HYPONORMALITY OF TRIGONOMETRIC TOEPLITZ OPERATORS ON HILBERT SPACES OF ANALYTIC FUNCTIONS

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Abstract

In this work we study the hyponormality property of Toeplitz operators T_φ. We have framed different sets of necessary and sufficient conditions under which a trigonometric Toeplitz operator T_φ is hyponormal. For our study we have considered three different spaces of analytic functions namely the Hardy-Hilbert space $H^2(\mathbb{T})$ consisting of analytic functions measurable on the unit circle $\mathbb{T}$, the Bergman space $A^2(\mathbb{D})$ consisting of analytic functions measurable on the unit disc $\mathbb{D}$, and the weighted Bergman space $A^2_\alpha(\mathbb{D})$ for $-1 < \alpha < \infty$.

DECLARATION

I, Ambeswar Phukon, hereby declare that the subject matter in this thesis entitled, **"Hyponormality of Trigonometric Toeplitz Operators on Hilbert Spaces of Analytic Functions"**, is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/institute.

This thesis is being submitted to the Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

Ambeswar Phukon

Place: Tezpur.

Date: 17/09/2012

Signature of the candidate

TEZPUR UNIVERSITY

CERTIFICATE

This is to certify that the thesis entitled **Hyponormality of Trigonometric Toeplitz Operators on Hilbert Spaces of Analytic Functions** submitted to the School of Science and Technology of Tezpur University in partial fulfillment for the award of the degree of Doctor of Philosophy in Mathematical Sciences is a record of research work carried out by **Mr. Ambeswar Phukon** under my supervision and guidance.

All help received by him from various sources have been duly acknowledged.

No part of this thesis have been submitted elsewhere for award of any other degree.

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Ambeswar Phukan
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Dedicated

To

My Parents

and

Dr. Hiri Kr. Sahoo,

Ex. Director of Higher Education, Govt. of Assam,

the architect of my life.

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Chapter 1

Introduction

1.1 Background and Motivation

In this thesis we investigate the hyponormality of trigonometric Toeplitz operators on Hilbert spaces of analytic functions, namely, the Hardy-Hilbert space $H^2(\mathbb{T})$, the Bergman space $A^2(\mathbb{D})$ and the weighted Bergman space $A^2_\alpha(\mathbb{D})$, where $\alpha > -1$, $\mathbb{T} = \partial\mathbb{D}$ and $\mathbb{D}$ is the unit disc of complex plane. A Toeplitz operator on these spaces is a multiplication operator followed by a projection onto the initial space. The main objective of our research work is to address the following question:

> "Is every trigonometric Toeplitz operator hyponormal? If not, which trigonometric polynomials induce a hyponormal Toeplitz operator and under what conditions?"

In [4], Cowen first characterized the hyponormality of Toeplitz operators on the Hardy space H^2 in terms of the coefficients of the symbol φ. His work was further extended by Nakazi and Takahashi. In [44], they reformulated Cowen's result as follows:

> "For $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator T_φ is hyponormal if and only if there exists $k \in H^\infty(\mathbb{T})$, $\|k\|_\infty \leq 1$ such that $\varphi - k\bar{\varphi} \in H^\infty(\mathbb{T})$."

This reformulation of Cowen's characterization gives a way to determine the hyponormality of T_φ for arbitrary trigonometric polynomial symbols φ. But the problem here is that for the hyponormality of T_φ one requires to solve a certain functional equation in the unit ball of H^∞ which is not always easy. In [53], Kehe Zhu shows that the hyponormality of Toeplitz operators with trigonometric polynomial symbols on the Hardy space can ultimately be reduced to the classical Schur's algorithm in function theory. Thus Zhu again reformulated Cowen's characterization via Schur's functions Φ_n for $n \geq 0$. By explicitly using Schur's functions Φ_n, Zhu has given a general criterion to determine the hyponormality of the Toeplitz operator T_φ with polynomial symbol $\varphi(z)$, where φ is of the form $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$.
Although Zhu's Theorem (refer to Theorem 2.4.1) gives a substantial amount of information to determine the hyponormality of T_φ, the main hurdle here is to find the Schur's functions Φ_n beyond $n \geq 2$. As there is no closed form to determine them so the process is quite laborious. In [36], Kim and Lee gave an alternate version of Zhu's theorem. Similarly in [25], Hwang and Kim studied the hyponormality of T_φ for special kinds of trigonometric polynomial symbols φ which are called circulant type symbols. Thus attempts are continuously being made to determine conditions under which a trigonometric polynomial will induce a hyponormal Toeplitz operator. We quote here a few significant references [6], [14], [26], [27], [28], [29], [30], [31], [32], [38] and [52].

The results obtained from the study of hyponormality of Toeplitz operators on the Hardy space do not automatically extend to the Bergman space. This is because Cowen's Theorem does not adapt to the Bergman space. The reason is that Cowen's theorem is based on a dilation theorem of Sarason [46]. In the Hardy space H^2, the functions in $H^{2\perp}$ are the conjugates of the functions in zH^2. But, for the Bergman space A^2, $A^{2\perp}$ is much larger than the conjugates of functions in zA^2. And so we can find no dilation theorem similar to Sarason in A^2. In his doctoral thesis, H. Sadraoui [45] was the first to study the hyponormality of trigonometric Toeplitz operators on A^2 space via Hankel operators. Hyponormality of Toeplitz operators was also studied in [40]. Then in [23], [24], [33] and [39], the hyponormality of Toeplitz operators was studied by considering some polynomial symbols which are similar to the polynomials considered in the H^2 space. Till date there is no concrete characterization in A^2 space by which the hyponormality of T_φ can be studied for an arbitrary trigonometric polynomial.

1.2 Definitions and Terminologies

Here we give a brief summary of the spaces and operators that are being considered in this work. We include the definitions and significant properties along with the references where these results are discussed in detail. For notational convenience throughout the thesis, $\mathbb{C}$ will always denote the complex plane. The open unit disk in the complex plane $\{z \in \mathbb{C} : |z| < 1\}$ will be denoted by $\mathbb{D}$, and the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ will be denoted by $\mathbb{T}$.

1.2.1 Hardy-Hilbert Space

The Hardy-Hilbert space is the set of all analytic functions whose power series have square summable coefficients. The Hilbert space of functions analytic on the disk is customarily denoted by H^2. Some other spaces of analytic functions are the Bergman and Dirichlet spaces.
Details of these spaces are found in references [22], [43], [54].

We begin with a formal definition of the H^2 space.

Definition 1.2.1. [43] The Hardy-Hibert space, denoted by H^2, consists of all analytic functions having power series representations with square summable complex coefficients. That is,

$$H^2 = \{f : f(z) = \sum_{n=0}^{\infty} a_n z^n \; and \; \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in H^2, the inner product is defined as

$$< f, g >= \sum_{n=0}^{\infty} a_n \bar{b_n}$$

Also the norm of the vector $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is

$$\|f\| = (\sum_{n=0}^{\infty} |a_n|^2)^{\frac{1}{2}}$$

For $z_0 \in \mathbb{D}$, the function k_{z_0} defined by

$$k_{z_0}(z) = \sum_{n=0}^{\infty} \bar{z_0}^n z^n = \frac{1}{1 - \bar{z_0} z}$$

is called the reproducing kernel for z_0 in H^2. Clearly, $k_{z_0} \in H^2$ and $\|k_{z_0}\| = (1 - |z_0|^2)^{-\frac{1}{2}}$. Also for $f \in H^2$, $f(z_0) =< f, k_{z_0} >$.

The Hardy-Hilbert space can also be viewed as a subspace of the well known Hilbert space L^2. $L^2 = L^2(\mathbb{T})$ is the Hilbert space of square-integrable functions on the unit circle $\mathbb{T}$ with respect to Lebesgue measure, normalized so that the measure of the entire circle is 1. The inner product is given by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta$$

where $d\theta$ denotes the ordinary(not normalized) Lebesgue measure on $[0, 2\pi]$. Therefore the norm of the function f in L^2 is given by

$$\|f\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \right)^2.$$

Though the elements of L^2 are equivalence classes, we consider them as functions with the understanding that f and g in L^2 are equal if f is equal to g almost everywhere with respect to normalized Lebesgue measure.

For each integer n, let $e_n(e^{i\theta}) := e^{in\theta}$, regarded as a function on $\mathbb{T}$. Then $\{e_n : n \in \mathbb{Z}\}$ forms an orthonormal basis for L^2. Thus, for $f \in L^2$, $< f, e_n >$ denotes the nth Fourier coefficient of f and so,

$$H^2 = \{f \in L^2 : \langle f, e_n \rangle = 0, \;\; for \;\; n \in \mathbb{Z} \;\; and \;\; n < 0\}$$

It is clear that H^2 is a closed subspace of L^2. Thus H^2 consists of square integrable analytic functions on $\mathbb{T}$.

If L^∞ denotes the space of all essentially bounded functions in L^2, then

$$H^\infty = \{f \in L^\infty : \langle f, e_n \rangle = 0, \;\; for \;\; all \;\; n \in \mathbb{Z} \;\; and \;\; n < 0\}$$

Equivalently, H^∞ consists of all functions that are analytic and bounded on the open unit disk $\mathbb{D}$. The norm of a function $f \in H^\infty$ is defined by

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$$

1.2.2 Toeplitz Operator

A bounded measurable function φ on the unit circle $\mathbb{T}$ induces in a natural way two operators, one on L^2 and one on H^2 as follows:

Definition 1.2.2. The **Laurent operator** $L = L_\varphi$ is the multiplication operator by φ defined as $L_\varphi f = \varphi \cdot f$ for every $f \in L^2$.

Definition 1.2.3. The **Toeplitz operator** is the compression of the multiplication operator to the subspace H^2. That is, for $\varphi \in L^\infty$, the Toeplitz operator with symbol φ is the operator T_φ defined by $T_\varphi f = P(\varphi \cdot f)$ for $f \in H^2$, where P is the orthogonal projection of L^2 onto H^2.

T_φ is linear and since $\varphi \in L^\infty$ so it is also bounded with norm $\|T_\varphi\| = \|\varphi\|_\infty$. If, in particular, $\varphi \in H^\infty$ then T_φ is called an **analytic Toeplitz operator**.

The Toeplitz operator T_φ is said to be co-analytic if T_φ^* is analytic, or equivalently if $\bar{\varphi} \in H^\infty$.

The Toeplitz operator T_φ induced by a trigonometric polynomial φ defined as $\varphi(z) := \sum_{n=-m}^{N} a_n z^n$, is called a **trigonometric Toeplitz operator**.

We recall here some of the important properties of Toeplitz operators as recorded in [2], [43] and [54].

(a) If φ and ψ are in L^∞ then $T_{\alpha\varphi+\psi} = \alpha T_\varphi + T_\psi$ for any scalar α. Also, $T_\varphi^* = T_{\bar{\varphi}}$.

(b) For φ and ψ are in L^∞, $T_\psi T_\varphi$ is a Toeplitz operator if and only if either T_ψ is co-analytic or T_φ is analytic. In both these cases, $T_\psi T_\varphi = T_{\psi\varphi}$.

(c) A Toeplitz operator is self adjoint if and only if its symbol is real valued almost everywhere.

(d) Let φ and ψ are in L^∞. Then $T_\varphi T_\psi = T_\psi T_\varphi$ if and only if atleast one of the following holds:

(i) Both φ and ψ are analytic.

(ii) Both φ and ψ are co-analytic.

(iii) There exists complex numbers α and β, not both zero, such that $\alpha\varphi + \beta\psi$ is a constant.

Since $e_n = e^{int}\,(n \geq 0)$ form an orthonormal basis for H^2, so if φ is a bounded function on $\mathbb{T}$ with Fourier coefficients $a_n\,(n \in \mathbb{N})$, then for $n, m \geq 0$ we have

$$
\begin{aligned}
< T_\varphi e_n, e_m > &= < \varphi e_n, e_m > \\
&= \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta}) e^{in\theta} e^{-im\theta} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta}) e^{-i(m-n)\theta} d\theta \\
&= < \varphi, e_{m-n} > = a_{m-n}
\end{aligned}
$$

Thus the matrix representation of the Toeplitz operator T_φ under the standard basis

$\{e_n\}$ has the following form

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The main characteristic of the above matrix (called the Toeplitz matrix) is that the entries on each main diagonal are constant. In fact this property actually characterizes Toeplitz operators. That is, if T is a bounded linear operator on H^2 with the above matrix under the standard basis, then $T = T_\varphi$ with $\varphi(t) = \sum_{n\in\mathbb{N}} a_n e^{\imath nt}$. Note that if T_φ is an analytic Toeplitz operator, then the matrix of T_φ with respect to the basis $\{e^{\imath n\theta}\}_{n=0}^{\infty}$ is

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\varphi(z) = \sum_0^\infty a_n z^n$.

1.2.3 Hankel Operator

Another operator that appears hand in hand with Toeplitz operators is the Hankel operator. To define Hankel operator we need to introduce the flip operator on L^2 defined as follows:

Definition 1.2.4. The *flip operator* is the operator J mapping L^2 into L^2 defined by $J(e^{\imath n\theta}) = e^{-\imath(n+1)\theta}$

Clearly the operator J is self adjoint and unitary.

Definition 1.2.5. [4] For $\varphi \in L^\infty$, the Hankel operator H_φ is the operator on H^2 given by

$$H_\varphi u = J(I-P)(\varphi \cdot u),$$

where J is the flip operator on L^2, and P is the projection of L^2 onto H^2.

With respect to the standard orthonormal basis for H^2, the matrix representation of the Hankel operator H_φ is as follows:

$$\begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_{-3} & a_{-4} & a_{-5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

The main characteristic of the above matrix is that the entries on each skew-diagonal are constants. In fact, this property determines a Hankel operator on the Hardy space.

Necessary facts about Hankel operators include

(i) $H_{\varphi_1} = H_{\varphi_2}$ if and only if $(I-P)\varphi_1 = (I-P)\varphi_2$;

(ii) $\|H_\varphi\| = inf\{\|\psi\|_\infty : (I-P)\varphi = (I-P)\psi\}$;

(iii) $H_\varphi^* = H_{\varphi^*}$;

(iv) $H_\varphi U = U^* H_\varphi$, where U is the unilateral shift operator;

(v) Either H_φ is one-to-one or $ker(H_\varphi) = \chi H^2$, where χ is an inner function. The closure of the range of H_φ is H^2 in the former case and $(\chi^* H^2)^\perp$ in the later.

1.2.4 Bergman Space

Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. In rectangular and polar coordinates,

$$dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} r dr d\theta.$$

Here the Hilbert space $L^2(\mathbb{D})$ is defined with respect to the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z).$$

Definition 1.2.6. The **Bergman space** $A^2 = A^2(\mathbb{D})$ is the closed subspace of $L^2(\mathbb{D})$ consisting of all analytic functions on $\mathbb{D}$.

The set

$$\{e_n(z) = \sqrt{(n+1)}\, z^n : n \in \mathbb{Z}, n \geq 0 \;\; and \;\; z \in \mathbb{D}\}$$

acts as an orthonormal basis for A^2. The Bergman kernel, to be denoted by $K(z, w)$, is the function

$$K(z,w) = \frac{1}{(1 - z\bar{w})^2}$$

for all $z, w \in \mathbb{D}$. If P denotes the Bergman projection from $L^2(\mathbb{D})$ onto A^2 then for any $f \in L^2(\mathbb{D})$ and $z \in \mathbb{D}$,

$$Pf(z) = \int_{\mathbb{D}} K(z,w)f(w)dA(w)$$

If $L^\infty(\mathbb{D})$ is the space of bounded area measurable functions on the unit disc $\mathbb{D}$, then for $\varphi \in L^\infty(D)$, the multiplication operator M_φ, on the Bergman space is defined by $M_\varphi(f) = \varphi.f$, where $f \in A^2(\mathbb{D})$. The Toeplitz operator T_φ on $A^2(\mathbb{D})$ is defined as $T_\varphi(f) = P(\varphi.f)$. Similarly, the Hankel operator on $A^2(\mathbb{D})$ is defined by $H_\varphi(f) = J(I - P)(\varphi.f)$.

If T_φ is the Toeplitz operator defined on $A^2(\mathbb{D})$ space, then we have

$$T_\varphi(f)(z) = P(\varphi.f)(z) = \int_{\mathbb{D}} \frac{\varphi(w)f(w)}{(1 - z\bar{w})^2}dA(w)$$

1.2.5 Weighted Bergman space

Let $\mathbb{D}$ denote the open unit disc in the complex plane. For $-1 < \alpha < \infty$, $L^2(\mathbb{D}, dA_\alpha)$ is the space of functions on $\mathbb{D}$ which are square integrable with respect to the measure

$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$, where dA denotes the normalized Lebesgue area measure on $\mathbb{D}$. $L^2(\mathbb{D}, dA_\alpha)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}dA_\alpha(z)$$

for $f, g \in L^2(\mathbb{D}, dA_\alpha)$.

Definition 1.2.7. The **weighted Bergman space** A^2_α is the closed subspace of $L^2(\mathbb{D}, dA_\alpha)$ consisting of analytic functions on $\mathbb{D}$. If $\alpha = 0$, A^2_0 is the Bergman space.

For any non negative integer n and $z \in \mathbb{D}$, let $e_n(z) = \frac{z^n}{\gamma_n}$ where $\gamma_n^2 = \frac{\Gamma(n+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}$. Here $\Gamma(s)$ stands for the usual Gamma function. Then $\{e_n\}$ is an orthonormal basis for $A^2_\alpha(\mathbb{D})$.

The reproducing kernel of $A^2_\alpha(\mathbb{D})$ is given by

$$K_z^{(\alpha)}(w) = \frac{1}{(1 - \bar{z}w)^{2+\alpha}} \quad for \quad z, w \in \mathbb{D}$$

The orthogonal projection P_α of $L^2(\mathbb{D}, dA_\alpha)$ onto $A^2_\alpha(\mathbb{D})$ is given by

$$(P_\alpha f)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^{2+\alpha}} dA_\alpha(w),$$

for $f \in L^2(\mathbb{D}, dA_\alpha)$.

If $L^\infty(\mathbb{D})$ denotes the space of all essentially bounded, measurable functions then for $\varphi \in L^\infty(\mathbb{D})$ the multiplication operator M_φ on $A^2_\alpha(\mathbb{D})$ is defined by $M_\varphi(f) = \varphi.f$. The Toeplitz operator T_φ with symbol φ is defined on $A^2_\alpha(\mathbb{D})$ by $T_\varphi f = P_\alpha(\varphi.f)$. Thus we have

$$T_\varphi f(z) = \int_{\mathbb{D}} \frac{\varphi(w)f(w)}{(1 - z\bar{w})^{2+\alpha}} dA_\alpha(w),$$

$f \in A^2_\alpha(\mathbb{D})$ and $z \in \mathbb{D}$.

Similarly, the Hankel operator H_φ on $A^2_\alpha(\mathbb{D})$ is defined by $H_\varphi f = J(I - P_\alpha)(\varphi.f)$. As $\varphi \in L^\infty(\mathbb{D})$, the operators T_φ and H_φ are bounded. For details on these results, we may refer [54].

1.2.6 Hyponormal Operator

Definition 1.2.8. A bounded linear operator T on a complex Hilbert space H is said to be hyponormal if its self commutator $[T^*, T] := T^*T - TT^*$ is positive semi definite. That is, T is hyponormal if $T^*T - TT^* \geq 0$.

The operator T is said to be normal if T commutes with T^*, that is $T^*T = TT^*$.
T is said to be subnormal if it has a normal extension.
It is well known from results discussed in [8] and [16] that every normal operator is subnormal, and every subnormal operator is hyponormal. The converse however is not true. For example, the unilateral shift operator on the ℓ^2 space is subnormal but not normal. Here ℓ^2 denotes the space of all square integrable complex sequences. Again for $0 < a < b < 1$, if $\{\alpha_n\}$ denotes the weight sequence defined as $\alpha_0 = a$, $\alpha_1 = b$ and $\alpha_n = 1\,(n \geq 2)$, then the weighted shift operator T defined on ℓ^2 as

$$T(x_0, x_1, x_2, \dots) = (0, \alpha_0 x_0, \alpha_1 x_1, \alpha_2 x_2, \dots)$$

is hyponormal but not subnormal.
Further, if T is a hyponormal operator, then so is $T - \lambda I$ for $\lambda \in \mathbb{C}$. However, powers of hyponormal operators need not be hyponormal.

1.3 Chapterwise brief outline

The thesis comprises of five chapters and has been organized as follows. The first chapter is introductory in nature. It includes a brief background leading to the problem in hand. The operators and spaces referred to in the sequel are also defined in this chapter.
Our main work begins with chapter 2. Throughout the thesis we have worked on trigonometric Toeplitz operators, denoted by T_φ, acting on different spaces of analytic functions. First we consider T_φ acting on the Hardy space $H^2(\mathbb{T})$. Subsequently

we consider T_φ acting on the Bergman space $A^2(\mathbb{D})$. And finally we consider T_φ acting on the weighted Bergman space $A^2_\alpha(\mathbb{D})$. Our objective is to determine necessary and sufficient conditions for T_φ to be hyponormal.

The second and third chapters deal with T_φ acting on the Hardy space $H^2(\mathbb{T})$. In our work we mainly refer to Kehe Zhu's reformulation of Cowen's theorem in terms of the Schur functions Φ_n. In chapter 2 we make explicit evaluations of Φ_0, Φ_1, Φ_2 and Φ_3. These results are then used in chapter 3 to determine hyponormality conditions of T_φ for situations where the coefficients of φ satisfy partial symmetry conditions. In chapter 4, we investigate hyponormality of T_φ in the Bergman space $A^2(\mathbb{D})$. In this case, J. Lee [39] gave the following necessary condition for the hyponormality of T_φ:

Theorem 1.3.1. *Let* $\varphi(z) = \overline{g(z)} + f(z)$, *where* $f(z) = a_1 z + a_2 z^2$ *and* $g(z) = a_{-1}z + a_{-2}z^2$. *If* T_φ *is hyponormal, then*

(i) $2(|a_2|^2 - |a_{-2}|^2) \geq 3(|a_{-1}|^2 - |a_1|^2)$

(ii) $\left(\frac{1}{2}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{3}(|a_2|^2 - |a_{-2}|^2)\right)\left(\frac{1}{3}(|a_1|^2 - |a_{-1}|^2) + (|a_2|^2 - |a_{-2}|^2)\right)$
$\geq \frac{4}{9}|\bar{a}_1 a_2 - \bar{a}_{-1}a_{-2}|^2$.

First we show that these conditions are not sufficient to guarantee hyponormality of T_φ. Subsequently, in Theorems 4.3.2 and 4.3.3 we give conditions which make T_φ hyponormal. Comparing these results with those obtained for the Hardy space $H^2(\mathbb{T})$ we have generated examples to bring out the fact that the hyponormality conditions of T_φ on $H^2(\mathbb{T})$ do not naturally extend to $A^2(\mathbb{D})$.

The Chapter 5 contains necessary and sufficient conditions for the hyponormality of T_φ in the weighted Bergman space. In particular, we have shown that the following result holds for specific values of m and N:

Theorem 1.3.2. *Let* $\varphi(z) = \overline{g(z)} + f(z)$, *where* $f(z) = a_m z^m + a_N z^N$, $g(z) = a_{-m}z^m + a_{-N}z^N$ $(1 \leq m < N)$. *If* $a_m \bar{a}_N = a_{-m}\bar{a}_{-N}$ *and* $\alpha > -1$, *then* T_φ *on* $A^2_\alpha(\mathbb{D})$

is hyponormal

$$\iff \begin{cases} \frac{\prod_{j=0}^{N-1}(\alpha+2+j)}{\prod_{j=0}^{N-(m+1)}(N-j)}(|a_{-m}|^2 - |a_m|^2) \leq (|a_N|^2 - |a_{-N}|^2) & if \;\; |a_{-N}| \leq |a_N| \\ N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2) & if \;\; |a_N| \leq |a_{-N}| \end{cases}$$

Though Toeplitz operators on spaces of analytic functions are quite well understood and the volume of work published in this area of research is immense, yet not much is known about the behavior of hyponormal Toeplitz operators. In our present work we attempt to carry forward the ongoing research; to plug some of the holes in the existing literature; to make particular case studies to gain better insight; with the aim of answering at least some of the queries regarding the various aspects of hyponormal Toeplitz operators.

Chapter 2

Schur's Function Φ_n and Kehe Zhu's Theorem

In this chapter we record Zhu's theorem and its different re-formulations. We give an explicit evaluation of Φ_3 and use this to give expressions of Φ_n for higher orders of n.

2.1 Introduction

In 1988, Cowen first characterized the hyponormality of Toeplitz operators on the Hardy space H^2 in terms of the coefficients of the symbol φ. He showed that:

Theorem 2.1.1. [4] *If φ is in $H^\infty(\mathbb{T})$, where $\varphi = f + \bar{g}$ for $f, g \in H^2(\mathbb{T})$, then T_φ is hyponormal if and only if $g = c + T_{\bar{h}} f$ for some constant c and some function h in $H^\infty(\mathbb{T})$ with $\|h\|_\infty \leq 1$.*

In 1993, Nakazi and Takahashi reformulated this theorem as follows:

Theorem 2.1.2. [44] *Suppose that $\varphi \in L^\infty(\mathbb{T})$ is arbitrary and write*

$$\mathcal{E}(\varphi) = \left\{k \in H^\infty(\mathbb{T}) : \|k\|_\infty \leq 1 \text{ and } \varphi - k\bar{\varphi} \in H^\infty(\mathbb{T})\right\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

This reformulation of Cowen's characterization gives a way to determine the hyponormality of T_φ for arbitrary trigonometric polynomial symbols φ. But the problem here is that for the hyponormality of T_φ one requires to solve a certain functional equation in the unit ball of H^∞ which is not always easy. In [53], Kehe Zhu again reformulated Cowen's characterization via Schur's functions Φ_n for $n \geq 0$. By using explicitly Schur's functions Φ_n, Zhu has given a general criterion to determine the hyponormality of the Toeplitz operator T_φ with polynomial symbol $\varphi(z)$, where φ is of the form $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$. We begin with a brief description of Zhu's idea.

2.2 Schur's Functions Φ_n

Suppose that $f(z) = \sum_{j=0}^{\infty} c_j z^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$ (i.e. $\|f\|_\infty \leq 1$). If $f_0 = f$, define by induction a sequence $\{f_n\}$ of functions in the closed unit ball of $H^\infty(\mathbb{T})$ as follows:

$$f_{n+1}(z) = \frac{f_n(z) - f_n(0)}{z(1 - \overline{f_n(0)} f_n(z))}, \quad |z| < 1, \quad n = 0, 1, 2, \ldots \tag{2.2.1}$$

As $f_n(0)$ only depends on the coefficients $c_0, c_1, \ldots, c_n$, we can write

$$f_n(0) = \Phi_n(c_0, \ldots, c_n) \quad n = 0, 1, 2, \ldots,$$

where Φ_n is a function of $n+1$ complex variables. We call the Φ_n's Schur's functions. No closed-form for the general Schur's function Φ_n is known. However, Schur's algorithm enables us to derive Φ_n for any desired values for $n \geq 0$. In [53], Zhu has listed the first three Schur's functions:

$$\Phi_0(c_0) = c_0 \tag{2.2.2}$$

$$\Phi_1(c_0, c_1) = \frac{c_1}{1 - |c_0|^2} \tag{2.2.3}$$

$$\Phi_2(c_0, c_1, c_2) = \frac{c_2(1 - |c_0|^2) + \bar{c}_0 {c_1}^2}{(1 - |c_0|^2)^2 - |c_1|^2} \tag{2.2.4}$$

In the next section we give the procedure to find the Schur's function Φ_3.

2.3 Evaluation of Schur's Function Φ_3

As $\Phi_n(c_0, c_1, ..., c_n) = f_n(0)$, so

$$\Phi_0(c_0) = f_0(0) = f(0) = c_0$$

$$f_1(z) = \frac{f_0(z) - f_0(0)}{z(1 - \overline{f_0(0)} f_0(z))} = \frac{\sum_{k=0}^{\infty} c_k z^k - c_0}{z(1 - \bar{c_0} \sum_{k=0}^{\infty} c_k z^k)} = \frac{\sum_{k=1}^{\infty} c_k z^{k-1}}{1 - |c_0|^2 - \bar{c_0} \sum_{k=1}^{\infty} c_k z^k} \tag{2.3.1}$$

Therefore,

$\Phi_1(c_0, c_1) = f_1(0) = \frac{c_1}{1-|c_0|^2}$

$$f_2(z) = \frac{f_1(z) - f_1(0)}{z(1 - \overline{f_1(0)} f_1(z))} = \frac{N}{zD} \tag{2.3.2}$$

where,

$$\begin{aligned} N &= \frac{\sum_{k=1}^{\infty} c_k z^{k-1}}{1 - \bar{c_0} \sum_{k=0}^{\infty} c_k z^k} - \frac{c_1}{1 - |c_0|^2} \\ &= \frac{(1 - |c_0|^2) \sum_{k=1}^{\infty} c_k z^{k-1} - c_1(1 - |c_0|^2 - \bar{c_0} \sum_{k=1}^{\infty} c_k z^k)}{(1 - \bar{c_0} \sum_{k=0}^{\infty} c_k z^k)(1 - |c_0|^2)} \\ &= \frac{(1 - |c_0|^2) \sum_{k=2}^{\infty} c_k z^{k-1} + \bar{c_0} c_1 \sum_{k=1}^{\infty} c_k z^k}{(1 - \bar{c_0} \sum_{k=0}^{\infty})(1 - |c_0|^2)} \end{aligned}$$

$$\begin{aligned} D &= 1 - (\frac{\bar{c_1}}{1 - |c_0|^2})(\frac{\sum_{k=1}^{\infty} c_k z^{k-1}}{1 - \bar{c_0} \sum_{k=0}^{\infty} c_k z^k}) \\ &= \frac{(1 - \bar{c_0} \sum_{k=0}^{\infty} c_k z^k)(1 - |c_0|^2) - (|c_1|^2 + \bar{c_1} \sum_{k=2}^{\infty} c_k z^{k-1})}{(1 - \bar{c_0} \sum_{k=o}^{\infty} c_k z^k)(1 - |c_0|^2)} \\ &= \frac{((1 - |c_0|^2)^2 - |c_1|^2) - \bar{c_0}(1 - |c_0|^2) \sum_{k=1}^{\infty} c_k z^k - \bar{c_1} \sum_{k=2}^{\infty} c_k z^{k-1}}{(1 - \bar{c_0} \sum_{k=0}^{\infty} c_k z^k)(1 - |c_0|^2)} \end{aligned}$$

Therefore,

$$f_2(z) = \frac{(1 - |c_0|^2) \sum_{k=2}^{\infty} c_k z^{k-2} + \bar{c_0} c_1 \sum_{k=1}^{\infty} c_k z^{k-1}}{((1 - |c_0|^2)^2 - |c_1|^2) - \bar{c_0}(1 - |c_0|^2) \sum_{k=1}^{\infty} c_k z^k - \bar{c_1} \sum_{k=2}^{\infty} c_k z^{k-1}} \tag{2.3.3}$$

which gives,

$$\Phi_2(c_0, c_1, c_2) = f_2(0) = \frac{c_2(1 - |c_0|^2) + \bar{c_0} {c_1}^2}{(1 - |c_0|^2)^2 - |c_1|^2} \tag{2.3.4}$$

Again,

$$f_3(z) = \frac{f_2(z) - f_2(0)}{z(1 - \overline{f_2(0)} f_2(z))} \tag{2.3.5}$$

Now, $f_2(z) - f_2(0)$

$= \frac{(1-|c_0|^2)\sum_{k=2}^{\infty} c_k z^{k-2} + \bar{c_0} c_1 \sum_{k=1}^{\infty} c_k z^{k-1}}{((1-|c_0|^2)^2 - |c_1|^2) - \bar{c_0}(1-|c_0|^2)\sum_{k=1}^{\infty} c_k z^k - \bar{c_1}\sum_{k=2}^{\infty} c_k z^{k-1}} - \frac{c_2(1-|c_0|^2) + \bar{c_0} c_1{}^2}{(1-|c_0|^2)^2 - |c_1|^2}$

$= \frac{1}{P}\Bigg[\Big\{(1-|c_0|^2)\sum_{k=2}^{\infty} c_k z^{k-2} + \bar{c_0} c_1 \sum_{k=1}^{\infty} c_k z^{k-1}\Big\}\Big\{(1-|c_0|^2)^2 - |c_1|^2\Big\} - \Big\{c_2(1 -$
$|c_0|^2) + \bar{c_0} c_1{}^2\Big\}\Bigg(\Big\{(1-|c_0|^2)^2 - |c_1|^2\Big\} - \bar{c_0}(1-|c_0|^2)\sum_{k=1}^{\infty} c_k z^k - \bar{c_1}\sum_{k=2}^{\infty} c_k z^{k-1}\Bigg)\Bigg]$

$= \frac{z}{P}\Bigg[\Big\{(1-|c_0|^2)^2 - |c_1|^2\Big\}\Big\{(1-|c_0|^2)\sum_{k=2}^{\infty} c_{k+1} z^{k-2} + \bar{c_0} c_1 \sum_{k=1}^{\infty} c_{k+1} z^{k-1}\Big\}$
$+ \Big\{c_2(1-|c_0|^2) + \bar{c_0} c_1{}^2\Big\}\Big\{\bar{c_0}(1-|c_0|^2)\sum_{k=0}^{\infty} c_{k+1} z^k + \bar{c_1}\sum_{k=1}^{\infty} c_{k+1} z^{k-1}\Big\}\Bigg]$

$= \frac{z}{P}\Bigg[\Big\{(1-|c_0|^2)^2 - |c_1|^2\Big\}\sum_{k=1}^{\infty}\Big\{(1-|c_0|^2) c_{k+2} + \bar{c_0} c_1 c_{k+1}\Big\} z^{k-1}$
$+ \Big\{c_2(1-|c_0|^2) + \bar{c_0} c_1{}^2\Big\}\sum_{k=0}^{\infty}\Big\{\bar{c_0}(1-|c_0|^2) c_{k+1} + \bar{c_1} c_{k+2}\Big\} z^k\Bigg]$

$= \frac{zN(z)}{P}$

and

$1 - \overline{f_2(0)} f_2(z)$

$= 1 - \Big(\frac{\bar{c_2}(1-|c_0|^2) + c_0 \bar{c_1}^2}{(1-|c_0|^2)^2 - |c_1|^2}\Big)\Big(\frac{(1-|c_0|^2)\sum_{k=2}^{\infty} c_k z^{k-2} + \bar{c_0} c_1 \sum_{k=1}^{\infty} c_k z^{k-1}}{((1-|c_0|^2)^2 - |c_1|^2) - \bar{c_0}(1-|c_0|^2)\sum_{k=1}^{\infty} c_k z^k - \bar{c_1}\sum_{k=2}^{\infty} c_k z^{k-1}}\Big)$

$= \frac{1}{P}\Bigg[\Big\{(1-|c_0|^2)^2 - |c_1|^2\Big\}^2 - \Big\{(1-|c_0|^2)^2 - |c_1|^2\Big\}\Big\{\bar{c_0}(1-|c_0|^2)\sum_{k=1}^{\infty} c_k z^k +$
$\bar{c_1}\sum_{k=2}^{\infty} c_k z^{k-1}\Big\} - \Big\{\bar{c_2}(1-|c_0|^2) + c_0\bar{c_1}^2\Big\}\Big\{c_2(1-|c_0|^2) + \bar{c_0} c_1{}^2\Big\}$
$- \Big\{\bar{c_2}(1-|c_0|^2) + c_0\bar{c_1}^2\Big\}\Big\{(1-|c_0|^2)\sum_{k=3}^{\infty} c_k z^{k-2} + \bar{c_0} c_1 \sum_{k=2}^{\infty} c_k z^{k-1}\Big\}\Bigg]$

$= \frac{D(z)}{P}$

where,

$$P = \left[\Big\{(1-|c_0|^2)^2 - |c_1|^2\Big\} - \bar{c_0}(1-|c_0|^2)\sum_{k=1}^{\infty} c_k z^k - \bar{c_1}\sum_{k=2}^{\infty} c_k z^{k-1}\right]\Big\{(1-|c_0|^2)^2 - |c_1|^2\Big\}$$

$$N(z) = \sum_{k=0}^{\infty}\Bigg[\Big\{(1-|c_0|^2)^2 - |c_1|^2\Big\}\Big\{(1-|c_0|^2) c_{k+3} + \bar{c_0} c_1 c_{k+2}\Big\}$$
$$+ \Big\{c_2(1-|c_0|^2) + \bar{c_0} c_1{}^2\Big\}\Big\{\bar{c_0}(1-|c_0|^2) c_{k+1} + \bar{c_1} c_{k+2}\Big\}\Bigg]$$

$D(z) = \big((1-|c_0|^2)^2 - |c_1|^2\big)^2\Big[1 - \sum_{k=1}^{\infty}\Big\{\bar{c_0}(1-|c_0|^2) c_k + \bar{c_1} c_{k+1}\Big\} z^k\Big] - |c_2(1-|c_0|^2) +$

$\bar{c_0}c_1{}^2|^2 - \{\bar{c_2}(1-|c_0|^2) + c_0\bar{c_1}{}^2\} \sum_{k=1}^{\infty} \{\bar{c_0}c_1c_{k+1} + (1-|c_0|^2)c_{k+2}\} z^k$

Therefore, $f_3(z) = \frac{N(z)}{D(z)}$

and hence,

$$\Phi_3(c_0, c_1, c_2, c_3) = f_3(0) = \frac{N(0)}{D(0)},$$

where

$N(0) = (1-|c_0|^2)^2 - |c_1|^2)((1-|c_0|^2)c_3 + \bar{c_0}c_1c_2) + (c_2(1-|c_0|^2) + \bar{c_0}c_1{}^2)(\bar{c_0}(1-|c_0|^2)c_1 + \bar{c_1}c_2)$

and

$D(0) = ((1-|c_0|^2)^2 - |c_1|^2)^2 - |c_2(1-|c_0|^2) + \bar{c_0}c_1{}^2|^2$

Thus, $\Phi_3(c_0, c_1, c_2, c_3) =$

$$\frac{(1-|c_0|^2)^2 - |c_1|^2)((1-|c_0|^2)c_3 + \bar{c_0}c_1c_2) + (c_2(1-|c_0|^2) + \bar{c_0}c_1{}^2)(\bar{c_0}(1-|c_0|^2)c_1 + \bar{c_1}c_2)}{((1-|c_0|^2)^2 - |c_1|^2)^2 - |c_2(1-|c_0|^2) + \bar{c_0}c_1{}^2|^2} \tag{2.3.6}$$

2.4 Application of Zhu's Theorem

By using explicitly Schur's functions Φ_n for $n \geq 0$, Zhu has given a general criterion to determine the hyponormality of the Toeplitz operator T_φ with trigonometric polynomial symbol $\varphi(z)$, where φ is of the form $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$. His theorem is as stated below:

Theorem 2.4.1. [53] *If $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$, where $a_N \neq 0$ and if*

$$\begin{pmatrix} \bar{c_0} \\ \bar{c_1} \\ \vdots \\ \bar{c}_{N-1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{N-1} & a_N \\ a_2 & a_3 & \cdots & a_N & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_N & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{a}_{-1} \\ \bar{a}_{-2} \\ \vdots \\ \bar{a}_{-N} \end{pmatrix}, \tag{2.4.1}$$

then T_φ is hyponormal if and only if $|\Phi_n(c_0, ..., c_n)| \leq 1$ for each $n = 0, 1, ..., N-1$.

