-q^7)}{\psi(-q)}. \tag{4.3.10}$$

Lemma 4.3.7. *[26, Lemma 3.1] We have*

$$S(q)T(q)=\frac{f_2f_8^2}{f_1f_4^2}, \quad \phi(q)=\frac{f_2^5}{f_1^2f_4^2}, \quad \textit{and} \quad \psi(q)=\frac{f_2^2}{f_1}. \tag{4.3.11}$$

Lemma 4.3.8. *We have*

$$\phi(-q)=\frac{f_1^2}{f_2}, \quad \psi(-q)=\frac{f_1f_4}{f_2}, \quad f(q)=\frac{f_2^3}{f_1f_4}, \quad \textit{and} \quad \chi(q)=\frac{f_2^2}{f_1f_4}. \tag{4.3.12}$$

Proof: These identities easily follow from Entries 24-25 [11, pp. 39-40].

Lemma 4.3.9. *We have*

$$f(-q^3,-q^5)=\frac{f_1f_4}{f_2}S(q), \tag{4.3.13}$$

and

$$f(-q,-q^7)=\frac{f_1f_4}{f_2}T(q). \tag{4.3.14}$$

Proof: By [22, Lemma 2.6] and (1.1.9), we note that

$$S(q)=\frac{\psi(-q^2)f(-q^3,-q^5)}{f_1f_8}, \tag{4.3.15}$$

and

$$T(q)=\frac{\psi(-q^2)f(-q,-q^7)}{f_1f_8}. \tag{4.3.16}$$

Using (4.3.12) we complete the proof.

Lemma 4.3.10. *[11, Corollary, p. 49] We have*

$$\psi(q)=f(q^6,q^{10})+qf(q^2,q^{14}). \tag{4.3.17}$$

Replacing q by $-q$ in (4 3 17), we have the following useful result

Lemma 4.3.11. *We have*

$$\psi(-q) = f(q^6, q^{10}) - qf(q^2, q^{14}) \tag{4 3.18}$$

In the following six Schroter's formulas, we assume that μ and ν are integers such that $\mu > \nu \geq 0$

Lemma 4.3.12. *[11, p. 67, (36.1)] We have*

$$\frac{1}{2}\{f(Aq^{\mu+\nu}, \frac{q^{\mu+\nu}}{A})f(Bq^{\mu-\nu}, \frac{q^{\mu-\nu}}{B}) + f(-Aq^{\mu+\nu}, -\frac{q^{\mu+\nu}}{A})f(-Bq^{\mu-\nu}, -\frac{q^{\mu-\nu}}{B})\}$$
$$= \sum_{m=0}^{\mu-1}\left(\frac{A}{B}\right)^m q^{2\mu m^2} f\left(\frac{A^{\mu-\nu}}{B^{\mu+\nu}}q^{(2\mu+4m)(\mu^2-\nu^2)}, \frac{B^{\mu+\nu}}{A^{\mu-\nu}}q^{(2\mu-4m)(\mu^2-\nu^2)}\right)$$
$$\times f\left(ABq^{2\mu+4\nu m}, \frac{q^{2\mu-4\nu m}}{AB}\right). \tag{4 3 19}$$

Lemma 4.3.13. *[11, p 68, (36 2)] We have*

$$\frac{1}{2}\{f(Aq^{\mu+\nu}, \frac{q^{\mu+\nu}}{A})f(Bq^{\mu-\nu}, \frac{q^{\mu-\nu}}{B}) - f(-Aq^{\mu+\nu}, -\frac{q^{\mu+\nu}}{A})f(-Bq^{\mu-\nu}, -\frac{q^{\mu-\nu}}{B})\}$$
$$= A\sum_{m=0}^{\mu-1}(AB)^m q^{(2m+1)(\mu+\nu)+2\mu m^2}$$
$$\times f\left(A^{\mu-\nu}B^{\mu+\nu}q^{(2\mu+4m+2)(\mu^2-\nu^2)}, \frac{q^{(2\mu-4m-2)(\mu^2-\nu^2)}}{A^{\mu-\nu}B^{\mu+\nu}}\right)$$
$$\times f\left(\frac{A}{B}q^{4\mu+2\nu+4\nu m}, \frac{B}{A}q^{-2\nu-4\nu m}\right). \tag{4.3 20}$$

Lemma 4.3.14. *[11, p 69, (36 7)] If μ is odd, then*

$$\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = \phi(q^{\mu(\mu^2-\nu^2)})\psi(q^{2\mu})$$
$$+ \sum_{m=1}^{(\mu-1)/2} q^{\mu m^2-\nu m} f(q^{(\mu+2m)(\mu^2-\nu^2)}, q^{(\mu-2m)(\mu^2-\nu^2)})$$
$$\times f(q^{2\nu m}, q^{2\mu-2\nu m}) \tag{4 3 21}$$

Lemma 4.3.15. *[11, p. 69, (36.8)] If μ is even, then*

$$\begin{aligned}
&\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) \\
&= \phi(q^{\mu(\mu^2-\nu^2)})\psi(q^{2\mu}) + \sum_{m=1}^{\mu/2-1} q^{\mu m^2-\nu m} \\
&\times f(q^{(\mu+2m)(\mu^2-\nu^2)}, q^{(\mu-2m)(\mu^2-\nu^2)})f(q^{2\nu m}, q^{2\mu-2\nu m}) \\
&+ q^{\mu^3/4-\mu\nu/2}\psi(q^{2\mu(\mu^2-\nu^2)})f(q^{\mu\nu}, q^{2\mu-\mu\nu}).
\end{aligned} \tag{4.3.22}$$

Lemma 4.3.16. *[11, p. 69, (36.9)] If μ is odd, then*

$$\begin{aligned}
\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) =& q^{\mu^3/4-\mu/4}\psi(q^{2\mu(\mu^2-\nu^2)})f(q^{\mu+\mu\nu}, q^{\mu-\mu\nu}) \\
&+ \sum_{m=0}^{(\mu-3)/2} q^{\mu m(m+1)} f(q^{(\mu+2m+1)(\mu^2-\nu^2)}, q^{(\mu-2m-1)(\mu^2-\nu^2)}) \\
&\times f(q^{\mu+\nu+2\nu m}, q^{\mu-\nu-2\nu m}).
\end{aligned} \tag{4.3.23}$$

Lemma 4.3.17. *[11, p. 69, (36.10)] If μ is even, then*

$$\begin{aligned}
&\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) \\
&= \sum_{m=0}^{\mu/2-1} q^{\mu m(m+1)} f(q^{(\mu+2m+1)(\mu^2-\nu^2)}, q^{(\mu-2m-1)(\mu^2-\nu^2)}) f(q^{\mu+\nu+2\nu m}, q^{\mu-\nu-2\nu m}).
\end{aligned} \tag{4.3.24}$$

4.4 Proofs of (4.2.1) and (4.2.2):

Proof of (4.2.1)**:** Adding (4.3.9) and (4.3.10), we find that

$$f^2(-q^3, -q^5) + qf^2(-q, -q^7) = \psi(-q)\phi(q) \tag{4.4.1}$$

Employing (4.3.13), (4.3.14), (4.3.11), and (4.3.12) in (4.4.1), we easily arrive at (4.2.1).

Proof of (4.2.2)**:** Subtracting (4.3.10) from (4.3.9), we obtain

$$f^2(-q^3, -q^5) - qf^2(-q, -q^7) = \psi(-q)\phi(q^2). \tag{4.4.2}$$

Employing (4.3.13), (4.3.14), (4.3.11), and (4.3.12) in (4.4.2), we easily deduce (4.2.2).

4.5 Proofs of (4.2.3) - (4.2.5):

Proof of (4.2.3): Putting $\mu = 4$, $\nu = 3$ in (4.3.22), it can be shown [11, p. 315, (19.1)] that

$$\psi(q)\psi(q^7) = \phi(q^{28})\psi(q^8) + q\psi(q^{14})\psi(q^2) + q^6\psi(q^{56})\phi(q^4). \tag{4.5.1}$$

Replacing q by $-q$ in (4.5.1), and then subtracting the resulting identity from (4.5.1), we find that

$$\psi(q)\psi(q^7) - \psi(-q)\psi(-q^7) = 2q\psi(q^{14})\psi(q^2). \tag{4.5.2}$$

Using (4.3.17) and (4.3.18) in (4.5.2), we obtain

$$f(q^2, q^{14})f(q^{42}, q^{70}) + q^6 f(q^6, q^{10})f(q^{14}, q^{98}) = \psi(q^{14})\psi(q^2). \tag{4.5.3}$$

Replacing q^2 by $-q$ in (4.5.3), we deduce that

$$f(-q, -q^7)f(-q^{21}, -q^{35}) - q^3 f(-q^3, -q^5)f(-q^7, -q^{49}) = \psi(-q^7)\psi(-q). \tag{4.5.4}$$

Employing (4.3.13), (4.3.14) and (4.3.12), we obtain (4.2.3). So, we complete the proof.

Proofs of (4.2.4) **and** (4.2.5): Putting $\mu = 2$ and $\nu = 1$ in (4.3.22), we find that

$$\psi(q)\psi(q^3) = \phi(q^6)\psi(q^4) + q\psi(q^{12})\phi(q^2). \tag{4.5.5}$$

Replacing q by $-q$ in (4 5.5), we obtain

$$\psi(-q)\psi(-q^3) = \phi(q^6)\psi(q^4) - q\psi(q^{12})\phi(q^2). \tag{4.5.6}$$

Adding (4.5.5) and (4.5.6), we arrive at

$$\psi(q)\psi(q^3) + \psi(-q)\psi(-q^3) = 2\phi(q^6)\psi(q^4).$$

Using (4.3.17) and (4.3.18), this can be written as

$$f(q^6, q^{10})f(q^{18}, q^{30}) + q^4 f(q^2, q^{14})f(q^6, q^{42}) = \phi(q^6)\psi(q^4). \tag{4.5.7}$$

Replacing q^2 by $-q$ in (4.5.7), we find that

$$f(-q^3, -q^5)f(-q^9, -q^{15}) + q^2 f(-q, -q^7)f(-q^3, -q^{21}) = \phi(-q^3)\psi(q^2). \tag{4.5.8}$$

Using (4.3.13), (4.3.14), (4.3.11) and (4.3.12) in (4.5.8), we easily obtain

$$S(q^3)S(q) + q^2T(q^3)T(q) = \frac{f_3f_4}{f_1f_{12}}.$$

Thus, we complete the proof of (4.2.4).

Now, subtracting (4.5.6) from (4.5.5), we find that

$$\psi(q)\psi(q^3) - \psi(-q)\psi(-q^3) = 2q\psi(q^{12})\phi(q^2) \tag{4.5.9}$$

Using (4.3.17) and (4.3.18) on the left hand side, this can be written as

$$f(q^{18}, q^{30})f(q^2, q^{14}) + q^2 f(q^6, q^{42})f(q^6, q^{10}) = \phi(q^2)\psi(q^{12}). \tag{4.5.10}$$

Replacing q^2 by $-q$ in (4.5.10), we obtain

$$f(-q^9, -q^{15})f(-q, -q^7) - qf(-q^3, -q^{21})f(-q^3, -q^5) = \phi(-q)\psi(q^6). \tag{4.5.11}$$

By use of (4.3.13), (4.3.14), (4.3.11), and (4.3.12) in (4.5.11), we easily arrive at (4.2.5) to complete the proof.

4.6 Proofs of (4.2.6) and (4.2.7):

Proof of (4.2.6): Using (2.2.7) in (2.3.8), we find that

$$\frac{\phi^2(-q^{10})}{\phi^2(-q^2)} + q\left(\frac{\phi(q^5)\psi(q^{10})}{\phi(q)\psi(q^2)} - \frac{\phi(-q^5)\psi(q^{10})}{\phi(-q)\psi(q^2)}\right) = 1. \tag{4.6.1}$$

Employing (2.2.5), this can be written as

$$\frac{q\psi(q^{10})}{\psi(q^2)}\left(\phi(-q)\phi(q^5) - \phi(q)\phi(-q^5)\right) = \phi^2(-q^2) - \phi^2(-q^{10}). \tag{4.6.2}$$

With the help of (2.3.9), we deduce from (4.6.2) that

$$\frac{\psi(q^{10})}{\psi(q^2)}\left(4q^2 f(-q^4)f(-q^{20})\right) = \phi^2(-q^{10}) - \phi^2(-q^2). \tag{4.6.3}$$

Replacing q^2 by q in (4.6.3), we find that

$$\frac{\psi(q^5)}{\psi(q)}\left(4qf(-q^2)f(-q^{10})\right) = \phi^2(-q^5) - \phi^2(-q). \tag{4.6.4}$$

Replacing q by $-q$ in (4 6.4), we obtain

$$-\frac{\psi(-q^5)}{\psi(-q)}\left(4qf(-q^2)f(-q^{10})\right)=\phi^2(q^5)-\phi^2(q). \tag{4.6.5}$$

Subtracting (4.6.5) from (4.6.4), and then using (2.2.6) and (4.3.3), we find that

$$\frac{f(-q^2)f(-q^{10})}{\phi(-q^2)\psi(q^2)}\left(\psi(q)\psi(-q^5)+\psi(-q)\psi(q^5)\right)=2\left(\psi^2(q^4)-q^4\psi^2(q^{20})\right) \tag{4.6.6}$$

Replacing q by q^4 in (2.3.16) and using in (4.6.6), we deduce that

$$\psi(q)\psi(-q^5)+\psi(-q)\psi(q^5)=2\frac{\phi(-q^2)\psi(q^2)\phi(-q^{20})f(-q^{20})}{f(-q^2)f(-q^{10})\chi(-q^4)}. \tag{4.6.7}$$

Employing (4.3.17) and (4.3.18) in (4.6.7), and then replacing q^2 by $-q$ in the resulting identity, we deduce that

$$\begin{aligned}f(-q^3,-q^5)f(-q^{15},-q^{25})+q^3f(-q,-q^7)f(-q^5,-q^{35})\\=\frac{\phi(q)\psi(-q)\phi(-q^{10})f(-q^{10})}{f(q)f(q^5)\chi(-q^2)}.\end{aligned} \tag{4.6.8}$$

Employing (4.3.13), (4.3.14), (4.3.11), and (4.3.12) in (4.6.8) we easily deduce (4.2.6) to complete the proof of (4.2.6).

Proof of (4.2.7): Adding (4.6.4) and (4.6.5), and then using (2.2.6) and (4.3.4), we find that

$$\frac{2qf(-q^2)f(-q^{10})}{\phi(-q^2)\psi(q^2)}\left(\psi(q)\psi(-q^5)-\psi(-q)\psi(q^5)\right)=\phi^2(q^2)-\phi^2(q^{10}). \tag{4.6.9}$$

Using (4.3.12) in (2.3.1), we find that

$$\phi^2(q)-\phi^2(q^5)=4q\chi(q)f(-q^5)f(-q^{20}). \tag{4.6.10}$$

Replacing q by q^2 in (4.6.10) and using in (4.6.9), we find that

$$\psi(q)\psi(-q^5)-\psi(-q)\psi(q^5)=2\frac{\chi(q^2)\psi(q^2)\phi(-q^2)f(-q^{40})}{f(-q^2)}. \tag{4.6.11}$$

Employing (4.3.17) and (4.3.18) in (4.6.11), and then replacing q^2 by $-q$ in the resulting identity, we obtain

$$f(-q,-q^7)f(-q^{15},-q^{25}) - q^2 f(-q^3,-q^5)f(-q^5,-q^{35}) \\ = \frac{\chi(-q)\psi(-q)\phi(q)f(-q^{20})}{f(q)}. \tag{4.6.12}$$

Employing (4.3.13), (4.3.14), (4.3.11), and (4.3.12), we easily deduce (4.2.7).

4.7 Proof of (4.2.8):

We recall from (4.3.18) that

$$\psi(-q) = f(q^6,q^{10}) - qf(q^2,q^{14}). \tag{4.7.1}$$

Using (4.3.8), we write (4.7.1) as

$$\psi(-q) = \left\{ \frac{f(-q^{12},-q^{20})}{f(-q^6,-q^{10})} - q\frac{f(-q^4,-q^{28})}{f(-q^2,-q^{14})} \right\} \phi(-q^{16}). \tag{4.7.2}$$

Employing (4.3.13), (4.3.14), (4.3.11), and (4.3.12) we easily arrive at (4.2.8).

4.8 Proofs of (4.2.9)-(4.2.10):

Proof of (4.2.9): Using (3.2.2), with q replaced by q^3, in (3.2.4), we find that

$$\psi(q) - q\psi(q^9) = \frac{\phi(-q^9)}{\chi(-q^3)}. \tag{4.8.1}$$

This can also be written as

$$\frac{\psi(q)}{q\psi(q^9)} = 1 + \frac{\phi(-q^9)}{q\psi(q^9)\chi(-q^3)}. \tag{4.8.2}$$

Employing (3.2.1) in (4.8.2), we obtain

$$\frac{\psi(q)}{q\psi(q^9)} = 1 + \frac{\phi(-q^9)}{qf(-q^3,-q^{15})}. \tag{4.8.3}$$

Replacing q by $-q$. we find that

$$-\frac{\psi(-q)}{q\psi(-q^9)} = 1 - \frac{\phi(q^9)}{qf(q^3,q^{15})}. \tag{4.8.4}$$

Adding (4.8.3) and (4.8.4), and then using (1.1.8), we deduce that

$$\begin{aligned}\frac{\psi(q)\psi(-q^9) - \psi(-q)\psi(q^9)}{\psi(q^9)\psi(-q^9)} \\ = 2q + \frac{f(q^3,q^{15})f(-q^9,-q^9) - f(-q^3,-q^{15})f(q^9,q^9)}{f(q^3,q^{15})f(-q^3,-q^{15})}.\end{aligned} \tag{4.8.5}$$

Employing (2.2.10) and (4.3.8), with $a = q^3$, $b = q^{15}$, $c = d = -q^9$, in (4.8.5), we deduce that

$$\frac{\psi(q)\psi(-q^9) - \psi(-q)\psi(q^9)}{\psi(q^9)\psi(-q^9)} = 2q + 2q^3\frac{f(-q^6,-q^{30})}{\phi(-q^{18})}. \tag{4.8.6}$$

Using (2.2.6), this can be written as

$$\psi(q)\psi(-q^9) - \psi(-q)\psi(q^9) = 2q\psi(q^{18})\{\phi(-q^{18}) + q^2 f(-q^6,-q^{30})\}. \tag{4.8.7}$$

Employing (3.2.3) in (4.8.7), we deduce that

$$\psi(q)\psi(-q^9) - \psi(-q)\psi(q^9) = q\psi(q^{18})\{3\phi(-q^{18}) - \phi(-q^2)\}. \tag{4.8.8}$$

Now, using (3.2.1) in (3.2.3), we deduce that

$$\phi(-q^9) - \phi(-q) = 2q\psi(q^9)\chi(-q^3). \tag{4.8.9}$$

Using (4.8.1), we rewrite this as

$$\phi(-q^9) - \phi(-q) = 2\psi(q)\chi(-q^3) - 2\phi(-q^9). \tag{4.8.10}$$

Thus,

$$3\phi(-q^9) - \phi(-q) = 2\psi(q)\chi(-q^3). \tag{4.8.11}$$

Replacing q by q^2 in (4.8.11), and then using it in (4.8.8), we find that

$$\psi(q)\psi(-q^9) - \psi(-q)\psi(q^9) = 2q\psi(q^{18})\psi(q^2)\chi(-q^6). \tag{4.8.12}$$

Invoking (4.3.17) and (4.3.18), and then replacing q^2 by $-q$, we deduce from (4.8.12) that

$$\begin{aligned} f(-q^{27},-q^{45})f(-q,-q^{7}) - q^{4}f(-q^{3},-q^{5})f(-q^{9},-q^{63}) \\ = \psi(-q^{9})\psi(-q)\chi(q^{3}). \end{aligned} \tag{4.8.13}$$

Employing (4.3.13), (4.3.14), and (4.3.12) in (4.8.13), we easily deduce (4.2.9).

Proof of (4.2.10): Subtracting (4.8.4) from (4.8.3), and then using (1.1.8), we obtain

$$\begin{aligned} &\frac{\psi(q)\psi(-q^{9})+\psi(-q)\psi(q^{9})}{\psi(q^{9})\psi(-q^{9})} \\ &= \frac{f(q^{3},q^{15})f(-q^{9},-q^{9})+f(-q^{3},-q^{15})f(q^{9},q^{9})}{f(q^{3},q^{15})f(-q^{3},-q^{15})}. \end{aligned} \tag{4.8.14}$$

Employing (2.2.9), with $a=q^3$, $b=q^{15}$, $c=d=-q^9$, and (3.2.1), in (4.8.14), we deduce that

$$\psi(q)\psi(-q^{9})+\psi(-q)\psi(q^{9}) = 2\frac{f^{2}(-q^{12})}{\chi(-q^{6})}, \tag{4.8.15}$$

where we have also used from [11, p. 39, Entry 24(iv)], that $\chi(q)\chi(-q) = \chi(-q^2)$. Now, employing (4.3.17) and (4.3.18), and then replacing q^2 by $-q$, we derive from (4.8.15) that

$$f(-q^{27},-q^{45})f(-q^{3},-q^{5}) + q^{5}f(-q,-q^{7})f(-q^{9},-q^{63}) = \frac{f^{2}(-q^{6})}{\chi(q^{3})}. \tag{4.8.16}$$

Employing (4.3.13), (4.3.14), and (4.3.12) in (4.8.16), we easily arrive at (4.2.10).

Remark: The sixth of Ramanujan's 40 identities is given by

$$G(q)G(q^{9}) + q^{2}H(q)H(q^{9}) = \frac{f^{2}(-q^{3})}{f(-q)f(-q^{9})}, \tag{4.8.17}$$

where $G(q)$ and $H(q)$ are as defined in (4.1.1) and (4.1.2).

