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FINITE ELEMENT METHODS FOR INTERFACE PROBLEMS

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Dedicated

To my Parents

Abstract

The main objective of this thesis is to study the convergence of finite element solutions to the exact solutions of elliptic, parabolic and hyperbolic interface problems in fitted finite element method. The emphasis is on the theoretical aspects of such methods.

Due to low global regularity of the true solution it is difficult to apply the classical finite element analysis to obtain optimal order of convergence for interface problems (cf. [5, 11]). In order to maintain the best possible convergence rate, a finite element discretization with straight interface triangles is considered and analyzed. More precisely, we have shown that the finite element solution converges to the exact solution at an optimal rate in L^2 and H^1 norms for elliptic problems. Then the results are extended for parabolic interface problems and optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are achieved. Further, optimal $L^\infty(H^1)$ and $L^\infty(L^2)$ norms error estimates for the parabolic interface problems have been derived under practical regularity assumption of the true solutions.

Although various finite element method for elliptic and parabolic interface problems have been proposed and studied in the literature, but finite element treatment of similar hyperbolic problems is mostly missing. In this work, we are able to prove optimal order pointwise-in-time error estimates in L^2 and H^1 norms for the hyperbolic interface problem with semidiscrete scheme. Fully discrete scheme based on a symmetric difference approximation is also analyzed and optimal H^1 norm error is obtained.

Finally, numerical results for two dimensional test problems are presented to illustrate our theoretical findings.

Declaration

I, **Tazuddin Ahmed**, hereby declare that the subject matter in this thesis entitled **Finite Element Methods for Interface Problems** is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/institute.

This thesis is being submitted to the Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

Place: Napaam
Date: 13.08.2012

(Tazuddin Ahmed)

TEZPUR UNIVERSITY

Certificate

This is to certify that the thesis entitled **Finite Element Methods for Interface Problems** submitted to the School of **Sciences and Technology** Tezpur University in partial fulfilment for the award of the degree of Doctor of Philosophy in **Mathematical Sciences** is a record of research work carried out by **Mr. Tazuddin Ahmed** under my supervision and guidance.

All help received by him from various sources have been dully acknowledged. No part of this thesis has been submitted elsewhere for award of any other degree.

December, 2011

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13/3/12

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Chapter 1

Introduction

The purpose of this thesis is to present some results on finite element Galerkin methods for linear elliptic, parabolic and hyperbolic interface problems.

1.1 Problem Description

Interface problems are often referred as differential equations with discontinuous coefficients. The discontinuity of the coefficients corresponds to the fact that the medium consists of two or more physically different materials. To begin with, we first introduce elliptic, parabolic and hyperbolic interface problems.

Elliptic interface problems: Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Further, let $\Omega_1 \subset \Omega$ be an open domain with C^2 smooth boundary Γ and $\Omega_2 = \Omega \backslash \Omega_1$ (see, Figure 1.1). We now consider the following linear elliptic interface problems of the form

$$\mathcal{L}u = f(x) \quad \text{in } \Omega \tag{1.1.1}$$

with Dirichlet boundary condition

$$u(x) = 0 \quad \text{on } \partial\Omega \tag{1.1.2}$$

and interface conditions

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = g(x) \quad \text{along } \Gamma. \tag{1.1.3}$$

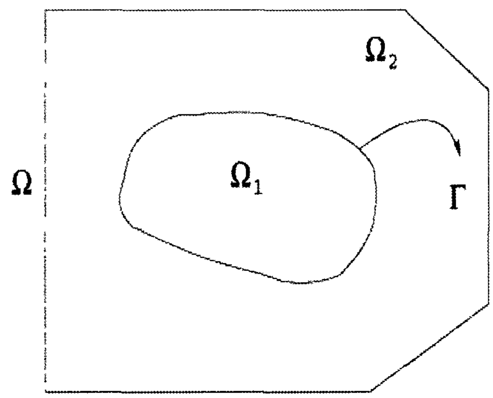

Figure 1.1: *Domain* Ω *and its sub domains* Ω_1, Ω_2 *with interface* Γ.