Note that each $\Phi_n(c_0, ..., c_n)$ is a rational function of the form

$$\Phi_n(c_0, ..., c_n) = \frac{F_n(c_0, ..., c_n)}{G_n(c_0, ..., c_n)},$$

where F_n and G_n are polynomials. Thus the inequalities $|\Phi_n(c_0, ..., c_n)| \le 1$ should be understood as $|F_n(c_0, ..., c_n)| \le |G_n(c_0, ..., c_n)|$.

If $f(z) = \sum_{j=0}^{\infty} c_j z^j$ is a function in $H^\infty(\mathbb{T})$ such that $\varphi - f\bar{\varphi} \in H^\infty(\mathbb{T})$, then $c_0, c_1, ..., c_{N-1}$ are just the values given in (2.4.1). Thus Zhu's theorem says that if $f(z) = \sum_{j=0}^{\infty} c_j z^j$ satisfies $\varphi - f\bar{\varphi} \in H^\infty(\mathbb{T})$, then the hyponormality of T_φ is independent of the values of c_j's for $j \ge N$. In [36], Kim and Lee reformulated Zhu's theorem in a simpler form which is often easier to apply.

Proposition 2.4.2. [36] *If $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where $m \le N$ and $a_N \ne 0$, then T_φ is hyponormal if and only if*

$$|\Phi_n(c_0, c_1, ..., c_n)| \le 1 \text{ for each } n = 0, 1, ..., N-1.$$

where c_n's are given by the following recurrence relation:

$$\begin{cases} c_0 = c_1 = ... = c_{N-m-1} = 0 \\ c_{N-m} = \frac{a_{-m}}{\bar{a}_N} \\ c_n = (\bar{a}_N)^{-1}\left(a_{-N+n} - \sum_{j=N-m}^{n-1} c_j \bar{a}_{N-n+j}\right) \quad \text{for } n = N-m+1, ..., N-1. \end{cases} \tag{2.4.2}$$

Using these re-formulations of Zhu's theorem we now prove the following result.

Proposition 2.4.3. *Suppose that $k(z) = \sum_{j=0}^{\infty} c_j z^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$ and that $\{\Phi_n\}$ is a sequence of Schur's functions associated with $\{c_n\}$. If $c_1 = c_2 = ... = c_{n-1} = 0$ and $c_n \ne 0$, then we have*

$$\Phi_0 = c_0, \quad \Phi_1 = ... = \Phi_{n-1} = 0; \quad \Phi_n = \frac{c_n}{1 - |c_0|^2};$$

$$\Phi_{n+1} = \frac{c_{n+1}}{(1 - |c_0|^2)(1 - |\Phi_n|^2)};$$

$$\Phi_{n+2} = \frac{(1 - |\Phi_n|^2)c_{n+2}c_n + |\Phi_n|^2 c_{n+1}^2}{c_n(1 - |c_0|^2)(1 - |\Phi_n|^2)^2(1 - |\Phi_{n+1}|^2)}.$$

Proof. Suppose $k(z) = \sum_{j=0}^{\infty} c_j z^j$. Then $\Phi_0 = k(0) = c_0$ and

$$k_1(z) = \frac{k_0(z) - k_0(0)}{z(1 - \bar{c}_0 k_0(z))} = \frac{\sum_{j=1}^{\infty} c_j z^{j-1}}{1 - \bar{c}_0 k_0(z)},$$

so that

$$\Phi_1 = k_1(0) = \frac{c_1}{1 - |c_0|^2} = 0.$$

Also we have that

$$k_2(z) = \frac{k_1(z) - k_1(0)}{z(1 - \overline{k_1(0)} k_1(z))} = \frac{k_1(z)}{z} = \frac{\sum_{j=2}^{\infty} c_j z^{j-2}}{1 - \bar{c}_0 k_0(z)},$$

so that

$$\Phi_2 = k_2(0) = \frac{c_2}{1 - |c_0|^2} = 0.$$

Inductively,

$$k_m(z) = \frac{\sum_{j=m}^{\infty} c_j z^{j-m}}{1 - \bar{c}_0 k_0(z)} \quad for \quad m = 3, ..., n-1,$$

so that

$$\Phi_m = k_m(0) = \frac{c_m}{1 - |c_0|^2} = 0 \quad for \quad m = 3, ..., n-1$$

Then

$$k_n(z) = \frac{k_{n-1}(z) - k_{n-1}(0)}{z(1 - \overline{k_{n-1}(0)} k_{n-1}(z))} = \frac{\sum_{j=n}^{\infty} c_j z^{j-n}}{1 - \bar{c}_0 k_0(z)},$$

so that

$$\Phi_n = k_n(0) = \frac{c_n}{1 - |c_0|^2} \tag{2.4.3}$$

$$\begin{aligned} k_{n+1}(z) &= \frac{k_n(z) - k_n(0)}{z(1 - \overline{k_n(0)} k_n(z))} \\ &= \frac{\frac{1}{1 - \bar{c}_0 k_0(z)} \sum_{j=n}^{\infty} c_j z^{j-n} - \frac{c_n}{1 - |c_0|^2}}{z(1 - \overline{k_n(0)} k_n(z))} \\ &= \frac{(1 - |c_0|^2) \sum_{j=n+1}^{\infty} c_j z^{j-n-1} + c_n \bar{c}_0 \sum_{j=1}^{\infty} c_j z^{j-1}}{(1 - |c_0|^2)(1 - \bar{c}_0 k_0(z))(1 - \overline{k_n(0)} k_n(z))} \\ &= \frac{(1 - |c_0|^2) \sum_{j=n+1}^{\infty} c_j z^{j-n-1} + c_n \bar{c}_0 \sum_{j=n}^{\infty} c_j z^{j-1}}{(1 - |c_0|^2)(1 - \bar{c}_0 k_0(z))(1 - \overline{k_n(0)} k_n(z))} \end{aligned} \tag{2.4.4}$$

so that

$$\Phi_{n+1} = k_{n+1}(0) = \frac{(1-|c_0|^2)c_{n+1}}{(1-|c_0|^2)^2(1-|k_0(0)|^2)}$$
$$= \frac{c_{n+1}}{(1-|c_0|^2)(1-|\Phi_n|^2)}. \tag{2.4.5}$$

Schur's function for k_{n+2} is,

$$k_{n+2}(z) = \frac{k_{n+1}(z) - k_{n+1}(0)}{z(1-\overline{k_{n+1}(0)}k_{n+1}(z))} \tag{2.4.6}$$

Now,

$$k_{n+1}(z) - k_{n+1}(0)$$
$$= \frac{(1-|c_0|^2)\sum_{j=n+1}^{\infty} c_j z^{j-n-1} + c_n\bar{c}_0 \sum_{j=n}^{\infty} c_j z^{j-1}}{(1-|c_0|^2)(1-\bar{c}_0 k_0(z))(1-\overline{k_n(0)}k_n(z))} - \frac{c_{n+1}(1-|c_0|^2)}{(1-|c_0|^2)^2-|c_n|^2} \tag{2.4.7}$$

Let $D(z) = (1-|c_0|^2)(1-\bar{c}_0 k_0(z))(1-\overline{k_n(0)}k_n(z))((1-|c_0|^2)^2-|c_n|^2)$

Then

$$k_{n+2}(z) = \frac{A(z)+B(z)-C(z)}{zD(z)\left(1-\overline{k_{n+1}(0)}k_{n+1}(z)\right)}$$

where,

$$A(z) =(1-|c_0|^2)((1-|c_0|^2)^2-|c_n|^2)\left(c_{n+1} + \sum_{j=n+2}^{\infty} c_j z^{j-n-1}\right)$$
$$=(1-|c_0|^2)((1-|c_0|^2)^2-|c_n|^2)c_{n+1} + z(1-|c_0|^2)\times$$
$$((1-|c_0|^2)^2-|c_n|^2)\sum_{j=n+2}^{\infty} c_j z^{j-n-2}$$

$$B(z) = c_n\bar{c}_0\left((1-|c_0|^2)^2 - |c_n|^2\right)\sum_{j=n}^{\infty} c_j z^{j-1}$$

and

$$C(z) = c_{n+1}\left(1-|c_0|^2\right)^2\left(1-\bar{c}_0\left(c_0+\sum_{j=1}^{\infty}c_j z^j\right)\right)\left(1-\frac{\overline{k_n(0)}\sum_{j=n}^{\infty}c_j z^{j-n}}{1-\bar{c}_0 k_0(z)}\right)$$

$$=c_{n+1}\left(1-|c_0|^2\right)^2\left(1-|c_0|^2-\bar{c}_0\sum_{j=1}^{\infty}c_j z^j\right)\left(1-\frac{\overline{k_n(0)}\left(c_n+\sum_{j=n+1}^{\infty}c_j z^{j-n}\right)}{1-\bar{c}_0\left(c_0+\sum_{j=1}^{\infty}c_j z^j\right)}\right)$$

$$=c_{n+1}\left(1-|c_0|^2\right)^2\left(1-|c_0|^2-\bar{c}_0\sum_{j=1}^{\infty}c_j z^j\right)\left(1-\frac{\frac{|c_n|^2}{1-|c_0|^2}+\frac{\bar{c}_n}{1-|c_0|^2}\sum_{j=n+1}^{\infty}c_j z^{j-n}}{1-|c_0|^2-\bar{c}_0\sum_{j=1}^{\infty}c_j z^j}\right)$$

$$=c_{n+1}\left(1-|c_0|^2\right)^2\left(1-|c_0|^2-\bar{c}_0\sum_{j=1}^{\infty}c_j z^j\right)\times$$

$$\left(\frac{1-|c_0|^2-\bar{c}_0\sum_{j=1}^{\infty}c_j z^j-\frac{|c_n|^2}{1-|c_0|^2}-\frac{\bar{c}_n}{1-|c_0|^2}\sum_{j=n+1}^{\infty}c_j z^{j-n}}{1-|c_0|^2-\bar{c}_0\sum_{j=1}^{\infty}c_j z^j}\right)$$

$$=\frac{\left(c_{n+1}(1-|c_0|^2)^3-c_{n+1}(1-|c_0|^2)^2\bar{c}_0\sum_{j=1}^{\infty}c_j z^j\right)}{1-|c_0|^2-\bar{c}_0\sum_{j=1}^{\infty}c_j z^j}\times$$

$$\left(1-|c_0|^2-\frac{|c_n|^2}{1-|c_0|^2}-\bar{c}_0\sum_{j=1}^{\infty}-\frac{\bar{c}_n}{1-|c_0|^2}\sum_{j=n+1}^{\infty}c_j z^{j-n}\right)$$

$$=\frac{(c_{n+1}(1-|c_0|^2)^3\left(1-|c_0|^2-\frac{|c_n|^2}{1-|c_0|^2}\right)}{1-|c_0|^2-\bar{c}_0\sum_{j=1}^{\infty}c_j z^j}-$$

$$\frac{c_{n+1}(1-|c_0|^2)^3\left(\bar{c}_0\sum_{j=1}^{\infty}c_j z^j+\frac{\bar{c}_n}{1-|c_0|^2}\sum_{j=n+1}^{\infty}c_j z^{j-n}\right)}{1-|c_0|^2-\bar{c}_0\sum_{j=1}^{\infty}c_j z^j}$$

$$-\frac{\left(1-|c_0|^2-\frac{|c_n|^2}{1-|c_0|^2}\right)c_{n+1}(1-|c_0|^2)^2\bar{c}_0\sum_{j=1}^{\infty}c_j z^j}{1-|c_0|^2-\bar{c}_0\sum_{j=1}^{\infty}c_j z^j}$$

$$+\frac{c_{n+1}(1-|c_0|^2)^2\bar{c}_0\sum_{j=1}^{\infty}c_j z^j\left(\bar{c}_0\sum_{j=1}^{\infty}c_j z^j+\frac{\bar{c}_n}{1-|c_0|^2}\sum_{j=n+1}^{\infty}c_j z^{j-n}\right)}{1-|c_0|^2-\bar{c}_0\sum_{j=1}^{\infty}c_j z^j}$$

So $A(z)+B(z)-C(z)=$

$$(1-|c_0|^2)^2((1-|c_0|^2)^2-|c_n|^2)c_{n+1}-(1-|c_0|^2)((1-|c_0|^2)^2-|c_n|^2)c_{n+1}\bar{c}_0 z\times$$
$$\sum_{j=1}^{\infty}c_j z^{j-1}+(1-|c_0|^2)^2 z(1-|c_0|^2)^2-|c_n|^2)\sum_{j=n+2}^{\infty}c_j z^{j-n-2}$$
$$-z^2(1-|c_0|^2)(1-|c_0|^2)^2-|c_n|^2)\sum_{j=n+2}^{\infty}c_j z^{j-n-2}\bar{c}_0\sum_{j=1}^{\infty}c_j z^{j-1}+$$
$$c_n\bar{c}_0 c_1(1-|c_0|^2)((1-|c_0|^2)^2-|c_n|^2)-c_n c_1\bar{c}_0^2 z((1-|c_0|^2)^2-|c_n|^2)\sum_{j=1}^{\infty}c_j z^{j-1}+$$
$$(1-|c_0|^2)c_n\bar{c}_0 z((1-|c_0|^2)^2-|c_n|^2)\sum_{j=2}^{\infty}c_j z^{j-2}-$$
$$z^2 c_n\bar{c}_0^2((1-|c_0|^2)^2-|c_n|^2)\sum_{j=2}^{\infty}c_j z^{j-2}\sum_{j=1}^{\infty}c_j z^{j-1}-\left(c_{n+1}(1-|c_0|^2)^3\left(1-|c_0|^2-\right.\right.$$
$$\left.\left.\frac{|c_n|^2}{1-|c_0|^2}\right)\right)+c_{n+1}(1-|c_0|^2)^3 z\left(\bar{c}_0\sum_{j=1}^{\infty}c_j z^{j-1}+\frac{\bar{c}_n}{1-|c_0|^2}\sum_{j=n+1}^{\infty}c_j z^{j-n-1}\right)+$$
$$\left(1-|c_0|^2-\frac{|c_n|^2}{1-|c_0|^2}\right)c_{n+1}(1-|c_0|^2)^2\bar{c}_0 z\sum_{j=1}^{\infty}c_j z^{j-1}-$$
$$c_{n+1}(1-|c_0|^2)^2\bar{c}_0 z^2\sum_{j=1}^{\infty}c_j z^{j-1}\left(\bar{c}_0\sum_{j=1}^{\infty}c_j z^{j-1}-\frac{\bar{c}_n}{1-|c_0|^2}\sum_{j=n+1}^{\infty}c_j z^{j-n-1}\right)\qquad(2.4.8)$$

The term free from z in the equation (2.4.8) is

$$\begin{aligned}&(1-|c_0|^2)((1-|c_0|^2)^2-|c_n|^2)c_{n+1}+c_n c_1\bar{c}_0(1-|c_0|^2)((1-|c_0|^2)^2-|c_n|^2)\\&-c_{n+1}(1-|c_0|^2)^3\left(1-|c_0|^2-\frac{|c_n|^2}{1-|c_0|^2}\right)\\&=0\ \ [\text{because } c_1=0]\end{aligned}\qquad(2.4.9)$$

The coefficient of z in the equation (2.4.8) is

$$\begin{aligned}
&-(1-|c_0|^2)((1-|c_0|^2)^2-|c_n|^2)c_{n+1}\bar{c}_0c_1+(1-|c_0|^2)^2((1-|c_0|^2)^2-|c_n|^2)c_{n+2}\\
&-c_n\bar{c}_0((1-|c_0|^2)^2-|c_n|^2)c_1^2c_0+(1-|c_0|^2)c_nc_0((1-|c_0|^2)^2-|c_n|^2)c_2c_{n+1}\times\\
&(1-|c_0|^2)^3\left(\bar{c}_0c_1+\frac{\bar{c}_n}{1-|c_0|^2}c_{n+1}\right)+\left(1-|c_0|^2-\frac{|c_n|^2}{1-|c_0|^2}\right)c_{n+1}(1-|c_0|^2)^2\bar{c}_0c_1\\
=&(1-|c_0|^2)((1-|c_0|^2)^2-|c_n|^2)c_{n+2}+\bar{c}_nc_{n+1}^2(1-|c_0|^2)^2 \quad [\text{because } c_1=c_2=0]\\
=&(1-|c_0|^2)^2\left(((1-|c_0|^2)^2-|c_n|^2)c_{n+2}+\bar{c}_nc_{n+1}^2\right) \qquad (2.4.10)
\end{aligned}$$

Therefore,

$$k_{n+2}=\frac{(1-|c_0|^2)^2\left(((1-|c_0|^2)^2-|c_n|^2)\,c_{n+2}+\bar{c}_nc_{n+1}^2\right)+O(z)}{D(z)\left(1-\overline{k_{n+1}(0)}k_{n+1}(z)\right)}$$

where $O(z)$ denotes the terms involving $z, z^2, z^3, \ldots$.

Therefore,

$$\begin{aligned}
\Phi_{n+2}=&\frac{(1-|c_0|^2)^2\left(((1-|c_0|^2)^2-|c_n|^2)\,c_{n+2}+\bar{c}_nc_{n+1}^2\right)}{D(0)\,(1-|k_{n+1}(0)|^2)}\\
=&\frac{((1-|c_0|^2)^2-|c_n|^2)c_{n+2}+\bar{c}_nc_{n+1}^2}{(1-|c_0|^2)(1-|\Phi_n|^2)((1-|c_0|^2)^2-|c_n|^2)(1-|\Phi_{n+1}|^2)}\\
=&\frac{(1-|\Phi_n|^2)c_{n+2}+\frac{|c_n|^2}{(1-|c_0|^2)^2c_n}c_{n+1}^2}{(1-|c_0|^2)(1-|\Phi_n|^2)^2(1-|\Phi_{n+1}|^2)}
\end{aligned}$$

Dividing both numerator and denominator by $(1-|c_0|^2)^2$ and writing $\Phi_n=\frac{c_n}{1-|c_0|^2}$, we get

$$\Phi_{n+2}=\frac{(1-|\Phi_n|^2)c_{n+2}c_n+|\Phi_n|^2c_{n+1}^2}{c_n(1-|c_n|^2)(1-|\Phi_n|^2)^2(1-|\Phi_{n+1}|^2)} \qquad (2.4.11)$$

□

Remark 2.4.1. Proposition 2.4.3 is an extension of Proposition 3 [36] where Φ_0, $\Phi_1, \ldots, \Phi_{n+1}$ were determined. Here we have also determined Φ_{n+2} using the explicit evaluation of Φ_3 given in section 2.3. We shall be using Proposition 2.4.3 in Chapter 3 to determine hyponormality conditions for different trigonometric Toeplitz operators T_φ.

Chapter 3

Hyponormality in Hardy Space

In this Chapter we provide some necessary and sufficient conditions for the hyponormality of Toeplitz operator T_φ, where φ is a trigonometric polynomial whose coefficients satisfy certain partial symmetry conditions.

3.1 Significant Prior Results

In this section we discuss some significant corollaries of the results established by Cowen, Nakazi & Takahashi, Zhu and others that have already been stated in chapter 2. These are included to give the necessary background leading to the significance of the work done in this chapter. Simplified versions of the original proofs are also included to make the work self contained.

Theorem 3.1.1. [53] *Suppose $\varphi = f+\bar{g}$, where f is an analytic polynomial of degree n, and g is in H^∞. If T_φ is hyponormal, then g must be an analytic polynomial of degree less than or equal to n.*

Proof. As T_φ is hyponormal, so by Theorem 2.1.1, there exists a constant c and $h \in H^\infty$ with $\|h\|_\infty \leq 1$ such that $g = c + T_{\bar{h}}f$. If P denotes the projection of L^2 onto H^2, then we have $c = P(g - \bar{h}f)$, which implies $\bar{g} - h\bar{f} = k$, for some $k \in H^2$.

For $z \in \mathbb{T}$, let $f(z) = \sum_{i=0}^{n} a_i z^i$, $a_n \neq 0$.

Since $h(z)f(\bar{z}) = \bar{z}^n h(z) \sum_{i=0}^{n} \bar{a}_i z^{n-i}$ so $z^n g(\bar{z}) = h(z) \sum_{i=0}^{n} \bar{a}_i z^{n-i} + z^n k(z) \in H^2$.

This implies that $g(z)$ is a polynomial of degree $\leq n$. □

Theorem 3.1.2. [12] *Suppose φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ where a_{-m} and a_N are non zero. If T_φ is hyponormal then $|a_{-m}| \leq |a_N|$.*

Proof. As T_φ is hyponormal, so by Theorem 2.1.2, there exists $k \in H^\infty$ with $\|k\|_\infty \leq 1$ such that $\varphi - k\bar{\varphi} \in H^\infty$. This implies that

$$k \sum_{n=1}^{N} \bar{a_n} z^{-n} - \sum_{n=1}^{m} a_{-n} z^{-n} \in H^\infty$$

So if $k(z) = \sum_{n=0}^{\infty} c_n z^n$, then we get

$$\begin{cases} c_0 = c_1 = \ldots = c_{N-m-1} = 0 \\ c_{N-m} = \frac{a_{-m}}{\bar{a}_N} \\ c_n = (\bar{a}_N)^{-1} \left(a_{-N+n} - \sum_{j=N-m}^{n-1} c_j \bar{a}_{N-n+j} \right) \quad for \;\; n = N-m+1, \ldots, N-1. \end{cases} \tag{3.1.1}$$

This relation uniquely determines the Fourier coefficients $\hat{k}(j)$ of k as $\hat{k}(j) = c_j$ for $j = 0, 1, \ldots, N-1$.

Therefore, $1 \geq \|k\|_\infty \geq |c_{N-m}| = |\frac{a_{-m}}{a_N}|$

and so, $|a_{-m}| \leq |a_N|$ □

Remark 3.1.1. From the above result we can conclude that for a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ where a_{-m} and a_N are non zero, the question of whether or not the Toeplitz operator T_φ is hyponormal is completely independent of the values of the coefficients $a_0, \ldots, a_{N-m}$ of φ.

In fact the following result was also proved independently:

Theorem 3.1.3. [36] *Suppose φ is a trigonometric polynomial such that $\varphi = \bar{f} + g$, where f and g are analytic polynomials of degree m and N $(m \leq N)$, respectively. If $\psi := \bar{f} + T_{\bar{z}^{N-m}} g$, then T_φ is hyponormal if and only if T_ψ is hyponormal.*

The necessary condition for hyponormality as given in Theorem 3.1.2 shows that the cases where $|a_{-m}| = |a_N|$ are, in some sense, extremal among all possibilities for hyponormality. In [36], Kim and Lee treated such cases and their result shows that under such extremal situation a certain symmetry property holds. We quote their result for reference:

Theorem 3.1.4. [36] *Suppose that φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where $m \leq N$ and $|a_{-m}| = |a_N| \neq 0$. Then T_φ is hyponormal if and only if the following equation in $\mathbb{C}^m$ is satisfied:*

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m} \begin{pmatrix} \bar{a}_{N-m+1} \\ \bar{a}_{N-m+2} \\ \vdots \\ \bar{a}_N \end{pmatrix} \tag{3.1.2}$$

In this work we try to relax the condition that $|a_{-m}| = |a_N|$ and yet retain some symmetry. In the following sections we shall discuss hyponormality under these partial symmetry conditions.

3.2 Hyponormality under partial symmetry conditions

We consider the trigonometric polynomial $\varphi(z) := \sum_{n=-m}^{N} a_n z^n$. In view of Theorem 3.1.3 we assume $m = N$. Our objective is to consider hyponormality when $|a_{-N}| \leq |a_N|$. In this regard Kim and Lee in [36] studied the hyponormality of T_φ when either $\bar{a}_N a_{-1} \neq a_{-N}\bar{a}_1$ or $\bar{a}_N a_{-2} \neq a_{-N}\bar{a}_2$ while $\bar{a}_N a_{-i} = a_{-N}\bar{a}_i$ for all other $i = 1, \ldots, N$. Here we make an exhaustive study of the situations where symmetry may not hold at any three of the initial entries and determine necessary and sufficient conditions under which T_φ will still be hyponormal.

We begin with the case where symmetry holds everywhere except at the third place, that is when $\bar{a}_N a_{-3} \neq a_{-N}\bar{a}_3$.

Theorem 3.2.1. *Suppose* $\varphi(z) = \sum_{-N}^{N} a_n z^n$ *is such that*

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-4} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_4 \\ \vdots \\ \bar{a}_N \end{pmatrix},$$

and put $\alpha := \frac{\det\begin{pmatrix} a_{-3} & a_{-N} \\ \bar{a}_3 & \bar{a}_N \end{pmatrix}}{|a_N|^2-|a_{-N}|^2}$, *where* $\alpha \neq 0$.

1. If $N = 4$, *then* T_φ *is hyponormal if and only if*

(a) $|\alpha| \leq 1$,

(b) $|\bar{\alpha}a_{-4} - a_3| \leq |a_4|(\frac{1}{|\alpha|} - |\alpha|)$

(c) $|(1-|\alpha|^2)(\bar{a}_3^2 - \bar{a}_2\bar{a}_4 - \bar{a}_3\bar{a}_{-4}\alpha) + \alpha(\bar{a}_{-4}\alpha - \bar{a}_3)(\bar{a}_{-4} - \bar{a}_3\bar{\alpha})|$
$\leq |\alpha|\left(|a_4|^2\left(\frac{1}{|\alpha|} - |\alpha|\right)^2 - |a_{-4}\bar{\alpha} - a_3|^2\right)$

2. If $N \geq 5$, *then* T_φ *is hyponormal if and only if*

(a) $|\alpha| \leq 1$

(b) $|\frac{a_{N-1}}{a_N}| \leq \frac{1}{|\alpha|} - |\alpha|$

(c) $\left|\frac{1}{\alpha}\overline{\left(\frac{a_{N-1}}{a_N}\right)}^2 - (\frac{1}{\alpha} - \bar{\alpha})\overline{\left(\frac{a_{N-2}}{a_N}\right)}\right| \leq \left(\frac{1}{|\alpha|} - |\alpha|\right)^2 - \left|\frac{a_{N-1}}{a_N}\right|^2$

Proof. (1) Let $N = 4$

Let c_0, c_1, c_2, c_3 be the solutions of of the recurrence relation (2.4.2). Then

$$c_0 = \frac{a_{-4}}{\bar{a}_4} \tag{3.2.1}$$

$$c_1 = (\bar{a}_4)^{-1}(a_{-3} - c_0\bar{a}_3) \tag{3.2.2}$$

$$c_2 = (\bar{a}_4)^{-1}(a_{-2} - c_0\bar{a}_2 - c_1\bar{a}_3) \tag{3.2.3}$$

$$c_3 = (\bar{a}_4)^{-1}(a_{-1} - c_0\bar{a}_1 - c_1\bar{a}_2 - c_2\bar{a}_3) \tag{3.2.4}$$

Now,

$$c_1 = (\bar{a}_4)^{-1}(a_{-3} - \frac{a_{-4}}{\bar{a}_4}\bar{a}_3) = (\bar{a}_4)^{-2}(a_{-3}\bar{a}_4 - \bar{a}_3)$$

As

$$\alpha = \frac{det \begin{pmatrix} a_{-3} & a_{-4} \\ \bar{a}_3 & \bar{a}_4 \end{pmatrix}}{|a_4|^2 - |a_{-4}|^2}.$$

So,

$$c_1 = (\bar{a}_4)^{-2}(|a_4|^2 - |a_{-4}|^2)\alpha \tag{3.2.5}$$

Also,

$$1 - |c_0|^2 = 1 - \frac{|a_{-4}|^2}{|a_4|^2} = |a_4|^{-2}(|a_4|^2 - |a_{-4}|^2) \tag{3.2.6}$$

From (3.2.5) and (3.2.6),

$$\frac{|c_1|}{1 - |c_0|^2} = |\alpha|$$

Again,

$$c_2 = (\bar{a}_4)^{-1}(a_{-2} - \frac{a_{-4}}{\bar{a}_4}\bar{a}_2 - c_1\bar{a}_3) = -\frac{c_1\bar{a}_3}{\bar{a}_4}$$

and

$$\begin{aligned} c_3 &= (\bar{a}_4)^{-1}\left(a_{-2} - c_0\bar{a}_1 - c_1\bar{a}_2 - c_2\bar{a}_3\right) \\ &= (\bar{a}_4)^{-1}\left(\frac{a_{-1}\bar{a}_4 - a_{-4}\bar{a}_1}{\bar{a}_4} - c_1\bar{a}_2 + \frac{\bar{a}_3^2}{\bar{a}_4}c_1\right) \\ &= \left(\frac{\bar{a}_3^2 - \bar{a}_2\bar{a}_4}{\bar{a}_4^2}\right)c_1 \end{aligned}$$

Now, from the equations (2.2.2), (2.2.3), (2.2.4) and (2.3.6)

$$\begin{aligned} \Phi_0 &= c_0, \\ \Phi_1 &= \frac{c_1}{1 - |c_0|^2}, \\ \Phi_2 &= \frac{c_2(1 - |\, c_0\,|^2) + \bar{c}_0 c_1{}^2}{(1 - |c_0|^2)^2 - |c_1|^2} \\ \Phi_3 &= \frac{N(0)}{D(0)} \end{aligned}$$

where,

$N(0) = (1-|c_0|^2)^2 - |c_1|^2)((1-|c_0|^2)c_3 + \bar{c}_0 c_1 c_2) + (c_2(1-|c_0|^2) + \bar{c}_0 {c_1}^2)(\bar{c}_0(1-|c_0|^2)c_1 + \bar{c}_1 c_2)$ and

$D(0) = ((1-|c_0|^2)^2 - |c_1|^2)^2 - |c_2(1-|c_0|^2) + \bar{c}_0 {c_1}^2|^2$

Applying the Proposition 2.4.2, T_φ is hyponormal if and only if

(i) $|\Phi_0| \leq 1$, i.e. if and only if $|a_{-4}| \leq |a_4|$, which is always true by our assumption.

(ii) $|\Phi_1| \leq 1$, i.e. if and only if $|\alpha| \leq 1$ (using (3.2.5) and (3.2.6)).

(iii) $|\Phi_2| \leq 1$, i.e. if and only if

$$
\begin{aligned}
& |c_2(1-|c_0|^2) + \bar{c}_0 c_1^2| \leq (1-|c_0|^2)^2 - |c_1|^2 \\
\iff & |-\frac{c_1 \bar{a}_3}{\bar{a}_4}(1-|c_0|^2) + \frac{\bar{a}_{-4}}{a_4} c_1^2| \leq \left(\frac{|c_1|}{|\alpha|}\right)^2 - |c_1|^2 \\
\iff & \left| \frac{\bar{a}_{-4}}{a_4} c_1 - \frac{\bar{a}_3}{\bar{a}_4}\left(1-|c_0|^2\right) \right| \leq |c_1| \left(\frac{1}{|\alpha|^2} - 1 \right) \\
\iff & \left| \frac{\bar{a}_{-4}}{a_4} c_1 - \frac{\bar{a}_3}{\bar{a}_4} \frac{c_1}{\phi_1} \right| \leq |c_1| \left(\frac{1}{|\alpha|^2} - 1 \right) \\
\iff & \left| \frac{\bar{a}_{-4}}{a_4} - \frac{\bar{a}_3}{\bar{a}_4 \Phi_1} \right| \leq \frac{1}{|\alpha|^2} - 1
\end{aligned}
$$

But,

$$
\Phi_1 = \frac{c_1}{1-|c_0|^2} = \frac{\left(\frac{|a_4|^2 - |a_{-4}|^2}{\bar{a}_4^2}\right)\alpha}{\frac{|a_4|^2-|a_{-4}|^2}{|a_4|^2}} = \frac{\alpha |a_4|^2}{\bar{a}_4^2} = \frac{\alpha a_4 \bar{a}_4}{\bar{a}_4 \bar{a}_4} = \left(\frac{a_4}{\bar{a}_4}\right)\alpha
$$

Therefore,

$|\Phi_2| \leq 1$ if and only if $\left|\frac{\bar{a}_{-4}}{a_4} - \frac{\bar{a}_3}{\bar{a}_4} . \frac{\bar{a}_4}{a_4 \alpha}\right| \leq \frac{1}{|\alpha|^2} - 1$,

i.e. if and only if $|\bar{a}_{-4}\alpha - \bar{a}_3| \leq |a_4| \left(\frac{1}{|\alpha|} - |\alpha|\right)$.

(iv) $|\Phi_3| \leq 1$ if and only if

$$
\begin{aligned}
& \Big|(1-|c_0|^2)^2 - |c_1|^2)((1-|c_0|^2)c_3 + \bar{c}_0 c_1 c_2) + (c_2(1-|c_0|^2) + \bar{c}_0 {c_1}^2) \times \\
& (\bar{c}_0(1-|c_0|^2)c_1 + \bar{c}_1 c_2)\Big| \leq \left((1-|c_0|^2)^2 - |c_1|^2\right)^2 - |c_2(1-|c_0|^2) + \bar{c}_0 {c_1}^2|^2 \qquad (3.2.7)
\end{aligned}
$$

Now,

$$\left(1-|c_0|^2\right)^2-|c_1|^2=\left(1-|c_0|^2\right)^2\left(1-\left(\frac{|c_1|}{1-|c_0|^2}\right)^2\right)$$
$$=\left(1-|c_0|^2\right)^2\left(1-|\alpha|^2\right);\ .$$

$$\begin{aligned}&\left(1-|c_0|^2\right)c_3+\bar{c}_0c_1c_2\\&=\left(1-|c_0|^2\right)\left(\frac{\bar{a}_3^2-\bar{a}_2\bar{a}_4}{\bar{a}_4^2}\right)c_1-\frac{\bar{a}_{-4}\bar{a}_3}{a_4\bar{a}_4}c_1^2\\&=\left(1-|c_0|^2\right)c_1\left(\frac{\bar{a}_3^2-\bar{a}_2\bar{a}_4}{\bar{a}_4^2}-\frac{\bar{a}_{-4}\bar{a}_3}{a_4\bar{a}_4}\Phi_1\right)\\&=\frac{\left(1-|c_0|^2\right)c_1}{\bar{a}_4^2}(\bar{a}_3^2-a_2\bar{a}_4-\bar{a}_3a_{-4}\alpha);\end{aligned}$$

$$c_2\left(1-|c_0|^2\right)+\bar{c}_0c_1^2=\left(1-|c_0|^2\right)c_1\left(\frac{\bar{a}_{-4}}{a_4}\Phi_1-\frac{\bar{a}_3}{\bar{a}_4}\right);$$

$$\begin{aligned}\bar{c}_0c_1\left(1-|c_0|^2\right)+\bar{c}_1c_2&=\frac{\bar{a}_{-4}}{a_4}c_1\left(1-|c_0|^2\right)+\bar{c}_1\left(-\frac{\bar{a}_3}{\bar{a}_4}\right)c_1\\&=\left(1-|c_0|^2\right)c_1\left(\frac{\bar{a}_{-4}}{\bar{a}_4}-\frac{\bar{a}_3}{\bar{a}_4}\bar{\Phi}_1\right);\end{aligned}$$

$$\begin{aligned}\bar{c}_0c_1\left(1-|c_0|^2\right)+\bar{c}_1c_2&=\frac{\bar{a}_{-4}}{a_4}c_1\left(1-|c_0|^2\right)+\bar{c}_1\left(-\frac{\bar{a}_3}{\bar{a}_4}\right)c_1\\&=\left(1-|c_0|^2\right)^2\left(\frac{\bar{a}_{-4}}{\bar{a}_4}\Phi_1-\frac{\bar{a}_3}{\bar{a}_4}\frac{|c_1|^2}{\left(1-|c_0|^2\right)^2}\right)\\&=\left(1-|c_0|^2\right)^2\left(\frac{\bar{a}_{-4}}{\bar{a}_4}\alpha-\frac{\bar{a}_3}{\bar{a}_4}|\alpha|^2\right)\\&=\frac{\left(1-|c_0|^2\right)^2\alpha}{\bar{a}_4}\left(\bar{a}_{-4}-\bar{a}_3\bar{\alpha}\right);\end{aligned}$$

$$\begin{aligned}c_2\left(1-|c_0|^2\right)+\bar{c}_0c_1^2&=\left(1-|c_0|^2\right)c_1\left(\frac{\bar{a}_{-4}}{a_4}\alpha-\frac{\bar{a}_3}{\bar{a}_4}\right)\\&=\frac{\left(1-|c_0|^2\right)c_1}{\bar{a}_4}\left(\bar{a}_{-4}\alpha-\bar{a}_3\right);\end{aligned}$$

Therefore,

$$
\begin{aligned}
&(1-|c_0|^2)^2-|c_1|^2)((1-|c_0|^2)c_3+\bar{c}_0c_1c_2)+(c_2(1-|c_0|^2)+\bar{c}_0{c_1}^2)\times\\
&(\bar{c}_0(1-|c_0|^2)c_1+\bar{c}_1c_2)\\
=&\left(1-|c_0|^2\right)^3c_1\left(\frac{(1-|\alpha|^2)(\bar{a}_3^2-a_2\bar{a}_4-\bar{a}_3a_{-4}\alpha)}{\bar{a}_4^2}+\frac{\alpha(\bar{a}_{-4}\alpha-\bar{a}_3)(\bar{a}_{-4}-\bar{a}_3\bar{\alpha})}{\bar{a}_4^2}\right)\\
=&\frac{\left(1-|c_0|^2\right)^3c_1}{\bar{a}_4^2}\left((1-|\alpha|^2)(\bar{a}_3^2-a_2\bar{a}_4-\bar{a}_3a_{-4}\alpha)+\alpha(\bar{a}_{-4}^2-\bar{a}_3\bar{a}_{-4}|\alpha|^2-\bar{a}_3\bar{a}_{-4}+\bar{a}_3^2\bar{\alpha})\right)\\
=&\frac{\left(1-|c_0|^2\right)^3c_1}{\bar{a}_4^2}\left((\bar{a}_3^2-a_2\bar{a}_4)-(2\bar{a}_3a_{-4}-|a_{-4}|^2)\alpha+(a_2\bar{a}_4+\bar{a}_3a_{-4}\alpha-\bar{a}_3\bar{a}_{-4}\alpha)|\alpha|^2\right)
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\left((1-|c_0|^2)^2-|c_1|^2\right)^2-|c_2(1-|c_0|^2)+\bar{c}_0{c_1}^2|^2\\
&=(1-|c_0|^2)^4(1-|\alpha|^2)^2-\frac{(1-|c_0|^2)^2|c_1|^2}{|a_4|^2}|\bar{a}_{-4}\alpha-\bar{a}_3|^2\\
&=(1-|c_0|^2)^4\left((1-|\alpha|^2)^2-\frac{|\alpha|^2}{|a_4|^2}|\bar{a}_{-4}\alpha-\bar{a}_3|^2\right);
\end{aligned}
$$