For proofs of (4.8.17), see [32] and [16]. With the help of (4.8.12) and (4.8.15), we now present a new proof of this identity.

First, putting $\mu = 5$, $\nu = 4$, $A = 1$, and $B = -1$ in (4.3.20), and then employing Entry 18 (iv) [11, p. 34], it can be deduced that

$$\begin{aligned}&\phi(q)\phi(-q^9) - \phi(-q)\phi(q^9)\\&= 4q\left\{f(-q^8, -q^{12})f(-q^{72}, -q^{108}) + q^8 f(-q^4, -q^{16})f(-q^{36}, -q^{144})\right\}.\end{aligned} \tag{4.8.18}$$

Using (2.2.7) this can be rewritten as

$$\begin{aligned}&\frac{\psi^2(q)\psi^2(-q^9) - \psi^2(-q)\psi^2(q^9)}{\psi(q^2)\psi(q^{18})}\\&= 4q\left\{f(-q^8, -q^{12})f(-q^{72}, -q^{108}) + q^8 f(-q^4, -q^{16})f(-q^{36}, -q^{144})\right\}\end{aligned} \tag{4.8.19}$$

Employing (4.8.12) and (4.8.15) in (4.8.19), we arrive at

$$f(-q^8, -q^{12})f(-q^{72}, -q^{108}) + q^8 f(-q^4, -q^{16})f(-q^{36}, -q^{144}) = f^2(-q^{12}). \tag{4.8.20}$$

Using (3.4.39) in (4.8.20), we obtain

$$G(q^4)G(q^{36}) + q^8 H(q^4)H(q^{36}) = \frac{f^2(-q^{12})}{f(-q^4)f(-q^{36})}. \tag{4.8.21}$$

Replacing q^4 by q in (4.8.21), we easily deduce (4.8.17).

4.9 Proofs of (4.2.11)- (4.2.13):

Proof of (4.2.11): From Entry 9(iv) [11, p. 377], we note that

$$\psi(q)\psi(q^{15}) + \psi(-q)\psi(-q^{15}) = 2\psi(q^6)\psi(q^{10}). \tag{4.9.1}$$

Using (4.3.17) and (4.3.18) in (4.9.1), and then replacing q^2 by $-q$. we obtain

$$f(-q^3, -q^5)f(-q^{45}, -q^{75}) + q^8 f(-q, -q^7)f(-q^{15}, -q^{105}) = \psi(-q^3)\psi(-q^5). \tag{4.9.2}$$

Employing (4.3.13), (4.3.14), and (4.3.12) in (4.9.2), we complete the proof of (4.2.11).

Proof of (4.2.12): From Entry 9(i) [11, p 377], we have

$$\psi(q^3)\psi(q^5) - \psi(-q^3)\psi(-q^5) = 2q^3\psi(q^2)\psi(q^{30}). \qquad (4.9.3)$$

The rest of the proof is same as the previous one.
Proof of (4.2.13): Putting $\mu = 4$, $\nu = 1$ in (4.3.22), we find that

$$\psi(q^5)\psi(q^3) = \psi(q^8)\phi(q^{60}) + q^3\psi(q^2)\phi(q^{30}) + q^{14}\psi(q^{120})\phi(q^4). \qquad (4.9.4)$$

Replacing q by $-q$ in (4.9.4), and then adding the resulting identity with (4.9.4), we deduce that

$$\psi(q^5)\psi(q^3) + \psi(-q^5)\psi(-q^3) = 2\psi(q^8)\phi(q^{60}) + 2q^{14}\psi(q^{120})\phi(q^4). \qquad (4.9.5)$$

Using (4.3.1) and (4.3.2) on the right of (4.9.5), we deduce that

$$\psi(q^5)\psi(q^3) + \psi(-q^5)\psi(-q^3) = \frac{1}{2q}\{\phi(q)\phi(q^{15}) - \phi(-q)\phi(-q^{15})\}. \qquad (4.9.6)$$

Employing (2.2.7) we can rewrite this as

$$\psi(q^5)\psi(q^3) + \psi(-q^5)\psi(-q^3) = \frac{\psi^2(q)\psi^2(q^{15}) - \psi^2(-q)\psi^2(-q^{15})}{2q\psi(q^2)\psi(q^{30})}. \qquad (4.9.7)$$

Thus,

$$\frac{\psi(q^5)\psi(q^3) + \psi(-q^5)\psi(-q^3)}{\psi(q)\psi(q^{15}) - \psi(-q)\psi(-q^{15})} = \frac{\psi(q)\psi(q^{15}) + \psi(-q)\psi(-q^{15})}{2q\psi(q^2)\psi(q^{30})}. \qquad (4.9.8)$$

Employing (4.3.17) and (4.3.18) in (4.9.8), replacing q^2 by $-q$ in the resulting identity, and then using (4.3.13), (4.3.14) and (4.3.12), we easily deduce that

$$\frac{S(q^5)S(q^3) + q^4T(q^5)T(q^3)}{S(q^{15})T(q) - q^7S(q)T(q^{15})} = \frac{\{S(q^{15})S(q) + q^8T(q^{15})T(q)\}f_2f_3f_5f_{12}f_{20}f_{30}}{f_1f_4f_6f_{10}f_{15}f_{60}}. \qquad (4.9.9)$$

Now, an application of (4.2.11) easily yields (4.2.13).

4.10 Proofs of (4.2.14)-(4.2.16):

Proofs of (4.2.14)-(4.2.16) are similar in nature. So, we give details only for (4.2.14).

Proof of (4.2.14): In (4.3.22), we put $\mu = 8$, $\nu = 5$ to obtain (See [11, p. 75, (37.4)] for details)

$$\begin{aligned}&\psi(q^{13})\psi(q^3) - \psi(-q^{13})\psi(-q^3)\\ &\qquad = 2q^3 f(q^{390}, q^{234})(q^6, q^{10}) + 2q^{43} f(q^{78}, q^{546}) f(q^2, q^{14}). \qquad (4.10.1)\end{aligned}$$

Using (4.3.17) and (4.3.18) in (4.10.1), replacing q^2 by $-q$, we find that

$$\frac{f(-q^{195}, -q^{117})(-q^3, -q^5) + q^{20} f(-q^{39}, -q^{273}) f(-q, -q^7)}{f(-q^{39}, -q^{65}) f(-q^3, -q^{21}) - q^5 f(-q^{13}, -q^{91}) f(-q^9, -q^{15})} = 1. \qquad (4.10.2)$$

Employing (4.3.13) and (4.3.14), we arrive at the desired result.
Proofs of (4.2.15) **and** (4.2.16): To prove (4.2.15) and (4.2.16), we put $(\mu, \nu) = (8, 3)$ and $(\mu, \nu) = (8, 1)$, respectively, in (4.3.22) and proceed as in the above proof.

4.11 Proofs of (4.2.17) and (4.2.18):

Proof of (4.2.17): Putting $\mu = 5$, $\nu = 2$, $A = 1$, $B = -1$ in (4.3.20), we find that

$$\begin{aligned}\{\phi(q^7)\phi(-q^3) - \phi(q^3)\phi(-q^7)\} &= -4q^3\{f(-q^{252}, -q^{168})(-q^{16}, -q^4)\\ &\quad - q^{16} f(-q^{84}, -q^{336}) f(-q^8, -q^{12})\}. \qquad (4.11.1)\end{aligned}$$

Using (2.2.7) this can be written as

$$\begin{aligned}&\{\psi(q^7)\psi(-q^3) + \psi(q^3)\psi(-q^7)\}\{\psi(q^7)\psi(-q^3) - \psi(q^3)\psi(-q^7)\}\\ &= -4q^3\psi(q^{14})\psi(q^6)\{f(-q^{252}, -q^{168})(-q^{16}, -q^4)\\ &\quad - q^{16} f(-q^{84}, -q^{336}) f(-q^8, -q^{12})\}. \qquad (4.11.2)\end{aligned}$$

We now employ (4.3.17) and (4.3.18) on the left hand side of (4.11.2) and then replace q^2 by $-q$. Then we use (4.3.13), (4.3.14), and (4.3.12), to obtain

$$\begin{aligned}&\{S(q^7)S(q^3) + q^5 T(q^7)T(q^3)\}\{S(q^7)T(q^3) - q^2 S(q^3)T(q^7)\}\\ &= \frac{f_6 f_{14}}{f_3 f_7 f_{12} f_{28}}\{f(-q^{126}, -q^{84})(-q^8, -q^2) - q^8 f(-q^{42}, -q^{168}) f(-q^4, -q^6)\}. \qquad (4.11.3)\end{aligned}$$

Employing (3.4.39) in (4.11.3), we obtain

$$\begin{aligned}&\{S(q^7)S(q^3)+q^5T(q^7)T(q^3)\}\{S(q^7)T(q^3)-q^2S(q^3)T(q^7)\}\\&=\frac{f_6f_{14}f_2f_{42}}{f_3f_7f_{12}f_{28}}\{G(q^{42})H(q^2)-q^8G(q^2)H(q^{42})\}.\end{aligned} \tag{4.11.4}$$

Now replacing x by q^2 in the fifteenth of Ramanujan's forty identities for the Rogers-Ramanujan functions [29, p. 236] (see also [26, (R.15) and (R.16)]), we note that

$$\begin{aligned}G(q^{14})H(q^6)+q^4H(q^{14})H(q^6)&=G(q^{42})H(q^2)-q^8G(q^2)H(q^{42})\\&=\frac{1}{2q}\left\{\frac{f_2f_2f_3f_{14}f_{14}f_{21}}{f_1f_4f_6f_7f_{28}f_{42}}-\frac{f_1f_6f_6f_7f_{42}f_{42}}{f_2f_3f_{12}f_{14}f_{21}f_{84}}\right\}.\end{aligned} \tag{4.11.5}$$

Using (4.11.5) in (4 11.4) we easily find (4.2.17) to complete the proof.

Proof of (4.2.18): Replacing q by $q^{1/3}$ in (4.11.1), we find that

$$\begin{aligned}&\phi(q^{7/3})\phi(-q)-\phi(q)\phi(-q^{7/3})\\&=-4q\{f(-q^{84},-q^{56})f(-q^{16/3},-q^{4/3})\\&\quad-q^{16/3}f(-q^{28},-q^{112})f(-q^{8/3},-q^4)\}.\end{aligned} \tag{4.11.6}$$

Now, replacing q by $\pm q^{7/3}$ in (3.2.3), we obtain

$$\phi(\pm q^{7/3})=\phi(\pm q^{21})\pm 2qf(q^7,q^{35}). \tag{4.11.7}$$

Again, from Entry 10(iii) and 10(iv) [11, p. 379], we note that

$$f(-q^{2/3},-q)=f(-q^7,-q^8)-qf(-q^2,-q^{13})-q^{2/3}f(-q^3,-q^{12}). \tag{4.11.8}$$

and

$$f(-q^{1/3},-q^{4/3})=f(-q^6,-q^9)-q^{1/3}\{f(-q^4,-q^{11})+qf(-q,-q^{14})\}. \tag{4.11.9}$$

Replacing q by q^4 in (4.11.8) and (4.11.9), we obtain

$$f(-q^{8/3},-q^4)=f(-q^{28},-q^{32})-q^4f(-q^8,-q^{52})-q^{8/3}f(-q^{12},-q^{48}). \tag{4.11.10}$$

and

$$f(-q^{4/3},-q^{16/3})=f(-q^{24},-q^{36})-q^{4/3}\{f(-q^{16},-q^{44})+q^4f(-q^4,-q^{56})\}, \tag{4.11.11}$$

respectively. Employing (4.11.7), (4.11.10), and (4.11.11) in (4.11 6), and then equating the rational parts from both sides of the resulting identity, we deduce that

$$\phi(-q)\phi(q^{21}) - \phi(q)\phi(-q^{21})$$
$$= -4q\{f(-q^{84},-q^{56})f(-q^{24},-q^{36}) + q^8 f(-q^{28},-q^{112})f(-q^{12},-q^{48})\}. \tag{4.11.12}$$

Using (2.2.7) this can be rewritten as

$$\{\psi(-q)\psi(q^{21}) + \psi(q)\psi(-q^{21})\}\{\psi(-q)\psi(q^{21}) - \psi(q)\psi(-q^{21})\}$$
$$= -4q\psi(q^2)\psi(q^{42})\{f(-q^{84},-q^{56})f(-q^{24},-q^{36})$$
$$+ q^8 f(-q^{28},-q^{112})f(-q^{12},-q^{48})\}. \tag{4.11.13}$$

Employing (4 3.17) and (4.3.18) on the left hand side of (4 11.13), and then replacing q^2 by $-q$ in the resulting identity, we arrive at

$$\{f(-q^3,-q^5)f(-q^{63},-q^{105}) + q^{11}f(-q,-q^7)f(-q^{21},-q^{147})\}$$
$$\times\{f(-q,-q^7)f(-q^{63},-q^{105}) - q^{10}f(-q^3,-q^5)f(-q^{21},-q^{147})\}$$
$$= \psi(-q)\psi(-q^{21})\{f(-q^{42},-q^{28})f(-q^{12},-q^{18})$$
$$+ q^4 f(-q^{14},-q^{56})f(-q^6,-q^{24})\}. \tag{4.11.14}$$

Using (4.3.13), (4.3.14), (4.3.12), and (3.4.39), we easily deduce that

$$\{S(q^{21})S(q) + q^{11}T(q^{21})T(q)\}\{S(q^{21})T(q) - q^{10}S(q)T(q^{21})\}$$
$$= \frac{f_2 f_6 f_{14} f_{42}}{f_1 f_4 f_{21} f_{84}}\{G(q^{14})G(q^6) + q^4 H(q^{14})H(q^6)\}. \tag{4.11.15}$$

Employing (4.11.5) in (4.11.15), we arrive at (4.2.18). Hence the proof is complete.

4.12 Proofs of (4.2.19) - (4.2.21):

Proof of (4.2.19): Putting $a = 1$, $b = -q^4$, $c = q$ and $d = -q^3$ in (2.2.9), we obtain

$$f(1,-q^4)f(q,-q^3) = 2f(q,q^7)f(-q^3,-q^5). \tag{4.12.1}$$

Replacing q by $-q$, we have

$$f(1,-q^4)f(-q,q^3) = 2f(-q,-q^7)f(q^3,q^5). \tag{4.12.2}$$

Adding (4.12.1) and (4.12.2), we find that

$$f(1,-q^4)\{f(q,-q^3)+f(-q,q^3)\} = 2\{f(q,q^7)f(-q^3,-q^5) + f(-q,-q^7)f(q^3,q^5)\}. \qquad (4.12.3)$$

Using (4.3.6) in (4.12.3), we obtain

$$f(1,-q^4)f(-q^6,-q^{10}) = \{f(q,q^7)f(-q^3,-q^5)+f(-q,-q^7)f(q^3,q^5)\}. \qquad (4.12.4)$$

Employing (1.1.9) in (4.12.4), we deduce that

$$\{f(q,q^7)f(-q^3,-q^5)+f(-q,-q^7)f(q^3,q^5)\} = 2\psi(-q^4)f(-q^6,-q^{14}). \qquad (4.12.5)$$

Using (4.3.13), (4.3.14) and (4.3.12), we deduce (4.2.19).
Proof of (4.2.20): Subtracting (4.12.2) from (4.12.1), we obtain

$$f(1,-q^4)\{f(q,-q^3)-f(-q,q^3)\} = 2\{f(q,q^7)f(-q^3,-q^5) - f(-q,-q^7)f(q^3,q^5)\}. \qquad (4.12\ 6)$$

Using (4.3.7) in (4.12.6), then using (1.1.9), we find that

$$\{f(q,q^7)f(-q^3,-q^5)-f(-q,-q^7)f(q^3,q^5)\} = 2q\psi(-q^4)f(-q^2,-q^{14}). \qquad (4.12.7)$$

Invoking (4.3.13), (4.3.14) and (4.3.12), we easily deduce (4.2.20). Thus, we complete the proof.
Proof of (4.2.21): Putting $a=q$, $b=-q^3$, $c=q^2$ and $d=-q^2$, in (2.2.9) and (2.2.10), we obtain

$$f(q,-q^3)f(q^2,-q^2)+f(-q,q^3)f(-q^2,q^2) = 2f(q^3,q^5)f(-q^3,-q^5). \qquad (4.12.8)$$

$$f(q,-q^3)f(q^2,-q^2)-f(-q,q^3)f(-q^2,q^2) = 2qf(-q,-q^7)f(q,q^7). \qquad (4.12.9)$$

Subtracting (4.12.9) from (4.12.9), we find that

$$f(-q,q^3)f(q^2,-q^2) = f(q^3,q^5)f(-q^3,-q^5)-qf(-q,-q^7)f(q,q^7). \qquad (4.12.10)$$

Using (4.3.5), we obtain

$$f(q^2,-q^2) = f(-q^8,-q^8) = \phi(-q^8). \qquad (4.12.11)$$

Employing (4.12.11) in (4.12.10), and then using (4.3.13), (4.3.14) and (4.3.12) we complete the proof.

4.13 New modular relations for the Göllnitz Gordon functions:

Making different choices for μ, ν, A, and B in the Schroter's formulas (4.3.19)-(4.3.24), and then using the methods as in the previous nine sections, one can find many other relations for $S(q)$ and $T(q)$. In the following, we give some examples.

$$S(q)S(q^{11}) + q^6T(q)T(q^{11}) = \frac{f_2f_{12}f_{12}f_{22}f_{33}f_{33}}{f_1f_4f_6f_{11}f_{44}f_{66}} - q^3\frac{f_6f_6f_{66}f_{66}}{f_1f_{12}f_{33}f_{44}}, \quad (4.13.1)$$

$$T(q)S(q^{11}) - q^5S(q)T(q^{11}) = \frac{f_6f_6f_{66}f_{66}}{f_3f_4f_{11}f_{132}} - q^7\frac{f_2f_3f_3f_{22}f_{132}f_{132}}{f_1f_4f_6f_{11}f_{44}f_{66}}, \quad (4.13.2)$$

$$\{S(q^7)S(q^5) + q^6T(q^7)T(q^5)\}\{S(q^7)T(q^5) - qT(q^7)S(q^5)\}$$
$$= \frac{1}{q}\left\{\frac{f_2f_2f_2f_{10}f_{14}f_{70}f_{70}f_{70}}{f_1f_4f_5f_7f_{20}f_{28}f_{35}f_{140}} - 1\right\}, \quad (4.13.3)$$

$$\{S(q^{35})S(q) + q^{18}T(q)T(q^{35})\}\{S(q^{35})T(q) - q^{17}T(q^{35})S(q)\}$$
$$= \frac{f_2f_{10}f_{10}f_{10}f_{14}f_{14}f_{14}f_{70}}{f_1f_4f_5f_7f_{20}f_{28}f_{35}f_{140}} + q^4, \quad (4.13.4)$$

$$\{S(q)S(q^{27}) + q^{14}T(q)T(q^{27})\}\{T(q)S(q^{27}) - q^{13}S(q)T(q^{27})\}$$
$$= \frac{f_2f_6f_6f_6f_{18}f_{18}f_{18}f_{54}}{f_1f_3f_4f_9f_{12}f_{27}f_{36}f_{108}} - q^3. \quad (4.13.5)$$

Proofs of (4.13.1)-(4.13.5): To prove (4.13.1) and (4.13.2), we set $\mu = 6$ and $\nu = 5$ in (4.3.22), and then proceed as in the proofs of (4.2.4) and (4.2.5).

To prove (4.13.3), (4.13.4) and (4.13.5), we use Entry 17(ii), (i) [11, p. 417] and Entry 4(iv) [11, p. 359], respectively, and proceed as in the proof of (4.2.17). It is worthwhile to note that Berndt [11] used Schröter's formulas (4.3.19) and (4.3.20), and also other theta-function identities of Ramanujan to establish these entries. One can also get many other analogous identities from (4.3.19)-(4.3.24).

Remark: I am grateful to my mathematical brother Nipen Saikia for giving the idea of proofs of identities (4.2.19)–(4.2.21).

Chapter 5

Nonic Analogues of the Rogers-Ramanujan Functions with Applications to Partitions

5.1 Introduction

As mentioned in Chapter 1, H. Hahn [24] – [25] defined the septic analogues of the Rogers-Ramanujan functions as

$$A(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7;q^7)_\infty(q^3;q^7)_\infty(q^4;q^7)_\infty}{(q^2;q^2)_\infty}, \qquad (5.1.1)$$

$$B(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7;q^7)_\infty(q^2;q^7)_\infty(q^5;q^7)_\infty}{(q^2;q^2)_\infty}, \qquad (5.1.2)$$

and

$$C(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}} = \frac{(q^7;q^7)_\infty(q;q^7)_\infty(q^6;q^7)_\infty}{(q^2;q^2)_\infty}, \qquad (5.1.3)$$

where the later equalities are due to Rogers [30], [31] (These appear in the list of L. J. Slater [34, p. 155, equations (33), (32) and (31)]). Hahn found many identities involving only $A(q)$, $B(q)$, and $C(q)$ as well as identities which are connected with the Rogers-Ramanujan and Göllnitz-Gordon functions.