The symbol $[v]$ is a jump of a quantity v across the interface Γ, i.e., $[v](x) = v_1(x) - v_2(x), \ \ x \in \Gamma$, where $v_i(x) = v(x) \mid_{\Omega_i}$, $i = 1, 2$ and $\mathbf{n}$ denotes the unit outward normal to the boundary $\partial\Omega_1$. Here, $\mathcal{L}$ is a second order elliptic partial differential operator of the form

$$\mathcal{L}v = -\nabla.(\beta(x)\nabla v).$$

We assume that the coefficient function β is positive and piecewise constant, i.e.,

$$\beta(x) = \beta_i \ \text{ in } \Omega_i, \ i = 1, 2.$$

Parabolic interface problems: We consider the following linear parabolic interface problems of the form

$$u_t + \mathcal{L}u = f(x,t) \ \text{ in } \Omega \times (0,T] \tag{1.1.4}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \ \text{ in } \Omega; \ \ u(x,t) = 0 \ \text{ on } \partial\Omega \times (0,T] \tag{1.1.5}$$

and interface conditions

$$[u] = 0, \ \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = g(x,t) \ \text{ along } \Gamma, \tag{1.1.6}$$

The domain Ω, operator $\mathcal{L}$, symbols $[v]$ and $\mathbf{n}$ are defined as before, and $T < \infty$.

Hyperbolic interface problems: We shall also consider the following hyperbolic interface problems of the form

$$u_{tt} + \mathcal{L}u = 0 \ \text{ in } \Omega \times (0,T] \tag{1.1.7}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \ \& \ u_t(x,0) = v_0(x) \ \text{ in } \Omega; \ \ u(x,t) = 0 \ \text{ on } \partial\Omega \times (0,T] \tag{1.1.8}$$

and interface conditions

$$[u] = 0, \ \ \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = g(x,t) \ \text{ along } \Gamma, \tag{1.1.9}$$

The domain Ω, operator $\mathcal{L}$, symbols $[v]$ and $\mathbf{n}$ are defined as before, and $T < \infty$.

The equations of the form (1.1.1)-(1.1.3) are often encountered in the theory of magnetic field, heat conduction theory, the theory of elasticity and in reaction diffusion problems (see, [23, 29, 49]). Many interface problems in material science and fluid dynamics are modeled after above problem when two or more distinct materials or fluids with different conductivities or densities or diffusions are involved. For the literature relating to applications of elliptic differential equations with discontinuous coefficients, one may refer to Ewing [22], Nielsen [37] or Peaceman [38] for the model of the pressure equation arising in reservoir simulation, Reddy [41] for reactor dynamics, Z. Li *et al.* [33] for the model of the potential in the computation of micromagnetics for the ferromagnetic materials or electrostatics for macromolecules.

The equations of the form (1.1.4)-(1.1.6) involving discontinuous coefficients are sometimes called diffraction problems of parabolic types. This type of interface problem is critical in many applications of engineering and sciences, including non-stationary heat conduction problems, electromagnetic problems, shape optimization problems to name just a few. For a detailed discussion on parabolic problems with discontinuous coefficients, see Dautry and Lions [14], Gilberg and Trudinzer [25], Hackbush [27], Ladyzhenskaya *et al.* [30], Li and Ito [32] and Marti [36].