Therefore, the inequality (3.2.7) becomes

$$
\begin{aligned}
&|\alpha|\left|(1-|\alpha|^2)(\bar{a}_3^2-a_2\bar{a}_4-\bar{a}_3a_{-4}\alpha)+\alpha(\bar{a}_{-4}\alpha-\bar{a}_3)(\bar{a}_{-4}-\bar{a}_3\bar{\alpha})\right|\\
&\le(1-|\alpha|^2)^2|a_4|^2-|\alpha|^2|\bar{a}_{-4}\alpha-\bar{a}_3|^2\\
&=|\alpha|^2\left(|a_4|^2\left(\frac{1}{|\alpha|}-|\alpha|\right)^2-|\bar{a}_{-4}\alpha-\bar{a}_3|^2\right)
\end{aligned}
$$

Thus, $|\Phi_3|\le 1$ if and only if

$$
\begin{aligned}
&\left|(1-|\alpha|^2)(\bar{a}_3^2-\bar{a}_2\bar{a}_4-\bar{a}_3\bar{a}_{-4}\alpha)+\alpha(\bar{a}_{-4}\alpha-\bar{a}_3)(\bar{a}_{-4}-\bar{a}_3\bar{\alpha})\right|\\
&\qquad\le|\alpha|\left(|a_4|^2\left(\frac{1}{|\alpha|}-|\alpha|\right)^2-|\bar{a}_{-4}\alpha-\bar{a}_3|^2\right).
\end{aligned}
$$

(2) The case where $N\ge 5$

Let $c_0, c_1, ..., c_{N-1}$ be the solutions of the recurrence relation (2.4.2). Then, $c_0=\frac{a_{-N}}{\bar{a}_N}$

$$c_1 = (\bar{a}_N)^{-1}(a_{-(N-1)} - c_0\bar{a}_{N-1}) = (\bar{a}_N)^{-2}(\bar{a}_N a_{-(N-1)} - a_{-N}\bar{a}_{N-1}) = 0$$

$$\vdots$$

$$c_{N-4} = (\bar{a}_N)^{-1}(a_{-4} - c_0\bar{a}_4 - c_1\bar{a}_5 - \dots - c_{N-5}\bar{a}_{N-1}) = 0$$

$$c_{N-3} = (\bar{a}_N)^{-1}(a_{-3} - c_0\bar{a}_3 - c_1\bar{a}_4 - \dots - c_{N-4}\bar{a}_{N-1}) = (\bar{a}_N)^{-2}(a_{-3}\bar{a}_N - a_{-N}\bar{a}_3)$$

$$\begin{aligned} c_{N-2} &= (\bar{a}_N)^{-1}(a_{-2} - c_0\bar{a}_2 - c_1\bar{a}_3 - c_2\bar{a}_4 - \dots - c_{N-3}\bar{a}_{N-1}) \\ &= -(\bar{a}_N)^{-2}(a_{-3}\bar{a}_N - a_{-N}\bar{a}_3)\left(\tfrac{\bar{a}_{N-1}}{\bar{a}_N}\right) \\ &= -\left(\tfrac{\bar{a}_{N-1}}{\bar{a}_N}\right)c_{N-3} \end{aligned}$$

$$\begin{aligned} c_{N-1} &= (\bar{a}_N)^{-1}(a_{-1} - c_0\bar{a}_1 - c_1\bar{a}_2 - \dots - c_{N-3}\bar{a}_{N-2} - c_{N-2}\bar{a}_{N-1}) \\ &= -(\bar{a}_N)^{-1}\left((\bar{a}_N)^{-2}(a_{-3}\bar{a}_N - a_{-N}\bar{a}_3)\bar{a}_{N-2} - (\bar{a}_N)^{-3}(a_{-3}\bar{a}_N - a_{-N}\bar{a}_3)\bar{a}^2_{N-1}\right) \\ &= -(\bar{a}_N)^{-2}\left((\bar{a}_N)^{-2}\bar{a}^2_{N-1} - (\bar{a}_N)^{-1}\bar{a}_{N-2}\right)(\bar{a}_N)^{-2}(a_{-3}\bar{a}_N - a_{-N}\bar{a}_3) \\ &= (\bar{a}_N)^{-2}\left(\left(\tfrac{\bar{a}_{N-1}}{\bar{a}_N}\right)^2 - \tfrac{\bar{a}_{N-2}}{\bar{a}_N}\right)(a_{-3}\bar{a}_N - a_{-N}\bar{a}_3) \\ &= \left(\left(\tfrac{\bar{a}_{N-1}}{\bar{a}_N}\right)^2 - \tfrac{\bar{a}_{N-2}}{\bar{a}_N}\right)c_{N-3} \end{aligned}$$

Thus, $b_p(z) = c_0 + c_{N-3}z^{N-3} + c_{N-2}z^{N-2} + c_{N-1}z^{N-1}$ is the unique analytic polynomial of degree less than N satisfying $\varphi - b_p\bar{\varphi} \in H^\infty(\mathbb{T})$. By our assumption we have $c_{N-3} \neq 0$. Thus, by Proposition 2.4.3

$\Phi_0 = c_0,$

$\Phi_1 = \Phi_2 = \dots = \Phi_{N-4} = 0.$

$\Phi_{N-3} = \frac{c_{N-3}}{1-|c_0|^2}$

$\Phi_{N-2} = \frac{c_{N-2}}{(1-|c_0|^2)(1-|\Phi_{N-3}|^2)}$

$\Phi_{N-1} = \frac{(1-|\Phi_{N-3}|^2)c_{N-1}c_{N-3} + |\Phi_{N-3}|^2 c^2_{N-2}}{(1-|\Phi_{N-3}|^2)^2(1-|\Phi_{N-2}|^2)(1-|c_0|^2)c_{N-3}}$

By using the Proposition 2.4.2, T_φ is hyponormal if and only if

(i) $|\Phi_0| \leq 1$

That is, if and only if $|c_0| \leq 1$

That is, if and only if $|a_{-N}| \leq |a_N|$

which is true since we are taking $|a_{-N}| < |a_N|$.

(ii) $|\Phi_{N-3}| \leq 1$

That is, if and only if $|c_{N-3}| \leq 1 - |c_0|^2$

That is, if and only if $\frac{\left|det\begin{pmatrix} a_{-N} & a_{-3} \\ \bar{a}_N & \bar{a}_3 \end{pmatrix}\right|}{|a_N|^2} \leq 1 - \frac{|a_{-N}|^2}{|a_N|^2}$

That is, if and only if $|\alpha| \leq 1$.

(iii) $|\Phi_{N-2}| \leq 1$

That is, if and only if $|c_{N-2}| \leq (1 - |c_0|^2)(1 - |\Phi_{N-3}|^2)$

That is, if and only if $\frac{\left|det\begin{pmatrix} a_{-3} & a_{-N} \\ \bar{a}_3 & \bar{a}_N \end{pmatrix}\right|}{|a_N|^2} \left|\frac{a_{N-1}}{a_N}\right| \leq \frac{|a_N|^2 - |a_{-N}|^2}{|a_N|^2}(1 - |\alpha|^2)$

That is, if and only if

$$|\frac{a_{N-1}}{a_N}| \leq \frac{1}{|\alpha|} - |\alpha|.$$

(iv)

$$|\Phi_{N-1}| \leq 1 \qquad (3.2.8)$$

Now,

$$\Phi_{N-1} = \frac{(1-|\alpha|^2)\left(\overline{(\frac{a_{N-1}}{a_N})}^2 - \overline{(\frac{a_{N-2}}{a_N})}\right)c_{N-3}^2 + |\alpha|^2\overline{(\frac{a_{N-1}}{a_N})}^2 c_{N-3}^2}{(1-|c_0|^2)(1-|\alpha|^2)^2\left(1 - \frac{|c_{N-2}|^2}{(1-|c_0|^2)^2(1-|\alpha|^2)^2}\right)c_{N-3}}$$

$$= \frac{\left(\overline{(\frac{a_{N-1}}{a_N})}^2 - (1-|\alpha|^2)\overline{(\frac{a_{N-2}}{a_N})}\right)\frac{c_{N-3}}{1-|c_0|^2}}{(1-|\alpha|^2)^2 - \left|\frac{a_{N-1}}{a_N}\right|^2\left|\frac{c_{N-3}}{1-|c_0|^2}\right|^2}$$

Therefore, the inequality (3.2.8) holds if and only if $\left(\overline{\left(\frac{a_{N-1}}{a_N}\right)}^2 - (1 - |\alpha|^2)\overline{\left(\frac{a_{N-2}}{a_N}\right)}\right)|\alpha| \leq (1 - |\alpha|^2)^2 - \left|\frac{a_{N-1}}{a_N}\right|^2 |\alpha|^2$ That is, if and only if $\left|\frac{1}{\alpha}\overline{\left(\frac{a_{N-1}}{a_N}\right)}^2 - (\frac{1}{\alpha} - \bar{\alpha})\overline{\left(\frac{a_{N-2}}{a_N}\right)}\right| \leq \left(\frac{1}{|\alpha|} - |\alpha|\right)^2 - \left|\frac{a_{N-1}}{a_N}\right|^2$ □

Remark 3.2.1. For $N \geq 6$, if $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ with $|a_N| > |a_{-N}|$, and also,

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-4} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_4 \\ \vdots \\ \bar{a}_N \end{pmatrix} \text{ and } \bar{a}_N a_{-3} \neq a_{-N}\bar{a}_3$$

then clearly hyponormality of T_φ is determined by Theorem 3.2.1. In this particular case we would like to mention the following result of Hwang and Lee which was established in 2004.

Theorem 3.2.2. [29] *Suppose that* $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ $(|a_N| > |a_{-N}|)$ *is such that*

$$\bar{a}_N \begin{pmatrix} a_{-m} \\ a_{-m-1} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_m \\ \bar{a}_{m+1} \\ \vdots \\ \bar{a}_N \end{pmatrix} \qquad (m-1 \le \frac{N}{2})$$

and $\bar{a}_N a_{-m+1} \neq a_{-N}\bar{a}_{m-1}$. *Let*

$$\psi(z) := \sum_{k=-m+1}^{-1} d_k z^k + \sum_{k=1}^{m-1} d_{N-m+1+k} z^k,$$

where the d_k *are given by*

$$d_k := det \begin{pmatrix} a_k & a_{-N} \\ \bar{a}_{-k} & \bar{a}_N \end{pmatrix} \qquad (-m+1 \le k \le N)$$

then T_φ *is hyponormal if and only if* T_ψ *is hyponormal.*

For very large N, application of Theorem 3.2.2 would significantly reduce the work load in determining the hyponormality of T_φ. However, this theorem does not give a ready set of conditions to finally determine hyponormality. In view of this, Theorem 3.2.1 offers an alternative method to determine hyponormality of T_φ under the given restrictions on φ. And hence our result is much more convenient from the point of view of application. This will be justified with examples in the last section of the chapter.

Next, we consider φ such that the symmetry holds everywhere except at the second and third entries.

Theorem 3.2.3. *Suppose* $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ $(|a_{-N}| < |a_N|)$ *is such that*

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-4} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_4 \\ \vdots \\ \bar{a}_N \end{pmatrix}, \ \bar{a}_N a_{-2} \neq a_{-N}\bar{a}_2, \ \bar{a}_N a_{-3} \neq a_{-N}\bar{a}_3$$

with $\alpha := \frac{\det\begin{pmatrix} a_{-2} & a_{-N} \\ \bar{a}_2 & \bar{a}_N \end{pmatrix}}{|a_N|^2 - |a_{-N}|^2}$ *and* $\beta := \frac{\det\begin{pmatrix} a_{-3} & a_{-N} \\ \bar{a}_3 & \bar{a}_N \end{pmatrix}}{|a_N|^2 - |a_{-N}|^2}$. *Then*

1. For $N = 4$, T_φ *is hyponormal if and only if*

(a) $|\beta| \leq 1$,

(b) $|\alpha\bar{a}_4 - \beta\bar{a}_3 + \beta^2\bar{a}_{-4}| \leq |a_4|\,(1 - |\beta|^2)$

(c) $|(1 - |\beta|^2)\,(\beta\bar{a}_{-4}(\alpha\bar{a}_4 - \beta\bar{a}_3) - \beta\bar{a}_2\bar{a}_4 - \alpha\bar{a}_3\bar{a}_4 + \beta(\bar{a}_3)^2) + (\alpha\bar{a}_4 - \beta\bar{a}_3 + \beta^2\bar{a}_{-4})$
$\left(\beta\bar{a}_{-4} + \bar{\beta}(\alpha\bar{a}_4 - \beta\bar{a}_3)\right)| \leq \left(|a_4|(1 - |\beta|^2)\right)^2 - |\alpha\bar{a}_4 - \beta\bar{a}_3 + \beta^2\bar{a}_{-4}|^2$

2. For $N \geq 5$, T_φ *is hyponormal if and only if*

(a) $|\beta| \leq 1$

(b) $\left|\alpha - \beta\overline{\left(\frac{a_{N-1}}{a_N}\right)}\right| \leq 1 - |\beta|^2$

(c) $\left|(1 - |\beta|^2)\left(\beta\overline{(\tfrac{a_{N-2}}{a_N})} + \alpha\overline{(\tfrac{a_{N-1}}{a_N})} - \beta\overline{(\tfrac{a_{N-1}}{a_N})}^2\right) - \bar{\beta}\left(\alpha - \beta\overline{(\tfrac{a_{N-1}}{a_N})}\right)^2\right|$
$\leq (1 - |\beta|^2)^2 - \left|\alpha - \beta\overline{(\tfrac{a_{N-1}}{a_N})}\right|^2$

Proof. (1) **When** $N = 4$

Using the recurrence relation (2.4.2) and writing $A = |a_4|^2 - |a_{-4}|^2$, we get

$$\begin{aligned}
c_0 &= \frac{a_{-4}}{\bar{a}_4} \\
c_1 &= (\bar{a}_4)^{-2}(a_{-3}\bar{a}_4 - a_{-4}\bar{a}_3) \\
&= A\beta(\bar{a}_4)^{-2} \\
c_2 &= (\bar{a}_4)^{-1}\left(a_{-2} - \sum_{j=0}^{1} c_j\bar{a}_{2+j}\right) \\
&= (\bar{a}_4)^{-1}\left(\frac{\bar{a}_4 a_{-2} - \bar{a}_2 a_{-4}}{\bar{a}_4} - A\beta(\bar{a}_4)^{-2}\bar{a}_3\right) \\
&= A(\bar{a}_4)^{-3}(\alpha\bar{a}_4 - \beta\bar{a}_3) \\
c_3 &= (\bar{a}_4)^{-1}\left(a_{-1} - \sum_{j=0}^{2} c_j\bar{a}_{1+j}\right) \\
&= (\bar{a}_4)^{-1}\left(a_{-1} - \frac{a_{-4}}{\bar{a}_4}\bar{a}_1 - A\beta(\bar{a}_4)^{-2}\bar{a}_2 - A(\bar{a}_4)^{-3}\bar{a}_3(\alpha\bar{a}_4 - \beta\bar{a}_3)\right) \\
&= (\bar{a}_4)^{-4}(-A\beta\bar{a}_2\bar{a}_4 - A\bar{a}_3(\alpha\bar{a}_4 - \beta\bar{a}_3)) \\
&= -A(\bar{a}_4)^{-4}\left(\beta\bar{a}_2\bar{a}_4 + \alpha\bar{a}_3\bar{a}_4 - \beta(\bar{a}_3)^2\right)
\end{aligned}$$

Now, using these values of c_0, c_1, c_2 and c_3, we simplify the L.H.S. of the following expressions as follows:

$$\begin{aligned}
c_2(1 - |c_0|^2) + \bar{c}_0 c_1^2 &= A^2|a_4|^{-2}(\bar{a}_4)^{-3}(\alpha\bar{a}_4 - \beta\bar{a}_3) + A^2\beta^2|a_4|^{-2}(\bar{a}_4)^{-3}\bar{a}_{-4} \\
&= A^2(\bar{a}_4)^{-3}|a_4|^{-2}(\alpha\bar{a}_4 - \beta\bar{a}_3 + \beta^2\bar{a}_{-4}) \qquad (3.2.9)
\end{aligned}$$

$$\begin{aligned}
(1 - |c_0|^2)^2 - |c_1|^2 &= |a|^{-4}A^2 - A^2|\beta|^2|a_4|^{-4} \\
&= A^2|a_4|^{-4}(1 - |\beta|^2) \qquad (3.2.10)
\end{aligned}$$

$$\begin{aligned}
&(1 - |c_0|^2)c_3 + \bar{c}_0 c_1 c_2 \\
=& -A^2|a_4|^{-2}(\bar{a}_4)^{-4}(\beta\bar{a}_2\bar{a}_4 + \alpha\bar{a}_3\bar{a}_4 - \beta(\bar{a}_3)^2) + A^2\beta|a_4|^{-2}(\bar{a}_4)^{-4}\bar{a}_{-4}(\alpha\bar{a}_4 - \beta\bar{a}_3) \\
=& A^2|a_4|^{-2}(\bar{a}_4)^{-4}\left(\beta\bar{a}_{-4}(\alpha\bar{a}_4 - \beta\bar{a}_3) - \beta\bar{a}_2\bar{a}_4 - \alpha\bar{a}_3\bar{a}_4 + \beta(\bar{a}_3)^2\right) \qquad (3.2.11)
\end{aligned}$$

$$\bar{c}_0(1-|c_0|^2)c_1+\bar{c}_1c_2 = A^2\beta|a_4|^{-4}(\bar{a}_4)^{-1}\bar{a}_{-4}+A^2\bar{\beta}|\bar{a}_4|^{-4}(\bar{a}_4)^{-1}(\alpha\bar{a}_4-\beta\bar{a}_3)$$
$$= A^2|a_4|^{-4}(\bar{a}_4)^{-1}\left(\beta\bar{a}_{-4}+\bar{\beta}(\alpha\bar{a}_4-\beta\bar{a}_3)\right) \quad (3.2.12)$$

Now, using the values of c_0 and c_1 in Φ_0, Φ_1 of the equations (2.2.2), (2.2.3) respectively, and the values of the equations (3.2.9) and (3.2.10) in Φ_2 of the equation (2.2.4) and applying the Proposition 2.4.2, we get

$|\Phi_0| \leq 1$ iff $|a_{-4}| \leq |a_4|$ which is true according to our assumption.

$$|\Phi_1| \leq 1 \iff |\beta| \leq 1 \quad (3.2.13)$$

$$|\Phi_2| \leq 1 \iff |\alpha\bar{a}_4-\beta\bar{a}_3+\beta^2\bar{a}_{-4}| \leq |a_4|(1-|\beta|^2) \quad (3.2.14)$$

Multiplying (3.2.10) and (3.2.11)

$$((1-|c_0|^2)^2-|c_1|^2)((1-|c_0|^2)c_3+\bar{c}_0c_1c_2)$$
$$=A^4|a_4|^{-6}(\bar{a}_4)^{-4}(1-|\beta|^2)(\beta\bar{a}_{-4}(\alpha\bar{a}_4-\beta\bar{a}_3)-\beta\bar{a}_2\bar{a}_4-\alpha\bar{a}_3\bar{a}_4+\beta(\bar{a}_3)^2) \quad (3.2.15)$$

Multiplying (3.2.9) and (3.2.12)

$$(c_2(1-|c_0|^2)+\bar{c}_0c_1^2)(\bar{c}_0(1-|c_0|^2)c_1+\bar{c}_1c_2)$$
$$=A^4|a_4|^{-6}(\bar{a}_4)^{-4}(\alpha\bar{a}_4-\beta\bar{a}_3+\beta^2\bar{a}_{-4})(\beta\bar{a}_{-4}+\bar{\beta}(\alpha\bar{a}_4-\beta\bar{a}_3)) \quad (3.2.16)$$

Adding (3.2.15) and (3.2.16), we get

$$((1-|c_0|^2)^2-|c_1|^2)((1-|c_0|^2)c_3+\bar{c}_0c_1c_2)+(c_2(1-|c_0|^2)+\bar{c}_0c_1^2)\times$$
$$(\bar{c}_0(1-|c_0|^2)c_1+\bar{c}_1c_2)$$
$$=A^4|a_4|^{-6}(\bar{a}_4)^{-4}(\bar{a}_4)^{-4}((1-|\beta|^2)(\beta\bar{a}_{-4}(\alpha\bar{a}_4-\beta\bar{a}_3)-\beta\bar{a}_2\bar{a}_4-\alpha\bar{a}_3\bar{a}_4+\beta(\bar{a}_3)^2)+$$
$$(\alpha\bar{a}_4-\beta\bar{a}_3+\beta^2\bar{a}_{-4})(\beta\bar{a}_{-4}+\bar{\beta}(\alpha\bar{a}_4-\beta\bar{a}_3))) \quad (3.2.17)$$

From (3.2.9) and (3.2.10), we get

$$\left((1-|c_0|^2)^2-|c_1|^2\right)^2-\left|c_2(1-|c_0|^2)+\bar{c}_0c_1^2\right|^2$$
$$=A^4|a_4|^{-8}\left((1-|\beta|^2)^2-|a_4|^{-2}|\alpha\bar{a}_4-\beta\bar{a}_3+\beta^2\bar{a}_{-4}|^2\right) \quad (3.2.18)$$

Putting the values of the equations (3.2.17) and (3.2.18) in Φ_3 of the equation (2.3.6) and applying the Proposition 2.4.2, we get

$$\begin{aligned}&|(1-|\beta|^2)\left(\beta\bar{a}_{-4}(\alpha\bar{a}_4-\beta\bar{a}_3)-\beta\bar{a}_2\bar{a}_4-\alpha\bar{a}_3\bar{a}_4+\beta(\bar{a}_3)^2\right)+(\alpha\bar{a}_4-\beta\bar{a}_3+\beta^2\bar{a}_{-4})\\&\left(\beta\bar{a}_{-4}+\bar{\beta}(\alpha\bar{a}_4-\beta\bar{a}_3)\right)|\leq\left(|a_4|(1-|\beta|^2)\right)^2-|\alpha\bar{a}_4-\beta\bar{a}_3+\beta^2\bar{a}_{-4}|^2 \qquad (3.2.19)\end{aligned}$$

Hence, the result follows from the equations (3.2.13), (3.2.14) and (3.2.19), and from the Proposition 2.4.2.

(2) **When** $N \geq 5$

From the recurrence relation (2.4.2), $c_n = (\bar{a}_N)^{-1}\left(a_{-N+n} - \sum_{j=0}^{n-1} c_j\bar{a}_{N-n+j}\right)$ and writing $A = |a_N|^2 - |a_{-N}|^2$, we get

$$\begin{aligned}c_0 &= \frac{a_{-N}}{\bar{a}_N}\\ c_1 &= (\bar{a}_N)^{-1}(a_{-N+1} - c_0\bar{a}_{N-1})\\ &= (\bar{a}_N)^{-2}(a_{-N+1}\bar{a}_N - a_{-N}\bar{a}_{N-1})\\ &= 0\\ &\vdots\\ c_{N-4} &= 0\\ c_{N-3} &= (\bar{a}_N)^{-1}(a_{-3} - c_0\bar{a}_3 - c_1\bar{a}_{N-1})\\ &= (\bar{a}_N)^{-2}(a_{-3}\bar{a}_N - a_{-N}\bar{a}_3)\\ &= (\bar{a}_N)^{-2}A\beta\\ c_{N-2} &= (\bar{a}_N)^{-1}(a_{-2} - c_0\bar{a}_2 - c_1\bar{a}_3 - \ldots - c_{N-4}\bar{a}_{N-2} - c_{N-3}\bar{a}_{N-1})\\ &= (\bar{a}_N)^{-1}\left(\frac{a_{-2}\bar{a}_N - a_{-N}\bar{a}_N}{\bar{a}_N} - \frac{A\beta\bar{a}_{N-1}}{(\bar{a}_N)^2}\right)\\ &= (\bar{a}_N)^{-3}A\left(\alpha\bar{a}_N - \beta\bar{a}_{N-1}\right)\\ c_{N-1} &= (\bar{a}_N)^{-1}(a_{-1} - c_0\bar{a}_1 - c_1\bar{a}_2 - \ldots - c_{N-4}\bar{a}_{N-3} - c_{N-3}\bar{a}_{N-3} - c_{N-2}\bar{a}_{N-1})\\ &= (\bar{a}_N)^{-1}\left(\frac{a_{-1}\bar{a}_N - a_{-N}\bar{a}_1}{\bar{a}_N} - \frac{A\beta\bar{a}_{N-2}}{(\bar{a}_N)^2} - \frac{A(\alpha\bar{a}_N - \beta\bar{a}_{N-1})\bar{a}_{N-1}}{(\bar{a}_N)^3}\right)\\ &= -(\bar{a}_N)^{-4}\left(\beta\bar{a}_{N-2}\bar{a}_N + \alpha\bar{a}_{N-1}\bar{a}_N - \beta(\bar{a}_{N-1})^2\right)A\end{aligned}$$

Thus, $b_p(z) = c_0 + c_{N-3}z^{N-3} + c_{N-2}z^{N-2} + c_{N-1}z^{N-1}$ is the unique analytic polynomial of degree less than N satisfying $\varphi - b_p\bar{\varphi} \in H^\infty$ Thus, by Proposition 2.4.3,

$\Phi_0 = c_0$

$\Phi_1 = \Phi_2 = \ldots = \Phi_{N-4} = 0$

$\Phi_{N-3} = \frac{c_{N-3}}{1-|c_0|^2}$

$\Phi_{N-2} = \frac{c_{N-2}}{(1-|c_0|^2)(1-|\Phi_{N-3}|^2)}$

$\Phi_{N-1} = \frac{(1-|\Phi_{N-3}|^2)c_{N-1}c_{N-3}+|\Phi_{N-3}|^2 c_{N-2}^2}{c_{N-3}(1-|c_0|^2)(1-|\Phi_{N-3}|^2)^2(1-|\Phi_{N-2}|^2)}$

Now, by putting the values of c_0, c_{N-3}, c_{N-2} in $\Phi_0, \Phi_{N-3}, \Phi_{N-2}$ and simplifying, we get

$|\Phi_0| \leq 1$ iff $|a_{-N}| \leq |a_N|$, which is always true according to our assumption.

$$|\Phi_{N-3}| \leq 1 \Longleftrightarrow |\beta| \leq 1 \tag{3.2.20}$$

$$|\Phi_{N-2}| \leq 1 \Longleftrightarrow \left|\alpha - \beta\overline{\left(\frac{a_{N-1}}{a_N}\right)}\right| \leq 1 - |\beta|^2 \tag{3.2.21}$$

Next, by putting the values of $c_0, c_{N-3}, c_{N-2}, c_{N-1}$ in Φ_{N-1}, we get

$$\begin{aligned}
&\Phi_{N-1} \\
&= \frac{(1-|\Phi_{N-3}|^2)c_{N-1}c_{N-3}+|\Phi_{N-3}|^2 c_{N-2}^2}{c_{N-3}(1-|c_0|^2)(1-|\Phi_{N-3}|^2)^2(1-|\Phi_{N-2}|^2)} \\
&= \frac{-(\bar{a}_N)^{-6}A^2\beta((1-|\beta|^2)(\beta\bar{a}_{N-2}\bar{a}_N + \alpha\bar{a}_N\bar{a}_{N-1} - \beta(\bar{a}_{N-1})^2) - \bar{\beta}(\alpha\bar{a}_N - \beta\bar{a}_{N-1})^2)}{(\bar{a}_N)^{-2}A^2\beta|a_N|^{-2}(1-|\beta|^2)^2\left(1 - \frac{\left|\alpha-\beta\overline{\left(\frac{a_{N-1}}{a_N}\right)}\right|^2}{1-|\beta|^2}\right)} \\
&= \frac{-(\bar{a}_N)^{-6}A^2\beta((1-|\beta|^2)(\beta\bar{a}_{N-2}\bar{a}_N + \alpha\bar{a}_N\bar{a}_{N-1} - \beta(\bar{a}_{N-1})^2) - \bar{\beta}(\alpha\bar{a}_N - \beta\bar{a}_{N-1})^2)}{(\bar{a}_N)^{-2}A^2\beta|a_N|^2((1-|\beta|^2)^2 - \left|\alpha - \beta\overline{\left(\frac{a_{N-1}}{a_N}\right)}\right|^2)}
\end{aligned}$$

Therefore,

$|\Phi_{N-1}| \leq 1$ if and only if

$$\begin{aligned}
&\left|(1-|\beta|^2)\left(\beta\overline{\left(\frac{a_{N-2}}{a_N}\right)} + \alpha\overline{\left(\frac{a_{N-1}}{a_N}\right)} - \beta\overline{\left(\frac{a_{N-1}}{a_N}\right)}^2\right) - \bar{\beta}\left(\alpha - \beta\overline{\left(\frac{a_{N-1}}{a_N}\right)}\right)^2\right| \\
&\leq (1-|\beta|^2)^2 - \left|\alpha - \beta\overline{\left(\frac{a_{N-1}}{a_N}\right)}\right|^2
\end{aligned} \tag{3.2.22}$$

Hence, the result follows from the inequalities (3.2.20), (3.2.21), (3.2.22) and from the Proposition 2.4.2. □

The next theorem investigates the hyponormality of T_φ when the Fourier coefficients of φ do not satisfy the symmetry in the first two entries.

Theorem 3.2.4. *Suppose $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ ($|a_{-N}| < |a_N|$) is such that*

$$\bar{a}_N \begin{pmatrix} a_{-3} \\ a_{-4} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_3 \\ \bar{a}_4 \\ \vdots \\ \bar{a}_N \end{pmatrix}, \quad \bar{a}_N a_{-1} \neq a_{-N}\bar{a}_1, \quad \bar{a}_N a_{-2} \neq a_{-N}\bar{a}_2$$

with $\alpha := \frac{\bar{a}_N a_{-1} - a_{-N}\bar{a}_1}{|a_N|^2 - |a_{-N}|^2}$ and $\beta := \frac{\bar{a}_N a_{-2} - a_{-N}\bar{a}_2}{|a_N|^2 - |a_{-N}|^2}$. Then

1. For $N = 3$, T_φ is hyponormal if and only if

(a) $|\beta| \leq 1$

(b) $|\alpha\bar{a}_3 - \beta\bar{a}_2 + \beta^2\bar{a}_{-3}| \leq |a_3|(1 - |\beta|^2)$

2. For $N \geq 4$, T_φ is hyponormal if and only if

(a) $|\beta| \leq 1$

(b) $\left|\bar{\alpha} - \bar{\beta}\left(\frac{a_{N-1}}{a_N}\right)\right| \leq 1 - |\beta|^2$.

Remark 3.2.2. Theorem 5 in [36] says that if $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ and the coefficients of φ satisfies the condition $\bar{a}_N \begin{pmatrix} a_{-3} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_3 \\ \vdots \\ \bar{a}_N \end{pmatrix}$ then hyponormality of T_φ necessarily implies $\left| det \begin{pmatrix} a_{-2} & a_{-N} \\ \bar{a}_2 & \bar{a}_N \end{pmatrix} \right| \leq |a_N|^2 - |a_{-N}|^2$. Theorem 3.2.4 gives sufficient conditions under which the converse is also true.

Proof. (1) For $N = 3$.

Here we write $A = |a_3|^2 - |a_{-3}|^2$. By using the recurrence relation (2.4.2), we get

$c_0 = \frac{a_{-3}}{\bar{a}_3}$

$c_1 = (\bar{a}_3)^{-1}(a_{-2} - c_0\bar{a}_2) = (\bar{a}_3)^{-2}(a_{-2}\bar{a}_3 - \bar{a}_2 a_{-3}) = (\bar{a}_3)^{-2}A\beta$

$c_2 = (\bar{a}_3)^{-1}(a_{-1} - c_0\bar{a}_1 - c_1\bar{a}_2) = (\bar{a}_3)^{-1}\left(\frac{\bar{a}_3 a_{-1} - a_{-3}\bar{a}_1}{\bar{a}_3} - \frac{A\beta\bar{a}_2}{(\bar{a}_3)^2}\right) = (\bar{a}_3)^{-3}(\bar{a}_3\alpha - \bar{a}_2\beta)A$

Now, by using these values in Schur's functions Φ_n for $n = 0, 1, 2$, of the equations (2.2.2), (2.2.3) and (2.2.4) respectively, we get

$\Phi_0 = \frac{a_{-3}}{\bar{a}_3}$

$\Phi_1 = \frac{|a_3|^2}{(\bar{a}_3)^2}\beta$

$\Phi_2 = \frac{(\bar{a}_3)^{-3}|a_3|^{-2}(\alpha\bar{a}_3 - \beta\bar{a}_2 + \beta^2\bar{a}_{-3})}{|a_3|^{-4}(1-|\beta|^2)}$

Hence, by Proposition 2.4.2, T_φ is hyponormal if and only if $|\Phi_1| \leq 1$ and $|\Phi_2| \leq 1$, that is, if and only if $|\beta| \leq 1$ and $|\alpha\bar{a}_3 - \beta\bar{a}_2 + \beta^2\bar{a}_{-3}| \leq |a_3|(1 - |\beta|^2)$.

(2) For $N \geq 4$

Throughout here we write $A = |a_N|^2 - |a_{-N}|^2$. By using the recurrence relation (2.4.2), we get

$c_0 = \frac{a_{-N}}{\bar{a}_N}$

$c_1 = (\bar{a}_N)^{-1}(a_{-N+1} - c_0\bar{a}_{N-1}) = (\bar{a}_N)^{-2}(\bar{a}_N a_{-N+1} - a_{-N}\bar{a}_{N-1}) = 0$

$c_2 = \ldots = c_{N-3} = 0$

$$\begin{aligned} c_{N-2} &= (\bar{a}_N)^{-1}(a_{-2} - c_0\bar{a}_2 - c_1\bar{a}_3 - \ldots - c_{N-3}\bar{a}_{N-1}) \\ &= (\bar{a}_N)^{-2}(a_{-2}\bar{a}_N - \bar{a}_2 a_{-N}) \\ &= (\bar{a}_N)^{-2}A\beta \end{aligned}$$

$$\begin{aligned} c_{N-1} &= (\bar{a}_N)^{-1}(a_{-1} - c_0\bar{a}_1 - c_1\bar{a}_2 - \ldots - c_{N-3}\bar{a}_{N-2} - c_{N-2}\bar{a}_{N-1}) \\ &= (\bar{a}_N)^{-1}\left(\frac{a_{-1}\bar{a}_N - a_{-N}\bar{a}_1}{\bar{a}_N} - \frac{A\beta\bar{a}_{N-1}}{(\bar{a}_N)^2}\right) \\ &= (\bar{a}_N)^{-3}(\alpha\bar{a}_N - \beta\bar{a}_{N-1})A \end{aligned}$$

Thus, $b_p = c_0 + c_{N-2}z^{N-2} + c_{N-1}z^{N-1}$ is the unique analytic polynomial of degree less than N satisfying $\varphi - b_p\bar{\varphi} \in H^\infty$. Now, by using Proposition 2.4.3, and simplifying the expressions we get

$\Phi_0 = c_0 = \frac{a_{-N}}{\bar{a}_N}$

$\Phi_1 = \Phi_2 = \ldots = \Phi_{N-3} = 0$

$\Phi_{N-2} = \frac{c_{N-2}}{1-|c_0|^2} = \frac{(\bar{a}_N)^{-2}}{|a_N|^{-2}}\beta$

$\Phi_{N-1} = \frac{c_{N-1}}{(1-|c_0|^2)(1-|\Phi_{N-2}|^2)} = \frac{(\bar{a}_N)^{-3}(\alpha\bar{a}_N - \beta\bar{a}_{N-1})}{|a_N|^{-2}(1-|\beta|^2)}$

Hence, T_φ is hyponormal if and only if $|\Phi_{N-2}| \leq 1$ and $|\Phi_{N-1}| \leq 1$.

That is, if and only if $|\beta| \leq 1$ and $\left|\bar{\alpha} - \bar{\beta}(\frac{a_{N-1}}{a_N})\right| \leq 1 - |\beta|^2$. □

Finally, in the following theorem we determine the hyponormality of Toeplitz operators T_φ with the symbol φ not having the symmetry in the alternate positions.

Theorem 3.2.5. *Let $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ (with $|a_{-N}| \leq |a_N|$) be a trigonometric polynomial which satisfies the following partial symmetry condition:*

$$\bar{a}_N \begin{pmatrix} a_{-2} \\ a_{-4} \\ a_{-5} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_2 \\ \bar{a}_4 \\ \bar{a}_5 \\ \vdots \\ \bar{a}_N \end{pmatrix} \text{ with } \bar{a}_N a_{-1} \neq a_{-N}\bar{a}_1 \text{ and } \bar{a}_N a_{-3} \neq a_{-N}\bar{a}_3.$$

Let $\alpha = \frac{\bar{a}_N a_{-1} - a_{-N}\bar{a}_1}{|a_N|^2 - |a_{-N}|^2}$ and $\beta = \frac{\bar{a}_N a_{-3} - a_{-N}\bar{a}_3}{|a_N|^2 - |a_{-N}|^2}$. Then,

(a) For $N = 4$, T_φ is hyponormal if and only if

1. $|\beta| \leq 1$;

2. $|\frac{1}{a_4}(\bar{a}_{-4}\beta - \bar{a}_3)| \leq \frac{1}{|\beta|} - |\beta|$;

3. $|(1-|\beta|^2)(\alpha\bar{a}_4^2 - \beta\bar{a}_2\bar{a}_4 + \beta\bar{a}_3^2 - \beta^2\bar{a}_{-4}\bar{a}_3) + (\beta\bar{a}_{-4} - \bar{a}_3)(\bar{a}_{-4} - \bar{\beta}\bar{a}_3)\beta^2|$
$\leq |\beta|(|a_4|^2(\frac{1}{\beta} - |\beta|)^2 - |\bar{a}_3 - \beta\bar{a}_{-4}|^2)$.

(b) For $N \geq 5$, T_φ is hyponormal if and only if

1. $|\beta| \leq 1$;

2. $|\frac{a_{N-1}}{a_N}| \leq \frac{1}{|\beta|} - |\beta|$;

3. $\left|(1-|\beta|^2)\left(\alpha - \overline{(\frac{a_{N-2}}{a_N})}\beta\right) + \overline{(\frac{a_{N-1}}{a_N})^2}\beta\right| \leq \left((\frac{1}{|\beta|} - |\beta|^2)^2 - |\frac{a_{N-1}}{a_N}|^2\right)|\beta|^2$.