Now, we define the following nonic analogues of the Rogers-Ramanujan functions

$$D(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n} q^{3n^2}}{(q^3;q^3)_n (q^3;q^3)_{2n}} = \frac{(q^4;q^9)_\infty (q^5;q^9)_\infty (q^9;q^9)_\infty}{(q^3;q^3)_\infty}, \tag{5.1.4}$$

$$E(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n}(1-q^{3n+2}) q^{3n(n+1)}}{(q^3;q^3)_n (q^3;q^3)_{2n+1}} = \frac{(q^2;q^9)_\infty (q^7;q^9)_\infty (q^9;q^9)_\infty}{(q^3;q^3)_\infty}, \tag{5.1.5}$$

$$F(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n+1} q^{3n(n+1)}}{(q^3;q^3)_n (q^3;q^3)_{2n+1}} = \frac{(q;q^9)_\infty (q^8;q^9)_\infty (q^9;q^9)_\infty}{(q^3;q^3)_\infty}, \tag{5.1.6}$$

where the later equalities are due to W. N. Bailey [2, p. 422, equations (1.6), (1.8), and (1.7)]. It is worthwhile to mention that, Bailey used non-standard notation in the paper where these identities first appeared. All three of these identities appear in the list of Slater [34, p. 156] as equations (42), (41), and (40) in that order. However, all three contain misprints. These misprints are corrected as given in (5.1.4)-(5.1.6) by A. V. Sills [33]. In this chapter, we establish several modular relations involving $D(q)$, $E(q)$ and $F(q)$, which are analogues of Ramanujan's forty identities. We also establish several other modular relations involving quotients of $D(q)$, $E(q)$ and $F(q)$. Some of these are connected with the Rogers-Ramanujan functions, Göllnitz-Gordon functions and Septic Rogers-Ramanujan-type functions. Furthermore, by the notion of colored partitions, we are able to extract partition interpretations from some of these identities.

In Section 5.2, we list our modular relations.

In Sections 5.3-5.9, we present the proofs of the modular relations stated in Section 5.2.

In our last section of this chapter, we find applications of some of our modular relations to the theory of partitions.

5.2 Main Results

In this section, we present the modular relations for the functions $D(q)$, $E(q)$, and $F(q)$, which we prove in this chapter. It is worthwhile to note that by replacing q by $-q$ in each of the following relations one can get more relations. For simplicity, we define, for positive integer n, $D_n := D(q^n)$, $E_n := E(q^n)$, and $F_n := F(q^n)$. We also define $f_\alpha := f(-q^\alpha)$.

The identities (5.2.1)-(5.2.23) involve $D(q)$, $E(q)$, and $F(q)$.

$$D_1^2E_1 + qE_1^2F_1 - qD_1F_1^2 = 1, \tag{5.2.1}$$

$$D_1^2F_1 - E_1^2D_1 + qF_1^2E_1 = 0, \tag{5.2.2}$$

$$D_3 - qE_3 - q^2F_3 = f_1/f_9, \tag{5.2.3}$$

$$D_6E_3F_3 + qE_6D_3F_3 + q^2F_6D_3E_3 = \frac{f_2f_3^3f_{18}f_{27}f_{54}}{f_1f_6f_9^5}, \tag{5.2.4}$$

$$D_5D_4 + q^3E_5E_4 + q^6F_5F_4 = \frac{f_2^2f_{10}^2}{f_1f_{12}f_{15}f_{20}} - q, \tag{5.2.5}$$

$$D_6D_3 + q^3E_6E_3 + q^6F_6F_3 = \frac{f_2^2f_9}{f_1f_{18}^2} - q, \tag{5.2.6}$$

$$D_{20}E_1 - q^7E_{20}F_1 + q^{13}F_{20}D_1 = \frac{f_2^2f_{10}^2}{f_3f_4f_5f_{60}} + q^2, \tag{5.2.7}$$

$$D_2E_1 - qE_2F_1 + qF_2D_1 = 1, \tag{5.2.8}$$

$$F_1D_5 + qD_1E_5 - q^3E_1F_5 = 1, \tag{5.2.9}$$

$$D_1D_8 + q^3E_1E_8 + q^6F_1F_8 = \frac{f_2^2f_4^2}{f_1f_3f_8f_{24}} - q, \tag{5.2.10}$$

$$D_{11}E_1 - q^4E_{11}F_1 + q^7F_{11}D_1 = \frac{f_1f_{11}}{f_3f_{33}} + q, \tag{5.2.11}$$

$$D_2D_7 + q^3E_2E_7 + q^6F_2F_7 = \frac{f_2^2f_7^2}{f_1f_6f_{14}f_{21}} - q, \tag{5.2.12}$$

$$D_{14}F_1 + q^4D_1E_{14} - q^9E_1F_{14} = \frac{f_1^2f_{14}^2}{f_2f_3f_7f_{42}} + q, \tag{5.2.13}$$

$$D_{23}F_1 + q^7E_{23}D_1 - q^{15}F_{23}E_1 = \frac{f_1f_{23}}{f_3f_{69}} + q^2, \tag{5.2.14}$$

$$D_{32}F_1 + q^{10}E_{32}D_1 - q^{21}F_{32}E_1 = \frac{f_1f_4f_8f_{32}}{f_2f_3f_{16}f_{96}} + q^3, \tag{5.2.15}$$

$$D_1D_{35} + q^{12}E_1E_{35} + q^{24}F_1F_{35} = \frac{f_5f_7}{f_3f_{105}} - q^4, \tag{5.2.16}$$

$$q^5D_2E_{19} + D_{19}F_2 - q^{12}F_{19}E_2 = \frac{f_1f_{38}}{f_6f_{57}} + q, \tag{5.2.17}$$

$$D_{38}E_1 - q^{13}E_{38}F_1 + q^{25}F_{38}D_1 = \frac{f_2f_{19}}{f_3f_{114}} + q^4, \tag{5.2.18}$$

$$D_1D_{44} + q^{15}E_1E_{44} + q^{30}F_1F_{44} = \frac{f_4f_{11}}{f_3f_{132}} - q^5, \tag{5.2.19}$$

$$D_{56}E_1 - q^{19}E_{56}F_1 + q^{37}F_{56}D_1 = \frac{f_2f_7f_8f_{28}}{f_3f_4f_{14}f_{168}} + q^6, \tag{5.2.20}$$

$$D_{24}F_3 + q^6E_{24}D_3 - q^{15}F_{24}E_3 = \frac{f_1f_4f_{18}}{f_2f_9f_{36}} + q \tag{5.2.21}$$

$$D_1D_{80} + q^{27}E_1E_{80} + q^{54}F_{80}F_1 = \frac{f_4f_5f_{16}f_{20}}{f_3f_8f_{10}f_{240}} - q^9, \tag{5.2.22}$$

$$D_{1955}E_1 - q^{652}E_{1955}F_1 + q^{1303}F_{1955}D_1 = q^{217}. \tag{5.2.23}$$

The identities (5.2.24)-(5.2.32) involve quotients of the nonic analogues $D(q)$, $E(q)$, and $F(q)$.

$$\frac{D_3 - qE_3 - q^2F_3}{D_9 - q^3E_9 - q^6F_9} = \frac{f_1f_{27}}{f_3f_9}, \tag{5.2.24}$$

$$\frac{D_{11}E_1 - q - q^4E_{11}F_1 + q^7F_{11}D_1}{D_{33} - q^{11}E_{33} - q^{22}F_{33}} = \frac{f_1f_{99}}{f_3f_{33}}, \tag{5.2.25}$$

$$\frac{D_1D_{35} + q^4 + q^{12}E_{35}E_1 + q^{24}F_1F_{35}}{D_{21} - q^7E_{21} - q^{14}F_{21}} = \frac{f_5f_{63}}{f_3f_{105}}, \tag{5.2.26}$$

$$\frac{D_{23}F_1 - q^2 + q^7E_{23}D_1 - q^{15}F_{23}E_1}{D_{69} - q^{23}E_{69} - q^{46}F_{69}} = \frac{f_1f_{207}}{f_3f_{69}}, \tag{5.2.27}$$

$$\frac{D_2D_{25} + q^3 + q^9E_2E_{25} + q^{18}F_2F_{25}}{D_{50}F_1 - q^5 + q^{16}F_{50}D_1 - q^{33}F_{50}E_1} = \frac{f_2f_3f_{25}f_{150}}{f_1f_6f_{50}f_{75}}, \tag{5.2.28}$$

$$\frac{D_{73}F_2 - q^7 + q^{23}E_{73}D_2 - q^{48}F_{73}E_2}{D_{146}E_1 - q^{16} - q^{49}E_{146}F_1 + q^{97}F_{146}D_1} = \frac{f_3f_{438}}{f_6f_{219}}, \tag{5.2.29}$$

$$\frac{D_{49}F_8 - q + q^{11}E_{49}D_8 - q^{30}F_{49}E_8}{D_{392}F_1 - q^{43} + q^{130}E_{392}D_1 - q^{261}F_{392}E_1} = \frac{f_3f_{1176}}{f_{24}f_{147}}, \tag{5.2.30}$$

$$\frac{D_{68}F_1 - q^7 + q^{22}E_{68}D_1 - q^{45}F_{68}E_1}{D_{17}E_4 - q - q^7E_{17}F_4 + q^{10}F_{17}D_4} = \frac{f_{12}f_{51}}{f_3f_{204}}, \tag{5.2.31}$$

$$\frac{D_1D_{260} + q^{29} + q^{87}E_1E_{260} + q^{174}F_1F_{260}}{D_{65}F_4 - q^5 + q^{19}E_{65}D_4 - q^{42}F_{65}E_4} = \frac{f_{12}f_{195}}{f_3f_{780}}. \tag{5.2.32}$$

The following identities are relations involving some combinations of $D(q)$, $E(q)$ and $F(q)$ with the Rogers-Ramanujan functions $G(q)$ and $H(q)$. Here, for positive integer n, we define $G_n := G(q^n)$ and $H_n := H(q^n)$.

$$\frac{D_9 - q^3E_9 - q^6E_9}{G_9G_1 + q^2H_9H_1} = \frac{f_1f_9}{f_3f_{27}}, \tag{5.2.33}$$

$$\frac{D_{25}D_2+q^3+q^9E_{25}E_2+q^{18}F_{25}F_2}{G_5G_{10}+q^3H_5H_{10}}=\frac{f_5f_{10}}{f_6f_{75}}, \tag{5.2.34}$$

$$\frac{D_3D_{33}+q^4+q^{12}E_3E_{33}+q^{24}F_3F_{33}}{G_9G_{11}+q^4H_9H_{11}}=\frac{f_{11}}{f_{99}}, \tag{5.2.35}$$

$$\frac{D_3D_{42}+q^5+q^{15}E_3E_{42}+q^{30}F_3F_{42}}{G_{18}G_7+q^5H_{18}H_7}=\frac{f_{18}f_7}{f_9f_{126}}, \tag{5.2.36}$$

$$\frac{D_1D_{26}+q^3+q^9E_1E_{26}+q^{18}F_1F_{26}}{G_{13}G_2+q^3H_{13}H_2}=\frac{f_2f_{13}}{f_3f_{78}}, \tag{5.2.37}$$

$$\frac{D_{29}E_1-q^3-q^{10}E_{29}F_1+q^{19}F_{29}D_1}{G_{29}G_1+q^6H_{29}H_1}=\frac{f_1f_{29}}{f_3f_{87}}, \tag{5.2.38}$$

$$\frac{D_{74}E_1-q^8-q^{25}E_{74}F_1+q^{49}F_{74}D_1}{G_{37}H_2-q^7G_2H_{37}}=\frac{f_3f_{222}}{f_2f_{37}}, \tag{5.2.39}$$

$$\frac{D_1D_{116}+q^{13}+q^{39}E_1E_{116}+q^{78}F_1F_{116}}{G_{29}H_4-q^5G_4H_{29}}=\frac{f_8f_{58}}{f_3f_{348}}, \tag{5.2.40}$$

$$\frac{D_1D_{125}+q^{14}+q^{42}E_1E_{125}+q^{84}F_1F_{125}}{G_{25}H_5-q^4G_5H_{25}}=\frac{f_5f_{25}}{f_3f_{375}}. \tag{5.2.41}$$

The following identities are relations involving some combinations of $D(q)$, $E(q)$, and $F(q)$ with the Göllnitz-Gordon functions $S(q)$ and $T(q)$. For simplicity, for positive integer n, we define $S_n := S(q^n)$ and $T_n := T(q^n)$.

$$\frac{D_{15}-q^5E_{15}-q^{10}F_{15}}{S_5S_1+q^3T_5T_1}=\frac{f_1f_5f_{20}}{f_2f_{10}f_{45}}, \tag{5.2.42}$$

$$\frac{D_{60}-q^{20}E_{60}-q^{40}F_{60}}{S_5T_1-q^2T_5S_1}=\frac{f_4f_5f_{20}}{f_2f_{10}f_{180}}, \tag{5.2.43}$$

$$\frac{D_{68}F_1-q^7+q^{22}E_{68}D_1-q^{45}F_{68}E_1}{S_{17}T_1-q^8T_{17}S_1}=\frac{f_1f_4f_{17}f_{68}}{f_2f_3f_{34}f_{204}}, \tag{5.2.44}$$

$$\frac{D_{128}E_1-q^{14}+q^{43}E_{128}F_1-q^{85}F_{128}D_1}{S_{16}T_2-q^7S_2T_{16}}=\frac{f_2f_8f_{16}f_{64}}{f_3f_4f_{32}f_{384}}, \tag{5.2.45}$$

$$\frac{D_{60}F_3-q^5+q^{18}E_{60}D_3-q^{39}F_{60}E_3}{S_{45}S_1+q^{23}T_{45}T_1}=\frac{f_1f_4f_{45}}{f_2f_9f_{90}}, \tag{5.2.46}$$

$$\frac{\{S_{45}S_1+q^{23}T_{45}T_1\}\{S_{45}T_1-q^{22}T_{45}S_1\}}{D_6D_{30}+q^4+q^{12}E_6E_{30}+q^{24}F_6F_{30}}=\frac{f_2f_{18}f_{90}^2}{f_1f_4f_{45}f_{180}}, \tag{5.2.47}$$

$$\frac{D_3D_{60}+q^7+q^{21}E_3E_{60}+q^{42}F_3F_{60}}{S_9S_5+q^7T_9T_5}=\frac{f_5f_{20}f_{36}}{f_{10}f_{18}f_{180}}, \tag{5.2.48}$$

$$\frac{D_1D_{224}+q^{25}+q^{75}E_1E_{224}+q^{150}F_1F_{224}}{S_{14}T_4-q^5T_{14}S_4}=\frac{f_4f_{14}f_{16}f_{56}}{f_3f_8f_{28}f_{672}}, \tag{5.2.49}$$

$$\frac{D_{96}E_3-q^{10}-q^{33}E_{96}F_3+q^{63}F_{96}D_3}{S_{36}S_2+q^{19}T_{36}T_2}=\frac{f_2f_8f_{36}f_{144}}{f_4f_9f_{72}f_{288}}, \tag{5.2.50}$$

$$\frac{D_{44}F_7-q+q^{10}E_{44}D_7-q^{27}F_{44}E_7}{S_{77}T_1-q^{38}T_{77}S_1}=\frac{f_1f_4f_{77}f_{308}}{f_2f_{21}f_{132}f_{154}}, \tag{5.2.51}$$

$$\frac{D_{64}E_5-q^6-q^{23}E_{64}F_5+q^{41}F_{64}D_5}{S_2S_{40}+q^{21}T_2T_{40}}=\frac{f_2f_8f_{40}f_{160}}{f_4f_{15}f_{80}f_{192}}, \tag{5.2.52}$$

$$\frac{D_{320}F_1-q^{35}+q^{106}E_{320}D_1-q^{213}E_1F_{320}}{S_{10}T_8-q^2S_8T_{10}}=\frac{f_8f_{10}f_{32}f_{40}}{f_3f_{16}f_{20}f_{960}}. \tag{5.2.53}$$

The following identities are relation involving some combinations of $D(q)$, $E(q)$, and $F(q)$ with the Septic analogues $A(q)$, $B(q)$ and $C(q)$. Here also, for positive integer n, we define $A_n := A(q^n)$, $B_n := B(q^n)$ and $C_n := C(q^n)$.

$$\frac{D_9-q^3E_9-q^6F_9}{A_1A_{27}+q^4B_1B_{27}+q^{12}C_1C_{27}}=\frac{f_2f_{54}}{f_9f_{27}}, \tag{5.2.54}$$

$$\frac{D_{15}-q^5E_{15}-q^{10}F_9}{A_1A_{20}+q^3B_1B_{20}+q^9C_1C_{20}}=\frac{f_2f_{40}}{f_4f_{45}}, \tag{5.2.55}$$

$$\frac{D_{47}E_1-q^5-q^{16}E_{47}F_1+q^{31}F_{47}D_1}{A_{47}B_1-q^7B_{47}C_1-q^{20}C_{47}A_1}=\frac{f_2f_{94}}{f_3f_{141}}, \tag{5.2.56}$$

$$\frac{D_{59}F_1-q^6+q^{19}E_{59}D_1-q^{39}F_{59}E_1}{A_{59}C_1-q^8B_{59}C_1+q^{25}C_{59}B_1}=\frac{f_2f_{118}}{f_3f_{177}}, \tag{5.2.57}$$

$$\frac{E_2D_{31}-q^3-q^{11}E_{31}F_2+q^{20}F_{31}D_2}{A_{62}A_1+q^9B_{62}B_1+q^{27}C_1C_{62}}=\frac{f_2f_{124}}{f_6f_{93}}, \tag{5.2.58}$$

$$\frac{D_1D_{98}+q^{11}+q^{33}E_1E_{98}+q^{66}F_1F_{98}}{A_{14}B_7-q^4B_{14}C_7-q^5C_{14}A_7}=\frac{f_{14}f_{28}}{f_3f_{294}}, \tag{5.2.59}$$

$$\frac{D_6D_{39}+q^5+q^{15}E_6E_{39}+q^{30}F_6F_{39}}{A_{26}A_9+q^5B_{26}B_9+q^{15}C_{26}C_9}=\frac{f_{52}}{f_{117}}, \tag{5.2.60}$$

$$\frac{D_1D_{215}+q^{24}+q^{72}E_1E_{215}+q^{144}F_1F_{215}}{A_{43}C_5-q^4B_{43}A_5+q^{17}C_{43}B_5}=\frac{f_{10}f_{86}}{f_3f_{645}}, \tag{5.2.61}$$

$$\frac{D_2D_{115}+q^{13}+q^{39}E_2E_{115}+q^{78}F_2F_{115}}{A_{46}B_5-q^8B_{46}C_5-q^{19}C_{46}A_5}=\frac{f_{10}f_{92}}{f_6f_{345}}, \tag{5.2.62}$$

$$\frac{D_1D_{188}+q^{21}+q^{63}E_1E_{188}+q^{126}F_1F_{188}}{A_{47}C_4-q^5B_{47}A_4+q^{19}C_{47}B_4}=\frac{f_8f_{94}}{f_3f_{564}}, \tag{5.2.63}$$

$$\frac{D_{230}F_1 - q^{25} + q^{76}E_{230}D_1 - q^{153}F_{230}E_1}{A_{10}B_{23} - q^8 B_{10}C_{23} - qC_{10}A_{23}} = \frac{f_{20}f_{46}}{f_3 f_{690}}. \tag{5.2.64}$$

5.3 Proofs of (5.2.1)-(5.2.4)

First of all, invoking (1.1.10) and (1.1.6) in (5.1.4)-(5.1.6), we immediately arrive at the following lemma

Lemma 5.3.1. *We have*

$$D(q) = \frac{f(-q^4,-q^5)}{f(-q^3)}, \quad E(q) = \frac{f(-q^2,-q^7)}{f(-q^3)}, \quad \textit{and} \quad F(q) = \frac{f(-q,-q^8)}{f(-q^3)}. \tag{5.3.1}$$

Proof of (5.2.1): From Entry 2(viii) [11, p. 349], we find that

$$\frac{f(-q^4,-q^5)}{f(-q,-q^8)} + q\frac{f(-q^2,-q^7)}{f(-q^4,-q^5)} = q\frac{f(-q,-q^8)}{f(-q^2-q^7)} + \frac{f^4(-q^3)}{f(-q)f^3(-q^9)}. \tag{5.3.2}$$

Using (5.3.1) in (5.3.2), we obtain

$$D_1^2 E_1 + qE_1^2 F_1 = qD_1 F_1^2 + D_1 E_1 F_1 \frac{f_3^4}{f_1 f_9^3}. \tag{5.3.3}$$

Again, from Entry 2(vi) [11, p. 349], we note that

$$f(-q,-q^8)f(-q^2-q^7)f(-q^4,-q^5) = \frac{f(-q)f^3(-q^9)}{f(-q^3)}. \tag{5.3.4}$$

With the aid of (5.3.1), the above identity can be written as

$$D_1 E_1 F_1 = \frac{f_1 f_9^3}{f_3^4}. \tag{5.3.5}$$

Using (5.3.5) in (5.3.3) we easily arrive at (5.2.1).
Proof of (5.2.2): From Entry 2(vii) [11, p. 349]

$$\frac{f(-q^4,-q^5)}{f(-q^2,-q^7)} + q\frac{f(-q,-q^8)}{f(-q^4,-q^5)} = \frac{f(-q^2,-q^7)}{f(-q-q^8)}. \tag{5.3.6}$$

Using (5.3.1) and (5.3.5) in (5.3.6), we obtain (5.2.2) to complete the proof.

Proof of (5.2.3): Replacing q by q^3 in Entry 2(v) [11, p. 349], we obtain

$$f(-q^{12},-q^{15}) - qf(-q^6,-q^{21}) - q^2 f(-q^3,-q^{24}) = f(-q). \qquad (5.3.7)$$

Dividing both sides by $f(-q^9)$ and using (5.3.1), we complete the proof.

This result can also be obtained from Theorem 5.4 1 in Section 5 by setting $\epsilon_1 = 1$. $\epsilon_2 = 0$, $a = q = b$, $c = 1$, $d = q$, $\alpha = 1$, $\beta = 3$, and $m = 9$.

Proof of (5.2.4): Replacing q by q^3 in Entry 2(iv) [11, p 349] and using (3.2.2) and (3.2.4), we find that

$$f(q^{12},q^{15}) + qf(q^6,q^{21}) + q^2 f(q^3,q^{24}) = \phi(-q^3)/\chi(-q). \qquad (5.3.8)$$

Employing (4.3.8), (5.3.1), and (4.3.12), we complete the proof.