The model equations of the form (1.1.7)-(1.1.9) involving discontinuous coefficients are used in many applications such as ocean acoustics, elasticity, and seismology to model the propagation of small disturbances in fluids or solids. In electromagnetism, the equation (1.1.7) corresponds to a problem in which the material occupying the interior is a dielectric rather than a metal (cf. [2]). In the study of wave equations for some physical problems, such as acoustic or elastic waves travelling through heterogeneous media, there can be discontinuities in the coefficients of the equation. As a model, consider the problem of transverse vibrations of an infinite string, with a discontinuity in density ρ at a location $x = \alpha$. Let ψ represent the non dimensionalized displacement normal to the string. Then we have the equation

$$\rho\psi_{tt} - (\tau_0\psi_x)_x = 0$$

which is equivalent to the problem

$$\psi_{tt} - \beta(x)\psi_{xx} = 0$$

where

$$\beta(x) = \begin{cases} \beta_1 = \frac{\tau_0}{\rho_1} & \text{if} \quad x < \alpha \\ \beta_2 = \frac{\tau_0}{\rho_2} & \text{if} \quad x > \alpha \end{cases}$$

along with the initial condition

$$\psi(x,0) = f(x), \quad \psi_t(x,0) = 0.$$

For this physical model, we have the following jump conditions at the interface $x = \alpha$

$$[\psi] = 0, \quad [\psi_x] = 0.$$

The interface conditions correspond to the facts that displacement and normal component of the tension in the deflected string are continuous. The one dimensional acoustic wave equation is often used as a model in seismology. For example, consider the one-dimensional acoustic wave equation

$$\rho u_t + p_x = 0 \quad \& \quad p_t + ku_x = 0,$$

where ρ is the density, u is the velocity, p is the pressure and k is compression(bulk) modulus. At $x = \alpha$, the coefficients are given as

$$(\rho, k) = \begin{cases} (\rho^-, k^-) & \text{if} \quad x < \alpha \\ (\rho^+, k^+) & \text{if} \quad x > \alpha. \end{cases}$$

The velocity and pressure must be continuous across the interface, and therefore the jump conditions at the interface are

$$[u] = 0, \quad [p] = 0.$$

The above problem can also be rewritten as hyperbolic problems

$$\rho u_{tt} - k u_{xx} = 0, \quad p_{tt} - \frac{k}{\rho} p_{xx} = 0.$$

with discontinuous coefficients.

1.2 Notation and Preliminaries

In this section, we shall introduce some standard notation and preliminaries to be used throughout of this work.

All functions considered here are real valued. Let Ω be a bounded domain in $\mathbb{R}^d$, $d-$dimensional Euclidian space and $\partial\Omega$ denote the boundary of Ω. Let $x = (x_1, x_2, \ldots, x_d) \in \Omega$, and let $dx = dx_1, \ldots, dx_d$. Further, let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a $d-$tuple with nonnegative integer components and denote order of α as $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$. Then, by $D^\alpha \phi$, we shall mean the αth derivative of ϕ defined by

$$D^\alpha \phi = \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}.$$

We shall make frequent reference to the following well-known function spaces. For $1 \leq p < \infty$, $L^p(\Omega)$ denotes the linear space of equivalence classes of measurable functions ϕ in Ω such that $\int_\Omega |\phi(x)|^p dx$ exists and is finite. The norm on $L^p(\Omega)$ is given by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |\phi(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

For $p = \infty$, $L^\infty(\Omega)$ denotes the space of functions ϕ on Ω such that

$$\|\phi\|_{L^\infty(\Omega)} = \text{ess} \sup_{x \in \Omega} |\phi(x)| < \infty.$$

When $p = 2$, $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(\phi, \psi) = \int_\Omega \phi(x)\psi(x)dx.$$

By support of a function ϕ, supp ϕ, we mean the closure of all points x with $\phi(x) \neq 0$, i.e.,

$$\text{supp } \phi = \overline{\{x : \phi(x) \neq 0\}}.$$

For any nonnegative integer m, $C^m(\overline{\Omega})$ denotes the space of functions with continuous derivatives upto and including order m in $\overline{\Omega}$. $C_0^m(\Omega)$ is the space of all $C^m(\Omega)$ functions with compact support in Ω. Also, $C_0^\infty(\Omega)$ is the space of all infinitely differential functions with compact support in Ω.