Proof. (a) For **N=4** (Here we shall be using $A = |a_4|^2 - |a_{-4}|^2$).

Using the recurrence relation (2.4.2), c_0, c_1, c_2, and c_3 are determined uniquely as following:

$c_0 = \frac{a_{-4}}{\bar{a}_4}$;

$c_1 = (\bar{a}_4)^{-1}(a_{-3} - c_0\bar{a}_3)$;

$c_2 = -(\bar{a}_4)^{-1}(a_{-2} - c_0\bar{a}_2 - c_1\bar{a}_3)$;

$c_3 = (\bar{a}_4)^{-1}(a_{-1} - c_0\bar{a}_1 - c_1\bar{a}_2 - c_2\bar{a}_3)$.

Straightforward calculation gives that

$c_1 = (\bar{a}_N)^{-2}(\bar{a}_4 a_{-3} - \bar{a}_3 a_{-4}) = (\bar{a}_4)^{-2}A\beta$;

$c_2 = (\bar{a}_4)^{-1}\left(\frac{a_{-2}\bar{a}_4 - \bar{a}_2 a_{-4}}{\bar{a}_4} - \frac{A\beta\bar{a}_3}{(\bar{a}_4)^2}\right) = -(\bar{a}_4)^{-3}\bar{a}_3 A\beta$;

$c_3 = (\bar{a}_4)^{-1}\left(\frac{\bar{a}_4 a_{-1} - a_{-4}\bar{a}_1}{\bar{a}_4} - \frac{A\beta\bar{a}_2}{(\bar{a}_4)^2} - \frac{A\beta(\bar{a}_3)^2}{(\bar{a}_4)^3}\right) = (\bar{a}_4)^{-4}(\alpha\bar{a}_4^2 - \beta\bar{a}_2\bar{a}_4 + \beta\bar{a}_3^2)A$.

Now, putting the values of c_0, c_1, c_2, and c_3 in Schur functions Φ_n for $n = 0, 1, 2, 3$ in the equations (2.2.2), (2.2.3), (2.2.4) and (2.3.6), and simplifying we get,

$\Phi_0 = \frac{a_{-4}}{\bar{a}_4}$;

$\Phi_1 = \frac{(\bar{a}_4)^{-2}\beta}{|a_4|^{-2}}$;

$\Phi_2 = \frac{(\bar{a}_4)^{-4}(\beta\bar{a}_{-4} - \bar{a}_3)\beta}{a_4|a_4|^{-4}(1-|\beta|^2)}$;

$\Phi_3 = \frac{(\bar{a}_4)^{-4}((1-|\beta|^2)(\alpha\bar{a}_4{}^2 - \beta\bar{a}_2\bar{a}_4 + \beta\bar{a}_3^2 - \beta^2\bar{a}_{-4}\bar{a}_3) + (\beta\bar{a}_{-4} - \bar{a}_3)(\bar{a}_{-4} - \bar{\beta}\bar{a}_3)\beta^2)}{|a_4|^{-4}((1-|\beta|^2)^2|a_4|^2 - |\bar{a}_3 - \beta\bar{a}_{-4}|^2|\beta|^2)}$.

Thus, from the Proposition 2.4.2, T_φ is hyponormal if and only if $|\Phi_1| \le 1$, $|\Phi_2| \le 1$ and $|\Phi_3| \le 1$. That is, if and only if

(1) $|\beta| \le 1$;

(2) $|\frac{1}{a_4}(\bar{a}_{-4}\beta - \bar{a}_3)| \le \frac{1}{|\beta|} - |\beta|$, and

(3)$|(1-|\beta|^2)(\alpha\bar{a}_4^2 - \beta\bar{a}_2\bar{a}_4 + \beta\bar{a}_3^2 - \beta^2\bar{a}_{-4}\bar{a}_3) + (\beta\bar{a}_{-4} - \bar{a}_3)(\bar{a}_{-4} - \bar{\beta}\bar{a}_3)\beta^2|$

$\le |\beta|(|a_4|^2\left(\frac{1}{\beta} - |\beta|\right)^2 - |\bar{a}_3 - \beta\bar{a}_{-4}|^2)$.

(b) For $\mathbf{N \ge 5}$ (Here we write $A = |a_N|^2 - |a_{-N}|^2$)

By Cowen's Theorem 2.1.2, T_φ is hyponormal if and only if there is a function k in the closed unit ball of $H^\infty(\mathbb{T})$ such that $\varphi - k\bar{\varphi} \in H^\infty(\mathbb{T})$. Because $\varphi - k\bar{\varphi} \in H^\infty(\mathbb{T})$, so k necessarily satisfies the property that

$$k\left(\sum_{n=1}^{N}\bar{a}_n z^{-n}\right)-\sum_{n=1}^{N}a_{-n}z^{-n}\in H^{\infty}(\mathbb{T}) \tag{3.2.23}$$

From (3.2.23) we can compute the Fourier coefficients $\hat{k}(0),\ \hat{k}(1),...,\hat{k}(N-1)$ of k, to be denoted by $\hat{k}(n)=c_n$ for $n=0,1,...,N-1$, uniquely in terms of the coefficients of φ as follows:

$$\begin{cases} c_0\bar{a}_1+c_1\bar{a}_2+...+c_{N-1}\bar{a}_N-a_{-1}=0\\ c_0\bar{a}_2+c_1\bar{a}_3+...+c_{N-2}\bar{a}_N-a_{-2}=0\\ c_0\bar{a}_3+c_1\bar{a}_4+...+c_{N-3}\bar{a}_N-a_{-3}=0\\ \cdots\qquad\cdots\qquad\cdots\qquad\cdots\\ c_0\bar{a}_{N-1}+c_1\bar{a}_N-a_{-N+1}=0\\ c_0\bar{a}_N-a_{-N}=0\end{cases} \tag{3.2.24}$$

$c_0=\frac{a_{-N}}{\bar{a}_N}$;

$c_1=(\bar{a}_N)^{-1}(a_{-N+1}-c_0\bar{a}_{N-1})=(\bar{a}_N)^{-2}(\bar{a}_N a_{-N+1}-\bar{a}_{N-1}a_{-N})=0$;

$c_2=...=c_{N-5}=c_{N-4}=0$;

$c_{N-3}=(\bar{a}_N)^{-1}(a_{-3}-c_0\bar{a}_3)=(\bar{a}_N)^{-2}(\bar{a}_N a_{-3}-\bar{a}_3 a_{-N})=(\bar{a}_N)^{-2}A\beta$;

$$\begin{aligned}c_{N-2}&=(\bar{a}_N)^{-1}(a_{-2}-c_0\bar{a}_2-c_{N-3}\bar{a}_{N-1})\\&=(\bar{a}_N)^{-1}\left(\frac{\bar{a}_N a_{-2}-\bar{a}_2 a_{-N}}{\bar{a}_N}-\frac{A\beta\bar{a}_{N-1}}{\bar{a}_N)^2}\right)\\&=-(\bar{a}_N)^{-3}\bar{a}_{N-1}A\beta;\end{aligned}$$

$$\begin{aligned}c_{N-1}&=(a_{-1}-c_0\bar{a}_1-c_{N-3}\bar{a}_{N-2}-c_{N-2}\bar{a}_{N-1})\\&=(\bar{a}_N)^{-4}(\alpha\bar{a}_N^2-\beta\bar{a}_{N-2}\bar{a}_N+\beta\bar{a}_{N-1}^2)A.\end{aligned}$$

Thus, $k_p(z)=c_0+c_{N-3}z^{N-3}+c_{N-2}z^{N-2}+c_{N-1}z^{N-1}$ is the unique analytic polynomial of degree less than N satisfying $\varphi-k_p\bar{\varphi}\in H^{\infty}(\mathbb{T})$. Now, by using the values of c_0; c_{N-3}, c_{N-2} and c_{N-1} in Proposition (2.4.3) and simplifying we get,

$\Phi_0=c_0=\frac{a_{-N}}{\bar{a}_N}$

$\Phi_{N-3}=\frac{c_{N-3}}{1-|c_0|^2}\qquad=\frac{(\bar{a}_N)^{-2}\beta}{|a_N|^{-2}}$

$$\begin{aligned}
\Phi_{N-2} &= \frac{c_{N-2}}{(1-|c_0|^2)(1-|\Phi_{N-3}|^2)} \\
&= -\frac{(\bar{a}_N)^{-3}\bar{a}_{N-1}\beta}{|a_N|^{-2}(1-|\beta|^2)} \\
\Phi_{N-1} &= \frac{(1-|\Phi_{N-3}|^2)c_{N-1}c_{N-3}+|\Phi_{N-3}|^2 c_{N-2}^2}{c_{N-3}(1-|c_0|^2)(1-|\Phi_{N-3}|^2)^2(1-|\Phi_{N-2}|^2)} \\
&= \frac{(\bar{a}_N)^{-2}\left((1-|\beta|^2)\left(\alpha-\beta\overline{\left(\frac{a_{N-2}}{a_N}\right)}\right)+\beta\overline{\left(\frac{a_{N-1}}{a_N}\right)^2}\right)}{|a_N|^{-2}\left((1-|\beta|^2)^2-|\beta|^2\left|\frac{a_{N-1}}{a_N}\right|^2\right)}
\end{aligned}$$

Thus, Proposition 2.4.3 implies that T_φ is hyponormal if and only

(1) $|\beta| \le 1$;

(2) $|\frac{a_{N-1}}{a_N}| \le \frac{1}{|\beta|} - |\beta|$, and

(3) $\left|(1-|\beta|^2)\left(\alpha - \overline{(\frac{a_{N-2}}{a_N})}\beta\right) + \overline{(\frac{a_{N-1}}{a_N})}^2\beta\right| \le \left((\frac{1}{|\beta|}-|\beta|^2)^2 - |\frac{a_{N-1}}{a_N}|^2\right)|\beta|^2.$ □

3.3 An Alternate Approach

In [36], hyponormality of T_φ was studied when $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ satisfies a 'partial' symmetry condition. Their result is as stated below:

Theorem 3.3.1 ([36]). *Suppose $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ is such that*

$$\bar{a}_N \begin{pmatrix} a_{-(\ell+1)} \\ a_{-(\ell+2)} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_{\ell+1} \\ \bar{a}_{\ell+2} \\ \vdots \\ \bar{a}_N \end{pmatrix} \quad \text{and} \quad \bar{a}_N a_{-\ell} \ne a_{-N}\bar{a}_\ell$$

for $1 \le \ell \le N-1$.

1. *If $|\bar{a}_N a_{-\ell} - a_{-N}\bar{a}_\ell| = |a_N|^2 - |a_{-N}|^2$, then T_φ is hyponormal if and only if* $c_0 = \frac{a_{-N}}{\bar{a}_N}$, *and for $k = 1, 2, \ldots, N-1$*
$$c_k = \begin{cases} \left(-\frac{\bar{a}_{-N}}{\bar{a}_N}\cdot\frac{\bar{a}_N a_{-\ell}-a_{-N}\bar{a}_\ell}{|a_N|^2-|a_{-N}|^2}\right)^{j-1} \frac{\bar{a}_N a_{-\ell}-a_{-N}\bar{a}_\ell}{\bar{a}_N^2}, & if \ \ k = j(N-\ell) \\ 0, & otherwise \end{cases}$$

2. *If $|\bar{a}_N a_{-\ell} - a_{-N}\bar{a}_\ell| = |a_N|^2 - |a_{-N}|^2$ $(\ell \le \frac{N}{2})$, then T_φ is hyponormal if and only if $\left(\frac{\bar{a}_N a_{-k}-a_{-N}\bar{a}_k}{\bar{a}_N a_{-\ell}-a_{-N}\bar{a}_\ell}\right) = \overline{\left(\frac{a_{N-\ell+k}}{a_N}\right)}$ for $k = 1, 2, \ldots, \ell-1$*

3. *If* $\ell \leq \frac{N}{2}$, *and if* $\left(\frac{\bar{a}_N a_{-k} - a_{-N}\bar{a}_k}{\bar{a}_N a_{-\ell} - a_{-N}\bar{a}_\ell}\right) = \overline{\left(\frac{a_{N-\ell+k}}{a_N}\right)}$ *for* $k = 1, 2, ..., \ell - 1$, *then* T_φ *is hyponormal if and only if* $|\bar{a}_N a_{-\ell} - a_{-N}\bar{a}_\ell| \leq |a_N|^2 - |a_{-N}|^2$.

4. *If* $\varphi(z) = a_{-N}z^{-N} + a_{-\ell}z^{-\ell} + a_\ell z^\ell + a_N z^N$ $(\ell \leq \frac{N}{2})$ *then* T_φ *is hyponormal if and only if* $|\bar{a}_N a_{-\ell} - a_{-N}\bar{a}_\ell| \leq |a_N|^2 - |a_{-N}|^2$.

In view of this theorem, if for $\ell \leq \frac{N}{2}$ we have $|\bar{a}_N a_{-\ell} - a_{-N}\bar{a}_\ell| < |a_N|^2 - |a_{-N}|^2$, then does hyponormality of T_φ necessarily imply $\left(\frac{\bar{a}_N a_{-k} - a_{-N}\bar{a}_k}{\bar{a}_N a_{-\ell} - a_{-N}\bar{a}_\ell}\right) = \overline{\left(\frac{a_{N-\ell+k}}{a_N}\right)}$ for $k = 1, 2, ..., \ell - 1$? Equivalently, if $\left(\frac{\bar{a}_N a_{-k} - a_{-N}\bar{a}_k}{\bar{a}_N a_{-\ell} - a_{-N}\bar{a}_\ell}\right) \neq \overline{\left(\frac{a_{N-\ell+k}}{a_N}\right)}$ for $k = 1, 2, ..., \ell - 1$, then what can we say about the hyponormality of T_φ? Unfortunately, Theorem 3.3.1 does not answer these questions. Consider the following example:

Let $\varphi(z) = 5z^{-6} + 7z^{-3} + 3z^{-2} - z^{-1} + z - 2z^2 - 8z^3 - 6z^6$. Here $N = 6$, $\ell = 3$ and as $\bar{a}_N a_{-\ell} - a_{-N}\bar{a}_\ell = -2$, $|a_N|^2 - |a_{-N}|^2 = 11$ so $|\bar{a}_N a_{-\ell} - a_{-N}\bar{a}_\ell| < |a_N|^2 - |a_{-N}|^2$. Also for $k = 1$ and 2, we do not have $\left(\frac{\bar{a}_N a_{-k} - a_{-N}\bar{a}_k}{\bar{a}_N a_{-\ell} - a_{-N}\bar{a}_\ell}\right) = \overline{\left(\frac{a_{N-\ell+k}}{a_N}\right)}$. Yet, T_φ is hyponormal (shown in Example 3.4.5).

Thus, in the following theorem we try to frame alternate hyponormality conditions for trigonometric Toeplitz operator T_φ, under a situation similar to Theorem 3.3.1 but independent of the condition:

$$\left(\frac{\bar{a}_N a_{-k} - a_{-N}\bar{a}_k}{\bar{a}_N a_{-\ell} - a_{-N}\bar{a}_\ell}\right) = \overline{\left(\frac{a_{N-\ell+k}}{a_N}\right)} \text{ for } k = 1, 2, ..., \ell - 1.$$

Theorem 3.3.2. *Let* $\varphi(z) = a_{-N}z^{-N} + a_{-m}z^{-m} + a_{-m+1}z^{-m+1} + a_{-m+2}z^{-m+2} + a_{m-2}z^{m-2} + a_{m-1}z^{m-1} + a_m z^m + a_N z^N$ $(N \leq 7$ *and* $m \leq \frac{N}{2})$ *be a trigonometric polynomial with* $\bar{a}_N a_{-m+2} \neq a_{-N}\bar{a}_{m+2}$. *Let* $\alpha_i = \frac{\bar{a}_N a_{-m+i} - a_{-N}\bar{a}_{m-i}}{|a_N|^2 - |a_{-N}|^2}$, $i = 0, 1, 2$. *Then,* T_φ *is hyponormal if and only if*

(a) $|\alpha_0| \leq 1$,

(b) $|\alpha_1| \leq 1 - |\alpha_0|^2$, *and*

(c) $|(1 - |\alpha_0|^2)\alpha_2 + \bar{\alpha}_0\alpha_1^2| \leq (1 - |\alpha_0|^2)^2 - |\alpha_1|^2$

Proof. Suppose $k \in H^\infty(\mathbb{T})$ satisfies $\varphi - k\bar{\varphi} \in H^\infty(\mathbb{T})$. Then $k\bar{f} - g \in H^\infty(\mathbb{T})$ where,

$f = a_{m-2}z^{m-2} + a_{m-1}z^{m-1} + a_m z^m + a_N z^N$, and

$g = a_{-N}z^{-N} + a_{-m}z^{-m} + a_{-m+1}z^{-m+1} + a_{-m+2}z^{-m+2}$

Thus, the Fourier coefficients $\hat{k}(0),\ \hat{k}(1), ..., \hat{k}(N-1)$ of k, to be denoted by $\hat{k}(n) = c_n$ for $n = 0, 1, ..., N-1$, are determined uniquely from the coefficients of φ by the recurrence relation (2.4.2):

$$c_0 = \frac{a_{-N}}{\bar{a}_N}$$

$$c_1 = c_2 = ... = c_{N-m-2} = c_{N-m-1} = 0$$

$$\begin{aligned} c_{N-m} &= (\bar{a}_N)^{-1}(a_{-m} - c_0\bar{a}_m) \\ &= (\bar{a}_N)^{-2}(\bar{a}_N a_{-m} - a_{-N}\bar{a}_m) \\ &= (\bar{a}_N)^{-2}A\alpha_0, \quad where \quad A = |a_N|^2 - |a_{-N}|^2 \end{aligned}$$

$$\begin{aligned} c_{N-m+1} &= (\bar{a}_N)^{-1}(a_{-m+1} - c_0\bar{a}_{m-1}) \\ &= (\bar{a}_N)^{-2}(\bar{a}_N a_{-m+1} - a_{-N}\bar{a}_{m-1}) \\ &= (\bar{a}_N)^{-2}A\alpha_1, \quad where \quad A = |a_N|^2 - |a_{-N}|^2 \end{aligned}$$

$$\begin{aligned} c_{N-m+2} &= (\bar{a}_N)^{-1}(a_{-m+2} - c_0\bar{a}_{m-2}) \\ &= (\bar{a}_N)^{-2}(\bar{a}_N a_{-m+2} - a_{-N}\bar{a}_{m-2}) \\ &= (\bar{a}_N)^{-2}A\alpha_2, \quad where \quad A = |a_N|^2 - |a_{-N}|^2 \end{aligned}$$

$$c_{N-m+3} = ... = c_{N-1} = 0 \ \ [because \ \ m \le \tfrac{N}{2}]$$

Thus, $k_p(z) = c_0 + c_{N-m}z^{N-m} + c_{N-m+1}z^{N-m+1} + c_{N-m+2}z^{N-m+2}$ is the unique analytic polynomial of degree less than $(N - m + 3)$ satisfying $\varphi - k_p\bar{\varphi} \in H^\infty(\mathbb{T})$. Now, by using Proposition 2.4.3, we get

$$\Phi_0 = c_0$$

$$\Phi_1 = \Phi_2 = ... = \Phi_{N-m-1} = 0$$

$$\Phi_{N-m} = \frac{c_{N-m}}{1-|c_0|^2} = \frac{(\bar{a}_N)^{-2}}{|a_N|^{-2}}\alpha_0$$

$$\Phi_{N-m+1} = \frac{c_{N-m+1}}{(1-|c_0|^2)(1-|\Phi_{N-m}|^2)} = \frac{(\bar{a}_N)^{-2}}{|a_N|^{-2}}\left(\frac{\alpha_1}{1-|\alpha_0|^2}\right)$$

$$\Phi_{N-m+2} = \frac{(1-|\Phi_{N-m}|^2)c_{N-m+2}c_{N-m} + |\Phi_{N-m}|^2 c_{N-m+1}^2}{c_{N-m}(1-|c_0|^2)(1-|\Phi_{N-m}|^2)^2(1-|\Phi_{N-m+1}|^2)}$$

$$= \frac{(1-|\alpha_0|^2)(\bar{a}_N)^{-2}A\alpha_2(\bar{a}_N)^{-2}A\alpha_0+|\alpha_0|^2(\bar{a}_N)^{-4}A^2\alpha_1^2}{(\bar{a}_N)^{-2}A\alpha_0\left(1-|\frac{a_{-N}}{a_N}|^2\right)(1-|\alpha_0|^2)^2\left(1-\frac{|\alpha_1|^2}{(1-|\alpha_0|^2)^2}\right)}$$
$$= \frac{(\bar{a}_N)^{-2}\left((1-|\alpha_0|^2)\alpha_0\alpha_2+|\alpha_0|^2\alpha_1^2\right)}{|a_N|^{-2}\alpha_0((1-|\alpha_0|^2)^2-|\alpha_1|^2)}$$

As $N \leq 7$ so $m \leq 3$ and hence $N-1 \leq N-m+2$. Thus by Proposition 2.4.2, T_φ is hyponormal if and only if $|\Phi_{N-m}| \leq 1$, $|\Phi_{N-m+1}| \leq 1$ and $|\Phi_{N-m+2}| \leq 1$; that is, if and only if

(a) $|\alpha_0| \leq 1$,

(b) $|\alpha_1| \leq 1 - |\alpha_0|^2$, and

(c) $|(1-|\alpha_0|^2)\alpha_2 + \bar{\alpha}_0\alpha_1^2| \leq (1-|\alpha_0|^2)^2 - |\alpha_1|^2$. □

In Theorem 3.3.2, the condition $m \leq \frac{N}{2}$ is necessary for the conditions (a), (b), (c) to imply hyponormality of T_φ. To see this, let us consider the following example:

Example 3.3.1. $\varphi(z) = 2z^{-6}+2z^{-4}-z^{-3}-z^{-2}+z^2+2z^3-z^4-3z^6$. *Here following notations as in Theorem 3.3.2 we have* $m=4, N=6$ *and hence* $m > \frac{N}{2}$.

We also have $\alpha_0 = -\frac{4}{5}$; $\alpha_1 = -\frac{1}{5}$; $\alpha_2 = \frac{1}{5}$ and so the conditions (a), (b) and (c) of the Theorem 3.3.2 (1) are satisfied. The matrix represented by $[T_\varphi^*, T_\varphi]$ is:

$$\begin{pmatrix} 5 & 1 & 0 & -4 & -1 & 0 \\ 1 & 5 & 1 & 0 & -4 & -1 \\ 0 & 1 & 5 & 0 & -1 & -4 \\ -4 & 0 & 0 & 2 & 0 & -1 \\ -1 & -4 & -1 & 0 & 2 & 0 \\ 0 & -1 & -4 & -1 & 0 & 2 \end{pmatrix} \bigoplus 0_\infty.$$

However, as this matrix is not positive semi definite, so T_φ is not hyponormal. Hence the condition $m \leq \frac{N}{2}$ is necessary for the conditions (a), (b) and (c) to imply hyponormality in Theorem 3.3.2.

3.4 Application

Example 3.4.1. *For* $\varphi(z) = z^{-4} + \lambda z^{-3} + 2z^{-1} + 4z^3 + 2z^6$, *we want to determine the values of* λ *for which the Toeplitz operator* T_φ *is hyponormal.*

For this first using Theorem 3.1.3 we reduce this expression φ(z) into the form $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$. Then by this proposition, T_φ is hyponormal if and only if T_ψ is hyponormal where $\psi(z) = z^{-4} + \lambda z^{-3} + 2z^{-1} + 4z + 2z^4$. Expressing $\psi(z)$ as $\sum_{n=-N}^{N} a_n z^n$ and comparing with the given expression, we have, $N = 4$ and also,

$$\bar{a}_4 \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-4} \end{pmatrix} = a_{-4} \cdot \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_4 \end{pmatrix}.$$

Hence, we can apply Case 1 of the Theorem 3.2.1 to determine the values of λ for which T_φ is hyponormal.

As $\alpha = \frac{\bar{a}_4 a_{-3} - \bar{a}_3 a_{-4}}{|a_4|^2 - |a_{-4}|^2} = \frac{2\lambda}{3}$, thus referring to the notations used in the Theorem 3.2.1 we have, T_φ is hyponormal if and only if

(i) $|\alpha| \le 1$ i.e. $|\lambda| \le \frac{3}{2} = 1.5$.

(ii) $|\bar{\alpha} a_{-4} - a_3| \le |a_4| \left(\frac{1}{|\alpha|} - |\alpha|\right)$ i.e. $|\lambda| \le \sqrt{\frac{3}{2}} = 1.22$ (correct upto 2 decimal places).

(iii) $|(1 - |\alpha|^2)(\bar{a}_3^2 - \bar{a}_2\bar{a}_4 - \bar{a}_3\bar{a}_{-4}\alpha) + \alpha(\bar{a}_{-4}\alpha - \bar{a}_3)(\bar{a}_{-4} - \bar{a}_3\bar{\alpha})|$
$\le |\alpha| \left(|a_4|^2 \left(\frac{1}{|\alpha|} - |\alpha|\right)^2 - |a_{-4}\bar{\alpha} - a_3|^2 \right)$

i.e. $12|\lambda|^4 - 6|\lambda|^3 - 72|\lambda|^2 + 81 \ge 0$,

i.e. $|Re[\lambda]| \le 1.13$ and $|Im[\lambda]| \le 1.13$ (correct upto 2 decimal places).

i.e. when λ lies within the circle $|\lambda| \le 1.13$.

This can also be seen from the graph of $12|\lambda|^4 - 6|\lambda|^3 - 72|\lambda|^2 + 81 = 0$ in Figure 1:

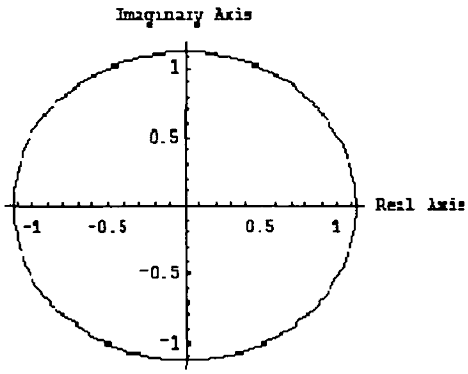

Figure 1

(*Detailed analysis using Mathematica is included in Appendix*)

Thus, T_φ is hyponormal if and only if $|\lambda| \leq 1.13$ (correct upto 2 decimal places).

Example 3.4.2. *For* $\varphi(z) = 2z^{-5}+4z^{-4}+\lambda z^{-3}+z^{-2}+2z^{-1}+3z+2z^2+6z^4+3z^5$ *we want to determine the values of* λ *for which the Toeplitz operator* T_φ *is hyponormal.*

Since $\bar{a}_5 \begin{pmatrix} a_{-1} \\ a_{-4} \\ a_{-5} \end{pmatrix} = a_{-5} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_4 \\ \bar{a}_5 \end{pmatrix}$, we can apply case 2 of Theorem 3.2.3.

Here, $\alpha = -\frac{1}{5}$ and $\beta = \frac{3\lambda}{5}$. Referring to the notations used in the Theorem 3.2.3, we get

(i) $|\beta| \leq 1$ iff $|\lambda| \leq \frac{5}{3}$(correct upto three decimal places). Note that $|\beta| = 1$ iff $|\lambda| = \frac{5}{3}$, for which condition (b) is not satisfied. Hence for hyponormality of T_φ we must have $|\beta| < 1$.

(ii) For $|\lambda| < \frac{5}{3}$, $|\alpha - \beta\overline{(\frac{a_4}{a_5})}| \leq 1 - |\beta|^2$

iff $(25 - 9|\lambda|^2) - 5|1 + 6\lambda| \geq 0$

iff $-0.805399 \leq Re[\lambda] \leq 0.569401$ and $|Im\lambda| \leq 0.68618$ (correct upto 6 decimal places)

iff λ lies inside and on the ellipse given in Figure 2.

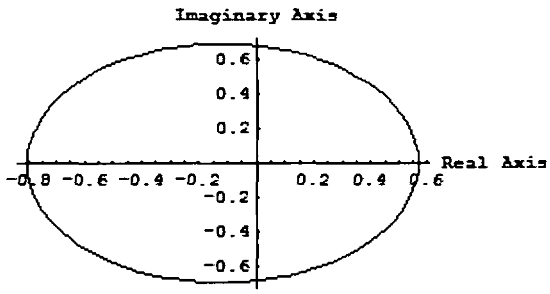

Figure 2

(iii) For λ, with $-0.805399 \le Re[\lambda] \le 0.569401$ and $|Im\lambda| \le 0.68618$,

$$\left|(1-|\beta|^2)\left(\beta\overline{(\tfrac{a_3}{a_5})}+\alpha\overline{(\tfrac{a_4}{a_5})}-\beta\overline{(\tfrac{a_4}{a_5})}^2\right)-\bar{\beta}\left(\alpha-\beta(\tfrac{a_4}{a_5})\right)\right| \le (1-|\beta|^2)^2-\left|\alpha-\beta\overline{(\tfrac{a_4}{a_5})}\right|^2$$

iff $\left|\left(1-\frac{9|\lambda|^2}{25}\right)\left(-\frac{2}{5}-\frac{12\lambda}{5}\right)-\frac{3\bar{\lambda}}{5}\left(\frac{1}{5}+\frac{6\lambda}{5}\right)^2\right| \le \left(1-\frac{9|\lambda|^2}{25}\right)^2-\left|\frac{1}{5}+\frac{6\lambda}{5}\right|^2$

iff $5|3\bar{\lambda}(1+6\lambda)^2+2(1+6\lambda)(25-9|\lambda|^2)| \le (25-9|\lambda|^2)^2-25|1+6\lambda|^2$

iff $-0.471929 \le Re[\lambda] \le 0.169957$ and $|Im\lambda| \le 0.32053$ (correct upto 6 decimal places)

iff λ lies inside and on the ellipse given in Figure 3.

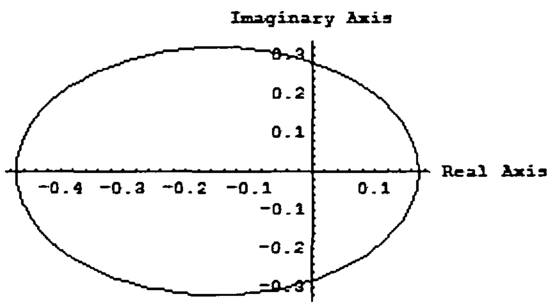

Figure 3

Thus, combining conditions (i), (ii) and (iii) we conclude that T_φ is hyponormal if and only if $-0.471929 \le Re[\lambda] \le 0.169957$ and $|Im\lambda| \le 0.32053$ (correct upto six decimal places) as is seen from the combined graphs of (ii) and (iii) shown below.

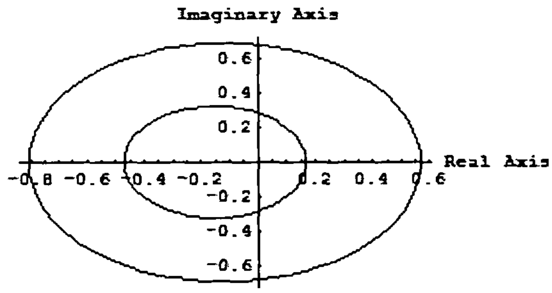

Figure 4

Example 3.4.3. *Let* $\varphi(z) = 2z^{-4} + \lambda z^{-3} - 3z^{-2} + z^{-1} + 3z + 5z^2 + 6z^4$. *We want to find the values of* λ *for which the Toeplitz operator* T_φ *is hyponormal.*

Here we can apply the case1 of Theorem 3.2.3 since $N = 4$. Referring to the notations used in this theorem, we get

$$\alpha = \frac{\bar{a}_4 a_{-2} - a_{-4}\bar{a}_2}{|a_4|^2 - |a_{-4}|^2} = -\frac{7}{8}; \quad \beta = \frac{\bar{a}_4 a_{-3} - a_{-4}\bar{a}_3}{|a_4|^2 - |a_{-4}|^2} = \frac{3\lambda}{16}$$

Then applying the theorem we get that T_φ is hyponormal if and only if

(i) $|\beta| \leq 1$ i.e. iff $|\lambda| \leq \frac{16}{3}$.

(ii) $|\alpha\bar{a}_4 - \beta\bar{a}_3 + \beta^2\bar{a}_{-4}| \leq |a_4|\,(1 - |\beta|^2)$

For $|\lambda| \leq \frac{16}{3}$, putting the values we get,

$\left|-\frac{21}{4} + \frac{9\lambda^2}{128}\right| \leq 6\left(1 - \frac{9|\lambda|^2}{256}\right)$

i.e. iff $|3\lambda^2 - 224| \leq 256 - 9|\lambda|^2$

Solving by using mathematica, we get

$|Re[\lambda]| \leq 2.309401$ and $|Im[\lambda] \leq 1.63299|$ (up to 6 decimal places). That is, if λ lies inside and on the ellipse given in Figure 5

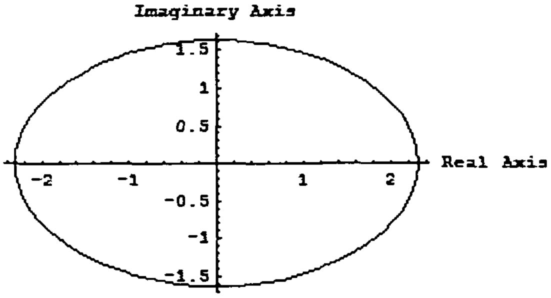

Figure 5

(iii) For λ with $|Re[\lambda]| \leq 2.309401$ and $|Im[\lambda] \leq 1.63299|$

$|(1-|\beta|^2)\left(\beta\bar{a}_{-4}(\alpha\bar{a}_4-\beta\bar{a}_3)-\beta\bar{a}_2\bar{a}_4-\alpha\bar{a}_3\bar{a}_4+\beta(\bar{a}_3)^2\right)+(\alpha\bar{a}_4-\beta\bar{a}_3+\beta^2\bar{a}_{-4})$

$\left(\beta\bar{a}_{-4}+\bar{\beta}(\alpha\bar{a}_4-\beta\bar{a}_3)\right)| \leq \left(|a_4|(1-|\beta|^2)\right)^2-|\alpha\bar{a}_4-\beta\bar{a}_3+\beta^2\bar{a}_{-4}|^2$

if and only if

$\left|\left(1-\frac{9|\lambda|^2}{256}\right)\left(-\frac{63\lambda}{32}-\frac{45\lambda}{8}\right)+\left(\frac{9|\lambda|^2}{128}-\frac{21}{4}\right)\left(\frac{3\lambda}{8}-\frac{63\bar{\lambda}}{64}\right)\right| \leq \left(6\left(1-\frac{9|\lambda|^2}{256}\right)\right)^2-\left|\frac{9\lambda^2}{128}-\frac{21}{4}\right|^2$

i.e. iff $\left|\left(\frac{256-9|\lambda|^2}{256}\right)\left(-\frac{243\lambda}{32}\right)+\left(\frac{9\lambda^2-672}{128}\right)\left(\frac{24\lambda-63\bar{\lambda}}{64}\right)\right| \leq \frac{9(256-9|\lambda|^2)^2}{16384}-\frac{|9\lambda^2-672|^2}{16384}$

i.e. iff $2|243\lambda(9|\lambda|^2-256)+(9\lambda^2-672)(24\lambda-63\bar{\lambda})| \leq 9(256-9|\lambda|^2)^2-|9\lambda^2-672|^2$

By solving this inequality and using mathematica, we get that T_φ is hyponormal if and only if $|Re[\lambda]| \leq 1.339494$ and $|Im[\lambda]| \leq 0.5115576$ (correct upto 6 decimal places). Graphically we can represent the values as follows: That is, if λ lies inside and on the ellipse given in Figure 6.

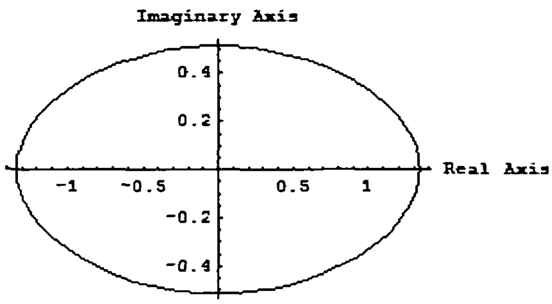

Figure 6

Combining the graphs in the figure 5 and figure 6, we get that T_φ is hyponormal if and only if λ lies inside and on the smaller ellipse given in the figure 7

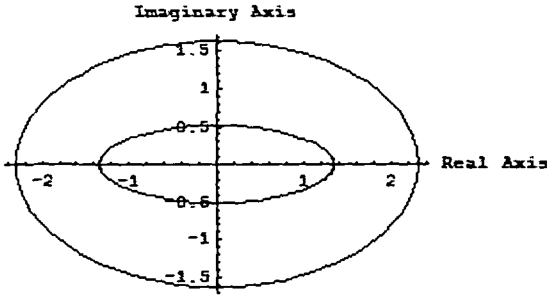

Figure 7

Example 3.4.4. *Let* $\varphi(z) = 2z^{-5} + z^{-4} + 3z^{-3} + 3z^{-1} + z + \lambda z^2 + \frac{15}{2}z^3 + \frac{5}{2}z^4 + 5z^5$. *To find the values of* λ *for which* T_φ *is hyponormal.*

We apply here Theorem 3.2.4. Using the notations of this theorem we get,

$$\alpha = \frac{\bar{a}_5 a_{-1} - \bar{a}_1 a_{-5}}{|a_5|^2 - |a_{-5}|^2} = \frac{13}{21}; \quad \beta = \frac{\bar{a}_5 a_{-2} - \bar{a}_2 a_{-5}}{|a_5|^2 - |a_{-5}|^2} = -\frac{2\bar{\lambda}}{21}$$

Using the theorem we get T_φ is hyponormal if and only if

(a) $|\beta| \leq 1$ i.e. if and only if $|\lambda| \leq 10.5$

(b) $\left|\bar{\alpha} - \bar{\beta}\left(\frac{a_4}{a_5}\right)\right| \leq 1 - |\beta|^2$

i.e. iff $21|\lambda + 13| \leq 441 - 4|\lambda|^2$ which when solved by using mathematica we get T_φ is hyponormal if and only if $-9.61718 \leq Re[\lambda] \leq 4.36718$ and $|Im[\lambda]| \leq 6.30541$ (correct upto 5 decimal places). That is, if λ lies inside and on the ellipse given in Figure 8.