5.4 Second proof of (5.2.3) and proofs of (5.2.5)-(5.2.7)

To present a second proof of (5.2.3) and proofs of (5.2.5) - (5.2.7), we use a formula of R. Bleckmith, J. Brillhart, and I. Gerst [18, Theorem 2], providing a representation for a product of two theta functions as a sum of m products of pair of theta functions, under certain conditions. This formula generalizes formulas of H. Schröter [11, p. 65-72]. Define, for $\epsilon \in \{0,1\}$ and $|ab| < 1$,

$$f_\epsilon(a,b) = \sum_{n=-\infty}^{\infty} (-1)^{\epsilon n} (ab)^{n^2/2} (a/b)^{n/2}. \qquad (5.4.1)$$

Theorem 5.4.1. *Let a, b, c, and d denote positive numbers with $|ab|$, $|cd| < 1$. Suppose that there exist positive integers α, β, and m such that*

$$(ab)^\beta = (cd)^{\alpha(m-\alpha\beta)}. \qquad (5.4.2)$$

Let ϵ_1, $\epsilon_2 \in \{0,1\}$, and define δ_1, $\delta_2 \in \{0,1\}$ by

$$\delta_1 \equiv \epsilon_1 - \alpha\epsilon_2 (mod\ 2) \quad and \quad \delta_2 \equiv \beta\epsilon_1 + p\epsilon_2 (mod\ 2), \qquad (5.4.3)$$

respectively, where $p = m - \alpha\beta$. Then if R denotes any complete residue system modulo m,

$$f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) = \sum_{r\in R}(-1)^{\epsilon_2 r}c^{r(r+1)/2}d^{r(r-1)/2} \times f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^\alpha}, \frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^\alpha}\right) \times f_{\delta_2}\left(\frac{(b/a)^{\beta/2}(cd)^{p(m+1-2r)/2}}{c^p}, \frac{(a\beta)^{\beta/2}(cd)^{p(m+1+2r)/2}}{d^p}\right). \quad (5\ 4\ 4)$$

Second proof of (5.2.3): Applying Theorem 5.4 1 with the parameters $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = 1$, $b = q^8$, $c = q$, $d = q^3$, $\alpha = 2$, $\beta = 3$, and $m = 9$, we find that

$$f(-q^{10}, -q^{14})\{f(-q^{69}, -q^{39}) - qf(-q^{33}, -q^{75}) - q^{11}f(-q^3, -q^{105})\} + qf(-q^2, -q^{22})\{f(-q^{57}, -q^{51}) - q^5f(-q^{21}, -q^{87}) - q^7f(-q^{15}, -q^{93})\} - \psi(-q^6)\{f(-q^{45}, -q^{63}) - q^9f(-q^9, -q^{99}) - q^3f(-q^{27}, -q^{81})\} = 0, \quad (5.4.5)$$

where we also used (1.1.9).

Again, applying Theorem 5.4.1 with $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = q^4$, $b = q^4$, $c = q$, $d = q^3$, $\alpha = 2$, $\beta = 3$, and $m = 9$, we obtain

$$\psi(q)\phi(-q^4) = f(-q^{10}, -q^{14})\{f(-q^{57}, -q^{51}) - q^5f(-q^{21}, -q^{87}) - q^7f(-q^{93}, -q^{15})\} + q^3f(-q^2, -q^{22})\{f(-q^{69}, -q^{39}) - q^{11}f(-q^3, -q^{105}) - qf(-q^{33}, -q^{75})\} + q\psi(-q^6)\{f(-q^{45}, -q^{63}) - q^9f(-q^9, -q^{99}) - q^3f(-q^{27}, -q^{81})\}. \quad (5.4\ 6)$$

Multiplying (5.4.5) by q and adding with (5.4.6), we deduce that

$$\psi(q)\phi(-q^4) = qf(-q^{10}, -q^{14})\{f(-q^{69}, -q^{39}) - q^6f(-q^{15}, -q^{93})\} - f(-q^{10}, -q^{14})[q^2\{f(-q^{33}, -q^{75}) + q^3f(-q^{21}, -q^{87})\} - \{f(-q^{51}, -q^{57}) - q^{12}f(-q^3, -q^{105})\}] + q^2f(-q^2, -q^{22})\{f(-q^{69}, -q^{39}) - q^6f(-q^{15}, -q^{93})\}q - q^2f(-q^2, -q^{22})[\{f(-q^{33}, -q^{75}) + q^3f(-q^{21}, -q^{87})\}q^2 - \{f(-q^{51}, -q^{57}) - q^{12}f(-q^3, -q^{105})\}]. \quad (5.4.7)$$

Employing in turn $a = -q^6$ and $b = q^{21}$; $a = -q^{12}$ and $b = q^{15}$; $a = q^3$ and $b = -q^{24}$ in (4.3.5), we find that

$$f(-q^6, q^{21}) = f(-q^{39}, -q^{69}) - q^6 f(-q^{15}, -q^{39}), \qquad (5.4.8)$$

$$f(-q^{12}, q^{15}) = f(-q^{51}, -q^{57}) - q^{12} f(-q^3, -q^{105}), \qquad (5.4.9)$$

$$f(q^3, -q^{24}) = f(-q^{33}, -q^{75}) + q^3 f(-q^{21}, -q^{87}) \qquad (5.4.10)$$

Applying (5.4.8), (5.4.9), (5.4.10) in (5.4.7), we obtain

$$\psi(q^3)\phi(-q^4) = \{f(-q^{10}, -q^{14}) + q^2 f(-q^2, -q^{22})\} \times \{q f(-q^6, q^{21}) - q^2 f(q^3, -q^{24}) + f(-q^{12}, q^{15})\} \qquad (5.4.11)$$

Again, putting $a = q^2$, $b = q^4$, $c = q$, $d = q^5$ in (2.2.9) and (2.2.10), we find that

$$f(q^2, q^4) f(q, q^5) + f(-q^2, -q^4) f(-q, -q^5) = 2 f(q^3, q^9) f(q^5, q^7), \qquad (5.4.12)$$

and

$$f(q^2, q^4) f(q, q^5) - f(-q^2, -q^4) f(-q, -q^5) = 2 f(q^3, q^9) f(q^{-1}, q^{13}). \qquad (5.4.13)$$

Employing (1.1.9), (1.1.10), (3.2.1), and (3.2.2), in the above two identities can be written as

$$2q f(q, q^{11}) = \psi^2(-q^3)/\psi(q^6)\chi(-q) - f(-q^2)\chi(-q), \qquad (5.4.14)$$

and

$$2 f(q^5, q^7) = \psi^2(-q^3)/\psi(q^6)\chi(-q) + f(-q^2)\chi(-q). \qquad (5.4.15)$$

Replacing q by $-q$ in (5.4.14) and (5.4.15), and then using (2.2.7) and (2.2.3), we find that

$$2 f(-q^5, -q^7) = \phi(q^3)/\chi(q) + f(q) \qquad (5.4.16)$$

and

$$2q f(-q, -q^{11}) = \phi(q^3)/\chi(q) - f(q). \qquad (5.4.17)$$

Adding (5.4.16) and (5.4.17), we obtain

$$f(-q^5, -q^7) + q f(-q, -q^{11}) = f(q). \qquad (5.4.18)$$

Replacing q by q^2 in (5.4.18), and then using the resulting identity (5.4.11), we deduce that

$$\psi(q^3)\phi(-q^4) = f(q^2)\{q f(-q^6, q^{21}) - q^2 f(q^3, -q^{24}) + f(-q^{12}, q^{15})\}. \qquad (5.4.19)$$

Dividing both sides by $f(q^9)$, using (4.3.11), (4.3.12), (5.3.1), and replacing q by $-q$, we arrive at (5.2.3) to finish the proof

Proof of (5.2.5): Applying Theorem 5.4.1 with the parameters $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = q^{10} = b$, $c = q$, $d = 1$, $\alpha = 5$, $\beta = 1$, and $m = 9$, we find that

$$\begin{aligned}\phi(-q^{10})\psi(q) &= f(-q^{20}, -q^{25})f(-q^{16}, -q^{20}) + qf_{15}f_{12} \\ &\quad + q^3 f(q^{10}, -q^{35})f(-q^8, -q^{28}) + q^6 f(-q^5, -q^{40})f(-q^4, -q^{32}).\end{aligned} \tag{5.4.20}$$

Using (4.3.11) and (4.3.12), in (5.4.20), we readily arrive at (5.2.5).

In a similar way, we can obtain the identities (5.2.7) and (5.2.6) by setting $m = 9$, $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = b = q^2$, $c = q^5$, $d = 1$, $\alpha = 1$, $\beta = 5$ and $m = 9$, $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = b = q^9$, $c = 1$, $d = q$, $\alpha = 6$, $\beta = 1$. respectively, in Theorem 5.4.1.

5.5 Proofs of (5.2.8)-(5.2.23)

In this section, we use the results of Rogers [32] and Bressoud [19]. We adopt Bressoud's notation, except that we use $q^{n/24}f(-q^n)$ instead of P_n, and the variable q instead of x. Let $g_\alpha^{(p,n)}$ and $\phi_{\alpha,\beta,m,p}$ be defined as

$$\begin{aligned}g_\alpha^{(p,n)}(q) &:= g_\alpha^{(p,n)}(q) \\ &= q^{\alpha(\frac{12n^2-12n+3-p}{24p})} \prod_{r=0}^{\infty} \frac{(1-(q^\alpha)^{pr+(p-2n+1)/2})(1-(q^\alpha)^{pr+(p+2n-1)/2})}{\prod_{k=1}^{p-1}(1-(q^\alpha)^{pr+k})}.\end{aligned} \tag{5.5.1}$$

For any positive odd integer p, integer n, and natural number α, let

$$\begin{aligned}\phi_{\alpha,\beta,m,p} &:= \phi_{\alpha,\beta,m,p}(q) \\ &= \sum_{n=1}^{p} \sum_{r,s=-\infty}^{\infty} (-1)^{r+s} q^{1/2\{p\alpha(r+m(2n-1)/2p)^2+p\beta(s+(2n-1)/2p)^2\}},\end{aligned} \tag{5.5.2}$$

where α, β, and p are natural numbers, and m is an odd positive integer. Then we can obtain immediately the following propositions.

Proposition 5.5.1. *[19, equations (2.12) and (2.13)] We have*

$$g_\alpha^{(5,1)} = q^{-\alpha/60}G_\alpha, \tag{5.5.3}$$

$$g_\alpha^{(5,2)} = q^{-11\alpha/60}H_\alpha. \tag{5.5.4}$$

Proposition 5.5.2. *[24, equations (6.3), (6.4), and (6.5)] We have*

$$g_\alpha^{(7,1)} = q^{-\alpha/42} f_{2\alpha} A_\alpha / f_\alpha, \tag{5.5.5}$$

$$g_\alpha^{(7,2)} = q^{5\alpha/42} f_{2\alpha} B_\alpha / f_\alpha, \tag{5.5.6}$$

$$g_\alpha^{(7,3)} = q^{17\alpha/42} f_{2\alpha} C_\alpha / f_\alpha \tag{5.5.7}$$

Proposition 5.5.3. *We have*

$$g_\alpha^{(9,1)} = q^{-\alpha/36} f_{3\alpha} D_\alpha / f_\alpha, \tag{5.5.8}$$

$$g_\alpha^{(9,2)} = q^{\alpha/12} f_{3\alpha} / f_\alpha, \tag{5.5.9}$$

$$g_\alpha^{(9,3)} = q^{11\alpha/36} f_{3\alpha} E_\alpha / f_\alpha, \tag{5.5.10}$$

$$g_\alpha^{(9,4)} = q^{23\alpha/36} f_{3\alpha} F_\alpha / f_\alpha \tag{5.5.11}$$

Proof: Taking $p = 9$, and $n = 1$ in (5.5.1), we find that

$$\begin{aligned} g_\alpha^{(9,1)} =& q^{-\alpha/36} \prod_{r=0}^{\infty} \frac{(1-(q^\alpha)^{9r+4})(1-(q^\alpha)^{9r+5})}{\prod_{k=1}^{8}(1-(q^\alpha)^{9r+k})} \\ =& \frac{q^{-\alpha/36}}{(q^\alpha;q^{9\alpha})(q^{2\alpha};q^{9\alpha})(q^{3\alpha};q^{9\alpha})(q^{6\alpha};q^{9\alpha})(q^{7\alpha};q^{9\alpha})(q^{8\alpha};q^{9\alpha})}. \end{aligned} \tag{5.5.12}$$

Therefore, we obtain $g_\alpha^{(9,1)} = q^{-\alpha/36} f_{3\alpha} D_\alpha / f_\alpha$

In a similar fashion, we can prove (5.5.9) - (5.5.11).

Lemma 5.5.4. *[19, Proposition 5.1] We have*

$$g_\alpha^{(p,n)} = g_\alpha^{(p,n-2p)}, \quad g_\alpha^{(p,n)} = g_\alpha^{(p,-n+1)}$$

$$g_\alpha^{(p,n)} = g_\alpha^{(p,2p-n+1)}, \quad g_\alpha^{(p,n)} = -g_\alpha^{(p,n-p)},$$

$$g_\alpha^{(p,n)} = -g_\alpha^{(p,p-n+1)}, \quad \text{and} \quad g_\alpha^{(p,(p+1)/2)} = 0.$$

Theorem 5.5.5. *[19, Proposition 5.4] For odd $p > 1$,*

$$\phi_{\alpha,\beta,m,p} = 2q^{\alpha+\beta/24} f_\alpha f_\beta) \left(\sum_{n=1}^{(p-1)/2} g_\beta^{(p,n)} g_\alpha^{(p,(2mn-m+1)/2)} \right). \tag{5.5.13}$$

Lemma 5.5.6. *[26, Lemma 5.1] We have*

$$\phi_{\alpha,\beta,1,4} = 2q^{(\alpha+\beta)/32}\frac{f_{2\alpha}f_{2\beta}f_{\alpha/2}f_{\beta/2}}{f_\alpha f_\beta} \times \{S(q^{\beta/2})S(q^{\alpha/2}) + q^{(\alpha+\beta)/4}T(q^{\beta/2})T(q^{\alpha/2})\}. \tag{5.5.14}$$

$$\phi_{\alpha,\beta,3,4} = 2q^{(9\alpha+\beta)/32}\frac{f_{2\alpha}f_{2\beta}f_{\alpha/2}f_{\beta/2}}{f_\alpha f_\beta} \times \{S(q^{\beta/2})T(q^{\alpha/2}) - q^{(\beta-\alpha)/2}S(q^{\alpha/2})T(q^{\beta/2})\}. \tag{5.5.15}$$

Lemma 5.5.7. *[19, Lemma 6.5] We have*

$$\phi_{\alpha,\beta,1,5} = 2q^{(\alpha+\beta)/40}f_\alpha f_\beta\{G_\beta G_\alpha + q^{(\alpha+\beta)/5}H_\beta H_\alpha\}. \tag{5.5.16}$$

$$\phi_{\alpha,\beta,3,5} = 2q^{(9\alpha+\beta)/40}f_\alpha f_\beta\{G_\beta H_\alpha - q^{(-\alpha+\beta)/5}H_\beta G_\alpha\} \tag{5.5.17}$$

Lemma 5.5.8. *[24, Lemma 6.6] We have*

$$\phi_{\alpha,\beta,1,7} = 2q^{(\alpha+\beta)/56}f_{2\alpha}f_{2\beta}\{A_\beta A_\alpha + q^{(\alpha+\beta)/7}B_\beta B_\alpha + q^{(3\alpha+3\beta)/7}C_\beta C_\alpha\}, \tag{5.5.18}$$

$$\phi_{\alpha,\beta,3,7} = 2q^{(9\alpha+\beta)/56}f_{2\alpha}f_{2\beta}\{A_\beta B_\alpha - q^{(2\alpha+\beta)/7}B_\beta C_\alpha - q^{(-\alpha+3\beta)/7}C_\beta A_\alpha\}, \tag{5.5.19}$$

$$\phi_{\alpha,\beta,5,7} = 2q^{(25\alpha+\beta)/56}f_{2\alpha}f_{2\beta}\{A_\beta C_\alpha - q^{(-3\alpha+\beta)/7}B_\beta A_\alpha + q^{(-2\alpha+3\beta)/7}C_\beta B_\alpha\}. \tag{5.5.20}$$

Lemma 5.5.9. *We have*

$$\phi_{\alpha,\beta,1,9} = 2q^{(\alpha+\beta)/72}f_{3\alpha}f_{3\beta}\{D_\alpha D_\beta + q^{(\alpha+\beta)/9} + q^{(\alpha+\beta)/3}E_\alpha E_\beta + q^{2(\alpha+\beta)/3}F_\alpha F_\beta\}. \tag{5.5.21}$$

$$\phi_{\alpha,\beta,3,9} = 2q^{(9\alpha+\beta)/72}f_{3\alpha}f_{3\beta}\{D_\beta - q^{\beta/3}E_\beta - q^{2\beta/3}F_\beta\}. \tag{5.5.22}$$

$$\phi_{\alpha,\beta,5,9} = 2q^{(25\alpha+\beta)/72}f_{3\alpha}f_{3\beta}\{D_\beta E_\alpha - q^{(\beta-2\alpha)/9} - q^{(\alpha+\beta)/3}E_\beta F_\alpha + q^{(2\beta-\alpha)/3}F_\beta D_\alpha\}. \tag{5.5.23}$$

$$\phi_{\alpha,\beta,7,9} = 2q^{(49\alpha+\beta)/72}f_{3\alpha}f_{3\beta}\{D_\beta F_\alpha - q^{(\beta-5\alpha)/9} + q^{(\beta-2\alpha)/3}E_\beta D_\alpha - q^{(2\beta-\alpha)/3}F_\beta E_\alpha\}. \tag{5.5.24}$$

Proof: Applying Theorem 5.5.5 with $m = 1$ and $p = 9$, we find that

$$\phi_{\alpha,\beta,1,9} = 2q^{(\alpha+\beta)/24} f(-q^{\alpha}) f(-q^{\beta}) \{ g_{\beta}^{(9,1)} g_{\alpha}^{(9,1)} + g_{\beta}^{(9,2)} g_{\alpha}^{(9,2)} + g_{\beta}^{(9,3)} g_{\alpha}^{(9,3)} + g_{\beta}^{(9,4)} g_{\alpha}^{(9,4)} \}. \tag{5.5.25}$$

Using (5.5.8) - (5.5.11) in (5.5.25) and then simplification, we arrive at (5.5.21). The identities (5.5.22), (5.5.23), and (5.5.24) can be proved in a similar way by setting $m = 3$, 5, and 7, respectively, and $p = 9$ in Theorem 5.5.5.

Corollary 5.5.10. *[19, Corollary 5.5 and 5.6] If $\phi_{\alpha,\beta,m,p}$ is defined by* (5.5.2), *then*

$$\phi_{\alpha,\beta,m,1} = 0, \tag{5.5.26}$$

$$\phi_{\alpha,\beta,1,3} = 2q^{(\alpha+\beta)/24} f(-q^{\alpha}) f(-q^{\beta}). \tag{5.5.27}$$

Corollary 5.5.11. *[19, Corollary 5.11] If α and β are even positive integers, then*

$$\phi_{\alpha,\beta,1,2} = 2q^{(\alpha+\beta)/16} \frac{f_{2\alpha} f_{2\beta} f_{\alpha/2} f_{\beta/2}}{f_{\alpha} f_{\beta}}. \tag{5.5.28}$$

Theorem 5.5.12. *[19, Corollary 7.3] Let α_i, β_i, m_i, p_i, where $i = 1,2$ be positive integers with m_1, m_2 be odd. Let $\lambda_1 := (\alpha_1 m_1^2 + \beta_1)/p_1$ and $\lambda_2 := (\alpha_2 m_2^2 + \beta_2)/p_2$. If the conditions*
$\lambda_1 = \lambda_2$, $\alpha_1\beta_1 = \alpha_2\beta_2$, and $\alpha_1 m_1 = \pm\alpha_2 m_2 (mod\ \lambda_1)$
hold, then $\phi_{\alpha_1,\beta_1,m_1,p_1} = \phi_{\alpha_2,\beta_2,m_2,p_2}$.

Next, let N denote the set of positive integers, and N_0 denote the set of non-negative integers.

Proposition 5.5.13. *For $p \epsilon N$, we have*

$$\phi_{p,2p,5,9} = \phi_{2p,p,1,1}. \tag{5.5.29}$$

Furthermore, the identity (5.2.8) *holds.*

Proof: By setting $\alpha_1 = p$, $\beta_1 = 2p$, $m_1 = 5$, $p_1 = 9$, $\alpha_2 = 2p$, $\beta = p$, $m_2 = 1$, and $p_2 = 1$, we see that the equality (5.5.29) holds by Theorem 5.5.12.

Using (5.5.26) in (5.5.29) we obtain

$$\phi_{p,2p,5,9} = 0. \tag{5.5.30}$$

In particular, by taking $p = 1$ in (5.5.30) and then using (5.5.23), we obtain the identity (5.2.8).

Proposition 5.5.14. *[19, Proposition (8.1)] Let p be an odd integer* ≥ 5, *then*

$$\phi_{1,p-4,p-2,p} = 0. \tag{5.5.31}$$

Furthermore, the identity (5.2.9) holds.

Proof: Setting $p = 9$ and using (5.5.24), we readily obtain (5.2.9).

This result can also be proved by setting $\epsilon_1 = 1$, $\epsilon_2 = 1$, $a = 1$, $b = q^5$, $c = 1$, $d = q$, $\alpha = 2$, $\beta = 2$, and $m = 9$ in Theorem 5.4.1 .

Proposition 5.5.15. *[26, Proposition (5.4)] For* $p > 1$,

$$\phi_{1,p-1,1,p} = q^{1/4} f(1, q^2) f(-q^{p-1}, -q^{p-1}). \tag{5.5.32}$$

Furthermore, the identity (5.2.10) holds.

Proof: Setting $p = 9$ and using (5.5.21), we readily obtain the identity (5.2.10).

This identity can also be proved by setting $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = q^4 = b$, $c = 1$, $d = q$, $\alpha = 1$, $\beta = 1$, and $m = 9$ in Theorem 5.4.1.