We now introduce the notion of Sobolev spaces. Let $m \geq 0$ and real p with $1 \leq p < \infty$. The Sobolev space of order (m, p) on Ω, denoted by $W^{m,p}(\Omega)$, is defined as a linear space of functions (or equivalence class of functions) in $L^p(\Omega)$ whose distributional derivatives upto order m are also in $L^p(\Omega)$, i.e.,

$$W^{m,p}(\Omega) = \{\phi : \ D^\alpha\phi \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}.$$

The space $W^{m,p}(\Omega)$ is endowed with the norm

$$\begin{aligned} \|\phi\|_{m.p} &= \left(\int_\Omega \sum_{0\leq|\alpha|\leq m} |D^\alpha\phi(x)|^p dx\right)^{\frac{1}{p}} \\ &= \left(\sum_{0\leq|\alpha|\leq m} \|D^\alpha\phi\|^p\right)^{\frac{1}{p}}, \ 1 \leq p < \infty. \end{aligned}$$

When $p = \infty$, the norm on the space $W^{m,\infty}(\Omega)$ is defined by

$$\|\phi\|_{m,\infty} = \max_{0\leq|\alpha|\leq m} \|D^\alpha\phi(x)\|_{L^\infty(\Omega)}.$$

For p=2, these spaces will be denoted by $H^m(\Omega)$. The space $H^m(\Omega)$ is a Hilbert space with natural inner product defined by

$$(\phi, \psi) = \sum_{0\leq|\alpha|\leq m} \int_\Omega D^\alpha\phi D^\alpha\psi dx, \ \ \phi, \psi \in H^m(\Omega).$$

The sobolev space $H^m(\Omega)$ (respectively, $H_0^m(\Omega)$) is also defined as the closure of $C^m(\Omega)$ (respectively, $C_0^\infty(\Omega)$) with respect to the norm $\|\phi\|_m = \|\phi\|_{m,2}$. This result is true under some smoothness assumption on the boundary $\partial\Omega$. Clearly, $L^2(\Omega) = H^0(\Omega)$ and $H^m(\Omega) = W^{m,2}(\Omega)$. We also need the fractional space $H^{\frac{1}{2}}(\Omega)$ equipped with the norm

$$\|\psi\|_{H^{\frac{1}{2}}(\Omega)} = \inf_{w\in H^1(\Omega)} \{\|w\|_{H^1(\Omega)} : \gamma_0 w = \psi\},$$

where γ_0 is a trace operator. For a more complete discussion on Sobolev spaces, see Adams [1].

We shall also use the following spaces in our error analysis. For a given Banach space $\mathcal{B}$, we define, for $m = 0, 1$ and $1 \leq p < \infty$

$$W^{m,p}(0,T;\mathcal{B}) = \left\{ u(t) \in \mathcal{B} \text{ for a.e. } t \in (0,T) \text{ and } \sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j u(t)}{\partial t^j} \right\|_{\mathcal{B}}^p dt < \infty \right\}$$

equipped with the norm

$$\|u\|_{W^{m,p}(0,T;\mathcal{B})} = \left(\sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j u(t)}{\partial t^j} \right\|_{\mathcal{B}}^p dt \right)^{\frac{1}{p}}.$$

We write $H^m(0,T;\mathcal{B}) = W^{m,2}(0,T;\mathcal{B})$ and $L^2(0,T;\mathcal{B}) = H^0(0,T;\mathcal{B})$. When no risk of confusion exists we shall write $L^2(\mathcal{B})$ for $L^2(0,T;\mathcal{B})$.

Further, we denote $L^\infty(0,T;\mathcal{B})$ to be the collection of all functions $v \in \mathcal{B}$ such that

$$\operatorname*{ess\ sup}_{t \in (0,T]} \|v(x,t)\|_{\mathcal{B}} < \infty.$$

Below, we shall discuss some preliminary materials which will be of frequent use in error analysis in the subsequent chapters. The bilinear form $A(\cdot,\cdot)$ associated with the operator $\mathcal{L}$, given by

$$A(u,v) = \int_\Omega \beta(x) \nabla u \cdot \nabla v dx,$$

satisfies the following boundedness and coercive properties: For $\phi, \psi \in H^1(\Omega)$, there exists positive constants C and c such that

$$A(\phi,\psi) \leq C \|\phi\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)}$$

and

$$A(\phi,\phi) \geq c \|\phi\|_{H^1(\Omega)}^2.$$

From time to time we shall also use the following inequalities (see, Hardy *et al.* [28]):

(*i*) Young's inequality: For $a, b \geq 0$ and $\epsilon > 0$, the following inequality

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$$

holds.