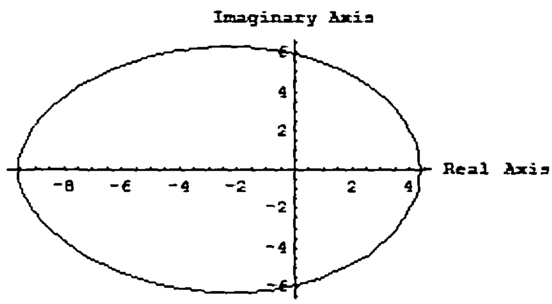

Figure 8

Example 3.4.5. *Let $\varphi(z) = 5z^{-6} + 7z^{-3} + 3z^{-2} - z^{-1} + z + \lambda z^2 - 8z^3 - 6z^6$. By using the Theorem 3.3.2 we investigate for what values of λ, T_φ will be hyponormal.*

Using the notations of this theorem, we get

$$\alpha_0 = \frac{\bar{a}_6 a_{-3} - a_{-6}\bar{a}_3}{|a_6|^2 - |a_{-6}|^2} = -\frac{2}{11}; \quad \alpha_1 = \frac{\bar{a}_6 a_{-2} - a_{-6}\bar{a}_2}{|a_6|^2 - |a_{-6}|^2} = -\frac{18+5\lambda}{11};$$
$$\alpha_2 = \frac{\bar{a}_6 a_{-1} - a_{-6}\bar{a}_1}{|a_6|^2 - |a_{-6}|^2} = \frac{1}{11}.$$

From this theorem T_φ is hyponormal if and only if

(a) $|\alpha_0| \le 1$,

(b) $|\alpha_1| \le 1 - |\alpha_0|^2$ and

(c) $|(1 - |\alpha_0|^2)\alpha_2 + \bar{\alpha}_0\alpha_1^2| \le (1 - |\alpha_0|^2)^2 - |\alpha_1|^2$.

Obviously, $|\alpha_0| < 1$.

Putting the values of α_0 and α_1 in (b) we get,

$11|18 + 5\lambda| \le 117$.

Solving this inequality we get T_φ is hyponormal if and only if

$-5.72727 \le Re[\lambda] \le -1.47273; \quad |Im[\lambda]| \le 2.12727$.

That is, T_φ is hyponormal if and only if λ lies inside and on the ellipse in Figure 9.

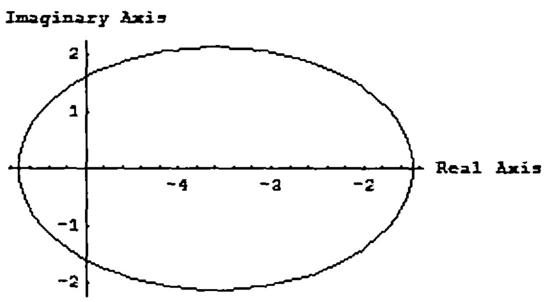

Figure 9

Putting the values of α_0, α_1 and α_2 in (c) we get,

$\left|\frac{117}{1331} - \frac{2(18+5\lambda)^2}{1331}\right| \le \frac{13689}{14641} - \frac{|18+5\lambda|^2}{121}$

That is for λ with $-5.72727 \le Re[\lambda] \le -1.47273; \quad |Im[\lambda]| \le 2.12727$, T_φ is hyponormal if and only if $|5841 + 3960\lambda + 550\lambda^2| \leq 13689 - 121|18 + 5\lambda|^2$. Solving this inequality we get T_φ is hyponormal if and only if

$-5.64673 \leq Re[\lambda] \leq -1.55327$ and $|Im[\lambda]| \leq 1.86255$ (correct upto 5 decimal places). That is, T_φ is hyponormal if and only λ lies inside and on the ellipse given in the Figure 10.

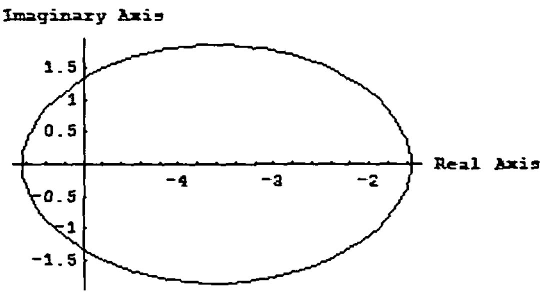

Figure 10

Combining the figures 9 and 10, we can conclude that T_φ is hyponormal if and only if λ lies inside and on the inner ellipse given in the figure 11.

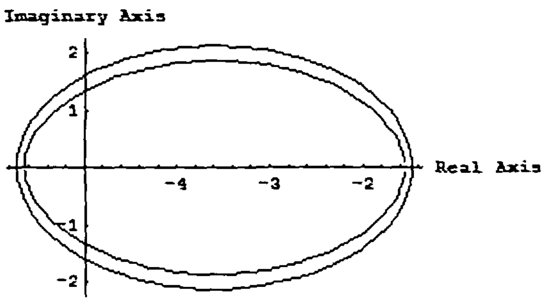

Figure 11

Chapter 4

Hyponormality in Bergman Space

In this Chapter we study the hyponormality of Toeplitz operators T_φ on the Bergman space $A^2(\mathbb{D})$ for the class of functions $\varphi = \bar{g} + f$ where f and g are bounded and analytic functions on $\mathbb{D}$.

4.1 Introduction

The Bergman space $A^2(\mathbb{D})$ and the definition of Toeplitz operator T_φ on it has already been discussed in Section 1.2.4 of Chapter 1 . For the case that $\varphi = \bar{g} + f$ with f, g bounded and analytic, H. Sadraoui [45] gave the following necessary and sufficient condition for the hyponormality of T_φ in the Bergman space $A^2(\mathbb{D})$.

Theorem 4.1.1. [45] *Let f, g be bounded and analytic in $L^2(\mathbb{D})$. Then the followings are equivalent:*

(i) $T_{f+\bar{g}}$ is hyponormal;

(ii) $H^*_{\bar{g}} H_{\bar{g}} \leq H^*_{\bar{f}} H_{\bar{f}}$;

(iii) $H_{\bar{g}} = C H_{\bar{f}}$, where C is of norm less than or equal to one.

Using this result, the following more specific result was established.

Theorem 4.1.2. [23] *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_m z^m + a_N z^N$ and*

$g(z) = a_{-m}z^m + a_{-N}z^N$ $(0 < m < N)$. *If* $a_{-m}\bar{a}_{-N} = a_m\bar{a}_N$, *then* T_φ *on* $A^2(\mathbb{D})$ *is hyponormal*

$$\iff \begin{cases} \frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) & if\ \ |a_{-N}| \leq |a_N| \\ N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2) & if\ \ |a_N| \leq |a_{-N}|. \end{cases}$$

The above result has the assumption $a_{-m}\bar{a}_{-N} = a_m\bar{a}_N$. In [39], a necessary condition for hyponormality of T_φ was established where this particular assumption is not required.

Theorem 4.1.3. [39] *Let* $\varphi(z) = \overline{g(z)} + f(z)$, *where* $f(z) = a_1z + a_2z^2$ *and* $g(z) = a_{-1}z + a_{-2}z^2$. *If* T_φ *is hyponormal, then*
(i) $2(|a_2|^2 - |a_{-2}|^2) \geq 3(|a_{-1}|^2 - |a_1|^2)$
(ii) $\left(\frac{1}{2}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{3}(|a_2|^2 - |a_{-2}|^2)\right)\left(\frac{1}{3}(|a_1|^2 - |a_{-1}|^2) + (|a_2|^2 - |a_{-2}|^2)\right)$
$\geq \frac{4}{9}|\bar{a}_1a_2 - \bar{a}_{-1}a_{-2}|^2$.

In Section 4.2 here, we show that the above set of conditions is not sufficient for T_φ to be hyponormal. In Theorem 4.3.2, we give a set of sufficient conditions for hyponormality of T_φ. Here we also discuss another sufficieny condition for hyponormality of T_φ keeping in view the following earlier results:

Theorem 4.1.4. [45] (i) *If* $n \geq m$, $T_{z^n+\alpha\bar{z}^m}$ *is hyponormal if and only if* $|\alpha| \leq \sqrt{\frac{m+1}{n+1}}$.
(ii) *If* $m \geq n$, $T_{z^n+\alpha\bar{z}^m}$ *is hyponormal if and only if* $|\alpha| \leq \frac{n}{m}$.

Theorem 4.1.5. [24] *If* $f(z) = \sum_{n=2}^{N} a_nz^n$ $(N \geq 2)$, $h(z) = az + f(z)$, *and* $A := max\{|a_i| : 2 \leq i \leq N\}$, *then* $T_{\bar{f}+h}$ *is hyponormal when* $|a| \geq 2N^2A$.

Here we establish the following result:

Theorem 4.1.6. *Let* $f(z) = \sum_{n=2}^{N} a_nz^n$ $(N \geq 2)$, $g(z) = az + f(z)$ *and* $h(z) = bz + f(z)$. *If* $\varphi(z) = \overline{g(z)} + h(z)$ *and* $|b| > |a|$, *then* T_φ *is hyponormal when*

$$\frac{|b|^2 - |a|^2}{|b - a|} \geq 2(N-1)NA,$$

where $A := max\{|a_i| : i = 2, 3, ..., N\}$

It may be noted that if we consider the above Toeplitz operator T_φ on the Hardy space $H^2(\mathbb{T})$, then by Theorem 1.4 [12], $T_{\bar{g}+h}$ on $H^2(\mathbb{T})$ is hyponormal if and only $a = b$, where g and h are as described in Theorem 4.1.6.

4.2 Preliminaries

We begin the section with an example showing that the conditions (i) and (ii) of Theorem 4.1.3 are not sufficient for T_φ to be hyponormal.

Example 2.1: Let $\varphi(z) = a_{-2}\bar{z}^2 + a_1 z$ where $|a_{-2}|^2 = \frac{7}{24}|a_1|^2 > 0$. Then by Theorem 4.1.2, T_φ is not hyponormal. However, the conditions (i) and (ii) of Theorem 4.1.3 are both satisfied.

We proceed to establish a set of conditions which are sufficient for hyponormality of T_φ where $\varphi = \bar{g} + f$ and f, g are polynomials of degree 2. But before that we recall a few specific properties of Toeplitz operators and work out some necessary lemmas. Since the hyponormality of operators is translation invariant, we may assume that $f(0) = g(0) = 0$. We have the following properties of Toeplitz operators:

If $f, g \in L^\infty(\mathbb{D})$, then

(i) $T_{f+g} = T_f + T_g$

(ii) $T_f^* = T_{\bar{f}}$

(iii) $T_{\bar{f}}T_g = T_{\bar{f}g}$ if f or g is analytic.

Also using the definitions given in Section 1.2.4, it follows directly that for any nonnegative integers s and t,

$$P(\bar{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & if s \geq t \\ 0 & if s < t. \end{cases} \tag{4.2.1}$$

Also,

$$\begin{aligned}\langle z^s, z^t\rangle &= \int_{\mathbb{D}} z^s \bar{z}^t dA(z)\\ &= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 r^{s+t+1} e^{i(s-t)\theta} dr d\theta = \frac{2}{s+t+2}\delta_{s,t}. \qquad (4.2.2)\end{aligned}$$

Lemma 4.2.1. [23] *If $k_i(z) := \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i}$ for $i = 0, 1, \dots, N-1$, then for $0 \le m \le N$ we have,*

$$(i)\ \ \|\bar{z}^m k_i(z)\|^2 = \sum_{n=0}^{\infty} \frac{|c_{Nn+i}|^2}{Nn+i+m+1}$$

$$(ii) \|P\big(\bar{z}^m k_i(z)\big)\|^2 = \begin{cases} \sum_{n=0}^{\infty} \frac{Nn+i-m+1}{(Nn+i+1)^2}|c_{Nn+i}|^2, & \text{if } m \le i;\\ \sum_{n=1}^{\infty} \frac{Nn+i-m+1}{(Nn+i+1)^2}|c_{Nn+i}|^2, & \text{if } m > i.\end{cases}$$

Lemma 4.2.2. *Let $k_i(z) := \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i}$ for $i = 0, 1, \dots, N-1$. Then*

$$(i)\ \ \sum_{i=0}^{N-1} \|H_{\bar{z}} k_i(z)\|^2 = \sum_{n=0}^{\infty} \frac{|c_n|^2}{(n+2)(n+1)^2}$$

$$(ii)\ \ \sum_{i=0}^{N-1} \|H_{\bar{z}^2} k_i(z)\|^2 = \frac{|c_0|^2}{3} + \sum_{n=1}^{\infty} \frac{4|c_n|^2}{(n+3)(n+1)^2}$$

Proof. (i)

$$\begin{aligned}&\sum_{i=0}^{N-1} \|M_{\bar{z}} k_i(z)\|^2 = \sum_{i=0}^{N-1} \|\bar{z} k_i(z)\|^2\\ &= \sum_{i=0}^{N-1}\sum_{n=0}^{\infty} \frac{|c_{Nn+i}|^2}{Nn+i+2}\\ &= \sum_{n=0}^{\infty} \frac{|c_{Nn}|^2}{Nn+2} + \sum_{n=0}^{\infty} \frac{|c_{Nn+1}|^2}{(Nn+1)+2} + \cdots + \sum_{n=0}^{\infty} \frac{|c_{Nn+(N-1)}|^2}{(Nn+(N-1))+2}\\ &= \sum_{n=0}^{\infty} \frac{|c_n|^2}{n+2}\end{aligned}$$

Again,

$$\begin{aligned}\|T_{\bar{z}} k_i(z)\|^2 &= \|P\big(\bar{z} k_i(z)\big)\|^2\\ &= \begin{cases} \sum_{n=1}^{\infty} \frac{Nn}{(Nn+1)^2}|c_{Nn}|^2, & \text{if } i = 0;\\ \sum_{n=0}^{\infty} \frac{Nn+i}{(Nn+i+1)^2}|c_{Nn+i}|^2, & \text{if } i > 0.\end{cases}\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{i=0}^{N-1}\|T_{\bar z}k_i(z)\|^2 &= \sum_{n=1}^{\infty}\frac{Nn}{(Nn+1)^2}|c_{Nn}|^2+\sum_{i=1}^{N-1}\sum_{n=0}^{\infty}\frac{Nn+i}{(Nn+i+1)^2}|c_{Nn+i}|^2\\ &= \sum_{i=0}^{N-1}\sum_{n=0}^{\infty}\frac{Nn+i}{(Nn+i+1)^2}|c_{Nn+i}|^2=\sum_{n=0}^{\infty}\frac{n}{(n+1)^2}|c_n|^2\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{i=0}^{N-1}\|H_{\bar z}k_i(z)\|^2 &= \sum_{n=0}^{\infty}\left(\frac{1}{n+2}-\frac{n}{(n+1)^2}\right)|c_n|^2\\ &= \sum_{n=0}^{\infty}\frac{|c_n|^2}{(n+2)(n+1)^2}\end{aligned}$$

(ii)

$$\sum_{i=0}^{N-1}\|M_{\bar z^2}k_i(z)\|^2=\sum_{i=0}^{N-1}\|\bar z^2k_i(z)\|^2=\sum_{i=0}^{N-1}\sum_{n=0}^{\infty}\frac{|c_{Nn+i}|^2}{Nn+i+3}=\sum_{n=0}^{\infty}\frac{|c_n|^2}{n+3},$$

and

$$\sum_{i=0}^{N-1}\|T_{\bar z^2}k_i(z)\|^2=\sum_{n=1}^{\infty}\frac{n-1}{(n+1)^2}|c_n|^2$$

so that we have

$$\begin{aligned}\sum_{i=0}^{N-1}\|H_{\bar z^2}k_i(z)\|^2 &= \frac{|c_0|^2}{3}+\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{n-1}{(n+1)^2}\right)|c_n|^2\\ &= \frac{|c_0|^2}{3}+\sum_{n=1}^{\infty}\frac{4|c_n|^2}{(n+3)(n+1)^2}\end{aligned}$$

□

Lemma 4.2.3. *Let* $k_i(z) = \sum_{n=0}^{\infty} c_{Nn+i}z^{Nn+i}$ *for* $i = 0, 1, \ldots, N-1$. *Then for* $s \geq 0$ *and* $i \neq j$, $\langle H_{\bar z^s}k_i(z), H_{\bar z^s}k_j(z)\rangle = 0$

Proof. We note that for $i, j = 0, 1, \ldots, N-1$, if $i \neq j$ then $Nn + i \neq Nt + j$ for all $n, t = 0, 1, 2, \ldots$, because if $n = t$ then $N(n-t) = 0 \neq j - i$, and if $n \neq t$ then $|N(n-t)| > N > |j-i|$ and so $N(n-t) \neq j - i$. So for $i \neq j$ we have

$$\langle M_{\bar z^s}k_i(z), M_{\bar z^s}k_j(z)\rangle=\sum_{n=0}^{\infty}\sum_{t=0}^{\infty}c_{Nn+i}\bar c_{Nt+j}\langle\bar z^sz^{Nn+i},\bar z^sz^{Nt+j}\rangle=0,$$

and,

$$\langle T_{\bar{z}^s}k_i(z), T_{\bar{z}^s}k_j(z)\rangle = \sum_{n=0}^{\infty}\sum_{t=0}^{\infty} c_{Nn+i}\bar{c}_{Nt+j}\langle P(\bar{z}^s z^{Nn+i}), P(\bar{z}^s z^{Nt+j})\rangle$$

$$= \sum_{n\geq\frac{s-i}{N}}\sum_{t\geq\frac{s-j}{N}} c_{Nn+i}\bar{c}_{Nt+j}\frac{Nn+i-s+1}{Nn+i+1}\frac{Nt+j-s+1}{Nt+j+1}\langle z^{Nn+i-s}, z^{Nt+j-s}\rangle$$

$$= 0$$

Hence, for $i \neq j$,

$$\langle H_{\bar{z}^s}k_i(z), H_{\bar{z}^s}k_j(z)\rangle = \langle M_{\bar{z}^s}k_i(z), M_{\bar{z}^s}k_j(z)\rangle - \langle T_{\bar{z}^s}k_i(z), T_{\bar{z}^s}k_j(z)\rangle = 0$$

□

Lemma 4.2.4. *Let* $k_i(z) := \sum_{n=0}^{\infty} c_{Nn+i}z^{Nn+i}$ *for* $i = 0, 1, \ldots, N-1$. *Then*

(i) $\langle H_{\bar{z}^2}k_i(z), H_{\bar{z}}k_i(z)\rangle = 0$

(ii) $\sum_{i,j=0}^{N-1}\langle H_{\bar{z}^2}k_i(z), H_{\bar{z}}k_j(z)\rangle = \sum_{n=0}^{\infty}\frac{2c_{n+1}\bar{c}_n}{(n+1)(n+2)(n+3)}$

Proof. (i)

$$\langle M_{\bar{z}^2}k_i(z), M_{\bar{z}}k_i(z)\rangle = \langle \bar{z}^2k_i(z), \bar{z}k_i(z)\rangle$$

$$= \sum_n\sum_t c_{Nn+i}\bar{c}_{Nt+i}\frac{1}{\pi}\int_0^1\int_0^{2\pi} r^{Nn+Nt+2i+4}e^{\iota\theta(Nn-Nt-1)}dr\, d\theta$$

$$= 0,$$

since $N(n-t) \neq 1$ for all n and t.

Similarly,

$$\langle T_{\bar{z}^2}k_i(z), T_{\bar{z}}k_i(z)\rangle = \langle P(\bar{z}^2k_i(z)), P(\bar{z}k_i(z))\rangle$$

$$= \sum_{n\geq\frac{2-i}{N}}\sum_{t\geq\frac{1-i}{N}} c_{Nn+i}\bar{c}_{Nt+i}\frac{(Nn+i-1)(Nt+i)}{(Nn+i+1)(Nt+i+1)}\langle z^{Nn+i-2}, z^{Nn+i-1}\rangle$$

$$= 0,$$

since $Nn+i-2 \neq Nt+i-1$ for any n or t.

Thus, $\langle H_{\bar{z}^2}k_i(z), H_{\bar{z}}k_i(z)\rangle = 0$

(ii) We have,

$$\langle M_{\bar{z}^2}k_i(z), M_{\bar{z}}k_j(z)\rangle = \sum_n \sum_t c_{Nn+i}\bar{c}_{Nt+j}\langle \bar{z}^2 z^{Nn+i}, \bar{z}z^{Nt+j}\rangle$$
$$= \sum_n \sum_t c_{Nn+i}\bar{c}_{Nt+j}\frac{1}{\pi}\int_0^1\int_0^{2\pi} r^{Nn+i+Nt+j+4}e^{\iota\theta(Nn+i-Nt-j-1)}dr\, d\theta$$

So,

$$\sum_{i,j=0}^{N-1}\langle M_{\bar{z}^2}k_i(z), M_{\bar{z}}k_j(z)\rangle = \sum_{j=0}^{N-1}\sum_{t=0}^{\infty}\frac{c_{Nt+j+1}\bar{c}_{Nt+j}}{Nt+j+3} = \sum_{n=0}^{\infty}\frac{c_{n+1}\bar{c}_n}{(n+3)}$$

Again,

$$\langle T_{\bar{z}^2}k_i(z), T_{\bar{z}}k_j(z)\rangle$$
$$= \sum_{n\geq\frac{2-i}{N}}\sum_{t\geq\frac{1-j}{N}} c_{Nn+i}\bar{c}_{Nt+j}\frac{(Nn+i-1)(Nt+j)}{(Nn+i+1)(Nt+j+1)}\langle z^{Nn+i-2}, z^{Nt+j-1}\rangle$$

and so,

$$\sum_{i,j=0}^{N-1}\langle T_{\bar{z}^2}k_i(z), T_{\bar{z}}k_j(z)\rangle = \sum_{j=0}^{N-1}\sum_{t\geq\frac{1-j}{N}} c_{Nt+j+1}\bar{c}_{Nt+j}\frac{Nt+j}{(Nt+j+2)(Nt+j+1)}$$
$$= \sum_{j=1}^{N-1}\sum_{t=0}^{\infty} c_{Nt+j+1}\bar{c}_{Nt+j}\frac{Nt+j}{(Nt+j+2)(Nt+j+1)}$$
$$= \sum_{n=1}^{\infty} c_{n+1}\bar{c}_n\frac{n}{(n+1)(n+2)}$$

Hence,

$$\sum_{i,j=0}^{N-1}\langle H_{\bar{z}^2}k_i(z), H_{\bar{z}}k_j(z)\rangle = \frac{\bar{c}_0 c_1}{3} + \sum_{n=1}^{\infty}\left(\frac{1}{n+3} - \frac{n}{(n+1)(n+2)}\right)c_{n+1}\bar{c}_n$$
$$= \sum_{n=0}^{\infty}\frac{2c_{n+1}\bar{c}_n}{(n+1)(n+2)(n+3)}$$

□

Lemma 4.2.5. *For a_n, $b_n \geq 0$, if $a_n a_{n+1} \geq b_n^2\ \forall n$, then $\sum_n a_n \geq \sum_n b_n$.*

Proof. As $a_n a_{n+1} \geq b_n^2\ \forall n$, so $(a_n + a_{n+1})^2 = (a_n - a_{n+1})^2 + 4a_n a_{n+1} \geq 4b_n^2$, and because a_n, $b_n \geq 0$ it follows that $a_n + a_{n+1} \geq 2b_n\ \forall n$.

Therefore, $2\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty}(a_n + a_{n+1}) + a_0 \geq \sum_{n=0}^{\infty}(a_n + a_{n+1}) \geq 2\sum_{n=0}^{\infty} b_n$ □

Lemma 4.2.6. *Let $k_\imath(z) := \Sigma_{n=0}^{\infty} c_{Nn+\imath} z^{Nn+\imath}$ for $\imath = 0, 1, \ldots, N-1$. Also let $\xi \in \{2, \ldots, N\}$, $\tau \in \{1, 2, \ldots, \xi - 1\}$ and $f(z) = \sum_{m=\xi}^{N} a_m z^m$. Then*

(ı) $\langle H_{\bar{f}} k_\imath(z), H_{\bar{z}^\tau} k_\imath(z)\rangle = 0$,

(ıı) $\sum_{\imath,\jmath=0}^{N-1} \langle H_{\bar{f}} k_\imath(z), H_{\bar{z}^\tau} k_\jmath(z)\rangle$

$$= \sum_{m=\xi}^{N} \bar{a}_m \left[\sum_{n=0}^{\tau-1} \frac{1}{n+m+1} + \sum_{n=\tau}^{\infty} \frac{m\tau}{(n+m-\tau+1)(n+m+1)(n+1)} \right] \bar{c}_n c_{n+m-\tau}$$

Proof. (i)

$$\langle M_{\bar{f}} k_\imath(z), M_{\bar{z}^\tau} k_\imath(z)\rangle = \left\langle \sum_{m=\xi}^{N} \bar{a}_m \bar{z}^m k_\imath(z), \bar{z}^\tau k_\imath(z) \right\rangle$$

$$= \sum_{m=\xi}^{N} \bar{a}_m \sum_n \sum_t c_{Nn+\imath} \bar{c}_{Nt+\imath} \langle \bar{z}^m z^{Nn+\imath}, \bar{z}^\tau z^{Nt+\imath} \rangle$$

$$= \sum_{m=\xi}^{N} \bar{a}_m \sum_n \sum_t c_{Nn+\imath} \bar{c}_{Nt+\imath} \int_{\mathbb{D}} z^{Nn+\imath+\tau} \bar{z}^{Nt+\imath+m} dA(z)$$

$$= 0,$$

since $1 \leq m - \tau \leq N-1$ implies $Nn + \tau \neq Nt + m$ for any n or t.

Again,

$$\langle T_{\bar{f}} k_\imath(z), T_{\bar{z}^\tau} k_\imath(z)\rangle = \sum_{m=\xi}^{N} \bar{a}_m \sum_n \sum_t c_{Nn+\imath} \bar{c}_{Nt+\imath} \langle P(\bar{z}^m z^{Nn+\imath}), P(\bar{z}^\tau z^{Nt+\imath})\rangle$$

$$= \sum_{m=\xi}^{N} \bar{a}_m \sum_{n \geq \frac{m-\imath}{N}} \sum_{t \geq \frac{\tau-\imath}{N}} c_{Nn+\imath} \bar{c}_{Nt+\imath} \frac{(Nn+\imath-m+1)(Nt+\imath-\tau+1)}{(Nn+\imath+1)(Nt+\imath+1)} \times$$

$$\langle z^{Nn+\imath-m}, z^{Nt+\imath-\tau} \rangle$$

$$=0,$$

since $Nn - m \neq Nt - \tau$ for any n or t.

Thus, $\langle H_{\bar{f}} k_i(z), H_{\bar{z}^\tau} k_i(z)\rangle = 0$

(ii)

$$\begin{aligned}
&\sum_{i,j=0}^{N-1} \langle M_{\bar{f}} k_i(z), M_{\bar{z}^\tau} k_j(z)\rangle \\
&= \sum_{i,j=0}^{N-1} \sum_{m=\xi}^{N} \bar{a}_m \sum_n \sum_t c_{Nn+i}\bar{c}_{Nt+j} \int_{\mathbb{D}} z^{Nn+i+\tau} \bar{z}^{Nt+j+m} dA(z) \\
&= \sum_{m=\xi}^{N} \bar{a}_m \sum_{j=0}^{N-1} \sum_{t=0}^{\infty} \frac{c_{Nt+j+m-\tau}\bar{c}_{Nt+j}}{Nt+j+m+1} \\
&= \sum_{m=\xi}^{N} \bar{a}_m \left(\sum_{n=0}^{\infty} \frac{\bar{c}_n\, c_{n+m-\tau}}{n+m+1} \right)
\end{aligned}$$

Also,

$$\begin{aligned}
&\sum_{i,j=0}^{N-1} \langle T_{\bar{f}} k_i(z), T_{\bar{z}^\tau} k_j(z)\rangle \\
&= \sum_{i,j=0}^{N-1} \sum_{m=\xi}^{N} \bar{a}_m \sum_{n \geq \frac{m-i}{N}} \sum_{t \geq \frac{\tau-j}{N}} c_{Nn+i}\bar{c}_{Nt+j} \frac{(Nn+i-m+1)(Nt+j-\tau+1)}{(Nn+i+1)(Nt+j+1)} \times \\
&\quad \langle z^{Nn+i-m}, z^{Nt+j-\tau}\rangle \\
&= \sum_{m=\xi}^{N} \bar{a}_m \sum_{j=0}^{N-1} \sum_{t \geq \frac{\tau-j}{N}} c_{Nt+j+m-\tau}\, \bar{c}_{Nt+j} \frac{Nt+j-\tau+1}{(Nt+j+m-\tau+1)(Nt+j+1)} \\
&= \sum_{m=\xi}^{N} \bar{a}_m \sum_{n=\tau}^{\infty} \frac{(n-\tau+1)\bar{c}_n\, c_{n+m-\tau}}{(n+m-\tau+1)(n+1)}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{i,j=0}^{N-1} \langle H_{\bar{f}} k_i(z), H_{\bar{z}^\tau} k_j(z)\rangle \\
&= \sum_{m=\xi}^{N} \bar{a}_m \left(\sum_{n=0}^{\infty} \frac{1}{n+m+1} - \sum_{n=\tau}^{\infty} \frac{n-\tau+1}{(n+m-\tau+1)(n+1)} \right) \bar{c}_n\, c_{n+m-\tau} \\
&= \sum_{m=\xi}^{N} \bar{a}_m \left(\sum_{n=0}^{\tau-1} \frac{1}{n+m+1} + \sum_{n=\tau}^{\infty} \frac{m\tau}{(n+m+1)(n+m-\tau+1)(n+1)} \right) \bar{c}_n\, c_{n+m-\tau}
\end{aligned}$$

□

4.3 Hyponormality Conditions

4.3.1 Necessary conditions for hyponormality

Theorem 4.3.1. *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = \sum_{n=1}^{N} a_n z^n$ and $g(z) = \sum_{n=1}^{N} a_{-n} z^n$. Let T_φ be hyponormal. Then*

(i) $\sum_{n=1}^{N} \frac{1}{n+1}(|a_n|^2 - |a_{-n}|^2) \geq 0$

(ii) $\left(\sum_{n=1}^{N} \frac{1}{n+1}(|a_n|^2 - |a_{-n}|^2)\right)\left(\sum_{n=1}^{N} \frac{1}{n+2}(|a_n|^2 - |a_{-n}|^2) + \frac{1}{4}(|a_{-1}|^2 - |a_1|^2)\right)$
$\geq \left|\sum_{n=1}^{N-1} \frac{1}{n+2}(a_n \bar{a}_{n+1} - a_{-n}\bar{a}_{-(n+1)})\right|^2$

Proof. Let T_φ be hyponormal. Then

$$\langle (H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}})(c_0 + c_1 z), (c_0 + c_1 z)\rangle \geq 0 \tag{4.3.1}$$

We have,

$$\begin{aligned}
&\langle M_{\bar{f}}(c_0 + c_1 z), M_{\bar{f}}(c_0 + c_1 z)\rangle \\
&= \left\langle \sum_{n=1}^{N} \bar{a}_n M_{\bar{z}^n}(c_0 + c_1 z), \sum_{n=1}^{N} \bar{a}_n M_{\bar{z}^n}(c_0 + c_1 z)\right\rangle \\
&= \left\langle \sum_{n=1}^{N} \bar{a}_n \bar{z}^n(c_0 + c_1 z), \sum_{n=1}^{N} \bar{a}_n \bar{z}^n(c_0 + c_1 z)\right\rangle \\
&= \left\langle \sum_{n=1}^{N} c_0 \bar{a}_n \bar{z}^n, \sum_{n=1}^{N} c_0 \bar{a}_n \bar{z}^n\right\rangle + \left\langle \sum_{n=1}^{N} c_0 \bar{a}_n \bar{z}^n, \sum_{n=1}^{N} c_1 \bar{a}_n \bar{z}^n z\right\rangle \\
&+ \left\langle \sum_{n=1}^{N} c_1 \bar{a}_n \bar{z}^n z, \sum_{n=1}^{N} c_0 \bar{a}_n \bar{z}^n\right\rangle + \left\langle \sum_{n=1}^{N} c_1 \bar{a}_n \bar{z}^n z, \sum_{n=1}^{N} c_1 \bar{a}_n \bar{z}^n z\right\rangle \\
&= |c_0|^2 \sum_{n=1}^{N} \frac{|a_n|^2}{n+1} + |c_1|^2 \sum_{n=1}^{N} \frac{|a_n|^2}{n+2} + 2Re\left(\bar{c}_0 c_1 \sum_{n=1}^{N-1} a_n \bar{a}_{n+1} \langle \bar{z}^{n+1} z, \bar{z}^n\rangle\right) \\
&= |c_0|^2 \sum_{n=1}^{N} \frac{|a_n|^2}{n+1} + |c_1|^2 \sum_{n=1}^{N} \frac{|a_n|^2}{n+2} + 2Re\left(\bar{c}_0 c_1 \sum_{n=1}^{N-1} \frac{a_n \bar{a}_{n+1}}{n+2}\right)
\end{aligned}$$

And,

$$\begin{aligned}
&\langle T_{\bar{f}}(c_0 + c_1 z), T_{\bar{f}}(c_0 + c_1 z)\rangle \\
&= \langle P(\bar{f}(c_0 + c_1 z)), P(\bar{f}(c_0 + c_1 z))\rangle \\
&= \left\langle P\left(\sum_{n=1}^{N} \bar{a}_n \bar{z}^n(c_0 + c_1 z)\right), P\left(\sum_{n=1}^{N} \bar{a}_n \bar{z}^n(c_0 + c_1 z)\right)\right\rangle \\
&= \left\langle \sum_{n=1}^{N} \bar{a}_n P(\bar{z}^n(c_0 + c_1 z)), \sum_{n=1}^{N} \bar{a}_n P(\bar{z}^n(c_0 + c_1 z))\right\rangle \\
&= \left\langle \sum_{n=1}^{N} c_0 \bar{a}_n P(\bar{z}^n) + \sum_{n=1}^{N} c_1 \bar{a}_n P(\bar{z}^n z), \sum_{n=1}^{N} c_0 \bar{a}_n P(\bar{z}^n) + \sum_{n=1}^{N} c_1 \bar{a}_n P(\bar{z}^n z)\right\rangle \\
&= \frac{1}{4}|c_1|^2 |a_1|^2, \qquad \text{(using 4.2.1)}
\end{aligned}$$

Therefore,

$\langle H_{\bar{f}}^{*}H_{\bar{f}}(c_0+c_1z), c_0+c_1z\rangle = |c_0|^2\sum_{n=1}^{N}\frac{|a_n|^2}{n+1} + |c_1|^2\sum_{n=1}^{N}\frac{|a_n|^2}{n+2} - \frac{1}{4}|c_1|^2|a_1|^2$
$+2Re\left(\bar{c}_0c_1\sum_{n=1}^{N-1}\frac{a_n\bar{a}_{n+1}}{n+2}\right)$.

Similarly,

$\langle H_{\bar{g}}^{*}H_{\bar{g}}(c_0+c_1z), c_0+c_1z\rangle = |c_0|^2\sum_{n=1}^{N}\frac{|a_{-n}|^2}{n+1} + |c_1|^2\sum_{n=1}^{N}\frac{|a_{-n}|^2}{n+2} - \frac{1}{4}|c_1|^2|a_{-1}|^2$
$+2Re\left(\bar{c}_0c_1\sum_{n=1}^{N-1}\frac{a_{-n}\bar{a}_{-(n+1)}}{n+2}\right)$

Therefore,

$$\begin{aligned}&\langle(H_{\bar{f}}^{*}H_{\bar{f}} - H_{\bar{g}}^{*}H_{\bar{g}})(c_0+c_1z), (c_0+c_1z)\rangle\\ =&|c_0|^2\sum_{n=1}^{N}\frac{1}{n+1}(|a_n|^2-|a_{-n}|^2) + |c_1|^2\sum_{n=1}^{N}\frac{1}{n+2}(|a_n|^2-|a_{-n}|^2)\\ &+\frac{1}{4}|c_1|^2(|a_{-1}|^2-|a_1|^2) + 2Re\left(\bar{c}_0c_1\sum_{n=1}^{N-1}\frac{1}{n+2}(a_n\bar{a}_{n+1} - a_{-n}\bar{a}_{-(n+1)})\right)\end{aligned} \tag{4.3.2}$$

If T_φ is hyponormal, then from (4.3.1) and (4.3.2) we get

$$\begin{aligned}&|c_0|^2\sum_{n=1}^{N}\frac{1}{n+1}(|a_n|^2-|a_{-n}|^2) + |c_1|^2\sum_{n=1}^{N}\frac{1}{n+2}(|a_n|^2-|a_{-n}|^2)+\\ &\frac{1}{4}|c_1|^2(|a_{-1}|^2-|a_1|^2) + 2Re\left(\bar{c}_0c_1\sum_{n=1}^{N-1}\frac{1}{n+2}(a_n\bar{a}_{n+1} - a_{-n}\bar{a}_{-(n+1)})\right) \geq 0\\ &\Longrightarrow\\ &|c_0|^2\sum_{n=1}^{N}\frac{1}{n+1}(|a_n|^2-|a_{-n}|^2) + |c_1|^2\sum_{n=1}^{N}\frac{1}{n+2}(|a_n|^2-|a_{-n}|^2)+\\ &\frac{1}{4}|c_1|^2(|a_{-1}|^2-|a_1|^2) + 2|c_0c_1|\left|\sum_{n=1}^{N-1}\frac{1}{n+2}(a_n\bar{a}_{n+1} - a_{-n}\bar{a}_{-(n+1)})\right| \geq 0\end{aligned}$$

Two cases arise:

Case 1: If $c_1 = 0$, then the hyponormality of T_φ gives that

$$\sum_{n=1}^{N}\frac{1}{n+1}(|a_n|^2-|a_{-n}|^2) \geq 0 \tag{4.3.3}$$

Case 2: If $c_1 \neq 0$, then we get

$$\left|\frac{c_0}{c_1}\right|^2 \sum_{n=1}^{N} \frac{1}{n+1}(|a_n|^2 - |a_{-n}|^2) + 2\left|\frac{c_0}{c_1}\right| \left|\sum_{n=1}^{N-1} \frac{1}{n+2}(a_n\bar{a}_{n+1} - a_{-n}\bar{a}_{-(n+1)})\right|$$
$$+ \left(\sum_{n=1}^{N} \frac{1}{n+2}(|a_n|^2 - |a_{-n}|^2) + \frac{1}{4}(|a_{-1}|^2 - |a_1|^2)\right) \geq 0 \tag{4.3.4}$$

If a quadratic polynomial $f(x) = ax^2 + bx + c$ (for a, b, c real and $a \geq 0$) takes a non negative value for all x, then it cannot have two distinct real roots. Hence for such a case we must necessarily have $4ac \geq b^2$. Thus from 4.3.3 and 4.3.4 it follows that

$$\left(\sum_{n=1}^{N} \frac{1}{n+1}(|a_n|^2 - |a_{-n}|^2)\right)\left(\sum_{n=1}^{N}(\frac{1}{n+2}(|a_n|^2 - |a_{-n}|^2) + \frac{1}{4}(|a_{-1}|^2 - |a_1|^2)\right)$$
$$\geq \left|\sum_{n=1}^{N-1} \frac{1}{n+2}(a_n\bar{a}_{n+1} - a_{-n}\bar{a}_{-(n+1)})\right|^2 \tag{4.3.5}$$

□

4.3.2 Sufficient conditions for hyponormality

Theorem 4.3.2. *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_1 z + a_2 z^2$ and $g(z) = a_{-1}z + a_{-2}z^2$. Also let $m_1 := |a_1|^2 - |a_{-1}|^2$ and $m_2 := 4(|a_{-2}|^2 - |a_2|^2)$. Then for $|a_{-2}| \geq |a_2|$, T_φ is hyponormal, if*

(i) $m_2 \leq m_1 \leq \sqrt{2}m_2$,

(ii) $4|a_1\bar{a}_2 - a_{-1}\bar{a}_{-2}| \leq m_1 - m_2$.