Proposition 5.5.16. *[19, Proposition (8.5)] Let p be an odd integer* ≥ 7, *then*

$$\phi_{1,3p-16,p-4,p} = \phi_{1,3p-16,1,3}. \tag{5.5.33}$$

Furthermore, the identity (5.2.11) holds.

Proof: Setting $p = 9$ in (5.5.33) and then using (5.5.23) and (5.5.27), we obtain the required identity.

Proposition 5.5.17. *[19, Proposition (8.11)] Let p be an odd integer* ≥ 3, *then*

$$\phi_{2,p-2,1,p} = 2q^{1/8} \prod_{n=0}^{\infty} (1 + q^{(n+1)})^2 (1 - q^{n+1})(1 - (q^{p-2})^{2n+1})^2 (1 - (q^{p-2})^{2n+2}). \tag{5.5.34}$$

Furthermore, the identity (5.2.12) holds.

Proof: Setting $p = 9$ in (5.5.34), we find that

$$\phi_{2,7,1,9} = q^{-1/4} \prod_{n=0}^{\infty} \frac{(1 - (q^7)^{2n+1})}{(1 - q^{(2n+1)})} = q^{-1/4} \chi(-q^7)/\chi(-q). \tag{5.5.35}$$

Employing (5.5.21), (2.2.3), in (5.5.35), we easily arrive at (5.2.12).

This result can also be proved by applying Theorem 5.4.1 with $m = 9$, $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = b = q^7$, $c = 1$, $d = q$, $\alpha = 2$, and $\beta = 1$.

Proposition 5.5.18. *[19, Proposition (8.8)] Let p be an odd integer* ≥ 3, *then*

$$\phi_{1,2p-4,p-2,p} = 2q^{(p-2)/8}\prod_{n=0}^{\infty}(1+q^{(p-2)(n+1)})^2(1-q^{(p-2)(n+1)})(1-q^{2n+1})^2(1-q^{2n+2}). \tag{5.5.36}$$

Furthermore, the identity (5.2.13) holds.

Proof: Setting $p = 9$ in (5.5.36), we find that

$$\begin{aligned}\phi_{1,14,7,9} &= 2q^{7/8}\prod_{n=0}^{\infty}(1+q^{7(n+1)})^2(1-q^{7(n+1)})(1-q^{2n+1})^2(1-q^{2n+2})\\ &= 2q^{7/8}f_2f_7\chi^2(-q)/\chi^2(-q^7).\end{aligned} \tag{5.5.37}$$

Invoking (5.5.24). (2.2.3). in (5.5 37), we deduce the required identity.

This result can also proved by employing Theorem 5 4 1 with $m = 9$, $\epsilon_1 = 0$, $\epsilon_2 = 1$, $a = 1$, $b = q^7$, $c = q$, $d = q$, $\alpha = 1$, and $\beta = 2$

Proposition 5.5.19. *[19, Proposition (8.3)] Let p be an odd integer* ≥ 5, *then*

$$\phi_{1,3p-4,p-2,p} = \phi_{1,3p-4,1,3}. \tag{5.5.38}$$

Furthermore, the identity (5.2.14) holds.

Proof: Setting $p = 9$ in (5.5.38) and using (5.5 24) and (5 5.27), we easily deduce (5.2.14).

Proposition 5.5.20. *For* $p\epsilon N$, *we have*

$$\phi_{p+14,p,1,2} = \phi_{1,p^2+14p,7,p+7}. \tag{5.5 39}$$

Furthermore, the identity (5.2.15) holds.

Proof: The equality (5.5.39) follows from Theorem 5.5.12 with $\lambda_1 = \lambda_2 = p + 7$. Furthermore, by setting $p = 2$, and using (5 5.24) and (5.5 28), we readily arrive at (5.2.15).

Proposition 5.5.21. *For* $p\epsilon N$, *we have*

$$\phi_{2p+2,p+4,1,3} = \phi_{2,p^2+5p+4,1,p+3} \tag{5 5.40}$$

Furthermore, the identity (5.2.16) holds.

Proof: The equality (5.5.40) follows from Theorem 5.5 12 with $\lambda_1 = \lambda_2 = p+2$. In particular, if we set $p = 6$ and use (5.5.21) and (5.5 27), we deduce the proffered identity.

Proposition 5.5.22. *[19, Equation (8.12)] Let p be an odd integer ≥ 5. Then*

$$\phi_{2,3p-8,p-2,p} = \phi_{1,16p-16,1,3}. \tag{5.5.41}$$

Furthermore, the identity (5.2.17) holds.

Proof: Setting $p = 9$ in (5.5.41), we derive the identity (5 2.17) by employing (5.5.24) and (5.5.27).

Proposition 5.5.23. *[19, Proposition (8.13)] Let p be an odd integer ≥ 5. Then*

$$\phi_{2,3p-8,1,3} = \phi_{1,6p-16,p-4,p}. \tag{5.5.42}$$

Furthermore, the identity (5.2.18) holds.

Proof: Setting $p = 9$ in (5.5.42), we obtain the identity (5.2.18) by using the identity (5.5.23) and (5.5.27).

Proposition 5.5.24. *[24, Proposition (6.19)] For $p \epsilon N$*

$$\phi_{2,p^2+3p,1,p+1} = \phi_{2p+6,p,1,3}, \tag{5.5.43}$$

Furthermore, the identity (5.2.19) holds.

Proof: Setting $p = 8$ in (5.5 43), we obtain the identity (5.2.19) by using (5.5.21) and (5.5.27).

Proposition 5.5.25. *[24, Proposition (6.19)] For $p \epsilon N$, we have*

$$\phi_{1,p^2+10p,5,p+5} = \phi_{p+10,p,1,2}. \tag{5.5.44}$$

Furthermore, the identity (5.2.20) holds.

Proof: Setting $p = 4$ in (5.5.44) and then using (5 5.23) and (5.5.28), we immediately obtain (5.2.20).

Proposition 5.5.26. *[19, Equation (8.14)] Let p be an odd integer ≥ 5. Then*

$$\phi_{3,4p-12,p-2,p} = \phi_{2,6p-18,1,2}. \tag{5.5.45}$$

Furthermore, the identity (5.2.21) holds.

Proof: Setting $p = 9$ in (5.5.45), we easily arrive at (5.2 21) with the help of (5.5.24) and (5.5.28).

Proposition 5.5.27. *For $p \epsilon N$, we have*

$$\phi_{p+1,6p^2,7,p+7} = \phi_{p,6p(p+1),1,p}. \qquad (5.5.46)$$

Furthermore, the identity (5.2.21) holds.

Proof: The equality (5.5.46) follows from Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 6p+7$. In particular, if we set $p = 2$ and proceed as in the above proposition, we obtain (5.2.21).

Proposition 5.5.28. *For $p \epsilon N$, we have*

$$\phi_{1,p^2+18p+80,1,p+9} = \phi_{p+8,p+10,1,2}. \qquad (5.5.47)$$

Furthermore, the identity (5.2.22) holds.

Proof: The equality (5.5.47) follows from Theorem 5.5.12 with $\lambda_1 = \lambda_2 = p+9$. In particular, if we let $p = 0$ and use (5.5.21) and (5.5 28). we deduce (5.2.22).

Proposition 5.5.29. *[24, Proposition (6.26)] For $p \epsilon N$, we have*

$$\phi_{1,16p^3+172p^2+472p+195,2p+1,p+7} = 0, \qquad (5.5\ 48)$$

Furthermore, the identity (5.2.23) holds.

Proof: Setting $p = 2$ in (5.5.48) and then using (5.5.23) and (5.5.26), we obtain (5.2.23).

5.6 Proofs of (5.2.24)-(5.2.32)

Proposition 5.6.1. *For an odd number p, we have*

$$\phi_{p+1,p^2+4p+4,p+2,(p+2)^2} = \phi_{1,(p+1)(p+2)^2,p+2,(p+2)^2}. \qquad (5.6.1)$$

Furthermore, the identity (5.2.24) holds.

Proof: The equality (5.6.1) follows from Theorem 5.5.12 with $\lambda_1 = \lambda_2 = p+2$. Furthermore, by setting $p = 1$ and using (5.5.22), we easily deduce (5.2.24).

Remark: The identity (5.2.24) also follows from (5.2.3).

Proposition 5.6.2. *Let p be an even number and $p > 3$. Then*

$$\phi_{1,3p^2-9,p-3,2p-3} = \phi_{3,p^2-3,p-1,2p-3}. \tag{5.6.2}$$

Furthermore, the identity (5.2.25) holds.

Proof: The equality (5.6.2) follows from Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 2p$. Furthermore by setting $p = 6$ and then employing (5.5.22) and (5.5.23), we deduce (5.2.25).

Remark: The identity (5.2.25) also follows from (5.2.3) auc (5.2.11).

Proposition 5.6.3. *[25, Proposition (3.4.1)] For $p \epsilon N$, we have*

$$\phi_{p+4,2p^2+9p,3,p+3} = \phi_{p,2p^2+17p+36,1,p+3}, \tag{5.6.3}$$

Furthermore, the identity (5.2.26) holds.

Proof: Setting $p = 6$, in (5.6.3) and then using (5.5.21) and (5.5.22), we obtain the identity (5.2.26).

Remark: The identity (5.2.26) also follows from (5.2.3) and (5.2.16).

Proposition 5.6.4. *[19, Proposition(8.16)] Let p be an odd integer ≥ 5. Then*

$$\phi_{1,3p^2-36,|p-6|,p} = \phi_{3,p^2-12,p-2,p}. \tag{5.6.4}$$

Furthermore, the identity (5.2.27) holds.

Proof: Setting $p = 9$ in (5.6.4) and then using (5.5.22) and (5.5.24), we obtain the identity (5.2.27).

Remark: The identity (5.2.27) also follows from (5.2.3) and (5.2.14).

Proposition 5.6.5. *[19, Corollary (9.2)] Let p be an odd integer ≥ 3. Then*

$$\phi_{1,3p-2,1,p} \cdot \phi_{1,6p-4,1,3} = \phi_{2,3p-2,1,3} \phi_{1,6p-4,p-2,p}. \tag{5.6.5}$$

Furthermore, the identity (5.2.28) holds.

Proof; Setting $p = 9$ in (5.6.5) and then using (5.5.21), (5.5.24), and (5.5.27), we easily arrive at (5.2.27).

Proposition 5.6.6. *[19, Propositin (8.17)] Let p be an odd integer ≥ 5. Then*

$$\phi_{1,2p^2-16,p-4,p} = \phi_{2,p^2-8,p-2,p}. \tag{5.6.6}$$

Furthermore, the identity (5.2.29) holds.
Proof: If we set $p = 9$ in (5.6.6), we arrive at (5.2.29) by means of (5.5 23) and (5.5.24).

Proposition 5.6.7. *[19, Propositin (8.18)] Let p be an odd integer ≥ 3. Then*

$$\phi_{p-1,p^2-4p+4,p-2,p} = \phi_{1,p^3-5p^2+8p-4,p-2,p}. \tag{5.6.7}$$

Furthermore, the identity (5.2.30) holds.
Proof: If we set $p = 9$ in (5.6.7), we arrive at (5.2 30) by using (5.5.24).

Proposition 5.6.8. *For p be an odd positive integer and $p > 4$, we have*

$$\phi_{p-3,p^2-8p+16,p-4,p-2} = \phi_{1,p^3-11p^2+40p-48,p-4,p-2} \tag{5.6.8}$$

Furthermore, the identity (5.2.30) holds.
Proof: Equality (5.6.8) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = (p-4)^2$. Taking $p = 11$ and proceeding in the same line as in the previous proposition, we obtain easily (5.2.30).

Proposition 5.6.9. *For $p \epsilon N$, we have*

$$\phi_{p,68p,7,9} = \phi_{4p,17p,5,9}. \tag{5.6.9}$$

Furthermore, the identity (5.2.31) holds.
Proof: Equality (5.6.9) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 13p$. If we set $p = 1$, we obtain (5.2.31) with the aid of (5 5.23) and (5 5.24).

Proposition 5.6.10. *For $p \epsilon N$, we have*

$$\phi_{p,260p,1,9} = \phi_{4p,65p,7,9}. \tag{5.6.10}$$

Furthermore, the identity (5.2.32) holds.
Proof: Equality (5.6.10) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 29p$. If we set $p = 1$, we obtain (5.2.32) with the aid of (5.5.21) and (5.5 24).

5.7 Proofs of (5.2.33)-(5.2.41)

Proposition 5.7.1. *For $p \epsilon N$, we have*

$$\phi_{5p,5p+40,3,5p+4} = \phi_{5p\ 5p+40,1,p+4}, \tag{5.7 1}$$

Furthermore, the identity (5 2.33) holds.

Proof: Equality (5.7.1) holds by Theorem 5.5 12 with $\lambda_1 = \lambda_2 = 10$. Setting $p = 1$ in (5.7.1), using (5.5.16) and (5.5 22), and then replacing q^5 by q, we arrive at (5.2.33).

Remark: From (5.2 3) and (5.2.33), we find that

$$G(q^9)G(q) + q^2 H(q^9)H(q) = \frac{f_3^2}{f_1 f_9},$$

which is the sixth identity of Ramanujan's forty identities [16].

Proposition 5.7.2. *For $p \in N$, we have*

$$\phi_{4,p(p+5),1,p+4} = \phi_{p+5,4p,1,5}. \tag{5 7 2}$$

Furthermore, the identity (5.2.34) holds.

Proof: Equality (5.7.2) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = p + 1$. Setting $p = 5$, using (5.5.16), and (5.5.21), and then replacing q^2 by q, we obtain (5.2.34).

Proposition 5.7.3. *[25, Proposition (3.4.11)] For $p \in N$, we have*

$$\phi_{6,p^2+5p,1,p+3} = \phi_{2p+10,3p,1,5}. \tag{5.7.3}$$

Furthermore, the identity (5.2.35) holds.

Proof: Setting $p = 6$ in (5.7.3), we obtain the identity (5.2.35) by means of (5.5.21) and (5.5.16), and finally replacing q^2 by q.

Proposition 5.7.4. *[25, Proposition (3.4.21)] For $p \in N$, we have*

$$\phi_{6,p^2+5p,1,p+2} = \phi_{3p+15,2p,1,5}, \tag{5.7.4}$$

Furthermore, the identity (5.2.36) holds.

Proof: Setting $p = 7$ in (5.7.4) and using (5.5.21) and (5.5 16), and then replacing q^2 by q, we finish the proof.

Proposition 5.7.5. *For $p \in N$, we have*

$$\phi_{p,4p+5,1,5} = \phi_{4p^2+5p,1,1,4p+1}. \tag{5.7.5}$$

Furthermore, the identity (5.2.37) holds.
Proof: Equality (5.7.5) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = p+1$. If we set $p=2$, we obtain (5.2.37) with the aid of (5.5.16) and (5.5.21).

Proposition 5.7.6. *For $p \epsilon N$, we have*

$$\phi_{p,5p+24,5,5p+4} = \phi_{5p+24,p,1,p+4} \tag{5 7.6}$$

Furthermore, the identity (5.2.38) holds.
Proof: Equality (5.7.6) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 6$. If we set $p=1$, we obtain (5.2.38) by means of (5.5.16) and (5.5.23).

Proposition 5.7.7. *For $p \epsilon N$, we have*

$$\phi_{p,74p,5,9} = \phi_{2p,37p,3,5}. \tag{5.7.7}$$

Furthermore, the identity (5 2.39) holds.
Proof: Equality (5.7.7) holds by Theorem 5 5 12 with $\lambda_1 = \lambda_2 = 11p$. If we set $p=1$, we obtain (5.2.39) by means of (5 5 17) and (5.5 23)

Proposition 5.7.8. *For $p \epsilon N$, we have*

$$\phi_{p,116p,1,9} = \phi_{4p,29p,3,5}. \tag{5.7.8}$$

Furthermore, the identity (5.2.40) holds.
Proof: Equality (5.7.8) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 13p$. If we set $p=1$, we obtain (5.2.40) by means of (5.5.17) and (5.5.21).

Proposition 5.7.9. *For $p \epsilon N$, we have*

$$\phi_{p,125p,1,9} = \phi_{5p,25p,3,5}. \tag{5.7.9}$$

Furthermore, the identity (5.2.41) holds.
Proof: Equality (5.7.9) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 14p$. If we set $p=1$, we obtain (5.2.41) by means of (5.5.17) and (5.5.21).

5.8 Proofs of (5.2.42)-(5.2.53)

Proposition 5.8.1. *[26, Proposition(7.7)] For $p \epsilon N$ and p even, we have*

$$\phi_{4,12p+21,p+1,2p+5} = \phi_{6,8p+14,1,4}. \tag{5.8.1}$$

Furthermore the identity (5.2.42) holds.
Proof: Setting $p = 2$ in (5.8.1) and employing (5.5.22) and (5.5.14), then replacing q^3 by q in the resulting identity, we complete the proof.

Remark: The identity (5.2.42) also follows from (5.2.3) and (4.2.6).

Proposition 5.8.2. *[26, Proposition(7.5)] For $p \epsilon N$ and p even, we have*

$$\phi_{1,24p+84,p-1,p+5} = \phi_{6,4p+14,3,4}. \tag{5.8.2}$$

Furthermore, the identity (5.2.43) holds.
Proof: Setting $p = 4$ in (5.8.2) and using (5.5.22) and (5.5.15), and replacing q^3 by q in the resulting identity, we deduce (5.2.43).

Remark: The identity (5.2.43) also follows from (5.2.3) and (4.2.7).

Proposition 5.8.3. *[26, Proposition(7.3)] For $p \epsilon N$ and p even, we have*

$$\phi_{1,8p+36,p+3,p+5} = \phi_{2,4p+18,3,4}. \tag{5.8 3}$$

Furthermore, the identity (5.2.44) holds.
Proof: Setting $p = 4$ in (5.8.3) and using (5.5.24) and (5.5.15), we readily arrive at (5.2.44).

Proposition 5.8.4. *[26, Proposition(7.4)] For $p \epsilon N$ and p even, we have*

$$\phi_{1,16p+64,p+1,p+5} = \phi_{4,4p+16,3,4}. \tag{5.8.4}$$

Furthermore, the identity (5.2.45) holds.
Proof: Setting $p = 4$ in (5.8.4), we obtain the identity (5 2 45) by means of (5.5.23) and (5.5.15).

Proposition 5.8.5. *[26, Propositin (7.6)] For $p \epsilon N$ and p even, we have*

$$\phi_{3,8p+28,p+3,p+5} = \phi_{2,12p+42,1,4}. \tag{5.8.5}$$

Furthermore, the identity (5.2.46) holds.
Proof: If we let $p = 4$ in (5.8.5), we arrive at (5.2.46) with the help of (5.5.24) and (5.5.14).

Proposition 5.8.6. *[26, Proposition(8.1)] For $p \epsilon N$ and p even, we have*

$$\phi_{2,12p+18,3,4} \cdot \phi_{3,8p+12,p+1,p+3} = \phi_{6,4p+6,1,p+3}\phi_{2,12p+18,1,2}. \tag{5.8.6}$$

Furthermore, the identity (5.2.47) holds.
Proof: Setting $p = 6$ in (5.8.6), we find that

$$\phi_{2,90,3,4} \cdot \phi_{3,60,7,9} = \phi_{6,30,1,9}\phi_{2,90,1,2}, \tag{5.8.7}$$

Using (5.8.5) with $p = 4$ in (5.8.7), we deduce that

$$\phi_{2,90,3,4} \cdot \phi_{2,90,1,4} = \phi_{6,30,1,9}\phi_{2,90,1,2}, \tag{5.8.8}$$

Employing (5.5.21), (5.5.28), (5.5.14) and (5.5.15), we deduce the required identity.

Proposition 5.8.7. *For $p \epsilon N$, we have*

$$\phi_{18p,10p,1,4} = \phi_{3p,60p,1,9}. \tag{5.8.9}$$

Furthermore, the identity (5.2.48) holds.
Proof: Equality (5.8.9) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 7p$. If we set $p = 1$, we obtain (5.2.48) via (5.5.14) and (5.5.21)

Proposition 5.8.8. *[26, Proposition (7.9)] For $p \epsilon N$ and p even, we have*

$$\phi_{1,32p+96,p-3,p+5} = \phi_{8,4p+12,3,4}. \tag{5.8.10}$$

Furthermore, the identity (5.2.49) holds.
Proof: Setting $p = 4$ in (5.8.10) and then using (5.5.21) and (5.5.15), we arrive at (5.2.49).

Proposition 5.8.9. *[26, Proposition (7.10)] For $p \epsilon N$ and p even, we have*

$$\phi_{3,16p+32,p+1,p+5} = \phi_{4,12p+24,1,4}. \tag{5.8.11}$$

Furthermore, the identity (5.2.50) holds.
Proof: If we let $p = 4$ in (5.8.11), we arrive at (5.2.50) with the help of (5.5.23) and (5.5.14).

Proposition 5.8.10. *[26, Proposition (7.8)] For $p \epsilon N$ and p even, we have*

$$\phi_{7,8p+12,p+3,p+5} = \phi_{2,28p+42,3,4}. \tag{5.8.12}$$

Furthermore, the identity (5.2.51) holds.
Proof: Setting $p = 4$ in (5.8.12), we obtain (5.2.51) by invoking (5.5.24) and (5.5.15).

Proposition 5.8.11. *[24, Proposition(6.23)] For $p \epsilon N$, we have*

$$\phi_{p+1,4p^2,5,p+5} = \phi_{p,4p(p+1),1,p}. \qquad (5.8.13)$$

Furthermore, the identity (5.2.52) holds.
Proof: Setting $p = 4$ in (5.8.13), we deduce (5.2.52) via (5.5.23) and (5.5.14).

Proposition 5.8.12. *For $p \epsilon N$, we have*

$$\phi_{p,320p,7,9} = \phi_{16p,20p,3,4}. \qquad (5.8.14)$$

Furthermore, the identity (5.2.53) holds.
Proof: Equality (5.8.14) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 41p$. If we set $p = 1$, we obtain (5.2.53) via (5.5.15) and (5.5.24).