(ii) Cauchy-Schwarz inequality: For $a, b \geq 0,\ 1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

In integral form , if ϕ and ψ are both real valued and $\phi \in L^p$ and $\psi \in L^q$, then

$$\int_\Omega \phi\psi \leq \|\phi\|_p \|\psi\|_q.$$

For $p = q = 2$, the above inequality is known as **Schwarz's inequality**. The discrete version of Schwarz's inequality may be stated as:

(iii) Let $\phi_j, \psi_j, j = 1, 2, \ldots, n$ be positive real numbers. Then

$$\sum_{j=1}^{n} \phi_j \psi_j \leq \left(\sum_{j=1}^{n} \phi_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \psi_j^2 \right)^{\frac{1}{2}}$$

Below, we state without proof, the following two versions of Grownwall's lemma. For a proof, see [40].

Lemma 1.2.1 **(Continuous Gronwall's Lemma)** *Let $G(t)$ be a continuous function and $H(t)$ a nonnegative continuous function on its interval $t_0 \leq t \leq t_0 + a$. If a continuous function $F(t)$ has the property*

$$F(t) \leq G(t) + \int_{t_0}^{t} F(s)H(s)ds \ \text{ for } t \in [t_0, t_0 + a],$$

then

$$F(t) \leq G(t) + \int_{t_0}^{t} G(s)H(s)exp\left[\int_{s}^{t} H(\tau)d\tau\right] ds \ \text{ for } t \in [t_0, t_0 + a].$$

In particular, when $G(t) = C$ a nonnegative constant, we have

$$F(t) \leq Cexp\left[\int_{t_0}^{t} H(s)ds\right] \ \text{ for } t \in [t_0, t_0 + a].$$

Lemma 1.2.2 **(Discrete Gronwall's Lemma)** *If $\langle y_n \rangle$, $\langle f_n \rangle$ and $\langle g_n \rangle$ are non-negative sequences and*

$$y_n \leq f_n + \sum_{0 \leq k < n} g_k y_k, \quad n \geq 0,$$

then

$$y_n \leq f_n + \sum_{0 \leq k < n} g_k f_k \exp\left(\sum_{k < j < n} g_j \right), \quad n \geq 0.$$

In addition, we shall also work on the following spaces:

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2) \quad \& \quad Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2)$$

equipped with the norms

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \sum_{i=1}^{2} \|v\|_{H^2(\Omega_i)} \;\; \& \;\; \|v\|_Y = \|v\|_{L^2(\Omega)} + \sum_{i=1}^{2} \|v\|_{H^1(\Omega_i)},$$

respectively.

We now turn to the literature concerning the regularity of elliptic, parabolic and hyperbolic problems with discontinuous coefficients. Due to the presence of discontinuous coefficients the solution u, in general, does not belong to $H^2(\Omega)$ even if the coefficients are sufficiently smooth in each individual subdomain Ω_i, $i = 1, 2$. Concerning the elliptic interface problems, we have the following regularity result. For a proof, see Chen and Zou [11], and Ladyzhenskaya *et al.* [30].

Theorem 1.2.1 *Let $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma)$. Then the problem (1.1.1)-(1.1.3) has a unique solution $u \in X \cap H_0^1(\Omega)$ and u satisfies a priori estimate*

$$\|u\|_X \leq C \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)} \right).$$

Regarding the parabolic interface problems (1.1.4)-(1.1.6), we have the following regularity result (cf. [11, 30]).