Proof. Let $K_\imath := \{k_\imath \in A^2(\mathbb{D}) : k_\imath(z) = \sum_{n=0}^{\infty} c_{2n+\imath} z^{2n+\imath}\}$, where i=0,1. By Theorem 4.1.1, T_φ is hyponormal if and only if

$$\left\langle (H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}}) \sum_{i=0}^{1} k_\imath(z), \sum_{\imath=0}^{1} k_\imath(z) \right\rangle \geq 0$$

i.e. if,

$(|a_1|^2 - |a_{-1}|^2) \left\langle H_{\bar{z}} \sum_{\imath=0}^{1} k_\imath(z), H_{\bar{z}} \sum_{\imath=0}^{1} k_\imath(z) \right\rangle + (|a_2|^2 - |a_{-2}|^2) \times$

$\left\langle H_{\bar{z}^2} \sum_{i=0}^1 k_i(z), H_{\bar{z}^2} \sum_{i=0}^1 k_i(z) \right\rangle + 2Re\Big[(a_1\bar{a}_2 - a_{-1}\bar{a}_{-2}) \times$

$\left\langle H_{\bar{z}^2} \sum_{i=0}^1 k_i(z), H_{\bar{z}} \sum_{i=0}^1 k_i(z) \right\rangle\Big] \geq 0$

i.e. if,

$m_1 \sum_{i=0}^1 \|H_{\bar{z}} k_i(z)\|^2 - \frac{m_2}{4} \sum_{i=0}^1 \|H_{\bar{z}^2} k_i(z)\|^2$

$+ 2Re\Big[(a_1\bar{a}_2 - a_{-1}\bar{a}_{-2}) \sum_{i,j=0}^1 \langle H_{\bar{z}^2} k_i(z), H_{\bar{z}} k_j(z)\rangle\Big] \geq 0$, (using lemma 4.2.3)

i.e. if,

$$m_1 \sum_{n=0}^{\infty} \frac{|c_n|^2}{(n+2)(n+1)^2} - \frac{m_2}{4}\left(\frac{|c_0|^2}{3} + \sum_{n=1}^{\infty} \frac{4|c_n|^2}{(n+3)(n+1)^2}\right) +$$
$$2Re\left[(a_1\bar{a}_2 - a_{-1}\bar{a}_{-2}) \sum_{n=0}^{\infty} \frac{2\bar{c}_n c_{n+1}}{(n+1)(n+2)(n+3)}\right] \geq 0,$$

(using Lemmas 4.2.2 and 4.2.4)

i.e. if,

$$4|a_1\bar{a}_2 - a_{-1}\bar{a}_{-2}| \sum_{n=0}^{\infty} \frac{|c_n||c_{n+1}|}{(n+1)(n+2)(n+3)} \leq$$
$$\sum_{n=0}^{\infty} \left[\frac{m_1}{(n+2)(n+1)^2} - \frac{m_2}{(n+3)(n+1)^2}\right] |c_n|^2 + \frac{m_2|c_0|^2}{4}$$

i.e. if,

$$4|a_1\bar{a}_2 - a_{-1}\bar{a}_{-2}| \sum_{n=0}^{\infty} \frac{|c_n||c_{n+1}|}{(n+1)(n+2)(n+3)} \leq$$
$$\sum_{n=0}^{\infty} \left[\frac{m_1}{(n+2)(n+1)^2} - \frac{m_2}{(n+3)(n+1)^2}\right] |c_n|^2$$

i.e. if,

$$16|a_1\bar{a}_2 - a_{-1}\bar{a}_{-2}|^2 \frac{|c_n|^2|c_{n+1}|^2}{(n+1)^2(n+2)^2(n+3)^2} \leq \left[\frac{m_1}{(n+2)(n+1)^2} - \frac{m_2}{(n+3)(n+1)^2}\right] \times$$
$$\left[\frac{m_1}{(n+3)(n+2)^2} - \frac{m_2}{(n+4)(n+2)^2}\right] |c_n|^2 |c_{n+1}|^2 \,\forall\, n \geq 0,$$

(using Lemma 4.2.5)

i.e. if,

$$16|a_1\bar{a}_2 - a_{-1}\bar{a}_{-2}|^2 \leq \left[\left(\frac{n+3}{n+2}\right)m_1 - m_2\right]\left[m_1 - \left(\frac{n+3}{n+4}\right)m_2\right] \;\forall n \geq 0 \quad (4.3.6)$$

For $x \in \mathbb{R}^+$, let

$$\xi(x) := \left[\left(\frac{x+3}{x+2}\right)m_1 - m_2\right]\left[m_1 - \left(\frac{x+3}{x+4}\right)m_2\right].$$

Then

$$\begin{aligned}
\xi'(x) &= \frac{m_2^2}{(x+4)^2} + \frac{2(x+3)m_1m_2}{(x+4)^2(x+2)^2} - \frac{m_1^2}{(x+2)^2} \\
&\leq \frac{m_2^2}{(x+4)^2} + \frac{2(x+3)m_1^2}{(x+4)^2(x+2)^2} - \frac{m_2^2}{(x+2)^2}, \quad \text{since } m_2 < m_1 \\
&= \frac{2(x+3)}{(x+4)^2(x+2)^2}\left[m_1^2 - 2m_2^2\right] \\
&< 0 \text{ if } 0 < m_1 < \sqrt{2}m_2 \qquad (4.3.7)
\end{aligned}$$

Thus, if $m_2 < m_1 < \sqrt{2}m_2$ then $\xi(x)$ is a decreasing sequence of positive values and

$$lim_{x\to\infty}\xi(x) = (m_1 - m_2)^2.$$

Hence if $16|a_1\bar{a}_2 - a_{-1}\bar{a}_{-2}|^2 \leq (m_1 - m_2)^2$ then Equation 4.3.6 holds for all $n \geq 0$, and so T_φ is hyponormal. □

Thus by the above theorem we can easily check that if $\varphi(z) = 10\sqrt{10}\bar{z}^2 + \sqrt{5}i\bar{z} + z + 10\sqrt{2}z^2$ then T_φ is hyponormal on the Bergman space $A^2(\mathbb{D})$. However, the conditions of the above theorem are not necessary for hyponormality as is seen by taking $\varphi(z) = \bar{z}^2 + 11\bar{z} + 15z + z^2$. In this case T_φ is hyponormal by Theorem 4.3.3, to be proved next.

Theorem 4.3.3. *Let $f(z) = \sum_{n=2}^{N} a_n z^n$ $(N \geq 2)$, $g(z) = az + f(z)$ and $h(z) = bz + f(z)$. If $\varphi(z) = \overline{g(z)} + h(z)$, and $|b| > |a|$ then T_φ is hyponormal when*

$$\frac{|b|^2 - |a|^2}{|b-a|} \geq 2(N-1)NA,$$

where $A := max\{|a_i| : i = 2, 3, ..., N\}$

Proof. Let $K_i := \{k_i \in A^2(\mathbb{D}) : k_i(z) = \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i}\}$, where i=0,1...,N-1. By the Theorem 4.1.1, T_φ is hyponormal if and only if

$$\left\langle (H_{\bar{h}}^* H_{\bar{h}} - H_{\bar{g}}^* H_{\bar{g}}) \sum_{i=0}^{N-1} k_i(z), \sum_{i=0}^{N-1} k_i(z) \right\rangle \geq 0 \tag{4.3.8}$$

Now,

$$\left\langle H_{\bar{h}} \sum_{i=0}^{N-1} k_i(z), H_{\bar{h}} \sum_{i=0}^{N-1} k_i(z) \right\rangle$$
$$= \left\langle (\bar{b} H_{\bar{z}} + H_{\bar{f}}) \sum_{i=0}^{N-1} k_i(z), (\bar{b} H_{\bar{z}} + H_{\bar{f}}) \sum_{i=0}^{N-1} k_i(z) \right\rangle$$
$$= |b|^2 \left\langle \sum_{i=0}^{N-1} H_{\bar{z}} k_i(z), \sum_{i=0}^{N-1} H_{\bar{z}} k_i(z) \right\rangle + 2Re\left(b \left\langle \sum_{i=0}^{N-1} H_{\bar{f}} k_i(z), \sum_{i=0}^{N-1} H_{\bar{z}} k_i(z) \right\rangle \right) +$$
$$\left\langle \sum_{i=0}^{N-1} H_{\bar{f}} k_i(z), \sum_{i=0}^{N-1} H_{\bar{f}} k_i(z) \right\rangle$$

Similarly,

$$\left\langle H_{\bar{g}} \sum_{i=0}^{N-1} k_i(z), H_{\bar{g}} \sum_{i=0}^{N-1} k_i(z) \right\rangle$$
$$= \left\langle (\bar{a} H_{\bar{z}} + H_{\bar{f}}) \sum_{i=0}^{N-1} k_i(z), (\bar{a} H_{\bar{z}} + H_{\bar{f}}) \sum_{i=0}^{N-1} k_i(z) \right\rangle$$
$$= |a|^2 \left\langle \sum_{i=0}^{N-1} H_{\bar{z}} k_i(z), \sum_{i=0}^{N-1} H_{\bar{z}} k_i(z) \right\rangle + 2Re\left(a \left\langle \sum_{i=0}^{N-1} H_{\bar{f}} k_i(z), \sum_{i=0}^{N-1} H_{\bar{z}} k_i(z) \right\rangle \right) +$$
$$\left\langle \sum_{i=0}^{N-1} H_{\bar{f}} k_i(z), \sum_{i=0}^{N-1} H_{\bar{f}} k_i(z) \right\rangle$$

Thus from the inequality (4.3.8), T_φ is hyponormal if and only if

$$(|b|^2 - |a|^2) \left\langle \sum_{i=0}^{N-1} H_{\bar{z}} k_i(z), \sum_{i=0}^{N-1} H_{\bar{z}} k_i(z) \right\rangle +$$
$$2Re\left((b-a) \left\langle \sum_{i=0}^{N-1} H_{\bar{f}} k_i(z), \sum_{i=0}^{N-1} H_{\bar{z}} k_i(z) \right\rangle \right) \geq 0$$

i.e. if

$$(|b|^2 - |a|^2)\sum_{n=0}^{\infty}\frac{1}{(n+2)(n+1)^2}|c_n|^2 +$$
$$2Re\left((b-a)\sum_{m=2}^{N}\bar{a}_m\sum_{n=0}^{\infty}\frac{m\bar{c}_n c_{n+m-1}}{(n+m+1)(n+m)(n+1)}\right) \geq 0$$

i.e. if for each $m = 2, 3, ..., N-1$ we have,

$$(|b|^2 - |a|^2)\sum_{n=0}^{\infty}\frac{|c_n|^2}{(n+2)(n+1)^2} \geq$$
$$2(N-1)|b-a||a_m|\sum_{n=0}^{\infty}\frac{m}{(n+m+1)(n+m)(n+1)}|c_n||c_{n+m-1}|$$

That is, if

$$\sum_{n=0}^{\infty}\frac{|c_n|^2}{(n+2)(n+1)^2} \geq \sum_{n=0}^{\infty}\frac{\alpha_m}{(n+m+1)(n+m)(n+1)}|c_n||c_{n+m-1}|,$$

where

$$\alpha_m = \frac{2(N-1)|b-a||a_m|m}{|b|^2 - |a|^2}$$

i.e. if for all $n \geq 0$ and for all $m = 2, 3, ..., N-1$

$$\frac{2\alpha_m|c_n||c_{n+m-1}|}{(n+m+1)(n+m)(n+1)} \leq \frac{|c_n|^2}{(n+2)(n+1)^2} + \frac{|c_{n+m-1}|^2}{(n+m+1)(n+m)^2}$$

i.e.if for each $m = 2, 3, ..., N-1$ and for each $n \geq 0$

$$\alpha_m^2 \leq \frac{n+m+1}{n+2}$$

As $m > 1$, so $\frac{n+m+1}{n+2} > 1$ for all $n \geq 0$. Thus if $A := max\{|a_i| : i = 2, 3, ..., N\}$ then T_φ is hyponormal if

$$\frac{|b|^2 - |a|^2}{|b-a|} \geq 2A(N-1)N$$

□

Remark 4.3.1. From the Theorem 1.4 of [12], it follows that T_φ on $H^2(\mathbb{T})$ is hyponormal if and only if $b = a$. But in the case of Bergman space T_φ can be hyponormal even if $b \neq a$ whenever $\frac{|b|^2 - |a|^2}{|b-a|}$ is sufficiently large.

Example 4.3.1. *Let* $\varphi(z) = 2\bar{z}^3 + \bar{z}^2 + 11\bar{z} + 15z + z^2 + 2z^3$*. So by the above Theorem 4.3.3,* T_φ *on the* A^2 *space is hyponormal, but it is not hyponormal on the Hardy space* $H^2(\mathbb{T})$.

Remark 4.3.2. In the Theorem 4.3.2, we have assumed that $|a_{-2}| \geq |a_2|$ and given a set of conditions sufficient to make T_φ hyponormal. However, no such sufficiency conditions have yet been established for the case $|a_{-2}| < |a_2|$.

Chapter 5

Hyponormality in Weighted Bergman Space

This Chapter is a continuation of the Chapter 4. Here we investigate the hyponormality of Toeplitz operator T_φ on $A^2_\alpha(\mathbb{D})$ space with the symbol $\varphi(z) = \bar{a}_{-N}\bar{z}^N + \bar{a}_{-m}\bar{z}^m + a_m z^m + a_N z^N$ $(0 < m < N)$ with some assumptions about the Fourier coefficients of φ.

5.1 Introduction

Recall that hyponormality of Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$ was characterized by Cowen [4], and subsequently by Nakazi and Takahashi [44]. The solution is based on a dilation theorem of Sarason [46]. As no such dilation theorem is available in the Bergman space, the question concerning the characterization of hyponormal Toeplitz operators on the Bergman space is still open. For $\varphi = f + \bar{g}$ with f, g bounded analytic, we can refer the result due to H. Sadraoui (Theorem 4.1.1):

Theorem 5.1.1. [45] *Let f, g be bounded and analytic in $L^2(\mathbb{D}, dA)$. Then the followings are equivalent:*

(i) $T_{f+\bar{g}}$ *is hyponormal;*

(ii) $H_{\bar{g}}^* H_{\bar{g}} \leq H_{\bar{f}}^* H_{\bar{f}}$*;*

(iii) $H_{\bar{g}} = CH_{\bar{f}}$*, where* C *is of norm less than or equal to one.*

The above result also holds on weighted Bergman space $A_\alpha^2(\mathbb{D})$ with $-1 < \alpha < \infty$. Also, for a certain trigonometric polynomial φ, I. S. Hwang [23] proved the following result:

Theorem 5.1.2. [23] *Let* $\varphi(z) = \overline{g(z)} + f(z)$*, where* $f(z) = a_m z^m + a_N z^N$*,* $g(z) = a_{-m} z^m + a_{-N} z^N$ $(0 < m < N)$*. If* $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$*, then* T_φ *on* $A^2(\mathbb{D})$ *is hyponormal*

$$\Longleftrightarrow \begin{cases} \frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) & if \ \ |a_{-N}| \leq |a_N| \\ N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2) & if \ \ |a_N| \leq |a_{-N}| \end{cases}$$

For the weighted Bergman space $A_\alpha^2(\mathbb{D})$, a similar result is the following:

Theorem 5.1.3. [34] *Let* $\varphi(z) = \overline{g(z)} + f(z)$*, where* $f(z) = a_1 z + a_2 z^2$ *and* $g(z) = a_{-1} z + a_{-2} z^2$*. If* $a_1 \bar{a}_2 = a_{-1} \bar{a}_{-2}$ *and* $\alpha \geq 0$*, then* T_φ *on* $A_\alpha^2(\mathbb{D})$ *is hyponormal*

$$\Longleftrightarrow \begin{cases} \frac{1}{\alpha+3}(|a_2|^2 - |a_{-2}|^2) \geq \frac{1}{2}(|a_{-1}|^2 - |a_1|^2) & if \ \ |a_{-2}| \leq |a_2| \\ 4(|a_{-2}|^2 - |a_2|^2) \leq |a_1|^2 - |a_{-1}|^2 & if \ \ |a_2| \leq |a_{-2}| \end{cases}$$

In this chapter we continue this line of investigation for T_φ on $A_\alpha^2(\mathbb{D})$ and establish hyponormality conditions for the following cases:

1. Where $\varphi = \bar{g} + f$ and $f(z) = a_1 z + a_3 z^3$ and $g(z) = a_{-1} z + a_{-3} z^3$ with $a_1 \bar{a}_3 = a_{-1} \bar{a}_{-3}$ and $\alpha \geq 0$.
2. Where $\varphi = \bar{g} + f$ and $f(z) = a_2 z^2 + a_3 z^3$ and $g(z) = a_{-2} z^2 + a_{-3} z^3$ with $a_2 \bar{a}_3 = a_{-2} \bar{a}_{-3}$ and $\alpha \geq 0$.

5.2 Few Significant Lemmas

The weighted Bergman space $A_\alpha^2(\mathbb{D})$ and related terms have already been introduced in section 1.2.5. It is also well known that Toeplitz operators on the Bergman space are related to Hankel operators by the following algebraic relation:

$$T_{fg} = T_f T_g + H_{\bar{f}}^* H_g \tag{5.2.1}$$

$$T_{gf} = T_g T_f + H_{\bar{g}}^* H_f \tag{5.2.2}$$

for every $f, g \in L^\infty(\mathbb{D})$. If $f, g \in H^\infty(\mathbb{D})$, then

$$T_{f+\bar{g}}^* T_{f+\bar{g}} - T_{f+\bar{g}} T_{f+\bar{g}}^* = H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}} \tag{5.2.3}$$

For studying the hyponormality of Toeplitz operators, we need to pay much attention to the Hankel operators $H_{\bar{f}}$ and $H_{\bar{g}}$.

Let $\gamma_k = \|z^k\|_\alpha$. Then a simple calculation shows that

$$\begin{aligned}\gamma_k^2 &= (\alpha+1)\int_0^1 |z|^{2k}(1-|z|^2)^\alpha dA(z)\\ &= (\alpha+1)\int_0^1 t^k(1-t)^\alpha dt\\ &= (\alpha+1)B(k+1,\alpha+1)\\ &= \frac{k!}{\prod_{j=1}^k(\alpha+1+j)} \le 1\end{aligned}$$

where $B(p,q)$ is the Beta function. It then follows that for $l \ge 0$, $\rho(k) = \frac{\gamma_k^2}{\gamma_{k+1}^2}$ is strictly decreasing and

$$\lim_{k\to\infty} \frac{\gamma_k^2}{\gamma_{k+1}^2} = 1 \tag{5.2.4}$$

The following lemma shows a few basic properties of Hankel operators.

Lemma 5.2.1. ([1], [42]) *Fix $m \ge 1$. Then for $\alpha \ge -1$,*

(i) $H_{\bar{z}^m}(z^k)(\xi) = \begin{cases} \bar{\xi}^m \xi^k & if\ \ 0 \le k < m \\ \bar{\xi}^m \xi^k - \frac{\gamma_k^2}{\gamma_{k-m}^2} & if\ \ m \le k; \end{cases}$

(ii) *the functions* $\{H_{\bar{z}^m}(z^k)\}_{k=0}^\infty$ *are orthogonal in* $L^2(\mathbb{D}, dA_\alpha)$*;*

(iii) $H_{\bar{z}^m}^* H_{\bar{z}^m}(z^k)(\xi) = \omega_{mk}^2 \xi^k$ $k = 0, 1, 2, \dots,$ *where*

$$\omega_{mk}^2 = \begin{cases} \frac{\gamma_{k+m}^2}{\gamma_k^2} & if\ \ 0 \le k < m \\ \frac{\gamma_{k+m}^2}{\gamma_k^2} - \frac{\gamma_k^2}{\gamma_{k-m}^2} & if\ \ m \le k; \end{cases}$$

(iv) $\|H_{\bar{z}^m}(z^k)\|_\alpha = \omega_{mk}\gamma_k$.

If P_α denotes the projection of $L^2(\mathbb{D}, dA_\alpha)$ onto $A^2(\mathbb{D})$, then we have the following useful result:

Lemma 5.2.2. [34] *For any s, t nonnegative integers,*

$$P_\alpha(\bar{z}^t z^s) = \begin{cases} \frac{\Gamma(s+1)\Gamma(s-t+\alpha+2)}{\Gamma(s+\alpha+2)\Gamma(s-t+1)} z^{s-t} & if \ \ s \geq t \\ 0 & if \ \ s < t \end{cases}$$

We now proceed to establish a few more lemmas which will help in proving the hyponormality results in the next section. In these lemmas, $k_i(z)$ for $i = 0, 1, \ldots, N-1$ is defined as follows:

$$k_i(z) := \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i}$$

Lemma 5.2.3. *For $0 < m < N$ and $i = 0, 1, \ldots, N-1$, we have*

(i) $\|\bar{z}^m k_i(z)\|_\alpha^2 = \sum_{n=0}^\infty \frac{(Nn+m+i)!\Gamma(\alpha+2)}{\Gamma(Nn+m+i+\alpha+2)} |c_{Nn+i}|^2$

(ii) $\|P_\alpha(\bar{z}^m k_i(z))\|_\alpha^2 = \begin{cases} \sum_{k=1}^\infty \frac{(Nk+i)!^2\Gamma(Nk+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(Nk+i-m+1)} |c_{Nk+i}|^2 & if \ \ m > i \\ \sum_{k=0}^\infty \frac{(Nk+i)!^2\Gamma(Nk+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(Nk+i-m+1)} |c_{Nk+i}|^2 & if \ \ m \leq i \end{cases}$

Proof. (i) $\|\bar{z}^m k_i(z)\|_\alpha^2 = \|\bar{z}^m \sum_{n=0}^\infty c_{Nn+i} z^{Nn+i}\|_\alpha^2$

$$\begin{aligned}
&= \left\langle \bar{z}^m \sum_{n=0}^\infty c_{Nn+i} z^{Nn+i}, \bar{z}^m \sum_{k=0}^\infty c_{Nk+i} z^{Nk+i} \right\rangle \\
&= \left\langle \sum_{n=0}^\infty c_{Nn+i} z^{Nn+i+m}, \sum_{k=0}^\infty c_{Nk+i} z^{Nk+i+m} \right\rangle \\
&= \sum_n \sum_k c_{Nn+i} \bar{c}_{Nk+i} \langle z^{Nn+i+m}, z^{Nk+i+m} \rangle \\
&= \sum_n \sum_k c_{Nn+i} \bar{c}_{Nk+i} \gamma_{Nn+i+m} \gamma_{Nk+i+m} \langle e_{Nn+i+m}(z), e_{Nk+i+m}(z) \rangle \\
&= \sum_n |c_{Nn+i}|^2 \gamma_{Nn+i+m}^2 \\
&= \sum_n \frac{(Nn+i+m)!\Gamma(\alpha+2)}{\Gamma(Nn+i+m+\alpha+2)} |c_{Nn+i}|^2
\end{aligned}$$

(ii) $\|P_\alpha(\bar{z}^m k_i(z))\|_\alpha^2 = \| \sum_{n=0}^\infty c_{Nn+i} P_\alpha(\bar{z}^m z^{Nn+i}) \|_\alpha^2$

$$\begin{aligned}
&= \begin{cases} \| \sum_{n=0}^\infty c_{Nn+i} \frac{\Gamma(Nn+i+1)\Gamma(Nn+i-m+\alpha+2)}{\Gamma(Nn+i+\alpha+2)\Gamma(Nn+i-m+1)} z^{Nn+i-m} \|_\alpha^2 & if \ \ m \leq i \\ \| \sum_{n=1}^\infty c_{Nn+i} \frac{\Gamma(Nn+i+1)\Gamma(Nn+i-m+\alpha+2)}{\Gamma(Nn+i+\alpha+2)\Gamma(Nn+i-m+1)} z^{Nn+i-m} \|_\alpha^2 & if \ \ m > i \end{cases} \\
&= \begin{cases} \| \sum_{n=0}^\infty \frac{(Nn+i)!^2\Gamma(Nn+i-m+\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-m+1)^2} \gamma_{Nn+i-m}^2 |c_{Nn+i}|^2 & if \ \ m \leq i \\ \| \sum_{n=1}^\infty \frac{(Nn+i)!^2\Gamma(Nn+i-m+\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-m+1)^2} \gamma_{Nn+i-m}^2 |c_{Nn+i}|^2 & if \ \ m > i \end{cases} \\
&= \begin{cases} \| \sum_{n=0}^\infty \frac{(Nn+i)!^2\Gamma(Nn+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-m+1)} |c_{Nn+i}|^2 & if \ \ m \leq i \\ \| \sum_{n=1}^\infty \frac{(Nn+i)!^2\Gamma(Nn+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-m+1)} |c_{Nn+i}|^2 & if \ \ m > i \end{cases}
\end{aligned}$$

□

Lemma 5.2.4. *For $m \geq 1$, $\langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^m} k_j(z)\rangle_\alpha = 0$ for $i \neq j$.*

Proof.

$$\langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^m} k_j(z)\rangle_\alpha = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_{Nn+i}\bar{c}_{Nk+j} \left\langle H_{\bar{z}^m} z^{Nn+i}, H_{\bar{z}^m} z^{Nk+j}\right\rangle_\alpha$$

Now,

$$Nn + i = Nk + j \iff N(n-k) = j - i.$$

As $0 \leq i, j \leq N-1$, so $0 \leq j - i \leq N-1$.

Thus, $N(n-k) = j - i \Longrightarrow n - k = 0$ i.e. $n = k$. This gives $i = j$, a contradiction.

Hence, $Nn + i \neq Nk + j$ for all n, k and so by Lemma 5.2.1,

$$\left\langle H_{\bar{z}^m} z^{Nn+i}, H_{\bar{z}^m} z^{Nk+j}\right\rangle_\alpha = 0.$$

Thus,

$$\langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^m} k_j(z)\rangle_\alpha = 0$$

if $i \neq j$. □

Lemma 5.2.5. *Let $f(z) = a_m z^m + a_N z^N$, $g(z) = a_{-m} z^m + a_{-N} z^N$ with $0 < m < N$). Let $\alpha > -1$ and $a_m \bar{a}_N = a_{-m}\bar{a}_{-N}$. Then for $i \neq j$ we have*

$$\langle H_{\bar{f}} k_i(z), H_{\bar{f}} k_j(z)\rangle_\alpha = \langle H_{\bar{g}} k_i(z), H_{\bar{g}} k_j(z)\rangle_\alpha = 0$$

Proof. $\langle H_{\bar{f}} k_i(z), H_{\bar{f}} k_j(z)\rangle_\alpha$

$= \langle \bar{a}_m H_{\bar{z}^m} k_i(z) + \bar{a}_N H_{\bar{z}^N} k_i(z), \bar{a}_m H_{\bar{z}^m} k_j(z) + \bar{a}_N H_{\bar{z}^N} k_j(z)\rangle_\alpha$

$= a_m \bar{a}_N \langle H_{\bar{z}^N} k_i(z), H_{\bar{z}^m} k_j(z)\rangle_\alpha + \bar{a}_m a_N \langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^N} k_j(z)\rangle_\alpha$

Lemma 5.2.4)

and similarly,

$$\langle H_{\bar{g}} k_i(z), H_{\bar{g}} k_j(z)\rangle_\alpha = a_{-m}\bar{a}_{-N} \langle H_{\bar{z}^N} k_i(z), H_{\bar{z}^m} k_j(z)\rangle_\alpha + \bar{a}_{-m} a_{-N} \langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^N} k_j(z)\rangle_\alpha.$$

Since $a_m \bar{a}_N = a_{-m}\bar{a}_{-N}$, hence we have the result. □

Lemma 5.2.6. *Let* $\varphi(z) = \overline{g(z)} + f(z)$*, where* $f(z) = a_m z^m + a_N z^N$*,* $g(z) = a_{-m} z^m + a_{-N} z^N$ $(0 < m < N)$*. If* $\alpha > -1$ *and* $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$ *then* T_φ *on* $A^2_\alpha(\mathbb{D})$ *is hyponormal if and only if*

$$(|a_m|^2 - |a_{-m}|^2)\left[\sum_{t=0}^{m-1} \frac{(m+t)!\Gamma(\alpha+2)}{\Gamma(m+t+\alpha+2)}|c_t|^2 + \sum_{t=m}^{\infty}\left\{\frac{(m+t)!\Gamma(\alpha+2)}{\Gamma(m+t+\alpha+2)} + \frac{(t!)^2\Gamma(t-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(t+\alpha+2)^2\Gamma(t-m+1)}|c_t|^2\right\}\right] + (|a_N|^2 - |a_{-N}|^2)\left[\sum_{t=0}^{N-1} \frac{(N+t)!\Gamma(\alpha+2)}{\Gamma(N+t+\alpha+2)}|c_t|^2 + \sum_{t=N}^{\infty}\left\{\frac{(N+t)!\Gamma(\alpha+2)}{\Gamma(N+t+\alpha+2)} + \frac{(t!)^2\Gamma(t-N+\alpha+2)\Gamma(\alpha+2)}{\Gamma(t+\alpha+2)^2\Gamma(t-N+1)}|c_t|^2\right\}\right] \geq 0$$

Proof. For $\imath = 0, 1, ..., N-1$, let $K_\imath := \{k_\imath \in A^2_\alpha : k_\imath(z) = \sum_{n=0}^{\infty} c_{Nn+\imath} z^{Nn+\imath}\}$. By Theorem 5.1.1, T_φ is hyponormal if and only if

$$\langle (H^*_{\bar{f}} H_{\bar{f}} - H^*_{\bar{g}} H_{\bar{g}}) \sum_{\imath=0}^{N-1} k_\imath(z), \sum_{\imath=0}^{N-1} k_\imath(z) \rangle_\alpha \geq 0$$

. That is, if and only if,

$$\left\langle H_{\bar{f}}\left(\sum_{\imath=0}^{N-1} k_\imath(z)\right), H_{\bar{f}}\left(\sum_{\imath=0}^{N-1} k_\imath(z)\right)\right\rangle_\alpha \geq \left\langle H_{\bar{g}}\left(\sum_{\imath=0}^{N-1} k_\imath(z)\right), H_{\bar{g}}\left(\sum_{\imath=0}^{N-1} k_\imath(z)\right)\right\rangle_\alpha$$

That is, if and only if

$$\sum_{\imath=0}^{N-1} \|H_{\bar{f}}(k_\imath(z))\|^2_\alpha \geq \sum_{\imath=0}^{N-1} \|H_{\bar{g}}(k_\imath(z))\|^2_\alpha,$$

(which we get by using Lemma 5.2.5)

That is, if and only if

$$\sum_{\imath=0}^{N-1} \left[\|M_{\bar{f}}(k_\imath(z))\|^2_\alpha - \|T_{\bar{f}}(k_\imath(z))\|^2_\alpha\right] \geq \sum_{\imath=0}^{N-1} \left[\|M_{\bar{g}}(k_\imath(z))\|^2_\alpha - \|T_{\bar{g}}(k_\imath(z))\|^2_\alpha\right] \quad (5.2.5)$$

We have,

$\|M_{\bar{f}}(k_\imath(z))\|^2_\alpha = \|\bar{f}(k_\imath(z))\|^2_\alpha$

$= \left\langle \left(\bar{a}_m \bar{z}^m + \bar{a}_N \bar{z}^N\right) k_\imath(z), \left(\bar{a}_m \bar{z}^m + \bar{a}_N \bar{z}^N\right) k_\imath(z)\right\rangle_\alpha$

$= |a_m|^2 \|\bar{z}^m k_\imath(z)\|^2_\alpha + |a_N|^2 \|\bar{z}^N k_\imath(z)\|^2_\alpha + \bar{a}_m a_N \left\langle \bar{z}^m k_\imath(z), \bar{z}^N k_\imath(z)\right\rangle_\alpha +$
$a_m \bar{a}_N \left\langle \bar{z}^N k_\imath(z), \bar{z}^m k_\imath(z)\right\rangle_\alpha$

Similarly,

$\|M_{\bar{g}}(k_\imath(z))\|_\alpha^2 = |a_{-m}|^2\|\bar{z}^m k_\imath(z)\|_\alpha^2 + |a_{-N}|^2\|\bar{z}^N k_\imath(z)\|_\alpha^2 + \bar{a}_{-m}a_{-N}\left\langle \bar{z}^m k_\imath(z), \bar{z}^N k_\imath(z)\right\rangle_\alpha$
$+a_{-m}\bar{a}_{-N}\left\langle \bar{z}^N k_\imath(z), \bar{z}^m k_\imath(z)\right\rangle_\alpha$

Also,

$\|T_{\bar{f}}(k_\imath(z))\|_\alpha^2$
$= \|\left(\bar{a}_m T_{\bar{z}^m} + \bar{a}_N T_{\bar{z}^N}\right) k_\imath(z)\|_\alpha^2$
$= |a_m|^2\|T_{\bar{z}^m} k_\imath(z)\|_\alpha^2 + |a_N|^2\|T_{\bar{z}^N} k_\imath(z)\|_\alpha^2 + \bar{a}_m a_N \langle T_{\bar{z}^m} k_\imath(z), T_{\bar{z}^N} k_\imath(z)\rangle_\alpha +$
$a_m\bar{a}_N \langle T_{\bar{z}^N} k_\imath(z), T_{\bar{z}^m} k_\imath(z)\rangle_\alpha$

and similarly,

$\|T_{\bar{g}}(k_\imath(z))\|_\alpha^2 = |a_{-m}|^2\|T_{\bar{z}^m} k_\imath(z)\|_\alpha^2 + |a_{-N}|^2\|T_{\bar{z}^N} k_\imath(z)\|_\alpha^2 +$
$\bar{a}_{-m}a_{-N}\langle T_{\bar{z}^m} k_\imath(z), T_{\bar{z}^N} k_\imath(z)\rangle_\alpha + a_{-m}\bar{a}_{-N}\langle T_{\bar{z}^N} k_\imath(z), T_{\bar{z}^m} k_\imath(z)\rangle_\alpha$

Using these in (5.2.5), we have, T_φ is hyponormal if and only if

$\sum_{\imath=0}^{N-1}\left[(|a_m|^2 - |a_{-m}|^2)\|\bar{z}^m k_\imath(z)\|_\alpha^2 + (|a_N|^2 - |a_{-N}|^2)\|\bar{z}^N k_\imath(z)\|_\alpha^2\right] \geq$
$\sum_{\imath=0}^{N-1}\left[(|a_m|^2 - |a_{-m}|^2)\|T_{\bar{z}^m} k_\imath(z)\|_\alpha^2 + (|a_N|^2 - |a_{-N}|^2)\|T_{\bar{z}^N} k_\imath(z)\|_\alpha^2\right]$

That is, if and only if

$(|a_m|^2 - |a_{-m}|^2)\sum_{\imath=0}^{N-1}\left[\|\bar{z}^m k_\imath(z)\|_\alpha^2 - \|P_\alpha\left(\bar{z}^m k_\imath(z)\right)\|_\alpha^2\right] +$
$(|a_N|^2 - |a_{-N}|^2)\sum_{\imath=0}^{N-1}\left[\|\bar{z}^N k_\imath(z)\|_\alpha^2 - \|P_\alpha\left(\bar{z}^N k_\imath(z)\right)\|_\alpha^2\right] \geq 0$

That is, if and only if

$(|a_m|^2 - |a_{-m}|^2)\Bigg[\sum_{\imath=0}^{N-1}\sum_{k=0}^{\infty}\frac{(Nk+\imath+m)!\Gamma(\alpha+2)}{\Gamma(Nk+\imath+m+\alpha+2)}|c_{Nk+\imath}|^2 -$
$\sum_{\imath=0}^{m-1}\sum_{k=1}^{\infty}\frac{(Nk+\imath)!^2\Gamma(Nk+\imath-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+\imath+\alpha+2)^2\Gamma(Nk+\imath-m+1)}|c_{Nk+\imath}|^2 +$
$\sum_{\imath=m}^{N-1}\sum_{k=0}^{\infty}\frac{(Nk+\imath)!^2\Gamma(Nk+\imath-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+\imath+\alpha+2)^2\Gamma(Nk+\imath-m+1)}|c_{Nk+\imath}|^2\Bigg] + (|a_N|^2 - |a_{-N}|^2)\times$
$\sum_{\imath=0}^{N-1}\left[\sum_{k=0}^{\infty}\frac{(Nk+\imath+N)!\Gamma(\alpha+2)}{\Gamma(Nk+\imath+N+\alpha+2)}|c_{Nk+\imath}|^2 - \sum_{k=1}^{\infty}\frac{(Nk+\imath)!^2\Gamma(Nk+\imath-N+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+\imath+\alpha+2)^2\Gamma(Nk+\imath-N+1)}|c_{Nk+\imath}|^2\right]$
≥ 0

That is, if and only if

$(|a_m|^2 - |a_{-m}|^2)\sum_{\imath=0}^{m-1}\Bigg[\frac{(m+\imath)!\Gamma(\alpha+2)}{\Gamma(m+\imath+\alpha+2)}|c_\imath|^2 + \sum_{k=1}^{\infty}\Bigg\{\frac{(Nk+\imath+m)!\Gamma(\alpha+2)}{\Gamma(Nk+\imath+m+\alpha+2)} -$
$\frac{(Nk+\imath)!^2\Gamma(Nk+\imath-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+\imath+\alpha+2)^2\Gamma(Nk+\imath-m+1)}\Bigg\}|c_{Nk+\imath}|^2\Bigg] + (|a_m|^2 - |a_{-m}|^2)\sum_{\imath=m}^{N-1}\sum_{k=0}^{\infty}\Bigg[\frac{(Nk+\imath+m)!\Gamma(\alpha+2)}{\Gamma(Nk+\imath+m+\alpha+2)}$

$$-\frac{(Nk+i)!^2\Gamma(Nk+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(Nk+i-m+1)}\Bigg]|c_{Nk+i}|^2 + (|a_N|^2 - |a_{-N}|^2)\sum_{i=0}^{N-1}\Bigg[\frac{(N+i)!\Gamma(\alpha+2)}{\Gamma(N+i+\alpha+2)}|c_i|^2$$
$$+\sum_{k=1}^{\infty}\left\{\frac{(N(k+1)+i)!\Gamma(\alpha+2)}{\Gamma(N(k+1)+i+\alpha+2)} - \frac{(Nk+i)!^2\Gamma(N(k-1)+i+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(N(k-1)+i+1)}\right\}|c_{Nk+i}|^2\Bigg] \geq 0$$