5.9 Proofs of (5.2.54)-(5.2.64)

Proposition 5.9.1. *[25, Proposition (3.4.12)] For $p \epsilon N$, we have*

$$\phi_{p+6,p^2+6p,3,p+6} = \phi_{p,(p+6)^2,1,p+4}. \qquad (5.9.1)$$

Furthermore, the identity (5.2.54) holds.
Proof: Setting $p = 3$ in (5.9.1) and then using (5.5.22) and (5.5.18), we easily obtain (5.2.54).

Remark: The identity (5.2.54) also follows from (5.2.3) and one of the septic identities of Hahn [24].

Proposition 5.9.2. *[25, Proposition (3.4.15)] For $p \epsilon N$, we have*

$$\phi_{p+2,2p^2+3p,3,p+3} = \phi_{p,2p^2+7p+6,1,p+1}. \qquad (5.9.2)$$

Furthermore, the identity (5.2.55) holds.
Proof: Setting $p = 6$ in (5.9.2), we obtain the identity (5.2.55) by means of (5.5.22) and (5.5.18).

Remark: The identity (5.2.55) also follows from (5.2.3) and another septic identity found by Hahn [24].

Proposition 5.9.3. *[24, Proposition (6.20)] For $p \epsilon N$, we have*

$$\phi_{1,8p+7,2p+3,p+4} = \phi_{1,8p+7,2p+1,p+2}. \qquad (5.9.3)$$

Furthermore, the identity (5.2.56) holds.
Proof: Setting $p = 5$ in (5.9.3), we obtain

$$\phi_{1,47,13,9} = \phi_{1,47,11,7}. \tag{5.9.4}$$

Using (5.5.13) and Lemma 7 1 in the above equation, we find that,

$$-g_{47}^{(9,1)}g_1^{(9,3)} + g_{47}^{(9,2)}g_1^{(9,2)} + g_{47}^{(9,3)}g_1^{(9,4)} - g_{47}^{(9,4)}g_1^{(9,1)} = -g_{47}^{(7,1)}g_1^{(7,2)} + g_{47}^{(7,2)}g_1^{(7,3)} + g_{47}^{(7,3)}g_1^{(7,1)}. \tag{5.9 5}$$

Now, using (5.5.5) - (5.5.11) in the above relation and multiplying both sides by q, we obtain (5.2.56).

Proposition 5.9.4. *For $p \epsilon N$ and p odd, we have*

$$\phi_{1,7p+10,p,p+2} = \phi_{1,7p+10,5,7}. \tag{5.9.6}$$

Furthermore, the identity (5.2.57) holds.
Proof: Equality (5.9 6) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = p + 5$. Setting $p = 7$, employing (5.5.20) and (5.5.24), and then replacing q^2 by q in the resulting identity, we obtain (5.2.57).

Proposition 5.9.5. *[25, Proposition (3.4.19)] For $p \epsilon N$, we have*

$$\phi_{p+4,4p^2+15p,5,p+5} = \phi_{p,4p^2+31p+60,1,p+3}. \tag{5.9.7}$$

Furthermore, the identity (5.2.58) holds.
Proof: Setting $p = 4$ in (5.9.7), we obtain the identity (5.2 58) via (5.5.23) and (5.5.18).

Proposition 5.9.6. *For $p \epsilon N$, we have*

$$\phi_{21p+154,p,1,p+7} = \phi_{7p,3p+22,3,3p+1}. \tag{5.9.8}$$

Furthermore, the identity (5.2.59) holds.
Proof: Equality (5.9.8) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 22$. Setting $p = 2$, using (5.5.19) and (5.5.21), and then replacing q^2 by q, we readily deduce the required identity.

Proposition 5.9.7. *[25, Proposition (3.4.14)] For $p \epsilon N$, we have*

$$\phi_{2p,3p+30,1,p+6} = \phi_{2p+20,3p,1,p+4}. \tag{5.9.9}$$

Furthermore, the identity (5.2.60) holds.

Proof: Putting $p = 3$ in (5.9.9), we obtain (5.2.60) via (5.5.21) and (5.5.18).

Proposition 5.9.8. *[25, Proposition (3.4.7)] For $p \epsilon N$, we have*

$$\phi_{2,5p^2+23p+24,1,p+2} = \phi_{p+3,10p+16,5,7}. \tag{5.9.10}$$

Furthermore, the identity (5.2.61) holds.

Proof: Setting $p = 7$ in (5.9.10), we obtain (5.2 61) with the help of (5.5.21) and (5.5.20).

Proposition 5.9.9. *[25, Proposition (3.4.23)] For $p \epsilon N$, we have*

$$\phi_{p,2p^2+27p+90,1,p+5} = \phi_{p+6,2p^2+15p,3,p+3}. \tag{5.9.11}$$

Furthermore, the identity (5.2.62) holds.

Proof: Putting $p = 4$ in (5.9.11), we obtain (5.2 62) via (5.5.21) and (5.5.19)

Proposition 5.9.10. *For $p \epsilon N$, we have*

$$\phi_{p,188p,1,9} = \phi_{4p,47p,5,7}. \tag{5.9.12}$$

Furthermore, the identity (5.2.63) holds.

Proof: Equality (5.9.12) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 21p$. If we set $p = 1$, we obtain (5.2.63) via (5.5.20) and (5.5.21).

Proposition 5.9.11. *For $p \epsilon N$, we have*

$$\phi_{p,230p,7,9} = \phi_{23p,10p,3,7}. \tag{5.9.13}$$

Furthermore, the identity (5.2.64) holds.

Proof: Equality (5.9 13) holds by Theorem 5 5.12 with $\lambda_1 = \lambda_2 = 31p$. If we set $p = 1$, we obtain (5.2.64) with the help of (5.5.19) and (5.5.24).

5.10 Applications to the Theory of Partitions

The identities (5.2.1) - (5.2 23) have applications to the theory of partitions. We demonstrate this by giving combinatorial interpretations for (5.2 1)-(5.2.3), (5.2.8), and (5 2 9). In the sequel, for simplicity, we adopt the standard notation

$$(a_1, a_2, \cdots, a_n; q)_\infty := \prod_{j=1}^{n} (a_j; q)_\infty$$

and define

$$(q^{r\pm};q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty,$$

where r and s are positive integers and $r < s$.

We also need the notion of colored partitions A positive integer n has k colors if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called *colored partitions*. For example, if 1 is allowed to have 2 colors, say r (*red*), and g (*green*), then all colored partitions of 2 are 2, $1_r + 1_r$, $1_g + 1_g$, and $1_r + 1_g$. An important fact is that

$$\frac{1}{(q^u;q^v)_\infty^k}$$

is the generating function for the number of partitions of n, where all the parts are congruent to u (mod v) and have k colors.

Theorem 5.10.1. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ±1, ±2, ±3 (mod 9) with ±1 (mod 9) having 2 colors and ±3 (mod 9) having 3 colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ±1, ±3, ±4 (mod 9) with ±3 (mod 9) having 3 colors and ±4 (mod 9) having 2 colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ±2, ±3, ±4 (mod 9) with ±2 (mod 9) having 2 colors and ±3 (mod 9) having 3 colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to ±1, ±2, ±4 (mod 9) having 2 colors each. Then, for any positive integer $n \geq 1$, we have*

$$p_1(n) + p_2(n-1) - p_3(n-1) = p_4(n).$$

Proof: The identity (5.2.1) is equivalent to

$$(q^{4\pm};q^9)^2(q^{2\pm};q^9) + q(q^{2\pm};q^9)^2(q^{1\pm};q^9) - q(q^{4\pm};q^9)(q^{1\pm};q^9)^2 = \frac{(q^3;q^3)^3}{(q^9;q^9)^3}. \tag{5.10.1}$$

Noting that $(q^3;q^3)_\infty = (q^{3\pm};q^9)_\infty(q^9;q^9)_\infty$, we can rewrite (5.10.1) as

$$\begin{aligned} &\frac{1}{(q^{1\pm};q^9)^2(q^{2\pm};q^9)(q^{3\pm};q^9)^3} + \frac{q}{(q^{1\pm};q^9)(q^{4\pm};q^9)^2(q^{3\pm};q^9)^3} \\ &- \frac{q}{(q^{2\pm};q^9)^2(q^{4\pm};q^9)(q^{3\pm};q^9)^3} = \frac{1}{(q^{1\pm,2\pm,4\pm};q^9)^2}. \end{aligned} \tag{5.10.2}$$

The four quotients of (5.10.2) represent the generating functions for $p_1(n)$, $p_2(n)$, $p_3(n)$, and $p_4(n)$, respectively. Hence, (5.10.2) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + q\sum_{n=0}^{\infty} p_2(n)q^n - q\sum_{n=0}^{\infty} p_3(n)q^n = \sum_{n=0}^{\infty} p_4(n)q^n,$$

where we set $p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1$. Equating coefficients of q^n on both sides yields the desired result.

Example: It can easily be seen that $p_1(5) = 24$, $p_2(4) = 6$, $p_3(4) = 4$, and $p_4(5) = 26$, which verifies the case $n = 5$ in Theorem 5.10.1.

Theorem 5.10.2. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 2 (mod 9) with ± 2 (mod 9) having 2 colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 4 (mod 9) with ± 4 (mod 9) having 2 colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 4 (mod 9) with ± 1 (mod 9) having 2 colors. Then, for any positive integer $n \geq 1$, we have*

$$p_1(n) + p_2(n-1) = p_3(n).$$

Proof: The identity (5.2.2) is equivalent to

$$\frac{1}{(q^{1\pm};q^9)(q^{2\pm};q^9)^2} + \frac{q}{(q^{2\pm};q^9)(q^{4\pm};q^9)^2} = \frac{1}{(q^{1\pm};q^9)^2(q^{4\pm};q^9)}. \tag{5.10.3}$$

Note that the three quotients of (5.10.3) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, we have

$$\sum_{n=0}^{\infty} p_1(n)q^n + q\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n,$$

where we set $p_2(0) = 0$. Equating coefficients of q^n on both sides yields the desired result.

Example: The following table illustrates the case $n = 5$ in Theorem 5.10.2.

$p_1(5)=6$	$p_2(4)=3$	$p_3(5)$=9
2_r+2_r+1	4_r	5
2_r+2_g+1	4_g	$4+1_r$
2_g+2_g+1	2+2	$4+1_g$
$2_r+1+1+1$		$1_r+1_r+1_r+1_r+1_r$
$2_g+1+1+1$		$1_r+1_r+1_r+1_r+1_g$
$1+1+1+1+1$		$1_r+1_r+1_r+1_g+1_g$
		$1_r+1_r+1_g+1_g+1_g$
		$1_r+1_g+1_g+1_g+1_g$
		$1_g+1_g+1_g+1_g+1_g$

Theorem 5.10.3. *Let $p_1(n)$ denote the number of partitions of n into parts not congruent to ±12, 27 (mod 27). Let $p_2(n)$ denote the number of partitions of n into parts not congruent to ±6, 27 (mod 27). Let $p_3(n)$ denote the number of partitions of n into parts congruent to ±3, 27 (mod 9). Then, for any positive integer $n \geq 2$, we have*

$$p_1(n) = p_2(n-1) + p_3(n-2).$$

Proof: The identity (5.2.3) is equivalent to

$$\frac{1}{(q^{1\pm,2\pm,\cdots,11\pm,13\pm};q^{27})} - \frac{q}{(q^{1\pm,2\pm,\cdots,5\pm,7\pm,\cdots,13\pm};q^{27})} - \frac{q^2}{(q^{1\pm,2\pm,4\pm,\cdots,13\pm};q^{27})} = 1. \qquad (5.10\,4)$$

Note that the three quotients of (5.10.4) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Thus, we have

$$\sum_{n=0}^{\infty} p_1(n)q^n - q\sum_{n=0}^{\infty} p_2(n)q^n - q^2\sum_{n=0}^{\infty} p_3(n)q^n = 1,$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating coefficients of q^n on both sides, we arrive at the desired result.
Example: We note that $p_1(7) = 15$, $p_2(6) = 10$, and $p_3(5) = 5$, which verifies the case $n = 5$ in the Theorem 5.10.3.

Theorem 5.10.4. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ±1, ±3, ±4, ±5, ±6 (mod 18) with ±6 (mod 18) having 2* *colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 3, ± 6, ± 7, ± 8 (mod 18) with ± 6 (mod 18) having 2 colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 3, ± 5, ± 6, ± 7 (mod 18) with ± 6 (mod 18) having 2 colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 2, ± 4, ± 5, ± 7, ± 8 (mod 18). Then, for any positive integer $n \geq 1$, we have*

$$p_1(n) + p_2(n-1) = p_3(n-1) + p_4(n).$$

Proof: The identity (5 2.8) can be written as

$$(q^{2\pm};q^9)(q^{8\pm};q^{18}) + q(q^{4\pm};q^9)(q^{2\pm};q^{18}) - q(q^{1\pm};q^9)(q^{4\pm};q^{18}) = \frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q^9;q^9)_\infty(q^{18};q^{18})_\infty}. \qquad (5.10.5)$$

Writing the products by the common base q^{18}, for examples, writing $(q;q^9)_\infty$ as $(q;q^{18})_\infty(q^{10};q^{18})_\infty$ and $(q^3;q^3)_\infty$ as $(q^{3\pm,6\pm,9,18};q^{18})_\infty$ and cancelling the common terms, we obtain

$$\frac{1}{(q^{1\pm,3\pm,4\pm 5\pm 6\pm 6\pm};q^{18})} + \frac{q}{(q^{1\pm,3\pm,6\pm 6\pm 7\pm 8\pm};q^{18})} - \frac{q}{(q^{2\pm,3\pm,5\pm,6\pm 6\pm 7\pm};q^{18})} = \frac{1}{(q^{1\pm,2\pm,4\pm,5\pm 7\pm 8\pm};q^{18})}. \qquad (5.10.6)$$

Note that the four quotients of (5.10.6) represent the generating functions for $p_1(n)$, $p_2(n)$, $p_3(n)$, and $p_4(n)$ respectively. Thus, we have

$$\sum_{n=0}^{\infty} p_1(n)q^n + q\sum_{n=0}^{\infty} p_2(n)q^n - q\sum_{n=0}^{\infty} p_3(n)q^n = \sum_{n=0}^{\infty} p_4(n)q^n,$$

where we set $p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1$. Equating coefficients of q^n on both sides, we arrive at the desired result.

Example: The following table illustrates the case $n = 7$ in Theorem 5.10.4.

$p_1(7) = 8$	$p_2(6) = 5$	$p_3(6)=4$	$p_4(7)=9$
$6_r + 1$	6_r	6_r	7
$6_g + 1$	6_g	6_g	$5 + 2$
$4 + 3$	3+3	$3 + 3$	$5 + 1 + 1$
$5 + 1 + 1$	3+1+1+1	$2 + 2 + 2$	$4 + 2 + 1$
4+1+1+1+1	1+1+1+1+1+1		$4 + 1 + 1 + 1$
$3 + 3 + 1$			$2 + 2 + 2 + 1$
3+1+1+1+1			$2 + 2 + 1 + 1 + 1$
1+1+1+1 +1+1+1			$2 + 1 + 1 + 1 + 1 + 1$
			1+1+1+1+1+1+1

Theorem 5.10.5. *Let $p_1(n)$ denote the number of partitions of n into parts not congruent to ±1, ±8, ±10, ±17, ±19, ±20, 45 (mod 45) with ±15 (mod 45) having 2 colors. Let $p_2(n)$ denote the number of partitions of n into parts not congruent to ±4,±5, ±10, ±13, ±14, ±22, 45 (mod 45) with ±15 (mod 45) having 2 colors. Let $p_3(n)$ denote the number of partitions of n into parts not congruent to ±2,±5, ±7, ±11, ±16, ±20, 45 (mod 45) with ±15 (mod 45) having 2 colors. Let $p_4(n)$ denote the number of partitions of n into parts not congruent to ±3,±6, ±10, ±11, ±15, ±21, 45 (mod 45). Then, for any positive integer $n \geq 3$, we have*

$$p_1(n) + p_2(n-1) = p_3(n-3) + p_4(n).$$

Proof: We proceed as in the proof of Theorem 5.10.4 to complete the proof.

Chapter 6

Another couple of functions analogous to the Rogers-Ramanujan Functions and Partitions

6.1 Introduction

In this chapter, we deal with another couple of functions analogous to the Rogers-Ramanujan functions. We recall from Chapter 1 that

$$X(q) := \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n(1-q^{n+1})q^{n(n+2)}}{(q;q)_{2n+2}} = \frac{(q;q^{12})_\infty(q^{11};q^{12})_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty}, \tag{6.1.1}$$

$$Y(q) := 1 + \sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1}(1+q^n)q^{n^2}}{(q;q)_{2n}} = \frac{(q^5;q^{12})_\infty(q^7;q^{12})_\infty(q^{12};q^{12})_\infty}{(q,q)_\infty}, \tag{6.1.2}$$

where the later equalities are due to Slater [34, pp. 156–157, equations (49) and (54)]. Note that, Sills [33] corrected a misprint in Slater's equation (54). Also, the formulation of (6.1.1) and Slater's equation (49) are equivalent.

Now, using (1.1.10) and (1.1.6), we can rewrite (6.1.1) and (6.1.2) as

$$X(q) = \frac{f(-q,-q^{11})}{f(-q)}, \qquad \text{and} \qquad Y(q) = \frac{f(-q^5,-q^7)}{f(-q)}. \tag{6.1.3}$$

By applying the same methods as in the previous chapter, we find several modular identities for $X(q)$ and $Y(q)$. Some of these relations are connected with the Rogers-Ramanujan functions and their analogues defined in the previous two chapters.

In Sections 6.2-6.5, we state and prove our modular relations involving $X(q)$ and $Y(q)$.

In Sections 6.6 and 6.7, we state and prove the identities involving quotients of $X(q)$ and $Y(q)$ as well as the Rogers-Ramanujan functions and their other analogues.

In our last section, we apply some of the modular relations to the theory of partitions.

6.2 Modular Relations For $X(q)$ and $Y(q)$

In this section, we present a list of modular relations for $X(q)$ and $Y(q)$. For simplicity, for positive integer n, we set $f_n = f(-q^n)$, $X_n := X(q^n)$, and $Y_n := Y(q^n)$. We also note that, some more relations can easily be obtained by replacing q by $-q$ in each of the following relations.

$$Y_1 + qX_1 = \frac{f_2^3}{f_1^2 f_4}, \tag{6.2.1}$$

$$Y_1 - qX_1 = \frac{f_4 f_6^5}{f_2^2 f_3^2 f_{12}^2}, \tag{6.2.2}$$

$$X_1Y_2 + qX_2Y_1 = \frac{f_3 f_{24}}{f_1 f_2}, \tag{6.2.3}$$

$$X_1Y_3 + q^2X_3Y_1 = \frac{f_4 f_6^5 f_9 f_{36}}{f_2^2 f_3^3 f_{12}^2 f_{18}}, \tag{6 2.4}$$

$$Y_1^3 + q^3X_1^3 = \frac{f_4^3 f_6^3}{f_1^3 f_3 f_{12}^2}, \tag{6.2.5}$$

$$Y_1Y_2 + q^3X_1X_2 = \frac{f_2 f_4 f_4}{f_1 f_1 f_8} - q\frac{f_3 f_{24}}{f_1 f_2}, \tag{6.2.6}$$

$$X_1Y_3 - q^2X_3Y_1 = \frac{f_2 f_{12}^3}{f_1 f_3 f_4 f_6} - q\frac{f_4 f_6 f_{18} f_{72}}{f_1 f_2 f_3 f_{36}}, \tag{6.2.7}$$

$$Y_3^2 + q^6X_3^2 = q\frac{f_9^2 f_4^2}{f_{18} f_2 f_3^2} - q^2\frac{f_{18}^4}{f_1^2 f_6^2}, \tag{6.2.8}$$

$$Y_1Y_5 + q^6X_1X_5 = \frac{f_4f_4f_5}{f_1f_2f_{10}} - q^2\frac{f_3f_{12}f_{15}f_{60}}{f_1f_5f_6f_{30}}, \tag{6.2.9}$$

$$X_1Y_8 + q^8X_8Y_1 = \frac{f_{32}}{f_8} + q^2\frac{f_3f_{12}f_{24}f_{72}}{f_1f_6f_8f_{48}}, \tag{6.2.10}$$

$$X_2Y_7 + q^5X_7Y_2 = \frac{f_1f_{56}}{f_2f_7} + q\frac{f_6f_{21}f_{24}f_{84}}{f_2f_7f_{12}f_{42}}, \tag{6.2.11}$$

$$X_1Y_{11} + q^{10}X_{11}Y_1 = \frac{f_4f_{44}}{f_2f_{22}} + q^3\frac{f_3f_{12}f_{33}f_{132}}{f_1f_6f_{22}f_{66}}, \tag{6.2.12}$$

$$Y_1Y_{14} + q^{15}X_1X_{14} = \frac{f_7f_8}{f_1f_{14}} - q^5\frac{f_3f_{12}f_{42}f_{168}}{f_1f_6f_{14}f_{84}}, \tag{6.2.13}$$

$$Y_1Y_{20} + q^{21}X_1X_{20} = \frac{f_5f_{16}}{f_1f_{20}} - q^7\frac{f_3f_{12}f_{60}f_{240}}{f_1f_6f_{20}f_{120}}, \tag{6.2.14}$$

$$q^3X_7Y_5 + qX_5Y_7 = \frac{f_{15}f_{21}f_{60}f_{84}}{f_5f_7f_{42}f_{30}} - \frac{f_1f_4f_{35}f_{140}}{f_2f_5f_7f_{70}}, \tag{6.2.15}$$

$$Y_1Y_{35} + q^{36}X_1X_{35} = q^{12}\frac{f_3f_{12}f_{105}f_{420}}{f_1f_6f_{35}f_{210}} - \frac{f_5f_7f_{20}f_{28}}{f_1f_{10}f_{14}f_{35}}. \tag{6.2.16}$$

6.3 Proofs of (6.2.1), (6.2.2), and (6.2.5):

Proof of (6.2.1): Putting $a = q$ and $b = q^2$ in (4.3.6) and (4.3.7), we find that

$$f(q, q^2) + f(-q, -q^2) = 2f(q^5, q^7) \tag{6.3.1}$$

and

$$f(q, q^2) - f(-q, -q^2) = 2qf(q, q^{11}), \tag{6.3.2}$$

respectively. Subtracting (6.3.1) and (6.3.2), and then replacing q by $-q$, we find that

$$f(-q^5, -q^7) + qf(-q, -q^{11}) = f(q, -q^2)) = f(q), \tag{6.3.3}$$

where the last equality follows from (1.1.10).
Dividing both sides of (6.3.3) by $f(-q)$, we arrive at

$$\frac{f(-q^5, -q^7)}{f(-q)} + q\frac{f(-q, -q^{11})}{f(-q)} = \frac{f(q)}{f(-q)}. \tag{6.3.4}$$

Employing (6.1.3) and (4.3.12) in (6.3.4), we easily arrive at (6.2.1).