Theorem 1.2.2 *Let $f \in H^1(0, T; L^2(\Omega))$, $g \in H^1(0, T; H^{\frac{1}{2}}(\Gamma))$ and $u_0 \in H_0^1(\Omega)$. Then the problem (1.1.4)-(1.1.6) has a unique solution $u \in L^2(0, T, X) \cap H^1(0, T, Y) \cap H_0^1(\Omega)$.*

We now recall the following regularity result for the solution u of the interface problem (1.1.7)-(1.1.9) (cf. [13, 30]).

Theorem 1.2.3 *Let u_0, $v_0 \in H_0^1(\Omega)$. Then the problem (1.1.7)-(1.1.9) has a unique solution $u \in L^2(0, T; X \cap H_0^1(\Omega)) \cap H^1(0, T; H^2(\Omega_1) \cap H^2(\Omega_2)) \cap H^2(0, T; Y)$.*

1.3 A Brief Survey on Numerical Methods

In this section, we shall discuss a brief survey of the relevant literature concerning of elliptic, parabolic and hyperbolic interface problems.

Solving differential equations with discontinuous coefficients by means of classical finite element methods usually leads to the loss in accuracy (cf. [5, 11]). One major difficulty is that the solution has low global regularity and the elements do not fit with the interface of general shape. For non-interface problems, one can assume full regularities of the solutions (at least $H^2(\Omega)$) on whole physical domain. But for the interface problems, the global regularity of the solution is low. So the classical analysis is difficult to apply for the convergence analysis of the interface problems. Thus the numerical solution to the interface problem is challenging as well as interesting also.

The standard finite difference and finite element methods may not be successful in giving satisfactory numerical results for such problems. Hence, many new methods have been developed. Some of them are developed with the modifications in the standard methods, so that they can deal with the discontinuities and the singularities. For the literature on the recent developments of the numerical methods for such problems, we refer to [15, 35] which includes extensive list of relevant literature. The numerical solutions of interface problems by means of finite element Galerkin procedures have been investigated by several authors. One of the first finite element methods treating interface problem has been studied by Babuška in [5]. In [5], the author has formulated the problem as an equivalent minimization problem and then finite element methods are used to solve the minimization problem. Under some approximation assumptions on finite element spaces, Babuška has obtained sub-optimal order error estimate in H^1 norm. The algorithm in [5] requires the exact evaluation of line integrals on the boundary of the domain and on the interface, and exact integrals on the interface finite elements are also needed. In the absence of variational crimes, finite element approximation of interface problem has been studied by Barrett and Elliott in [6]. They have shown that the finite element solution converges to the true solution at optimal rate in L^2 and H^1 norms over any interior subdomain. In [6], it is assumed that the solution and the normal derivatives of the solution are continuous along the interface, and fourth order differentiable on each subdomain. For the problems (1.1.1)-(1.1.3), Bramble and King [8] have considered a finite element method in which the domains Ω_1 and Ω_2 are replaced by polygonal domains $\Omega_{1,h}$ and $\Omega_{2,h}$, respectively. Then, the Dirichlet data and the interface function are transferred to the polygonal boundaries. Finally, discontinuous Galerkin finite element method is applied to the perturbed problem defined on the polygonal domains.

Optimal order error estimates are derived for rough as well as smooth boundary data. Under practical regularity assumptions on the true solution, the convergence of conforming finite element method is studied in [11], [37] and [43]. In [11], Chen and Zou have considered a practical piecewise linear finite element approximation for solving second order elliptic interface problem with $\mathcal{L}u = -\nabla.(\beta\nabla u)$ in a polygonal domain, where the coefficient β is assumed to be positive and piecewise constant in each subdomains. They have proved almost optimal order of convergence in L^2 and energy norms. More precisely, the error bounds obtained by Chen and Zou [11] are optimal up to the factor $\log h$. Under the assumptions on the source term $f|_{\Omega_1} = 0$ and the interface function $g = 0$, Neilsen [37] has proved optimal order of convergence in H^1 norm in the presence of arbitrarily small ellipticity. The algorithm in [37] requires that the interface triangles follow exactly the actual interface Γ. In [43], the finite element solution converges to the exact solution at an optimal rate in L^2 and H^1 norms if the grid lines coincide with the actual interface by allowing interface triangles to be curved triangles. Further, if the grid lines form an approximation to the actual interface, optimal order of convergence in H^1 norm and sub-optimal order in L^2 norm are derived for elliptic problems. More recently, in [16], the author has discussed quadrature finite element method for elliptic interface problems in a two dimensional convex polygonal domain. Optimal order error estimates in L^2 and H^1 norms are derived for a practical finite element discretization with straight interface triangles.