That is, if and only if

$$(|a_m|^2 - |a_{-m}|^2)\Bigg[\sum_{t=0}^{m-1}\frac{(t+m)!\Gamma(\alpha+2)}{\Gamma(t+m+\alpha+2)}|c_t|^2 + \sum_{t=m}^{\infty}\Bigg\{\frac{(t+m)!\Gamma(\alpha+2)}{\Gamma(t+m+\alpha+2)}$$
$$-\frac{t!^2\Gamma(t-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(t+\alpha+2)^2\Gamma(t-m+1)}\Bigg\}|c_t|^2\Bigg] + (|a_N|^2 - |a_{-N}|^2)\Bigg[\sum_{t=0}^{N-1}\frac{(t+N)!\Gamma(\alpha+2)}{\Gamma(t+N+\alpha+2)}|c_t|^2$$
$$+\sum_{t=N}^{\infty}\left\{\frac{(t+N)!\Gamma(\alpha+2)}{\Gamma(t+N+\alpha+2)} - \frac{t!^2\Gamma(t-N+\alpha+2)\Gamma(\alpha+2)}{\Gamma(t+\alpha+2)^2\Gamma(t-N+1)}\right\}|c_t|^2\Bigg] \geq 0$$ □

5.3 Hyponormality Conditions

Theorem 5.3.1. *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_1 z + a_3 z^3$, $g(z) = a_{-1}z + a_{-3}z^3$. If $\alpha \geq 0$ and $a_1\bar{a}_3 = a_{-1}\bar{a}_{-3}$ then T_φ on $A^2_\alpha(\mathbb{D})$ is hyponormal*

$$\Longleftrightarrow \begin{cases} 9(|a_{-3}|^2 - |a_3|^2) \leq (|a_1|^2 - |a_{-1}|^2) & if \ |a_3| \leq |a_{-3}| \\ 6(|a_3|^2 - |a_{-3}|^2) \geq (\alpha+3)(\alpha+4)(|a_{-1}|^2 - |a_1|^2) & if \ |a_3| \geq |a_{-3}| \end{cases}$$

Proof. By Lemma 5.2.6, T_φ on $A^2_\alpha(\mathbb{D})$ is hyponormal if and only if

$$(|a_1|^2 - |a_{-1}|^2)\left[\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+3)}|c_0|^2 + \sum_{t=1}^{\infty}\left\{\frac{(t+1)!\Gamma(\alpha+2)}{\Gamma(t+\alpha+3)} - \frac{t!^2\Gamma(t+\alpha+1)\Gamma(\alpha+2)}{\Gamma(t+\alpha+2)\Gamma t}\right\}|c_t|^2\right]$$
$$+(|a_3|^2 - |a_{-3}|^2)\left[\sum_{t=0}^{2}\frac{(t+3)!\Gamma(\alpha+2)}{\Gamma(t+\alpha+5)}|c_t|^2 + \sum_{t=3}^{\infty}\left\{\frac{(t+3)!\Gamma(\alpha+2)}{\Gamma(t+\alpha+5)} - \frac{t!^2\Gamma(t+\alpha-1)\Gamma(\alpha+2)}{\Gamma(t+\alpha+2)^2\Gamma(t-2)}\right\}|c_t|^2\right]$$
$$\geq 0$$

That is, if and only if

$$(|a_1|^2 - |a_{-1}|^2)\left[\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+3)}|c_0|^2 + \sum_{t=1}^{\infty}\left\{\frac{\Gamma(t+2)\Gamma(\alpha+2)}{\Gamma(t+\alpha+3)} - \frac{t\Gamma(t+1)\Gamma(t+\alpha+1)\Gamma(\alpha+2)}{\Gamma(t+\alpha+2)^2}\right\}|c_t|^2\right]$$
$$+(|a_3|^2 - |a_{-3}|^2)\Bigg[\sum_{t=0}^{2}\frac{\Gamma(t+4)\Gamma(\alpha+2)}{\Gamma(t+\alpha+5)}|c_t|^2 + \sum_{t=3}^{\infty}\Bigg\{\frac{\Gamma(t+4)\Gamma(\alpha+2)}{\Gamma(t+\alpha+5)}$$
$$-\frac{t(t-1)(t-2)\Gamma(t+1)\Gamma(t+\alpha-1)\Gamma(\alpha+2)}{\Gamma(t+\alpha+2)^2}\Bigg\}|c_t|^2\Bigg] \geq 0$$

That is, if and only if

$$(|a_1|^2 - |a_{-1}|^2)\sum_{t=0}^{\infty}\left\{\frac{\Gamma(t+2)}{\Gamma(t+\alpha+3)} - \frac{t\Gamma(t+1)\Gamma(t+\alpha+1)}{\Gamma(t+\alpha+2)^2}\right\}|c_t|^2 + (|a_3|^2 - |a_{-3}|^2)\times$$
$$\sum_{t=0}^{\infty}\left\{\frac{\Gamma(t+4)}{\Gamma(t+\alpha+5)} - \frac{t(t-1)(t-2)\Gamma(t+1)\Gamma(t+\alpha-1)}{\Gamma(t+\alpha+2)^2}\right\}|c_t|^2 \geq 0$$

For $n \in \mathbb{N}$, define ξ_α by

$$\begin{aligned}
\xi_\alpha(n) &:= \frac{\frac{\Gamma(n+2)}{\Gamma(n+\alpha+3)} - \frac{n\Gamma(n+1)\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+2)^2}}{\frac{\Gamma(n+4)}{\Gamma(n+\alpha+5)} - \frac{n(n-1)(n-2)\Gamma(n+1)\Gamma(n+\alpha-1)}{\Gamma(n+\alpha+2)^2}} \\
&= \frac{\frac{(n+1)\Gamma(n+1)}{(n+\alpha+2)\Gamma(n+\alpha+2)} - \frac{n\Gamma(n+\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+2)^2}}{\frac{(n+3)(n+2)(n+1)\Gamma(n+1)}{(n+\alpha+4)(n+\alpha+3)(n+\alpha+2)\Gamma(n+\alpha+2)} - \frac{n(n-1)(n-2)\Gamma(n+1)\Gamma(n+\alpha-1)}{\Gamma(n+\alpha+2)^2}} \\
&= \frac{\frac{(n+1)}{(n+\alpha+2)} - \frac{n}{(n+\alpha+1)}}{\frac{(n+3)(n+2)(n+1)}{(n+\alpha+4)(n+\alpha+3)(n+\alpha+2)} - \frac{n(n-1)(n-2)}{(n+\alpha+1)(n+\alpha)(n+\alpha-1)}} \\
&= \frac{\left\{\frac{\alpha+1}{(n+1)(n+\alpha+1)-n(n+\alpha+2)}\right\}(n+\alpha-1)(n+\alpha)(n+\alpha+3)(n+\alpha+4)}{(n+1)(n+2)(n+3)(n+\alpha-1)(n+\alpha)(n+\alpha+1)-n(n-1)(n-2)(n+\alpha+2)(n+\alpha+3)(n+\alpha+4)} \\
&= \frac{(\alpha+1)(n+\alpha-1)(n+\alpha)(n+\alpha+3)(n+\alpha+4)}{3(\alpha+1)(-2\alpha+2\alpha^2-18n-3\alpha n+3\alpha^2 n+3n^2+15\alpha n^2+12n^3+6\alpha n^3+3n^4)}
\end{aligned}$$

So, $\lim_{n\to\infty}\xi_\alpha(n) = \frac{1}{9}$ and $\xi'_\alpha = -\frac{N}{D}$ where,

$D = 3(-2\alpha + 2\alpha^2 - 18n - 3\alpha n + 3\alpha^2 n + 3n^2 + 15\alpha n^2 + 12n^3 + 6\alpha n^3 + 3n^4)^2$ and

$N = 192\alpha - 10\alpha^2 - 143\alpha^3 - 49\alpha^4 + 7\alpha^5 + 3\alpha^6 - 52\alpha n - 278\alpha^2 n + 66\alpha^3 n + 192\alpha^4 n +$
$66\alpha^5 n + 6\alpha^6 n + 54n^2 - 207\alpha n^2 + 267\alpha^2 n^2 + 558\alpha^3 n^2 + 240\alpha^4 n^2 + 30\alpha^5 n^2 -$
$72n^3 + 140\alpha n^3 + 592\alpha^2 n^3 + 360\alpha^3 n^3 + 60\alpha^4 n^3 - 12n^4 + 243\alpha n^4 + 255\alpha^2 n^4 +$
$60\alpha^3 n^4 + 24n^5 + 78\alpha n^5 + 30\alpha^2 n^5 + 6n^6 + 6\alpha n^6.$

$= 10\alpha^2(n^5 - 1) + 143\alpha^3(n^2 - 1) + 49\alpha^4(n^3 - 1) + 52\alpha n(n^4 - 1) + 278\alpha^2 n(n^2 -$
$1) + 207\alpha n^2(n^2 - 1) + 72n^3(\alpha^3 - 1) + 6n^2(n^4 + 4n^3 - 2n^2 - 12n + 9) + 192\alpha +$
$7\alpha^5 + 3\alpha^6 + 66\alpha^3 n + 192\alpha^4 n + 66\alpha^5 n + 6\alpha^6 n + 267\alpha^2 n^2 + 415\alpha^3 n^2 + 240\alpha^4 n^2 +$
$30\alpha^5 n^2 + 140\alpha n^3 + 314\alpha^2 n^3 + 360\alpha^3 n^3 + 11\alpha^4 n^3 + 36\alpha n^4 + 255\alpha^2 n^4 + 60\alpha^3 n^4 +$
$26\alpha n^5 + 20\alpha^2 n^5 + 6\alpha n^6,$

which is always positive for $n \geq 1$ and for $\alpha \geq 0$.

So $\xi_\alpha(n)$ is strictly decreasing function for all $\alpha \geq 0$.

Case 1: Let $|a_{-3}| \leq |a_3|$.

Then $\xi_\alpha(0) \geq \xi_\alpha(1)$.

Since $\xi_\alpha(0) = \frac{(\alpha+3)(\alpha+4)}{6}$ and $\xi_\alpha(1) = \frac{(\alpha+1)(\alpha+4)(\alpha+5)}{24(\alpha+2)}$ and as $4(\alpha+2)(\alpha+3) \geq (\alpha+1)(\alpha+5)$, so we have $\xi_\alpha(0) \geq \xi_\alpha(1)$.

Hence T_φ is hyponormal if and only if

$$\frac{|a_3|^2 - |a_{-3}|^2}{|a_1|^2 - |a_{-1}|^2} \geq \frac{(\alpha+3)(\alpha+4)}{6}.$$

That is, if and only if

$$6(|a_3|^2 - |a_{-3}|^2) \geq (\alpha+3)(\alpha+4)(|a_1|^2 - |a_{-1}|^2).$$

Case 2: Let $|a_3| \leq |a_{-3}|$.

Then since $\xi_\alpha(n) \geq \frac{1}{9}$ for all n, so T_φ is hyponormal if and only if

$$\frac{|a_3|^2 - |a_{-3}|^2}{|a_1|^2 - |a_{-1}|^2} \leq \frac{1}{9}.$$

That is, if and only if

$$9(|a_{-3}|^2 - |a_3|^2) \leq |a_1|^2 - |a_{-1}|^2.$$

□

Theorem 5.3.2. *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_2z^2 + a_3z^3$, $g(z) = a_{-2}z^2 + a_{-3}z^3$. If $\alpha \geq 0$ and $a_2\bar{a}_3 = a_{-2}\bar{a}_{-3}$ then T_φ on $A^2_\alpha(\mathbb{D})$ is hyponormal*

$$\Longleftrightarrow \begin{cases} 9(|a_{-3}|^2 - |a_3|^2) \leq 4(|a_2|^2 - |a_{-2}|^2) & if \ |a_3| \leq |a_{-3}| \\ 3(|a_3|^2 - |a_{-3}|^2) \geq (\alpha+4)(|a_{-2}|^2 - |a_2|^2) & if \ |a_3| \geq |a_{-3}| \end{cases}$$

Proof. T_φ on $A^2_\alpha(\mathbb{D})$ is hyponormal if and only if

$$(|a_2|^2 - |a_{-2}|^2)\left[\sum_{t=0}^{1} \frac{(t+2)!\Gamma(\alpha+2)}{\Gamma(t+\alpha+4)}|c_t|^2 + \sum_{t=2}^{\infty}\left\{\frac{(t+2)!\Gamma(\alpha+2)}{\Gamma(t+\alpha+4)} - \frac{t!^2\Gamma(t+\alpha)\Gamma(\alpha+2)}{\Gamma(t+\alpha+2)^2\Gamma(t-1)}\right\}|c_t|^2\right]$$
$$+(|a_3|^2 - |a_{-3}|^2)\left[\sum_{t=0}^{2} \frac{(t+3)!\Gamma(\alpha+2)}{\Gamma(t+\alpha+5)}|c_t|^2 + \sum_{t=3}^{\infty}\left\{\frac{(t+3)!\Gamma(\alpha+2)}{\Gamma(t+\alpha+5)} - \frac{t!^2\Gamma(t+\alpha-1)\Gamma(\alpha+2)}{\Gamma(t+\alpha+2)^2\Gamma(t-2)}\right\}|c_t|^2\right]$$
$$\geq 0$$

That is, if and only if

$$(|a_2|^2 - |a_{-2}|^2)\left[\sum_{t=0}^{1} \frac{(t+2)!}{\Gamma(t+\alpha+4)}|c_t|^2 + \sum_{t=2}^{\infty}\left\{\frac{(t+2)!}{\Gamma(t+\alpha+4)} - \frac{t(t-1)\Gamma(t+1)\Gamma(t+\alpha)}{\Gamma(t+\alpha+2)^2}\right\}|c_t|^2\right]$$
$$+(|a_3|^2 - |a_{-3}|^2)\left[\sum_{t=0}^{2} \frac{(t+3)!}{\Gamma(t+\alpha+5)}|c_t|^2 + \sum_{t=3}^{\infty}\left\{\frac{(t+3)!}{\Gamma(t+\alpha+5)} - \frac{t(t-1)(t-2)\Gamma(t+1)\Gamma(t+\alpha-1)}{\Gamma(t+\alpha+2)^2}\right\}|c_t|^2\right]$$
$$\geq 0$$

That is, if and only if

$$(|a_2|^2 - |a_{-2}|^2)\sum_{t=0}^{\infty}\left\{\frac{(t+2)(t+1)}{(t+\alpha+3)(t+\alpha+2)} - \frac{t(t-1)}{(t+\alpha+1)(t+\alpha)}\right\}\frac{\Gamma(t+1)}{\Gamma(t+\alpha+2)}|c_t|^2$$

$$+ (|a_3|^2 - |a_{-3}|^2) \sum_{t=0}^{\infty} \left\{ \frac{(t+3)(t+2)(t+1)}{(t+\alpha+4)(t+\alpha+3)(t+\alpha+2)} - \frac{t(t-1)(t-2)}{(t+\alpha+1)(t+\alpha)(t+\alpha-1)} \right\} \frac{\Gamma(t+1)}{\Gamma(t+\alpha+2)} |c_t|^2 \geq 0$$

For $n \in \mathbb{N}$, define ξ_α by

$$\xi_\alpha(n) := \frac{\frac{(n+2)(n+1)}{(n+\alpha+3)(n+\alpha+2)} - \frac{n(n-1)}{\Gamma(n+\alpha+2)(n+\alpha)}}{\frac{(n+3)(n+2)(n+1)}{(n+\alpha+4)(n+\alpha+3)(n+\alpha+2)} - \frac{n(n-1)(n-2)}{(n+\alpha+1)(n+\alpha)(n+\alpha-1)}}$$
$$= \frac{\{(n+1)(n+2)(n+\alpha)(n+\alpha+1) - n(n-1)(n+\alpha+2)(n+\alpha+3)\}(n+\alpha-1)(n+\alpha+4)}{(n+1)(n+2)(n+3)(n+\alpha-1)(n+\alpha)(n+\alpha+1) - n(n-1)(n-2)(n+\alpha+2)(n+\alpha+3)(n+\alpha+4)}$$
$$= \frac{2(n+\alpha-1)(n+\alpha+4)(2n^2+2\alpha n+4n+\alpha)}{3(-2\alpha+2\alpha^2-18n-3\alpha n+3\alpha^2 n+3n^2+15\alpha n^2+3\alpha^2 n^2+12n^3+6\alpha n^3+3n^4)}$$

So $\lim_{t\to\infty} \xi_\alpha(n) = \frac{4}{9}$. Also, $\xi'_\alpha(n) = -\frac{2N}{3D}$ where,

$$D = (-2\alpha + 2\alpha^2 - 18n - 3\alpha n + 3\alpha^2 n + 3n^2 + 15\alpha n^2 + 3\alpha^2 n^2 + 12n^3 + 6\alpha n^3 + 3n^4)^2,$$

and

$$N = 40\alpha + 4\alpha^2 - 29\alpha^3 - 14\alpha^4 - \alpha^5 - 8\alpha n - 34\alpha^2 n + 12\alpha^3 n + 24\alpha^4 n + 6\alpha^5 n + 24n^2 + 87\alpha n^2 + 264\alpha^2 n^2 + 216\alpha^3 n^2 + 66\alpha^4 n^2 + 6\alpha^5 n^2 - 24n^3 + 220\alpha n^3 + 368\alpha^2 n^3 + 168\alpha^3 n^3 + 24\alpha^4 n^3 - 18n^4 + 189\alpha n^4 + 168\alpha^2 n^4 + 36\alpha^3 n^4 + 12n^5 + 66\alpha n^5 + 24\alpha^2 n^5 + 6n^6 + 6\alpha n^6.$$

$$= 29\alpha^3(n^4 - 1) + 14\alpha^4(n - 1) + \alpha^5(n - 1) + 8\alpha n(n^2 - 1) + 34\alpha^2 n(n^3 - 1) + 6n^2\{n(n+1)(n^2+n-4) + 4\} + 40\alpha + 4\alpha^2 + 12\alpha^3 n + 10\alpha^4 n + 5\alpha^5 n + 87\alpha n^2 + 264\alpha^2 n^2 + 216\alpha^3 n^2 + 66\alpha^4 n^2 + 6\alpha^5 n^2 + 212\alpha n^3 + 368\alpha^2 n^3 + 168\alpha^3 n^3 + 24\alpha^4 n^3 + 189\alpha n^4 + 134\alpha^2 n^4 + 7\alpha^3 n^4 + 12n^5 + 66\alpha n^5 + 24\alpha^2 n^5 + 6\alpha n^6,$$

which is always positive for $n \geq 1$ and for $\alpha \geq 0$. Hence $\xi_\alpha(n)$ is a strictly decreasing sequence for all $\alpha \geq 0$.

Case 1: Let $|a_{-3}| \leq |a_3|$.

We have $\xi_\alpha(0) = \frac{2(\alpha-1)(\alpha+4)\alpha}{3(-2\alpha+2\alpha^2)} = \frac{\alpha+4}{3}$, $\xi_\alpha(1) = \frac{\alpha+5}{4}$ and since $4(\alpha+4) > 3(\alpha+5)$ for $\alpha \geq 1$, so $\xi_\alpha(0) \geq \xi_\alpha(1)$.

Hence T_φ is hyponormal if and only if

$$\frac{|a_3|^2 - |a_{-3}|^2}{|a_2|^2 - |a_{-2}|^2} \geq \frac{\alpha+4}{3}.$$

That is, if and only if

$$3(|a_3|^2 - |a_{-3}|^2) \geq (\alpha+4)(|a_2|^2 - |a_{-2}|^2).$$

Case 2: Let $|a_3| \leq |a_{-3}|$.

Since $\xi_\alpha(n) \geq \frac{4}{9}$ for all n, so T_φ is hyponormal if and only if

$$\frac{|a_{-3}|^2 - |a_3|^2}{|a_2|^2 - |a_{-2}|^2} \leq \frac{4}{9}.$$

That is, if and only if

$$9(|a_{-3}|^2 - |a_3|^2) \leq 4(|a_2|^2 - |a_{-2}|^2).$$

□

5.4 Conclusion

Looking at the Theorems 5.1.3, 5.3.1 and 5.3.2 it appears that we should be having a general result of the following kind:

Theorem 5.4.1. (Conjecture): *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_m z^m + a_N z^N$, $g(z) = a_{-m} z^m + a_{-N} z^N$ $(0 < m < N)$. If $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$ and α is sufficiently large, then T_φ on $A^2_\alpha(\mathbb{D})$ is hyponormal*

$$\iff \begin{cases} \frac{\prod_{j=0}^{N-1}(\alpha+2+j)}{\prod_{j=0}^{N-(m+1)}(N-j)}(|a_{-m}|^2 - |a_m|^2) \leq (|a_N|^2 - |a_{-N}|^2) & if\ \ |a_{-N}| \leq |a_N| \\ N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2) & if\ \ |a_N| \leq |a_{-N}| \end{cases}$$

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Appendix

Analysis of Example 3.4.1:

T_φ is hyponormal if and only if

(i) $|\lambda| \le 1.5$,

(ii) $|\lambda| \le 1.22$ and

(iii) $12|\lambda|^4 - 6|\lambda|^3 - 72|\lambda|^2 + 81 \ge 0$.

From (i) and (ii) we need to consider only those λ for which $|\lambda| \le 1.22$, and also satisfy the condition (iii). We first determine those λ which satisfy the equation $12|\lambda|^4 - 6|\lambda|^3 - 72|\lambda|^2 + 81 = 0$.

For convenience we replace λ by x.

Step 1: Determine x such that $12|x|^4 - 6|x|^3 - 72|x|^2 + 81 = 0$

Reduce [12Abs[x]^4-6Abs[x]^3-72Abs[x]^2+81= =0,x]

$$\left(-2.4777909793205577` \le \mathrm{Re}[x] < -1.1297552606843122` \;\&\& \right.$$

$$\left(\mathrm{Im}[x] == -1.` \sqrt{\begin{matrix}\mathrm{Root}[-81 + 108\,\mathrm{Re}[x]^2 - 40\,\mathrm{Re}[x]^4 + 4\,\mathrm{Re}[x]^6 + 108\,\#1 - 80\,\mathrm{Re}[x]^2\,\#1 + \\ 12\,\mathrm{Re}[x]^4\,\#1 - 40\,\#1^2 + 12\,\mathrm{Re}[x]^2\,\#1^2 + 4\,\#1^3\;\&,\;3]\end{matrix}} \;||\right.$$

$$\left.\left.\mathrm{Im}[x] == \sqrt{\begin{matrix}\mathrm{Root}[-81 + 108\,\mathrm{Re}[x]^2 - 40\,\mathrm{Re}[x]^4 + 4\,\mathrm{Re}[x]^6 + 108\,\#1 - 80\,\mathrm{Re}[x]^2\,\#1 + \\ 12\,\mathrm{Re}[x]^4\,\#1 - 40\,\#1^2 + 12\,\mathrm{Re}[x]^2\,\#1^2 + 4\,\#1^3\;\&,\;3]\end{matrix}}\right)\right) \;||$$

$$\left(-1.1297552606843122` \le Re[x] \le 1.1297552606843122` \;\&\&\; \left(Im[x] == -1.` \sqrt{Root[-81 + 108\,Re[x]^2 - 40\,Re[x]^4 + 4\,Re[x]^6 + 108\,\#1 - 80\,Re[x]^2\,\#1 + 12\,Re[x]^4\,\#1 - 40\,\#1^2 + 12\,Re[x]^2\,\#1^2 + 4\,\#1^3\,\&,\ 3]} \;||\; Im[x] == -1.` \sqrt{Root[-81 + 108\,Re[x]^2 - 40\,Re[x]^4 + 4\,Re[x]^6 + 108\,\#1 - 80\,Re[x]^2\,\#1 + 12\,Re[x]^4\,\#1 - 40\,\#1^2 + 12\,Re[x]^2\,\#1^2 + 4\,\#1^3\,\&,\ 1]} \;||\; Im[x] == \sqrt{Root[-81 + 108\,Re[x]^2 - 40\,Re[x]^4 + 4\,Re[x]^6 + 108\,\#1 - 80\,Re[x]^2\,\#1 + 12\,Re[x]^4\,\#1 - 40\,\#1^2 + 12\,Re[x]^2\,\#1^2 + 4\,\#1^3\,\&,\ 1]} \;||\; Im[x] == \sqrt{Root[-81 + 108\,Re[x]^2 - 40\,Re[x]^4 + 4\,Re[x]^6 + 108\,\#1 - 80\,Re[x]^2\,\#1 + 12\,Re[x]^4\,\#1 - 40\,\#1^2 + 12\,Re[x]^2\,\#1^2 + 4\,\#1^3\,\&,\ 3]}\right)\right) \;||$$

$$\left(1.1297552606843122` < Re[x] \le 2.4777909793205577` \;\&\&\; \left(Im[x] == -1.` \sqrt{Root[-81 + 108\,Re[x]^2 - 40\,Re[x]^4 + 4\,Re[x]^6 + 108\,\#1 - 80\,Re[x]^2\,\#1 + 12\,Re[x]^4\,\#1 - 40\,\#1^2 + 12\,Re[x]^2\,\#1^2 + 4\,\#1^3\,\&,\ 3]} \;||\; Im[x] == \sqrt{Root[-81 + 108\,Re[x]^2 - 40\,Re[x]^4 + 4\,Re[x]^6 + 108\,\#1 - 80\,Re[x]^2\,\#1 + 12\,Re[x]^4\,\#1 - 40\,\#1^2 + 12\,Re[x]^2\,\#1^2 + 4\,\#1^3\,\&,\ 3]}\right)\right)$$

Step 2 :

The graphical representation of the solutions of

$$12\,(\,|\,x\,|\,)^4 - 6\,(\,|\,x\,|\,)^3 - 72\,(\,|\,x\,|\,)^2 + 81 = 0$$

is as follows (To draw the graph we write y = Re[x]) :

```
Plot[Evaluate[{-1.` √Root[-81 + 108 y^2 - 40 y^4 + 4 y^6 + 108 #1 - 80 y^2 #1 +
     12 y^4 #1 - 40 #1^2 + 12 y^2 #1^2 + 4 #1^3 &, 3],
   √Root[-81 + 108 y^2 - 40 y^4 + 4 y^6 + 108 #1 - 80 y^2 #1 + 12 y^4 #1 - 40 #1^2 +
     12 y^2 #1^2 + 4 #1^3 &, 3],
   -1.` √(Root[-81 + 108 y^2 - 40 y^4 + 4 y^6 + 108 #1 - 80 y^2 #1 + 12 y^4 #1 -
       40 #1^2 + 12 y^2 #1^2 + 4 #1^3 &, 3]),
   √(Root[-81 + 108 y^2 - 40 y^4 + 4 y^6 + 108 #1 - 80 y^2 #1 + 12 y^4 #1 - 40 #1^2 +
       12 y^2 #1^2 + 4 #1^3 &, 3]),
   -1.` √(Root[-81 + 108 y^2 - 40 y^4 + 4 y^6 + 108 #1 - 80 y^2 #1 + 12 y^4 #1 -
       40 #1^2 + 12 y^2 #1^2 + 4 #1^3 &, 1]),
   √(Root[-81 + 108 y^2 - 40 y^4 + 4 y^6 + 108 #1 - 80 y^2 #1 + 12 y^4 #1 - 40 #1^2 +
       12 y^2 #1^2 + 4 #1^3 &, 1]),
   -1.` √(Root[-81 + 108 y^2 - 40 y^4 + 4 y^6 + 108 #1 - 80 y^2 #1 + 12 y^4 #1 -
       40 #1^2 + 12 y^2 #1^2 + 4 #1^3 &, 3]),
   √(Root[-81 + 108 y^2 - 40 y^4 + 4 y^6 + 108 #1 - 80 y^2 #1 + 12 y^4 #1 - 40 #1^2 +
       12 y^2 #1^2 + 4 #1^3 &, 3])},
  {y, -2.4777909793205577`, 2.4777909793205577`},
  AxesLabel → {"Real Axis ", "Imaginary Axis"}]]
```

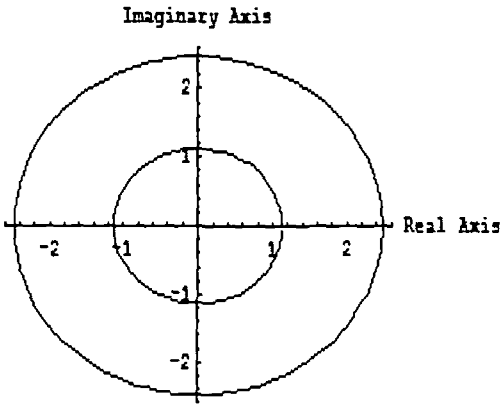

Figure 3.4.1 (a)

Step 3: We show that the condition (iii) is satisfied by all points x inside the smaller circle. Consider any point x such that y=Re[x] =0.5.

Im[x] = √Root[-81 + 108 y² - 40 y⁴ + 4 y⁶ + 108 #1 - 80 y² #1 + 12 y⁴ #1 - 40 #1² + 12 y² #1² + 4 #1³ &, 1] /. y→0.5

=1.01309

Then x=0.5+Im[x]i is a point inside the inner circle for -1.01309<Im[x] <1.01309.

Im[x] = √Root[-81 + 108 y² - 40 y⁴ + 4 y⁶ + 108 #1 - 80 y² #1 + 12 y⁴ #1 - 40 #1² + 12 y² #1² + 4 #1³ &, 3] /. y→0.5

=2.42682

Then x=0.5+Im[x]i is a point inside the outer circle for -2.42682<Im[x] <2.42682 [From the Figure 3.4.1 (a)]. To draw the graph we write Im[x] = z.

```
Plot [{12Abs [0.5+z I] ^4-6Abs [0.5+z I] ^3-72Abs [0.5+z I] ^2+81}, {z,

-2.4268185216868456`, 2.4268185216868456`}, AxesLabel→{"Im(x)","f(x)"}, AxesOrigin→{0,0}]
```

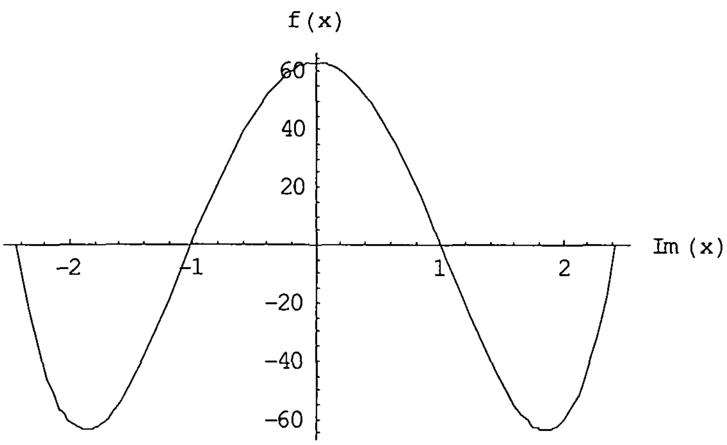

Figure 3.4.1 (b)

Where $f(x) = 12|x|^4 - 6|x|^3 - 72|x|^2 + 81$

Thus, if Re[x] = 0.5 then f(x) > 0 if and only if x lies inside the smaller circle in Figure 3.4.1 (a). Similarly, it can be shown that if -1.13 < Re[x] < 1.13, then f(x) > 0 if and only if x lies inside the inner circle in Figure 3.4.1 (a).

Step 4: To show that condition (iii) is not satisfied for points outside the inner circle in Figure 3.4.1 (a). Here we note that because of condition (ii) we need not consider points with Re[x] > 1.22 and Re[x] < -1.22.

Let y=Re[x] = -1.22. Then,

$$Im[x] = \sqrt{Root[-81+108y^2-40y^4+4y^6+108\#1-80y^2\#1+12y^4\#1-40\#1^2+12y^2\#1^2+4\#1^3 \&, 3]} \;/.\; y \to -1.22$$

=2.15663

Then x= -1.22+Im[x]i is a point inside the outer circle for -2.15663< Im[x] <2.15663

```
Plot [{12Abs[-1.22+z I ]^4-6Abs[-1.22+z I]^3-72Abs[-1.22+z I]^2+81},{z,-2.15662888258558`,
2.15662888258558`},AxesLabel→{"Im(x)","f(x)"}, AxesOrigin→{0,0}]
```

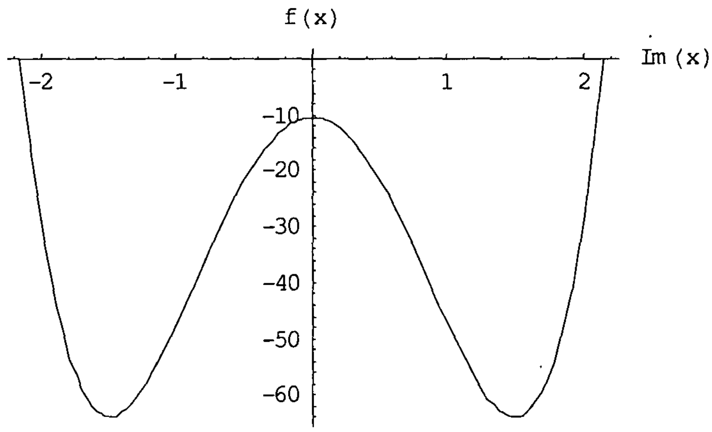

Figure 3.4.1 (c)

Where $f(x) = 12|x|^4 - 6|x|^3 - 72|x|^2 + 81$.

Thus f(x) <0 if x = -1.22+Im[x]i for -2.15663 < Im[x] < 2.15663.

Similarly, considering y=Re[x] =1.13 we see that x=1.13+Im[x]i is a point inside the outer circle for -2.202512< Im[x] <2.20512 and f(x) <0 as seen from the following figure 3.4.1 (d).

$$Im[x] = \sqrt{Root[-81+108y^2-40y^4+4y^6+108\#1-80y^2\#1+12y^4\#1-40\#1^2+12y^2\#1^2+4\#1^3 \&, 3]} \;/.\; y \to 1.13$$

=2.20512

```
Plot [{12Abs [1.13+z I]^4-6Abs[1.13+z I]^3-72Abs[1.13+z I]^2+81},{z,-2.205118622025202`,
  2.205118622025202`}, AxesLabel→ {"Im(x)","f(x)"}, AxesOrigin→ {0, 0}]
```

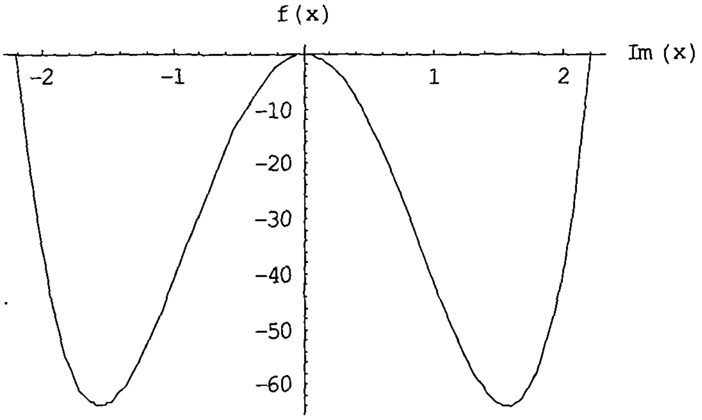

Figure 3.4.1 (d)

Thus, conditions (i), (ii) and (iii) are satisfied or equivalently T_φ is hyponormal iff $|\lambda| \leq 1.13$ (correct upto 2 decimal places).

Analysis of Example 3.4.2:

T_φ is hyponormal if and only if the following conditions are satisfied:

(i) $|\lambda| \leq \frac{5}{3}$,

(ii) $25 - 9|\lambda|^2 - 5|1+6\lambda| \geq 0$ and

(iii) $\left(25 - 9|\lambda|^2\right)^2 - 25|1+6\lambda|^2 - 5\left|3\bar{\lambda}(1+6\lambda)^2 + 2(1+6\lambda)\left(25-9|\lambda|^2\right)\right| \geq 0$.

We need to consider only those λ for which $|\lambda| \leq \frac{5}{3}$, and also satisfy the conditions (ii) and (iii).

We first determine those λ which satisfy the condition $25 - 9|\lambda|^2 - 5|1+6\lambda| \geq 0$. For convenience we replace λ by x.

Step 1: Here we show that $f(x) = 25 - 9|x|^2 - 5|1+6x| = 0$ if and only if x lies on the ellipse given by figure 3.4.2 (a).

Reduce [25-9Abs[x] ^2-5Abs [1+6x] = =0, x]

```
Plot[{-1.` √(-0.9622504486493763` √(67.` + 4.` y) + 0.3333333333333333` (25.` - 3.` y²)) ,
  1.` √(-0.9622504486493763` √(67.` + 4.` y) + 0.3333333333333333` (25.` - 3.` y²)) },
  {y, -0.8053994956985543`, 0.5694013108331231`}, AxesLabel→ {"Real Axis ",
   "Imaginary Axis"}, AxesOrigin→ {0, 0}]
```

For convenience we write y=Re[x].

Plot [{25-9Abs [0.4+z I] ^2-5Abs [1+6(0.4+z I)]}, {z,-0.451091, 0.451091},

AxesLabel→ {"Im[x] =z ","f(x)"}, AxesOrigin→ {0, 0}]

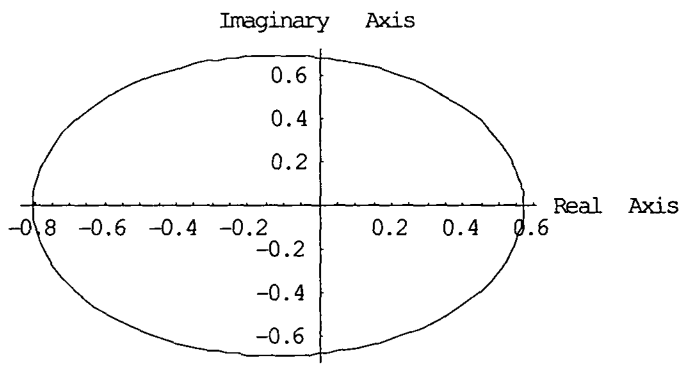

Figure 3.4.2(a)

Step 2: Here we show that f(x)>0 if and only if x lies inside the ellipse given by Figure 3.4.2(a). For instance, let us consider the point whose real part, Re[x] =0.4.