Proof of (6.2.2): Adding (6.3.1) from (6.3.2), and then replacing q by $-q$, we obtain

$$f(-q^5,-q^7) - qf(-q,-q^{11}) = f(-q,q^2) = \frac{\phi(q^3)}{\chi(q)}, \tag{6.3.5}$$

where the last equality follows from (3.2 2)
Employing (6.1.3), (4.3.11), and (4.3.12) in (6 3.5), we easily deduce (6.2.2).
Alternative Proof of (6.2.2): From [11, Entry 31, Corollary(ii)], we have

$$f(q^{15},q^{21}) + q^3 f(q^3,q^{33}) = \psi(q) - q\psi(q^9). \tag{6.3.6}$$

Replacing q^3, by $-q$, and employing (3.2.4), (3.2 2), (6.1 3) in (6.3.6), we easily arrive at (6.2.2).
Proof of (6.2.5): From [35, p. 306]. we have, for $|ab| < 1$,

$$f^3(ab^2,a^2b) - bf^3(a,a^2b^3) = \frac{f(-b^2,-a^3b)}{f(b,a^3b^2)} f^3(-ab). \tag{6 3.7}$$

Putting $a = q$, $b = q^3$, in (6.3.7), we obtain

$$f^3(q^5,q^7) - q^3 f^3(q,q^{11}) = \frac{\phi(-q^6)}{\psi(q^3)} f^3(-q^4). \tag{6.3.8}$$

Replacing q, by $-q$, in (6.3.8), we find that

$$f^3(-q^5,-q^7) + q^3 f^3(-q,-q^{11}) = \frac{\phi(-q^6)}{\psi(-q^3)} f^3(-q^4). \tag{6.3.9}$$

Using (4.3.12) and (6.1.3) in (6.3.9), we easily arrive at (6.2.5).

6.4 Proofs of (6.2.6) - (6.2.8):

Proof of (6.2.6): We apply Theorem 5.4.1 with the parameters $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = b = q^4$, $c = 1$, $d = q$, $\alpha = 2$, $\beta = 1$, $m = 6$. Consequently, we find that

$$\begin{aligned} 2\phi(-q^4)\psi(q) = 2\{&f(-q^7,-q^5)f(-q^{14},-q^{10}) \\ &+ q\psi(-q^3)\psi(-q^6) + q^3 f(-q,-q^{11})f(-q^2,-q^{22})\}. \end{aligned} \tag{6.4.1}$$

Now, using (6.1 3) and (4.3.12), we deduce (6.2.6).

In similar way, we prove the identities (6 2 7) and (6.2 8) To prove (6.2.7), we apply Theorem 5.4.1 with the parameters $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = 1$, $b = q^9$, $c = q$, $d = q^2$, $\alpha = 1$, $\beta = 1$, $m = 4$ and to prove (6.2.8), we again apply Theorem 5.4.1 with the parameters $\epsilon_1 = 1$, $\epsilon_2 = 0$ $a = b = q^{18}$, $c = q^4$, $d = 1$, $\alpha = 3$, $\beta = 1$, $m = 6$.

6.5 Proofs of (6.2.3), (6.2.4), and (6.2.9) - (6.2.16):

We will apply the method given by Bressoud in his thesis [19]. Here, we use f_n instead of P_n, and the variable q instead of x which stands for q^2 in [19]. The letters α, β, m, n, p always denote positive integers, and m must be odd. Following Bressoud [19], we define

$$\underline{g}_\alpha^{(p,n)} = \left\{q^{(12n^2-12n+3-p(p-1)/2)/(12p)\alpha}\right\} \frac{(q^{(p+1-2n)\alpha}; q^{2p\alpha})_\infty (q^{(p-1+2n)\alpha}; q^{2p\alpha})_\infty}{(q^\alpha; q^{2\alpha})_\infty}. \tag{6.5.1}$$

Proposition 6.5.1. *[19, Proposition 5.9]*

$$\underline{g}_\alpha^{(2,1)} = 1, \tag{6.5.2}$$

$$\underline{g}_\alpha^{(4,1)} = q^{-11\alpha/48} \frac{f_{4\alpha}}{f_{8\alpha}} S(q^\alpha), \qquad \text{and} \qquad \underline{g}_\alpha^{(4,2)} = q^{13\alpha/48} \frac{f_{4\alpha}}{f_{8\alpha}} T(q^\alpha), \tag{6.5.3}$$

where $S(q)$ and $T(q)$ are as defined in (1 1.17) and (1.1.18), respectively.

Proposition 6.5.2.

$$\underline{g}_\alpha^{(6,1)} = q^{-5\alpha/12} Y_\alpha \frac{f_{2\alpha}}{f_{12\alpha}}, \tag{6.5 4}$$

$$\underline{g}_\alpha^{(6,2)} = q^{-\alpha/12} \frac{f_{2\alpha} f_{3\alpha}}{f_\alpha f_{6\alpha}}, \tag{6.5.5}$$

$$\underline{g}_\alpha^{(6,3)} = q^{7\alpha/12} X_\alpha \frac{f_{2\alpha}}{f_{12\alpha}}. \tag{6.5.6}$$

Proof: Take $p = 6$, and $n = 1$ in (6.5.1). Then

$$\underline{g}_\alpha^{(6,1)} = q^{-5\alpha/12} \frac{(q^{5\alpha}; q^{12\alpha})_\infty (q^{7\alpha}; q^{12\alpha})_\infty (q^{12\alpha}; q^{12\alpha})_\infty}{(q^\alpha; q^{2\alpha})(q^{12\alpha}; q^{12\alpha})_\infty}. \tag{6.5.7}$$

Using (1.1.6) and (6.1.3), in (6.5.7) we obtain the result. Similarly we can prove (6.5.5) and (6.5.6).

Proposition 6.5.3. *[19, Proposition 5.8]*

$$\underline{g}_\alpha^{(p,n)} = \underline{g}_\alpha^{(p,-n+1)}, \quad \underline{g}_\alpha^{(p,n)} = -\underline{g}_\alpha^{(p,n-p)}, \quad \textit{and} \quad \underline{g}_\alpha^{(p,n)} = -\underline{g}_\alpha^{(p,p-n+1)}. \tag{6.5.8}$$

Theorem 6.5.4. *[19, Proposition 5.10] For even p,*

$$\phi_{\alpha,\beta,m,p} = 2q^{(2p-1)(\alpha+\beta)/24}\left\{\sum_{n=1}^{p/2} \underline{g}_\beta^{(p,n)}\underline{g}_\alpha^{(p,mn-((m-1)/2))}\right\}\frac{f_{2p\alpha}f_{2p\beta}f_\alpha f_\beta}{f_{2\alpha}f_{2\beta}}. \tag{6.5.9}$$

Proposition 6.5.5.

$$\begin{aligned}\phi_{\alpha,\beta,1,6} &= 2q^{(\alpha+\beta)/24}f_\alpha f_\beta \\ &\quad\times\left\{Y_\alpha Y_\beta + q^{(\alpha+\beta)/3}\frac{f_{3\alpha}f_{3\beta}f_{12\alpha}f_{12\beta}}{f_\alpha f_\beta f_{2\alpha}f_{2\beta}} + q^{(\alpha+\beta)}X_\alpha X_\beta\right\}. \qquad (6.5.10)\\ \phi_{\alpha,\beta,3,6} &= 2q^{(9\alpha+\beta)/24}\frac{f_\beta f_{3\alpha}f_{12\alpha}}{f_{6\alpha}}\left\{Y_\beta - q^{\beta/3}\frac{f_{3\beta}f_{12\beta}}{f_\beta f_{6\beta}} - q^\beta X_\beta\right\}. \qquad (6.5.11)\\ \phi_{\alpha,\beta,5,6} &= 2q^{(\alpha+\beta)/24}f_\alpha f_\beta \\ &\quad\times\left\{q^\alpha X_\alpha Y_\beta - q^{(\alpha+\beta)/3}\frac{f_{3\alpha}f_{3\beta}f_{12\alpha}f_{12\beta}}{f_\alpha f_\beta f_{6\alpha}f_{6\beta}} + q^\beta X_\beta Y_\alpha\right\}. \qquad (6.5.12)\end{aligned}$$

Proof: Applying equation (6.5.9) with $m = 1$ and $p = 6$, we have

$$\begin{aligned}\phi_{\alpha,\beta,1,6} &= 2q^{(\alpha+\beta)/24}\frac{f_{2p\alpha}f_{2p\beta}f_\alpha f_\beta}{f_{2\alpha}f_{2\beta}} \\ &\quad\times\left\{\underline{g}_\beta^{(6,1)}\underline{g}_\alpha^{(6,1)} + \underline{g}_\alpha^{(6,2)}\underline{g}_\alpha^{(6,2)} + \underline{g}_\alpha^{(6,3)}\underline{g}_\alpha^{(6,3)}\right\}. \qquad (6.5.13)\end{aligned}$$

Now, using (6.5.4), (6.5.5), (6.5.6), in (6.5.13), we obtain the result after simplification. The equation (6.5.11) and (6.5.12) can be proved in a similar way applying equation (6.5.8) with $m = 3$, 5 respectively and $p = 6$.

Theorem 6.5.6. *[19, Proposition 5.10]*

$$\phi_{\alpha,\beta,5,2} = -2q^{(\alpha+\beta)/8}\frac{f_{4\alpha}f_{4\beta}f_\alpha f_\beta}{f_{2\alpha}f_{2\beta}}. \tag{6.5.14}$$

Proof: Applying equation (6.5.9) with $m = 5$ and $p = 2$, we have

$$\phi_{\alpha,\beta,5,2} = 2q^{(\alpha+\beta)/8}\underline{g}_{\beta}^{(2,5)}\underline{g}_{\alpha}^{(2,3)}\frac{f_{4\alpha}f_{4\beta}f_{\alpha}f_{\beta}}{f_{2\alpha}f_{2\beta}}. \tag{6.5.15}$$

Now, using (6.5.8) and (6.5.2) in (6.5.15), we obtain the result.

Proof of (6.2.3): We set $p = 1$, in (5.8.13), we easily arrive at (6.2.3) by employing (6.5.12) and (5.5.26).

In the sequel, let N_0 denote the set of nonnegative integers.

Proposition 6.5.7. *[26, Proposition 6.3] For $p\epsilon N_0$, and p even,*

$$\phi_{6,4p+10,p+3,p+4} = \phi_{2,12p+30,1,2}, \tag{6.5.16}$$

Furthermore, the identity (6.2.4) holds.

Proof: If we set $p = 2$, in (6.5.16), we obtain (6.2.4) by using (6.5.12) and (5.5.28).

Proof of (6.2.9): If we set $p = 6$, in (5.5.32), we obtain (6.2.9) by using (6.5.10) and (4.3.12).

Proposition 6.5.8. *[26, Proposition 6.2] For $p\epsilon N_0$, and p even,*

$$\phi_{2,3p+10,p+3,p+4} = \phi_{1,6p+20,1,3}, \tag{6.5.17}$$

Furthermore, the identity (6.2.10) holds.

Proof: If we set $p = 2$, in (6.5.17), we obtain (6.2.10) by using (6.5.12) and (5.5.27).

Proposition 6.5.9. *[26, Proposition 6.3] For $p\epsilon N_0$, and p even,*

$$\phi_{4,3p+8,p+3,p+4} = \phi_{1,12p+32,1,3}, \tag{6.5.18}$$

Furthermore, the identity (6.2.11) holds.

Proof: If we set $p = 2$, in (6.5.18), we obtain (6.2.11) by using (6.5.12) and (5.5.27).

Proof of (6.2.12): If we set $p = 1$, in (5.5.44) we obtain (6.2 12) by using (6.5.12) and (5.5.28).

Alternative proof of (6.2.12): Applying (4 3.24), with $\mu = 6$, $\nu = 5$ and using (1.1.5), we find that

$$2q\psi(q)\psi(q^{11}) = 2q\{f(q,q^{11})f(q^{55},q^{77}) + q^3\psi(q^3)\psi(q^{33}) + q^{10}f(q^5,q^7)f(q^{11},q^{121})\}. \quad (6.5\ 19)$$

Replacing q, by $-q$, and dividing both sides by $f(-q)f(-q^{11})$, and using (6.1.3) and (4.3.12), we obtain the result.

Proposition 6.5.10. *[24, Proposition 6.13] For $p\epsilon N$,*

$$\phi_{2,p(p+3),1,p+2} = \phi_{p+3,2p,1,3}, \quad (6.5.20)$$

Furthermore, the identity (6.2.13) holds.

Proof: If we set $p = 4$, in (6.5.20) we obtain (6.2.13) by using (6.5.10) and (5.5.27).

Proof of (6.2.14): If we set $p = 5$, in (5.5.43), we obtain (6 2.14) by using (6.5.10) and (5.5.27).

Proposition 6.5.11. *For $p\epsilon N$,*

$$\phi_{7p,5p,5,6} = \phi_{p,35p,5,2}, \quad (6.5\ 21)$$

Furthermore, the identity (6.2.15) holds.

Proof : Equality (6.5.21) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 30p$. Furthermore, by putting $p = 1$, we obtain (6.2.15) by using (6.5.12) and (6.5.14).

Proposition 6.5.12. *For $p\epsilon N$,*

$$\phi_{p+1,p+3,1,2} = \phi_{1,p^2+4p+3,1,p+2}, \quad (6.5.22)$$

Furthermore, the identity (6.2.16) holds.

Proof : Equality (6.5.22) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = p + 2$. Furthermore, by putting $p = 4$, we obtain (6.2.16) by using (6.5.10) and (5.5.28).

6.6 Further identities for quotients of the functions $X(q)$ and $Y(q)$

In this section, we derive further identities for quotients of the functions $X(q)$ and $Y(q)$.

$$\frac{Y_{15}-q^5(f_{27}f_{108})/(f_{15}f_{54})-q^{15}X_{15}}{Y_3-q(f_9f_{36})/(f_3f_{18})-q^3X_3}=\frac{f_1f_4f_{10}f_{15}}{f_2f_3f_5f_{20}}, \tag{6.6.1}$$

$$\frac{X_5Y_{31}-q^7(f_{15}f_{60}f_{93}f_{372})/(f_5f_{30}f_{31}f_{186})+q^{26}X_{31}Y_5}{Y_1Y_{155}+q^{52}(f_3f_{12}f_{465}f_{1860})/(f_1f_6f_{155}f_{930})+q^{156}X_1X_{155}}=\frac{f_1f_{155}}{f_5f_{31}}, \tag{6.6.2}$$

$$\frac{X_7Y_{29}-q^5(f_{21}f_{84}f_{87}f_{348})/(f_7f_{29}f_{42}f_{174})+q^{22}X_{29}Y_7}{Y_1Y_{203}+q^{68}(f_3f_{12}f_{609}f_{2436})/(f_1f_6f_{203}f_{1218})+q^{204}X_1X_{203}}=\frac{f_1f_{203}}{f_7f_{29}}, \tag{6.6.3}$$

$$\frac{X_1Y_{275}-q^{91}(f_1f_{12}f_{825}f_{3300})/(f_1f_6f_{275}f_{1650})+q^{274}X_{275}Y_1}{X_{11}Y_{25}-q(f_{33}f_{75}f_{132}f_{300})/(f_{11}f_{25}f_{66}f_{150})+q^{14}X_{25}Y_{11}}=\frac{f_{11}f_{25}}{f_1f_{275}}. \tag{6.6.4}$$

The following identities are relations involving some combinations of $X(q)$ and $Y(q)$ with the Rogers-Ramanujan functions:

$$\frac{G_7G_8+q^3H_7H_8}{Y_2Y_7+q^3(f_6f_{21}f_{24}f_{84})/(f_2f_7f_{12}f_{42})+q^9X_2X_7}=\frac{f_2}{f_8}, \tag{6.6.5}$$

$$\frac{G_9G_{16}+q^5H_9H_{16}}{Y_3Y_{12}+q^5(f_9f_{36}^2f_{144})/(f_3f_{12}f_{18}f_{72})+q^{15}X_3X_{12}}=\frac{f_9f_{16}}{f_3f_{12}}, \tag{6.6.6}$$

$$\frac{G_8G_{27}+q^7H_8H_{27}}{Y_3Y_{18}+q^7(f_9f_{36}f_{54}f_{216})/(f_3f_{18}^2f_{72})+q^{21}X_3X_{18}}=\frac{f_3f_{18}}{f_8f_{27}}. \tag{6.6.7}$$

The following identities are relations involving some combinations of $X(q)$ and $Y(q)$ with the Göllnitz-Gordon functions:

$$\frac{Y_8Y_1+q^3(f_3f_{24})/(f_1f_2)+q^9X_1X_8}{S_4S_2+q^3T_4T_2}=\frac{f_1f_8}{f_2f_{16}}, \tag{6.6.8}$$

$$\frac{Y_{15}-q^5(f_{45}f_{180})/(f_{15}f_{90})-q^{15}X_{15}}{S_{15}S_1+q^8T_{15}T_1}=\frac{f_1f_4f_6f_{60}}{f_2f_3f_{12}f_{30}}, \tag{6.6.9}$$

$$\frac{X_1Y_{23}-q^7(f_3f_{12}f_{69}f_{276})/(f_1f_6f_{23}f_{138})+q^{22}Y_1X_{23}}{T_1S_{23}-qT_{23}S_1}=\frac{f_4f_{12}f_{92}f_{176}}{f_2f_8f_{46}f_{184}}, \tag{6.6.10}$$

$$\frac{Y_{95}Y_1 + q^{32}(f_3f_{12}f_{195}f_{780})/(f_1f_6f_{95}f_{390}) + q^{96}X_1X_{95}}{S_{19}T_5 - q^7S_5T_{19}} = \frac{f_5f_{19}f_{20}f_{76}}{f_1f_{10}f_{38}f_{95}}, \tag{6.6.11}$$

$$\frac{Y_1Y_{119} + q^{40}(f_3f_{12}f_{357}f_{1428})/(f_1f_6f_{119}f_{714}) + q^{120}X_1X_{119}}{S_{17}T_7 - q^5T_{17}S_7} = \frac{f_7f_{17}f_{28}f_{68}}{f_1f_{14}f_{34}f_{119}}, \tag{6.6.12}$$

$$\frac{X_5Y_{19} - q^3(f_{15}f_{60}f_{57}f_{228})/(f_5f_{19}f_{30}f_{114}) + q^{19}Y_5X_{19}}{S_{95}S_1 + q^{48}T_{95}T_1} = \frac{f_1f_4f_{95}f_{380}}{f_2f_5f_{19}f_{190}}, \tag{6.6.13}$$

$$\frac{X_3Y_{21} - q^5(f_9f_{36}f_{63}f_{252})/(f_3f_{18}f_{21}f_{126}) + q^{18}Y_3X_{21}}{S_{63}S_1 + q^{32}T_{63}T_1} = \frac{f_1f_4f_{63}f_{252}}{f_2f_3f_{21}f_{126}}, \tag{6.6.14}$$

$$\frac{X_7Y_{17} - q(f_{21}f_{51}f_{84}f_{204})/(f_7f_{17}f_{42}f_{102}) + q^{10}Y_7X_{17}}{S_{119}T_1 - q^{59}S_1T_{119}} = \frac{f_1f_4f_{119}f_{476}}{f_2f_7f_{17}f_{238}}. \tag{6.6.15}$$

The following identities are relations involving some combinations of $X(q)$ and $Y(q)$ with the septic analogues $A(q)$, $B(q)$, and $C(q)$:

$$\frac{Y_5Y_4 + q^3(f_{12}f_{15}f_{48}f_{60})/(f_4f_5f_{24}f_{30}) + q^9X_5X_4}{A_5A_{16} + q^3B_5B_{16} + q^9C_5C_{16}} = \frac{f_{32}f_{10}}{f_4f_5}, \tag{6.6.16}$$

$$\frac{Y_1Y_{98} + q^{33}(f_3f_{12}f_{294}f_{1176})/(f_1f_6f_{98}f_{588}) + q^{99}X_1X_{98}}{A_{56}C_7 - q^5B_{56}A_7 + q^{22}C_{56}B_7} = \frac{f_{14}f_{112}}{f_1f_{98}}. \tag{6.6.17}$$

The following identities are relations involving some combinations of $X(q)$ and $Y(q)$ with the nonic analogues $D(q)$, $E(q)$, and $F(q)$:

$$\frac{Y_1Y_2 + q(f_3f_{24})/(f_1f_2) + q^3X_1X_2}{D_1D_8 + q + q^3E_1E_8 + q^6F_1F_8} = \frac{f_3f_{24}}{f_1f_2}, \tag{6.6.18}$$

$$\frac{X_1Y_{50} - q^{16}(f_3f_{12}f_{150}f_{600})/(f_1f_6f_{50}f_{300}) + q^{49}Y_1X_{50}}{D_{25}E_8 - q - q^{11}E_{25}F_8 + q^{14}F_{25}D_8} = \frac{f_{24}f_{75}}{f_1f_{50}}. \tag{6.6.19}$$

6.7 Proofs of (6.6.1)-(6.6.19):

Proposition 6.7.1. *For $p \epsilon N$,*

$$\phi_{5p,9p,3,6} = \phi_{p,45p,3,6}, \tag{6.7.1}$$

Furthermore, the identity (6.6.1) holds.