We now turn to the finite element Galerkin approximation to parabolic interface problems (1.1.4)-(1.1.6). Although a good number of articles is devoted to the finite element approximation of elliptic interface problems, the literature seems to lack concerning the convergence of finite element solutions to the true solutions of parabolic interface problems (1.1.4)-(1.1.6). For the backward Euler time discretization, Chen and Zou [11] have studied the convergence of fully discrete solution to the exact solution using fitted finite element methods. They have proved almost optimal error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms when global regularity of the solution is low. Then an essential improvement was made in [21]. The authors of [21] have used a finite element discretization where interface triangles are assumed to be curved triangles instead of straight triangles like classical finite element methods. Optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are shown to hold for both semi discrete and fully discrete scheme in [21]. More recently, for similar triangulation, Deka and Sinha ([19]) have studied the pointwise-in-time convergence in finite element method for parabolic interface problems. They have shown optimal error estimates in $L^\infty(H^1)$ and $L^\infty(L^2)$ norms under the assumption that grid line exactly follow the actual interface. Similar results are also obtained by Attanayake and Senaratne in [4] for immersed finite element method.

Finally, we turn to the numerical methods for hyperbolic interface problems (1.1.7)-(1.1.9). Numerical solutions of hyperbolic equations with discontinuous coefficients draws significant attention in a variety of fields such as the oil exploration industry and mineral finding as well as the study of earthquakes. Numerical simulation of seismic wave propagation problems in heterogeneous media can be traced back to as early as Alterman and Karal([3]) in 1968 and Boore([7]) in 1972. Alterman and Karal developed a finite difference scheme to solve the equations of elasticity in one spatial dimension and they applied their scheme to the problem of a layered half space with a buried point source emitting a compressional pulse. The interface between different layers was placed at $z = h$ on the grid line, where z is the coordinate representing the depth below the surface of the Earth. A general introduction on the numerical treatment for hyperbolic interface problems by means of finite difference method can be found in Le Veque's Book [31]. Three numerical schemes namely Wendroff, TVD and WENO have been discussed in [31]. These schemes use values of the sound speed on discrete points or averaged values on grid cells. As a consequence, they do not describe accurately the position and the shape of interfaces cutting grid cells. Furthermore, due to low regularity of the true solution the method leads to the loss in accuracy near the interface. It is then a new approach called explicit jump immersed interface method was introduced in [48]. These numerical methods ensure a given accuracy at grid points near interface, but they are difficult to implement with higher order schemes. To overcome this difficulty an explicit simplified interface method was introduced by Piraux *et al.* in [39] for one dimensional acoustic velocity and acoustic pressure.

1.4 Objectives

This section elucidates our contributions and motivation for the present study. The physical world is replete with examples of free surfaces, material interface and moving boundaries that interact with a surrounding fluid. There are interfaces that separate air and water (in the case of bubbles or free surface flows) and boundaries between two materials of different physical properties (in porous media flow or mixing layers). While the mathematical modelling of the interaction is a difficult problem in itself, another formidable task is developing a numerical method that solves these problems effectively and efficiently.