To draw the graph we write Im[x] =z

Plot [{25-9Abs [0.4+z I] ^2-5Abs [1+6(0.4+z I)]}, {z,-0.451091, 1.67},

AxesLabel→ {" I m[x] =z ","f(x)"}, AxesOrigin→ {0, 0}]

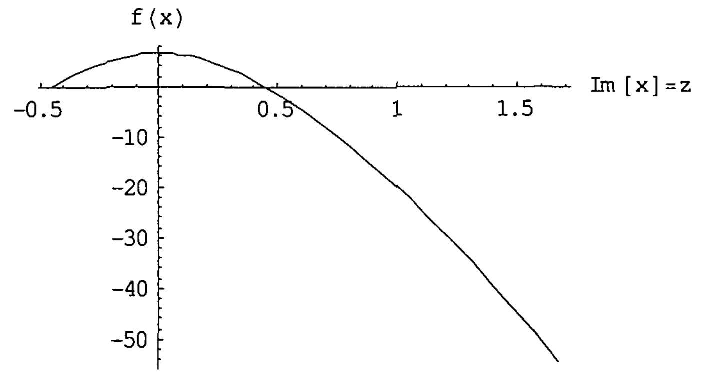

Figure 3.4.2 (b)

Figure 3.4.2 (b) shows that for Re[x] =0.4, $f(x)\geq 0$ if |Im[x]| $\leq$ 0.451091. On the other hand $f(x)<0$ if x=0.4+Im[x]i for 0.451091< Im[x]<1.67 as seen from the Figure 3.4.2(b). Thus, for $|x|\leq\frac{5}{3}$ and Re[x]=0.4, we see that $f(x)>0$ if and only if $|\text{Im}[x]|\leq 0.451091$. Similarly, it can be checked that $f(x)\geq 0$ if and only if x lies inside and on the ellipse given by the Figure 3.4.2 (a). Thus, condition (i) and (ii) are satisfied if and only if λ lies inside and on the ellipse given by Figure 3.4.2 (a) (i.e. for all λ such that $-0.805339\leq \text{Re}[\lambda]\leq 0.569401$ and $|\text{Im}[\lambda]|\leq 0.68618$, correct upto 6 decimal places). Out of this set, finally we need to determine those λ which will also satisfy the condition (iii).

Step 3: If $g(x)=\left(25-9|x|^2\right)^2-25|1+6x|^2-5\left|3\bar{x}(1+6x)^2+2(1+6x)\left(25-9|x|^2\right)\right|$, then here we determine x for which g(x)=0.

```
Reduce[(25-9Abs[x]^2)^2-25Abs[1+6x]^2-5Abs[3Conjugate[x](1+6x)^2+2(1+6x)(25-
9Abs[x]^2)]= =0,x]
```

-4.298511575384078` ≤ Re[x] < -0.4719292953470058` &&

(Im[x] == -1.` Sqrt[Root[-11900 + 44700 Re[x] + 150525 Re[x]^2 - 5400 Re[x]^3 - 22275 Re[x]^4 + 729 Re[x]^6 + 150525 #1 - 5400 Re[x] #1 - 44550 Re[x]^2 #1 + 2187 Re[x]^4 #1 - 22275 #1^2 + 2187 Re[x]^2 #1^2 + 729 #1^3 &, 3]] ||

Im[x] == Sqrt[Root[-11900 + 44700 Re[x] + 150525 Re[x]^2 - 5400 Re[x]^3 - 22275 Re[x]^4 + 729 Re[x]^6 + 150525 #1 - 5400 Re[x] #1 - 44550 Re[x]^2 #1 + 2187 Re[x]^4 #1 - 22275 #1^2 + 2187 Re[x]^2 #1^2 + 729 #1^3 &, 3]])) ||

-0.4719292953470058` ≤ Re[x] ≤ 0.16995729472822219` &&

(Im[x] == -1.` Sqrt[Root[-11900 + 44700 Re[x] + 150525 Re[x]^2 - 5400 Re[x]^3 - 22275 Re[x]^4 + 729 Re[x]^6 + 150525 #1 - 5400 Re[x] #1 - 44550 Re[x]^2 #1 + 2187 Re[x]^4 #1 - 22275 #1^2 + 2187 Re[x]^2 #1^2 + 729 #1^3 &, 3]] ||

Im[x] == -1.` Sqrt[Root[-11900 + 44700 Re[x] + 150525 Re[x]^2 - 5400 Re[x]^3 - 22275 Re[x]^4 + 729 Re[x]^6 + 150525 #1 - 5400 Re[x] #1 - 44550 Re[x]^2 #1 + 2187 Re[x]^4 #1 - 22275 #1^2 + 2187 Re[x]^2 #1^2 + 729 #1^3 &, 1]] ||

Im[x] == Sqrt[Root[-11900 + 44700 Re[x] + 150525 Re[x]^2 - 5400 Re[x]^3 - 22275 Re[x]^4 + 729 Re[x]^6 + 150525 #1 - 5400 Re[x] #1 - 44550 Re[x]^2 #1 + 2187 Re[x]^4 #1 - 22275 #1^2 + 2187 Re[x]^2 #1^2 + 729 #1^3 &, 1]] ||

Im[x] == Sqrt[Root[-11900 + 44700 Re[x] + 150525 Re[x]^2 - 5400 Re[x]^3 - 22275 Re[x]^4 + 729 Re[x]^6 + 150525 #1 - 5400 Re[x] #1 - 44550 Re[x]^2 #1 + 2187 Re[x]^4 #1 - 22275 #1^2 + 2187 Re[x]^2 #1^2 + 729 #1^3 &, 3]])) ||

```
(0.16995729472822219`<Re[x] ≤4.725088447721589`&&
(Im[x] == -1.` Sqrt[Root[-11900 + 44700 Re[x] + 150525 Re[x]^2 - 5400 Re[x]^3
    - 22275 Re[x]^4 + 729 Re[x]^6 + 150525 #1 - 5400 Re[x] #1 -
    44550 Re[x]^2 #1 + 2187 Re[x]^4 #1 - 22275 #1^2 + 2187 Re[x]^2 #1^2 + 729 #1^3 &, 3]] ||

  Im[x] == Sqrt[Root[-11900 + 44700 Re[x] + 150525 Re[x]^2 - 5400 Re[x]^3
    - 22275 Re[x]^4 + 729 Re[x]^6 + 150525 #1 - 5400 Re[x] #1 -
    44550 Re[x]^2 #1 + 2187 Re[x]^4 #1 - 22275 #1^2 + 2187 Re[x]^2 #1^2 + 729 #1^3 &, 3]]))
```

Graphs represented by the solutions of the above equation:

```
Plot[{-1.` Sqrt[Root[-11900 + 44700 y + 150525 y^2 - 5400 y^3 - 22275 y^4 + 729 y^6 + 150525 #1 -
    5400 y #1 - 44550 y^2 #1 + 2187 y^4 #1 - 22275 #1^2 + 2187 y^2 #1^2 + 729 #1^3 &, 3]],
  Sqrt[Root[-11900 + 44700 y + 150525 y^2 - 5400 y^3 - 22275 y^4 + 729 y^6 + 150525 #1 -
    5400 y #1 - 44550 y^2 #1 + 2187 y^4 #1 - 22275 #1^2 + 2187 y^2 #1^2 + 729 #1^3 &, 3]],
  -1.` Sqrt[Root[-11900 + 44700 y + 150525 y^2 - 5400 y^3 - 22275 y^4 + 729 y^6 + 150525 #1 -
    5400 y #1 - 44550 y^2 #1 + 2187 y^4 #1 - 22275 #1^2 + 2187 y^2 #1^2 + 729 #1^3 &, 3]],
  Sqrt[Root[-11900 + 44700 y + 150525 y^2 - 5400 y^3 - 22275 y^4 + 729 y^6 + 150525 #1 -
    5400 y #1 - 44550 y^2 #1 + 2187 y^4 #1 - 22275 #1^2 + 2187 y^2 #1^2 + 729 #1^3 &, 3]],
  -1.` Sqrt[Root[-11900 + 44700 y + 150525 y^2 - 5400 y^3 - 22275 y^4 + 729 y^6 + 150525 #1 -
    5400 y #1 - 44550 y^2 #1 + 2187 y^4 #1 - 22275 #1^2 + 2187 y^2 #1^2 + 729 #1^3 &, 1]],
  Sqrt[Root[-11900 + 44700 y + 150525 y^2 - 5400 y^3 - 22275 y^4 + 729 y^6 + 150525 #1 -
    5400 y #1 - 44550 y^2 #1 + 2187 y^4 #1 - 22275 #1^2 + 2187 y^2 #1^2 + 729 #1^3 &, 1]],
  -1.` Sqrt[Root[-11900 + 44700 y + 150525 y^2 - 5400 y^3 - 22275 y^4 + 729 y^6 + 150525 #1 -
    5400 y #1 - 44550 y^2 #1 + 2187 y^4 #1 - 22275 #1^2 + 2187 y^2 #1^2 + 729 #1^3 &, 3]],
  Sqrt[Root[-11900 + 44700 y + 150525 y^2 - 5400 y^3 - 22275 y^4 + 729 y^6 + 150525 #1 -
    5400 y #1 - 44550 y^2 #1 + 2187 y^4 #1 - 22275 #1^2 + 2187 y^2 #1^2 + 729 #1^3 &, 3]]},
 {y, -4.298511575384078`, 4.725088447721589`},
 AxesLabel → {"Real Axis ", "Imaginary Axis"}]
```

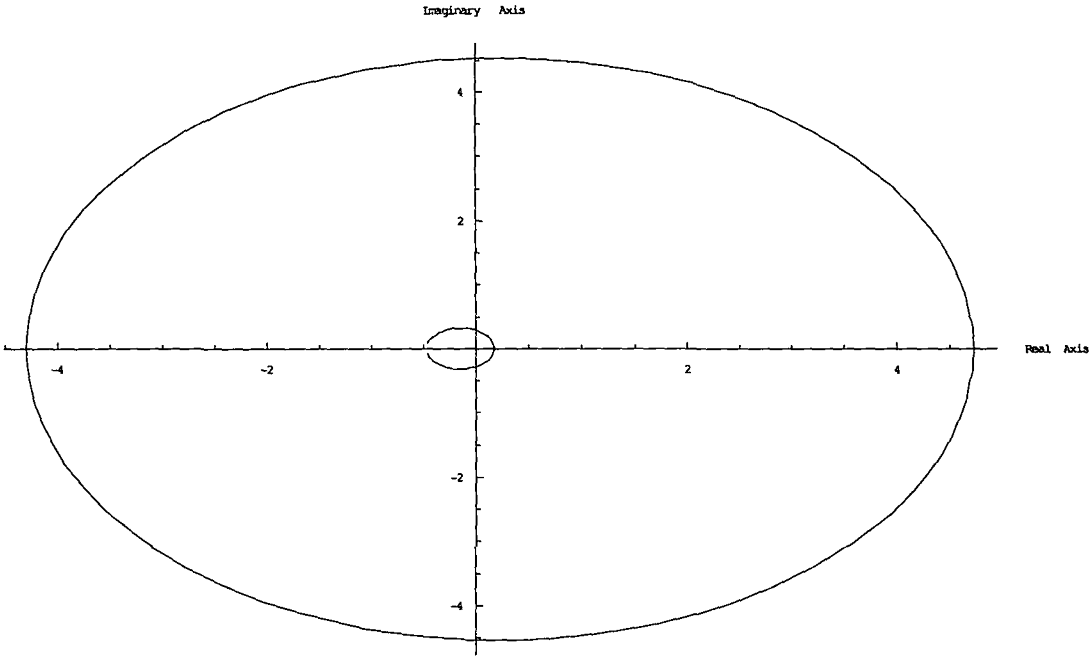

Figure 3.4.2 (c)

The inner ellipse of Figure 3.4.2 (c) can be better depicted in Figure 3.4.2 (d).

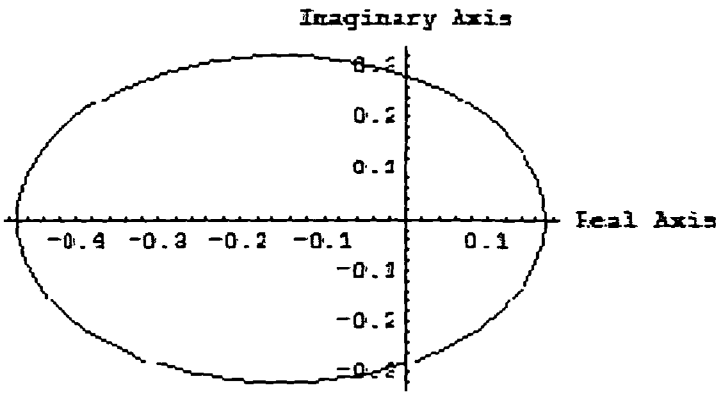

Figure 3.4.2 (d)

Step 4: Here we showed that the inner elliptical region will give the values of λ for which T_φ is hyponormal. That is, any point inside and on the inner ellipse will satisfy the condition (iii).

Consider a point x such that y =Re[x]= -0.2.

Im[x]=

$$\sqrt{\text{Root}[-11900+44700\,y+150525\,y^2-5400\,y^3-22275\,y^4+729\,y^6+150525\,\#1-5400\,y\,\#1-44550\,y^2\,\#1+2187\,y^4\,\#1-22275\,\#1^2+2187\,y^2\,\#1^2+729\,\#1^3\,\&,\,1]}\ /.\ y\to -0.2$$

=0.31677

Then x= -0.2+Im[x]i is a point inside the inner ellipse for -0.31677< Im[x]<0.31677

Im[x] =

$$\sqrt{\text{Root}[-11900+44700\,y+150525\,y^2-5400\,y^3-22275\,y^4+729\,y^6+150525\,\#1-5400\,y\,\#1-44550\,y^2\,\#1+2187\,y^4\,\#1-22275\,\#1^2+2187\,y^2\,\#1^2+729\,\#1^3\,\&,\,3]}\ /.\ y\to -0.2$$

=4.51874

Then x=-0.2+Im[x]i is a point inside the outer ellipse for -4.51874<Im[x]<4.51874 [From the Figure 3.4.2 (c)].

To draw the graph we write Im[x] =z.

```
Plot[{(25-9Abs[(-0.2+zI)]^2)^2-25Abs[1+6(-0.2+zI)]^2-5Abs[3Conjugate[-0.2+zI](1+6(-0.2+zI))^2+2(1+6(0.2+zI))(25-9Abs[(0.2+zI)]^2)]},{z,4.518736159707949`,
4.518736159707949`},AxesLabel→{"Im[x]=z ","g(x)"},AxesOrigin→{0,0}]
```

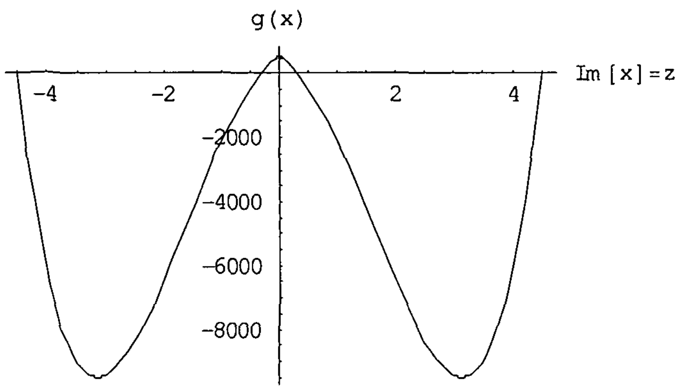

Figure 3.4.2 (e)

Thus if Re[x]=-0.2 then g(x)>0 if and only if x lies inside the inner ellipse in Figure 3.4.2 (c). Similarly, it can be shown that if -0.471929<Re[x]<0.169957, then g(x)>0 if and only if x lies inside the inner ellipse in Figure 3.4.2 (c).

Step 5: To show that condition (iii) is not satisfied for points outside the inner ellipse in Figure 3.4.2 (c). Here we note that because of condition (ii) we need not consider points with Re[x]>0.569401 and Re[x] <-0.805399.

Let y=Re[x] =0.2. Then

$$\text{Im}[x] = \sqrt{\begin{array}{l}\text{Root}[-11900+44700\,y+150525\,y^2-5400\,y^3-22275\,y^4+729\,y^6+150525\,\#1- \\ 5400\,y\,\#1-44550\,y^2\,\#1+2187\,y^4\,\#1-22275\,\#1^2+2187\,y^2\,\#1^2+729\,\#1^3\,\&,\,3]\end{array}}\ /.\ y\to 0.2$$

=4.53722

Then x=0.2+Im[x]i is a point inside the outer ellipse for -4.53722<Im[x]<4.53722.

To draw the graph we write Im[x] =z

```
Plot[{(25-9Abs[(0.2+zI)]^2)^2-25Abs[1+6(0.2+zI)]^2-5Abs[3Conjugate[0.2+zI](1+6(0.2+z
I))^2+2(1+6(0.2+zI))(25-9Abs[(0.2+zI)]^2)]},{z,4.537219565513397`,4.537219565513397`},

AxesLabel→{"Im[x]=z ","g(x)"},AxesOrigin→{0,0}]
```

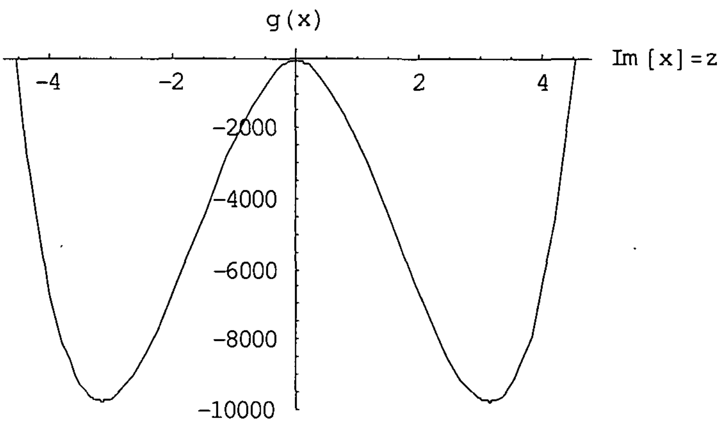

Figure 3.4.2 (f)

Thus $g(x) < 0$ if x=0.2+Im[x]i for -4.53722<Im[x]<4.53722. Similarly, we can show that $g(x) < 0$ for any point x outside the inner ellipse but inside the outer ellipse of of Figure 3.4.2 (c). Hence , T_φ is hyponormal if and only if λ lies inside and on the inner ellipse given by Figure 3.4.2 (c).

Example 3.4.3:

T_φ is hyponormal iff

(i) $\left|\lambda\right| \leq \frac{16}{3}$

(ii) $256-9\left|\lambda\right|^2-\left|3\bar{\lambda}^2-224\right| \geq 0$ and

(iii) $9\left(256-9\left|\lambda\right|^2\right)^2-\left|9\bar{\lambda}^2-672\right|^2-2\left|243\lambda\left(9\left|\lambda\right|^2-256\right)+\left(9\bar{\lambda}^2-672\right)\left(24\lambda-63\bar{\lambda}\right)\right| \geq 0$

Step 1: Reduce [256-9Abs[x] ^2-Abs [3x^2-224] = =0, x]

$$-2.309401076758503` \leq \text{Re}[x] \leq 2.309401076758503` \;\&\&\; \left(\text{Im}[x] == -1.` \sqrt{\left(-1.3333333333333333` \sqrt{841.` - 21.` \text{Re}[x]^2} + 0.3333333333333333` \left(124.` - 3.` \text{Re}[x]^2\right)\right)} \;||\; \text{Im}[x] == \sqrt{\left(-1.3333333333333333` \sqrt{841.` - 21.` \text{Re}[x]^2} + 0.3333333333333333` \left(124.` - 3.` \text{Re}[x]^2\right)\right)}\right)$$

Plot[{-1.` $\sqrt{\left(-1.3333333333333333` \sqrt{841.` - 21.` y^2} + 0.3333333333333333` \left(124.` - 3.` y^2\right)\right)}$,
$\sqrt{\left(-1.3333333333333333` \sqrt{841.` - 21.` y^2} + 0.3333333333333333` \left(124.` - 3.` y^2\right)\right)}$},
{y, -2.309401076758503`, 2.309401076758503`}, AxesLabel→ {"Real Axis ", "Imaginary Axis"},
AxesOrigin→ {0, 0}]

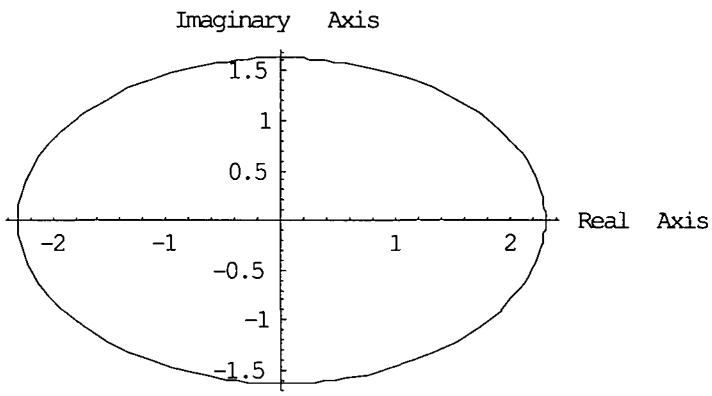

Figure 3.4.3 (a)

Conditions (i) and (ii) are satisfied by all points lying inside and on the ellipse in Figure 3.4.3 (a).

Step 2: Reduce[2Abs [243x(9Abs [x]^2-256)+(9x^2-672)(24x-63 Conjugate [x])]-9(256-9Abs [x]^2)^2+Abs [9x^2-672]^2= =0,x]

```
(-9.033968554320527` ≤ Re[x] < -4.692305189335676` &&
  (Im[x] == -1.` √Root[-14400 + 9520 Re[x]^2 - 849 Re[x]^4 + 9 Re[x]^6 + 54544 #1 -
        2250 Re[x]^2 #1 + 27 Re[x]^4 #1 - 1401 #1^2 + 27 Re[x]^2 #1^2 + 9 #1^3 &, 1] ||
    Im[x] == √Root[-14400 + 9520 Re[x]^2 - 849 Re[x]^4 + 9 Re[x]^6 + 54544 #1 -
        2250 Re[x]^2 #1 + 27 Re[x]^4 #1 - 1401 #1^2 + 27 Re[x]^2 #1^2 + 9 #1^3 &, 1])) ||

(-4.692305189335676` ≤ Re[x] < -1.3394943062215976` && (Im[x] ==
     -1.` - √Root[-14400 + 9520 Re[x]^2 - 849 Re[x]^4 + 9 Re[x]^6 + 54544 #1 -
        250 Re[x]^2 #1 + 27 Re[x]^4 #1 - 1401 #1^2 + 27 Re[x]^2 #1^2 + 9 #1^3 &, 3] ||
    Im[x] == √Root[-14400 + 9520 Re[x]^2 - 849 Re[x]^4 + 9 Re[x]^6 + 54544 #1
        - 2250 Re[x]^2 #1 + 27 Re[x]^4 #1 - 1401 #1^2 + 27 Re[x]^2 #1^2 + 9 #1^3 &, 3])) ||

(-1.3394943062215976` ≤ Re[x] ≤ 1.3394943062215976` && (Im[x] ==
     -1.` √Root[-14400 + 9520 Re[x]^2 - 849 Re[x]^4 + 9 Re[x]^6 + 54544 #1 -
        2250 Re[x]^2 #1 + 27 Re[x]^4 #1 - 1401 #1^2 + 27 Re[x]^2 #1^2 + 9 #1^3 &, 3] ||
    Im[x] == -1.` √Root[-14400 + 9520 Re[x]^2 - 849 Re[x]^4 + 9 Re[x]^6 + 54544 #1 -
        2250 Re[x]^2 #1 + 27 Re[x]^4 #1 - 1401 #1^2 + 27 Re[x]^2 #1^2 + 9 #1^3 &, 1] ||
    Im[x] == √Root[-14400 + 9520 Re[x]^2 - 849 Re[x]^4 + 9 Re[x]^6 + 54544 #1 -
        2250 Re[x]^2 #1 + 27 Re[x]^4 #1 - 1401 #1^2 + 27 Re[x]^2 #1^2 + 9 #1^3 &, 1] ||
    Im[x] == √Root[-14400 + 9520 Re[x]^2 - 849 Re[x]^4 + 9 Re[x]^6 + 54544 #1 -
        2250 Re[x]^2 #1 + 27 Re[x]^4 #1 - 1401 #1^2 + 27 Re[x]^2 #1^2 + 9 #1^3 &, 3])) ||

(1.3394943062215976` < Re[x] ≤ 4.692305189335676` && (Im[x] ==
     -1.` √Root[-14400 + 9520 Re[x]^2 - 849 Re[x]^4 + 9 Re[x]^6 + 54544 #1 -
        2250 Re[x]^2 #1 + 27 Re[x]^4 #1 - 1401 #1^2 + 27 Re[x]^2 #1^2 + 9 #1^3 &, 3] ||
    Im[x] == √Root[-14400 + 9520 Re[x]^2 - 849 Re[x]^4 + 9 Re[x]^6 + 54544 #1 -
        2250 Re[x]^2 #1 + 27 Re[x]^4 #1 - 1401 #1^2 + 27 Re[x]^2 #1^2 + 9 #1^3 &, 3])) ||
```

```
(4.692305189335676` < Re[x] ≤ 9.033968554320527` && (Im[x] ==
    -1.` √Root[-14400 + 9520 Re[x]^2 - 849 Re[x]^4 + 9 Re[x]^6 + 54544 #1 -
        2250 Re[x]^2 #1 + 27 Re[x]^4 #1 - 1401 #1^2 + 27 Re[x]^2 #1^2 + 9 #1^3 &, 1] ||
   Im[x] == √Root[-14400 + 9520 Re[x]^2 - 849 Re[x]^4 + 9 Re[x]^6 + 54544 #1 -
        2250 Re[x]^2 #1 + 27 Re[x]^4 #1 - 1401 #1^2 + 27 Re[x]^2 #1^2 + 9 #1^3 &, 1]))

Plot[{-1.` √Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 -
        1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1], √Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 +
      54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1],
   -1.` √Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 -
        1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 3], √Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 +
      54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 3],
   -1.` √Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 -
        1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 3], √Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 +
      54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 3],
   √Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 -
      1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1], -√Root[-14400 + 9520 y^2 - 849 y^4 +
        9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1],
   -1.` √Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 -
        1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 3], √Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 +
      54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 3],
   -1.` √Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 -
        1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1], √Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 +
      54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1]},
  {y, -9.033968554320527`, 9.033968554320527`}, AxesLabel → {"Real Axis ",
    "Imaginary Axis"}, AxesOrigin → {0, 0}]
```

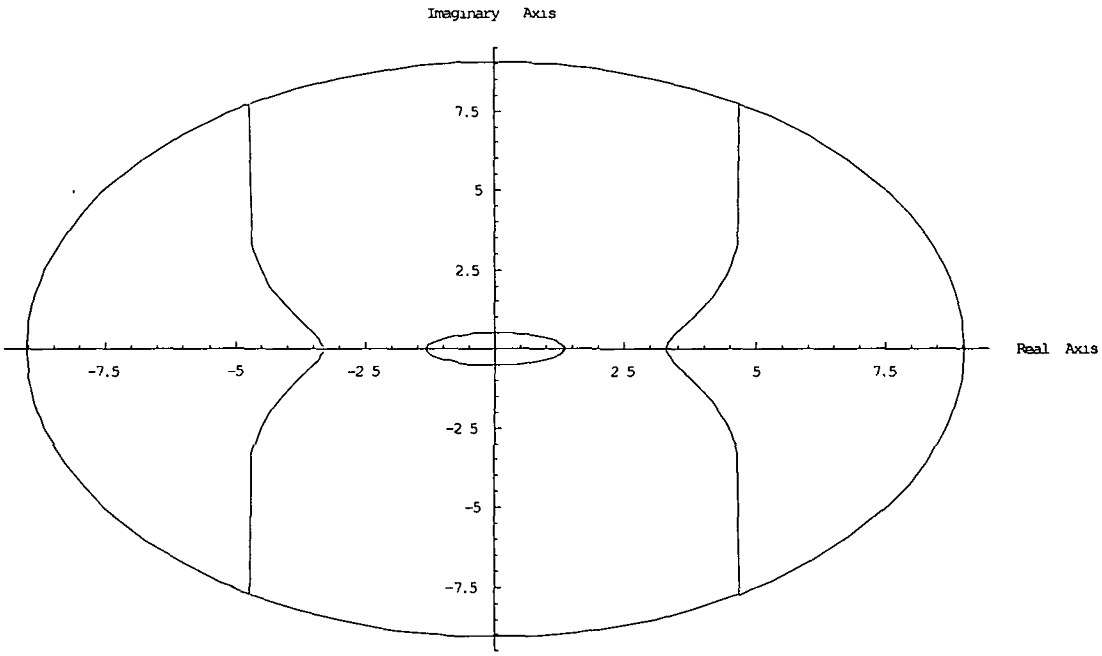

Figure 3.4.3 (b)

Step 3: Combined graph of Steps 1 and 2:

```
Plot[Evaluate[{-1.` Sqrt[(-1.3333333333333333` Sqrt[841.` - 21.` y^2] + 0.3333333333333333` (124.` - 3.` y^2))],
  Sqrt[(-1.3333333333333333` Sqrt[841.` - 21.` y^2] + 0.3333333333333333` (124.` - 3.` y^2))],
  Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1]],
  -Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1]],
  -1.` Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1]],
  Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1]],
  -1.` Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 3]],
  Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 3]],
  -1.` Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 3]],
  Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 3]],
  Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1]],
  -Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1]],
  -1.` Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 3]],
  Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 3]],
  -1.` Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1]],
  Sqrt[Root[-14400 + 9520 y^2 - 849 y^4 + 9 y^6 + 54544 #1 - 2250 y^2 #1 + 27 y^4 #1 - 1401 #1^2 + 27 y^2 #1^2 + 9 #1^3 &, 1]]},
  {y, -9.033968554320527`, 9.033968554320527`}, AxesLabel -> {"Real Axis ", "Imaginary Axis"},
  AxesOrigin -> {0, 0}]]
```

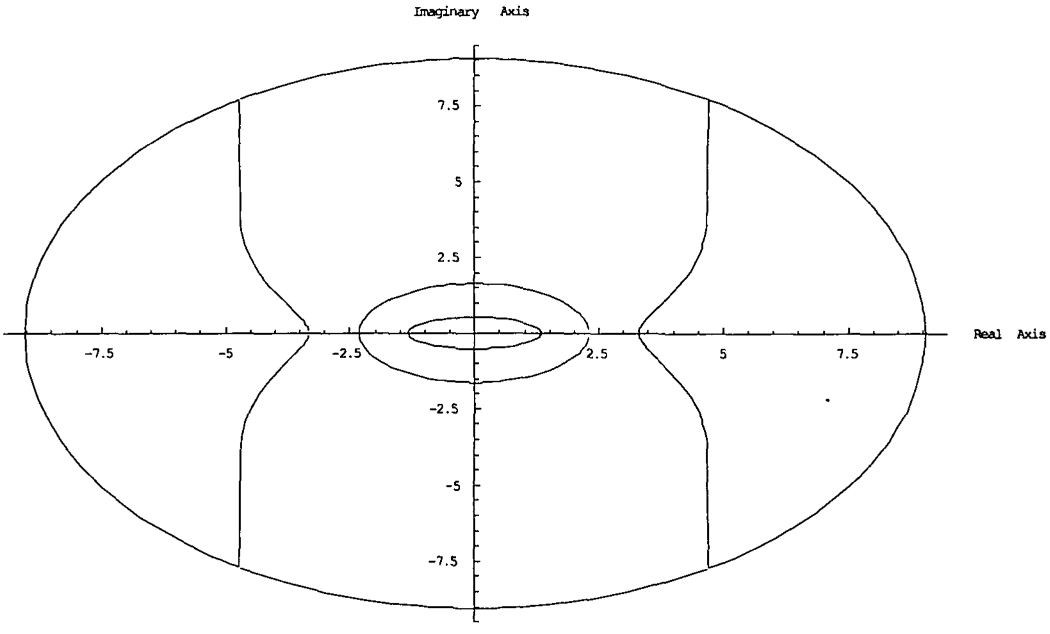

Figure 3.4.3 (c)

As in the earlier examples, it can be shown that T_φ is hyponormal if and only if λ lies inside and on the innermost ellipse in Figure 3.4.3 (c).

Example 3.4.4:

Reduce [441-4Abs[x] ^2-21Abs[x+13] = =0, x]

```
-9.6171831354734 98` ≤ Re[x] ≤ 4.3671831354734 98` &&
 (Im[x] == -1.` √(-0.65625` √(18313.` + 1664.` Re[x]) + 0.03125` (3969.` - 32.` Re[x]^2)) ||
  Im[x] == √(-0.65625` √(18313.` + 1664.` Re[x]) + 0.03125` (3969.` - 32.` Re[x]^2)))
```

```
Plot[{-1.` √(-0.65625` √(18313.` + 1664.` y) + 0.03125` (3969.` - 32.` y^2)),
  √(-0.65625` √(18313.` + 1664.` y) + 0.03125` (3969.` - 32.` y^2))},
 {y, -9.617183135473498`, 4.367183135473498`},
 AxesLabel→ {"Real Axis ", "Imaginary Axis"}, AxesOrigin→ {0, 0}]
```

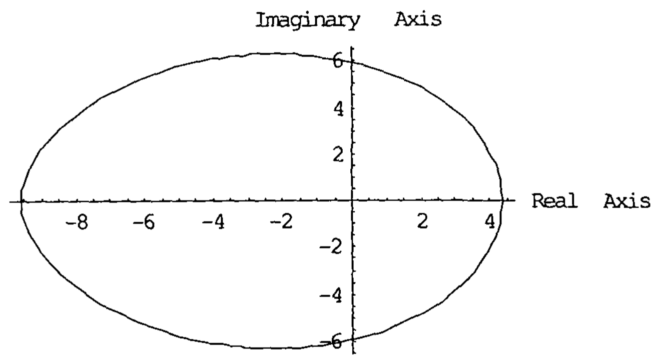

Figure 3.4.4

Example 3.4.5:

Step 1: Reduce [117-11Abs [18+5x] = = 0, x]

```
-5.72727272727272275` ≤ Re[x] ≤ -1.47272727272727272727` &&
 (Im[x] == -0.0406557814090870 86` √(-5103.` - 4356.` Re[x] - 605.` Re[x]^2) ||
  Im[x] == 0.040655781409087086` √(-5103.` - 4356.` Re[x] - 605.` Re[x]^2))
```

```
Plot[{0.040655781409087086` √(-5103.` - 4356.` y - 605.` y^2), -0.040655781409087086`
  √(-5103.` - 4356.` y - 605.` y^2)}, {y, -5.72727272727272275`, -1.47272727272727272727`},
 AxesLabel→ {"Real Axis ", "Imaginary Axis"}]
```

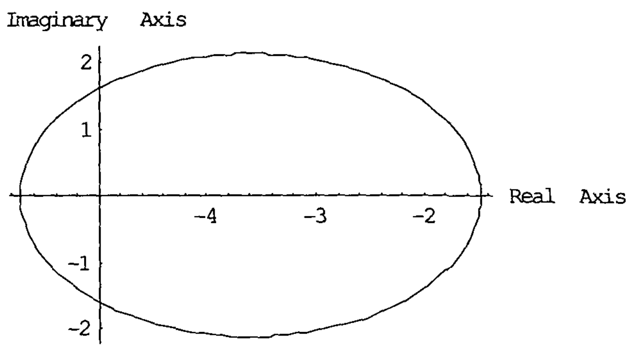

Figure 3.4.5 (a)

Step 2: Reduce [13689-121Abs [18+5x] ^2-Abs [5841+3960x+550 x^2] = =0, x]

```
-5.646726877013861` ≤ Re[x] ≤ -1.5532731229861396` &&
 (Im[x] == -1.` √((-0.0036363636363636364` √(-187607.` - 174240.` Re[x] - 24200.` Re[x]^2) +
        0.2` (-41.` - 36.` Re[x] - 5.` Re[x]^2))) ||
   Im[x] == √((-0.0036363636363636364` √(-187607.` - 174240.` Re[x] - 24200.` Re[x]^2) +
        0.2` (-41.` - 36.` Re[x] - 5.` Re[x]^2))))
```

```
Plot[{√((-0.0036363636363636364` √(-187607.` - 174240.` y - 24200.` y^2) +
      0.2` (-41.` - 36.` y - 5.` y^2))), -√((-0.0036363636363636364`
        √(-187607.` - 174240.` y - 24200.` y^2) + 0.2` (-41.` - 36.` y - 5.` y^2)))},
  {y, -5.646726877013861`, -1.5532731229861396`},
  AxesLabel → {"Real Axis ", "Imaginary Axis"}]
```

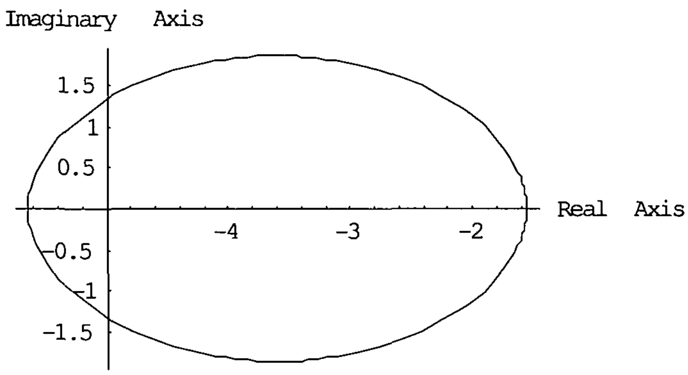

Figure 3.4.5 (b)

Step 3: Combined graph of Steps 1 and 2.

```
Plot[Evaluate[{0.040655781409087086` √(-5103.` - 4356.` y - 605.` y^2),
   -0.040655781409087086` √(-5103.` - 4356.` y - 605.` y^2),
   √((-0.0036363636363636364` √(-187607.` - 174240.` y - 24200.` y^2) +
      0.2` (-41.` - 36.` y - 5.` y^2))), -√((-0.0036363636363636364`
        √(-187607.` - 174240.` y - 24200.` y^2) + 0.2` (-41.` - 36.` y - 5.` y^2)))},
  {y, -5.7272727272727275`, -1.4727272727272727`},
  AxesLabel → {"Real Axis ", "Imaginary Axis"}]]
```

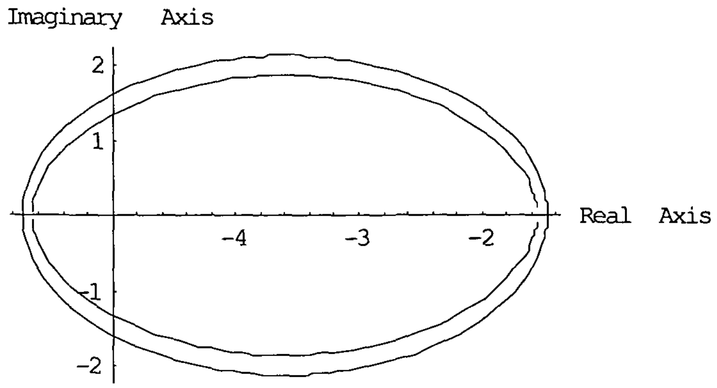

Figure 3.4.5 (c)