Proof: Equality (6.7.1) holds by Theorem 5.5 12 with $\lambda_1 = \lambda_2 = 9p$. Furthermore, by putting $p = 1$, in (6.7.1), we obtain (6.6.1) by using (6.5.11).

Proposition 6.7.2. *For $p \epsilon N$,*

$$\phi_{p,155p,1,6} = \phi_{5p,31p,5,6}, \tag{6.7 2}$$

Furthermore, the identity (6.6.2) holds.

Proof: Equality (6.7.2) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 26p$ Furthermore, by putting $p = 1$, in (6.7.2), we obtain (6.6 2) by using (6.5 10) and (6.5.12).

Proposition 6.7.3. *For $p \epsilon N$,*

$$\phi_{p,203p,1,6p} = \phi_{7p,29p,5,6p}, \tag{6.7.3}$$

Furthermore, the identity (6.6.3) holds.

Proof: Equality (6.7.3) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 34$. Furthermore, by putting $p = 1$, in (6.7.3), we obtain (6 6.3) by using (6.5.10) and (6.5.12).

Proposition 6.7.4. *For $p \epsilon N$,*

$$\phi_{p,275p,5,6} = \phi_{11p,25p,5,6}, \tag{6.7 4}$$

Furthermore, the identity (6.6.4) holds.

Proof: Equality (6.7.4) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 50p$. Furthermore, by putting $p = 1$, in (6.7.4), we obtain (6.6.4) by using (6.5.12).

Proof of (6.6.5): Setting $p = 2$ in (5.7.2), we obtain (6.6.5) by using (6.5.10) and (5.5.16).

Proof of (6.6.6): If we set $p = 3$ in (5.7.3), we arrive at (6.6.6) by employing (6.5.10) and (5.5.16).

Proof of (6.6.7): We set $p = 4$ in (5.7.4), to obtain (6.6.7) with the help of (6.5.10) and (5.5.16).

Proposition 6.7.5. *For $p\epsilon N$,*

$$\phi_{16p,8p,1,4} = \phi_{32p,4p,1,6}, \tag{6.7.5}$$

Furthermore, the identity (6.6.8) holds.

Proof: Equality (6.7.5) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 6p$. Furthermore, by putting $p = 1$, in (6.7.5) we obtain (6 6.8) by using (6.5.10) and (5.5.14).

Proposition 6.7.6. *For $p\epsilon N$,*

$$\phi_{1,4p+3,p,p+3} = \phi_{1,4p+3,1,4}, \tag{6.7.6}$$

Furthermore, the identity (6.6.9) holds.

Proof: Equality (6.7.6) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = p + 1$. Furthermore, by putting $p = 3$, we obtain (6.6.9) with the help of (6.5.11) and (5.5.14).

Proof of (6.6.10) : If we set $p = 2$ in (5.9.3), we obtain

$$\phi_{1,23,7,6} = \phi_{1,23,5,4}. \tag{6.7.7}$$

Employing (6.5.9) and (6.5.8) in (6.7.7), we find that

$$2q^{11}\{-\underline{g}_{23}^{(6,1)}\underline{g}_{1}^{(6,3)} + \underline{g}_{23}^{(6,2)}\underline{g}_{1}^{(6,2)} - \underline{g}_{23}^{(6,3)}\underline{g}_{1}^{(6,1)}\} = q^7\{-\underline{g}_{23}^{(4,1)}\underline{g}_{1}^{(4,2)} + \underline{g}_{23}^{(4,2)}\underline{g}_{1}^{(4,1)}\}. \tag{6.7.8}$$

Applying (6.5.4), (6.5.5), (6.5.6), and (6.5.3) in (6.7.8), we readily arrive at (6.6.10).

Proposition 6.7.7. *[25, Proposition 3.4.3] For $p\epsilon N$, we have*

$$\phi_{15p+80,p,1,p+5} = \phi_{5p,3p+16,3,3p+1}, \tag{6.7.9}$$

Furthermore, the identity (6.6.11) holds.

Proof: If we set $p = 1$, in (6.7.9), we obtain

$$\phi_{95,1,1,6} = \phi_{5,19,3,4}. \tag{6.7.10}$$

Now, using (6.5.10) and (5.5.15) in (6.7.10), we easily obtain (6 6.11).

Proof of (6.6.12): If we set $p = 1$ in (5.9.11), we find that

$$\phi_{1,119,1,6} = \phi_{7,17,3,4}. \tag{6.7.11}$$

We deduce (6.6.12) with the help of (6.5.10) and (5.5.15).

Proposition 6.7.8. *[26, Proposition 6.8] For p even, we have*

$$\phi_{5,4p+11,p+3,p+4} = \phi_{1,20p+55,1,4}, \tag{6.7.12}$$

Furthermore, the identity (6.6.13) holds.
Proof: If we set $p = 2$, in (6.7.12), we obtain

$$\phi_{5,19,5,6} = \phi_{1,95,1,4}. \tag{6.7.13}$$

Employing (6.5.12) and (5.5.14) in (6.7.13), we deduce (6.6.13).

Proposition 6.7.9. *[26, Proposition 6.7.] For $p \epsilon N_0$ and p even, we have*

$$\phi_{3,4p+13,p+3,p+4} = \phi_{1,12p+39,1,4}, \tag{6.7.14}$$

Furthermore, the identity (6.6.14) holds.
Proof: If we set $p = 2$, in (6.7.14), we obtain

$$\phi_{3,21,5,6} = \phi_{1,63,1,4}. \tag{6.7.15}$$

Applying (6.5.12) and (5.5.14) in (6.7.15), we deduce (6.6.14).

Proposition 6.7.10. *[26, Proposition 6.9] For $p \epsilon N$, we have*

$$\phi_{7,4p+19,p+3,p+4} = \phi_{1,28p+63,3,4}, \tag{6.7.16}$$

Furthermore, the identity (6.6.15) holds.
Proof: If we set $p = 2$, in (6.7.16), we obtain

$$\phi_{7,17,5,6} = \phi_{1,119,3,4}, \tag{6.7.17}$$

Employing (6.5.12) and (5.5.15) in the above identity, we readily arrive at (6.6.15).

Proposition 6.7.11. *For $p \epsilon N$,*

$$\phi_{10p,8p,1,6p} = \phi_{16p,5p,1,7p}, \tag{6.7.18}$$

Furthermore, the identity (6.6.16) holds.

Proof: Equality (6.7.18) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 3$. Furthermore, setting $p = 1$ in (6.7.18), we obtain (6.6.16) with the help of (6.5.10) and (5.5.18).

Proof of (6.6.17)**:** If we set $p = 3$ in (5.9.10), we find that

$$\phi_{7,56,5,7} = \phi_{2,196,1,6}. \tag{6.7.19}$$

Employing (5.5.20) and (6.5.10) in the above identity, we immediately arrive at (6.6.17).

Proposition 6.7.12. *[25, Proposition 3.4.25] For $p \epsilon N$, we have*

$$\phi_{p,2p+18,1,p+6} = \phi_{p+9,2p,1,p+3}, \tag{6.7.20}$$

Furthermore, the identity (6.6.18) holds.

Proof: If we set $p = 3$ in (6.7.20), we obtain

$$\phi_{3,24,1,9} = \phi_{12,6,1,6}. \tag{6.7.21}$$

Using (5.5.21) and (6.5.10) in (6.7.21), we readily obtain (6.6.18).

Proposition 6.7.13. *For $p \epsilon N$,*

$$\phi_{2p,100p,5,6} = \phi_{8p,25p,5,9}, \tag{6.7.22}$$

Furthermore, the identity (6.6.19) holds.

Proof: Equality (6.7.22) holds by Theorem 5.5.12 with $\lambda_1 = \lambda_2 = 30p$. Setting $p = 1$ in (6.7.22), we obtain (6.6.19) with the help of (6.5.12) and (5.5.23).

6.8 Applications to the Theory of Partitions

In this section, by the notion of colored partitions, we extract some partition theoretic results from some of our identities. We recall from Chapter 5 that

$$\frac{1}{(q^u; q^v)_\infty^k}$$

is the generating function for the number of partitions of n, where all the parts are congruent to $u \pmod v$ and have k colors.

Theorem 6.8.1. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ±1, ±2, ±4, 6 (mod 12) with ±2, 6 (mod 12) having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ±2, ±4, ±5, 6 (mod 12) with ±2, 6 (mod 12) having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ±1, ±3, ±5 (mod 12) with ±1, ±5 (mod 12) having two colors. Then, for any positive integer $n \geq 1$, $p_1(n) + p_2(n-1) = p_3(n)$.*

Proof: The identity (6.2.1) is equivalent to

$$\frac{(q^{5\pm};q^{12})(q^{12};q^{12})}{(q;q)} + q\frac{(q^{1\pm};q^{12})(q^{12};q^{12})}{(q;q)} = \frac{(q^2;q^2)^3}{(q;q)^2(q^4;q^4)}. \qquad (6.8.1)$$

Rewriting the products of the above identity subject to the common base q^{12}, we deduce that

$$\frac{1}{(q^{1\pm,2\pm,2\pm,4\pm,6,6};q^{12})} + \frac{q}{(q^{2\pm,2\pm,4\pm,5\pm,6,6};q^{12})} = \frac{1}{(q^{1\pm,1\pm,3\pm,5\pm,5\pm};q^{12})}. \qquad (6.8.2)$$

The three quotients of (6.8.2) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, (6.8.4) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + q\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n,$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating coefficients of q^n on both sides yields the desired result.

Example: The following table illustrates the case $n = 9$ in Theorem 6.8.1.

$p_1(5) = 7$	$p_2(4) = 4$	$p_3(5) = 11$
$4+1$	$2_r + 2_r$	5_r
$2_r + 2_r + 1$	$2_r + 2_g$	5_g
$2_r + 2_g + 1$	$2_g + 2_g$	$3 + 1_r + 1_r$
$2_g + 2_g + 1$	4	$3 + 1_r + 1_g$
$2_g + 1 + 1 + 1$		$3 + 1_g + 1_g$
$2_r + 1 + 1 + 1$		$1_r + 1_r + 1_r + 1_r + 1_r$
$1+1+1+1+1$		$1_r + 1_r + 1_r + 1_r + 1_g$
		$1_r + 1_r + 1_r + 1_g + 1_g$
		$1_r + 1_r + 1_g + 1_g + 1_g$
		$1_r + 1_g + 1_g + 1_g + 1_g$
		$1_g + 1_g + 1_g + 1_g + 1_g$

Theorem 6.8.2. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 3, ± 5, ± 6, ± 7, ± 9 (mod 24). Let $p_2(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 3, ± 6, ± 9, ± 10, ± 11 (mod 24). Let $p_3(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 2, ± 5, ± 7, ± 10, ± 11, (mod 24). Then, for any positive integer $n \geq 2$, $p_1(n) + p_2(n-2) = p_3(n)$.*

Proof: The identity (6 2 3) is equivalent to

$$\frac{(q^{1\pm};q^{12})(q^{12};q^{12})(q^{10\pm};q^{24})(q^{24};q^{24})}{(q;q)(q^2;q^2)} + \frac{q^2(q^{5\pm};q^{12})(q^{12};q^{12})(q^{2\pm};q^{24})(q^{24};q^{24})}{(q;q)(q^2;q^2)} = \frac{(q^3,q^3)(q^{24};q^{24})}{(q,q)(q^2;q^2)}. \tag{6 8.3}$$

This identity can be written as

$$\frac{(q^{1\pm},q^{12};q^{12})(q^{10\pm};q^{24})}{(q^3;q^3)} + q^2\frac{(q^{5\pm},q^{12};q^{12})(q^{2\pm};q^{24})}{(q^3;q^3)} = 1. \tag{6.8.4}$$

Rewriting all the products by the common base q^{24}, for examples, writing $(q^{1\pm};q^{12})_\infty$ as $(q^{1\pm};q^{24})_\infty(q^{11\pm};q^{24})_\infty$ and $(q^3;q^3)_\infty$ as $(q^{3\pm,6\pm,9\pm,12,24};q^{24})$ and cancelling the common terms, we obtain

$$\frac{1}{(q^{2\pm,3\pm,5\pm,6\pm,7\pm,9\pm};q^{24})} + \frac{q^2}{(q^{1\pm,3\pm,6\pm,9\pm,10\pm,11\pm},q^{24})} = \frac{1}{(q^{1\pm,2\pm,5\pm,7\pm,10\pm,11\pm};q^{24})}. \tag{6 8.5}$$

The three quotients of (6.8.5) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, (6.8.5) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + q^2\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n, \tag{6.8.6}$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating the coefficients of q^n on both sides of (6.8.6), we arrive at the desired result.

Example: The following table illustrates the case $n = 9$ in Theorem 6.8.2.

$p_1(9) = 5$	$p_2(7) = 4$	$p_3(9) = 9$
9	$1+1+1+1+1+1+1$	$2+2+2+2+1$
$5+2+2$	$3+1+1+1+1$	$5+2+2$
$7+2$	6+1	$7+2$
$3+3+3$	3+3+1	$2+2+2+1+1+1$
$3+2+2+2$		$7+1+1$
		$5+1+1+1+1$
		$5+2+1+1$
		$2+2+1+1+1+1+1$
		2+1+1+1+1+1+1+1

Theorem 6.8.3. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ± 3, ± 5, ± 6, ± 7, ± 17, 18 (mod 36) with parts congruent to ± 6, 18 (mod 36) having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 6, ± 11, ± 13, ± 15, 18 (mod 36) and parts congruent to ± 6, 18 (mod 36)having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 3, ± 10, ± 14, ± 15 (mod 36) with parts congruent to ± 3, ± 15 (mod 36) having two colors. Then, for any positive integer $n \geq 2$, $p_1(n) + p_2(n-2) = p_3(n)$.*

Proof: The identity (6.2.4) is equivalent to

$$\frac{(q^{1\pm};q^{12})(q^{12};q^{12})(q^{15\pm};q^{36})(q^{36};q^{36})}{(q;q)(q^3;q^3)} + \frac{q^2(q^{5\pm};q^{12})(q^{12};q^{12})(q^{3\pm};q^{36})(q^{36};q^{36})}{(q;q)(q^3;q^3)}$$
$$= \frac{(q^4;q^4)(q^6;q^6)^5(q^9;q^9)(q^{36};q^{36})}{(q^2;q^2)^2(q^3;q^3)^3(q^{12};q^{12})^2(q^{18};q^{18})}. \tag{6.8.7}$$

Rewriting all the products in (6.8.7) by the common base q^{36}, for examples, writing $(q^{1\pm},q^{12})_\infty$ as $(q^{1\pm},q^{11\pm},q^{13\pm};q^{36})_\infty$ and $(q^3;q^3)_\infty$ as $(q^3,q^6)_\infty(q^6;q^6)_\infty$ and cancelling the common terms, we obtain

$$\frac{1}{(q^{3\pm,5\pm,6\pm,6\pm,7\pm,17\pm,18,18};q^{36})} + \frac{q^2}{(q^{1\pm,6\pm,6\pm,11\pm,13\pm,15\pm,18,18};q^{36})}$$
$$= \frac{1}{(q^{2\pm,3\pm,3\pm,10\pm,14\pm,15\pm,15\pm};q^{36})}. \tag{6.8.8}$$

The three quotients of (6.8.8) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, (6.8.8) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + q^2 \sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n, \tag{6.8.9}$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating the coefficients of q^n on both sides of (6.8.9), we obtain the required result.

Example: The following table illustrates the case $n = 16$ in Theorem 6.8.3.

$p_1(16) = 6$	$p_2(14) = 8$	$p_3(16) = 14$
$3+3+3+7$	$6_r + 6_r + 1 + 1$	$14 + 2$
$5+5+3+3$	$6_r + 6_g + 1 + 1$	$10 + 3_r + 3_r$
$6_r + 5 + 5$	$6_y + 6_y + 1 + 1$	$10 + 3_y + 3_y$
$6_y + 5 + 5$	$11 + 1 + 1 + 1$	$10 + 3_y + 3_r$
$6_r + 7 + 3$	$13 + 1$	$10 + 2 + 2 + 2$
$6_g + 7 + 3$	$6_r + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$	$3_r + 3_r + 3_r + 3_r + 2 + 2$
	$6_g + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$	$3_r + 3_r + 3_r + 3_g + 2 + 2$
	$1+1+1+1+1+1+1+1+1+1+1+1+1+1$	$3_r + 3_r + 3_y + 3_y + 2 + 2$
		$3_r + 3_y + 3_g + 3_g + 2 + 2$
		$3_g + 3_g + 3_g + 3_g + 2 + 2$
		$3_r + 3_r + 2 + 2 + 2 + 2 + 2$
		$3_r + 3_g + 2 + 2 + 2 + 2 + 2$
		$3_g + 3_g + 2 + 2 + 2 + 2 + 2$
		$2+2+2+2+2+2+2+2$

Theorem 6.8.4. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 4 (mod 12) having three colors and parts congruent to 6 (mod 12) having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ± 4, ± 5 (mod 12) having three colors and 6 (mod 12) having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 3, ± 5 (mod 12) with ± 1, ± 5 (mod 12) having three colors. Then, for any positive integer $n > 3$, $p_1(n) + p_2(n-3) = p_3(n)$.*

Proof: The identity (6.2.5) is equivalent to

$$\frac{(q^{5\pm};q^{12})^3(q^{12};q^{12})^3}{(q;q)^3} + q^3\frac{(q^{1\pm};q^{12})^3(q^{12};q^{12})^3}{(q;q)^3} = \frac{(q^6;q^6)^3(q^4;q^4)^3}{(q;q)^3(q^3;q^3)(q^{12};q^{12})^2}. \tag{6.8.10}$$

Noting that $(q^6;q^6)_\infty = (q^6;q^{12})_\infty(q^{12};q^{12})_\infty$, and rewriting all the products by the common base q^{12}, and cancelling the common terms, we can rewrite (6.8.10) as

$$\frac{1}{(q^{1\pm,1\pm,1\pm,4\pm,4\pm,4\pm,6,6};q^{12})} + \frac{q^3}{(q^{4\pm,4\pm,4\pm,5\pm,5\pm,5\pm,6,6};q^{12})} = \frac{1}{(q^{1\pm,1\pm,1\pm,3\pm,5\pm,5\pm,5\pm};q^{12})}. \tag{6.8.11}$$

The three quotients of (6.8.11) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, (6.8.11) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + q^3\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n, \tag{6.8.12}$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating the coefficients of q^n on both sides of (6.8.12), we obtain the desired result.

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CONTRIBUTIONS TO RAMANUJAN'S THETA-FUNCTIONS AND MODULAR EQUATIONS

Doctoral thesis by JONALI BORA

Corrigendum

1. The theta-function identities which we proved in Chapter 2 are in fact different from those proved by Baruah and Bhattacharyya [4] (i.e. [3] in the previous version). We have rewritten the first paragraph of the introduction of Chapter 2 in support of this fact.

2. The idea of proofs of the three identities (4.2.19)–(4.2.21) are due to Mr. N. Saikia and for this reason we have written our paper [8] ([5] in the previous version) together. The other proofs came out from a joint work with the author's supervisor Dr. N. D. Baruah. We have added a remark at the end of Chapter 4 to acknowledge the help received from Mr. Saikia.

3. In fact, the identities (5.1.1), (5.1.2), (5.1.3) are due to L. J. Rogers [30], [31]. These appear in L. J. Slater's list [34] of 130 Rogers-Ramanujan type identities. We have properly given reference of Rogers' papers and as suggested by one of the referees, marked these identities according to Slater's list. Also the identities (5.1.4), (5.1.5), (5.1.6) are due to W. N. Bailey [2]. These also appear in Slater's list, but all three contain misprints. These misprints are corrected by A. V. Sills [33]. We have incorporated these changes in the rewritten Introduction of Chapter 5. Sills also corrected one misprint in Slater's formulation of (6.1.2). It is to be noted that the formulation of (6.1.1) with that of Slater are equivalent. In summary, we have corrected all the misprints in p.68, (5.1.1)-(5.1.3), p.69, (5.1.4)-(5.1.6) and p.100, (6.1.2). It is worthwhile to note that these misprints/ corrections do not change our modular relations.

4. p.2 , In Eq. (1.1.7), we have defined $(a; q)_0$.

5. p.6, In Eqs. (1.1.26) and p.100, In Eq (6.1.2) we have changed $\sum_{n=0}^{\infty}$ to $1+\sum_{n=1}^{\infty}$

6. p.4, p.49, We have changed all $\lim q \to 1-$ to $\lim q \to 1^-$

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7. We have corrected the misprints pointed out by one of the referees in his comments are listed below :

a) Abstract page, L_6 : Knowledge $\rightarrow$ Knowledge of
b) Declaration page, L_2 : has $\rightarrow$ have
c) p.4, L_1 : Ramaujan's $\rightarrow$ Ramanujan's
d) p.4, L_3 : Proofs all of $\rightarrow$ Proofs of all
e) p.4, L_6 : is $\rightarrow$ are

8. We have also corrected the misprints pointed out by one of the other two referees.[p.83, Eq (5.5.36) & Eq (5.5.37)]

9. Since the submission of our previous version of the thesis, the contents of Chapters 5 and 6 have been accepted for publication by the Journal of Number Theory and Integers, respectively. We have added these papers [6] and [7] in the Bibliography of the thesis.

We are extremely grateful to the referees for their helpful comments.

Signature: Jonali Bora

(Jonali Bora)

Date: 6.11.06

6/11/2006

Professor & Head