The analysis of finite element methods for interface problem has become an active research area over the years. The main objective of this work is to establish some new optimal a priori error estimates in fitted finite element method for interface problem with straight interface triangles. The achieved estimates are analogous to the case with a regular solution, however, due to low regularity, the proof requires a careful technical work coupled with a approximation result for the linear interpolant. Other technical tools used in this work are Sobolev embedding inequality, approximations properties for modified elliptic projection, modified duality arguments and some known results on elliptic interface problems.

In the present work, optimal order error estimates in L^2 and H^1 norms are derived for the linear elliptic interface problems (c.f. [17]) and which improve the earlier results in the articles [11] and [43]. Then the results are extended for parabolic interface problems and optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are achieved (c.f. [18]).

Due to low global regularity of the solutions, the error analysis of the standard finite element method for parabolic problems is difficult to adopt for parabolic interface problems. In this work, we are able to fill a theoretical gap between standard energy technique of finite element method for non interface problems and parabolic interface problems. Optimal $L^\infty(H^1)$ and $L^\infty(L^2)$ norms error estimates have been derived under practical regularity assumption of the true solution (c.f. [20]).

Although various FEM for elliptic and parabolic interface problems have been proposed and studied in the literature, but FEM treatment of similar hyperbolic problems is mostly missing. In this work, we are able to prove optimal order pointwise-in-time error estimates in L^2 and H^1 norms for the hyperbolic interface problem with semidiscrete scheme. Fully discrete scheme based on a symmetric difference approximation is also analyzed and optimal $L^\infty(H^1)$ norm error is obtained.

1.5 Organization of the Thesis

The organization of the thesis is as follows: Chapter 2 deals with the error analysis for elliptic interface problems in two dimensional convex polygonal domains. Optimal order error estimates in L^2 and H^1 norms are derived for a practical finite element discretization.

Chapter 3 is devoted to the convergence of finite element method for parabolic interface problems with straight interface triangles. The proposed method yields optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms for semi-discrete scheme. Convergence of fully discrete solution is also discussed and optimal error estimate in $L^2(H^1)$ norm is achieved.

In Chapter 4, we analyze the continuous time Galerkin method for spatially discrete scheme for parabolic interface problems. Optimal $L^\infty(H^1)$ and $L^\infty(L^2)$ norms error estimates have been derived under practical regularity assumption of the true solution. Further, the fully discrete scheme based on backward Euler method is also proposed and analyzed. Optimal L^2 norm error estimate is obtained for fully discrete scheme.

Chapter 5 is concerned with a priori error estimates for hyperbolic interface problems. Optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$ norms are established for continuous time discretization. Further, the fully discrete scheme based on a symmetric difference approximation is considered and optimal order convergence in H^1 norm is established.

Finally, numerical results are presented for two dimensional test problems in Chapter 6 for the completeness of this work.

For clarity of presentation we have repeatedly given equations (1.1.1) – (1.1.3) or (1.1.4) – (1.1.6) or (1.1.7) – (1.1.9) at the beginning of subsequent chapters.

Chapter 2

Finite Element Methods for Elliptic Interface Problems

In this chapter, we have discussed the convergence of finite element solution to the exact solution of elliptic interface problem. For a finite element discretization based on a mesh which involve the approximation of the interface, optimal order error estimates in L^2 and H^1 norms are achieved under practical regularity assumptions of the true solution.

2.1 Introduction

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Let Γ be the C^2 smooth boundary of the open domain $\Omega_1 \subset \Omega$ and $\Omega_2 = \Omega \setminus \Omega_1$. We recall the following linear elliptic interface problems of the form

$$\mathcal{L}u = f(x) \quad \text{in } \Omega \tag{2.1.1}$$

with Dirichlet boundary condition

$$u(x) = 0 \quad \text{on } \partial\Omega \tag{2.1.2}$$

and interface conditions

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = 0 \quad \text{along } \Gamma. \tag{2.1.3}$$

Here, $f = f(x)$ is a real valued function in Ω. The operator $\mathcal{L}$, symbols $[v]$ and $\mathbf{n}$ are defined as in Chapter 1.

As a first step towards finite element approximation for the elliptic interface problem (2.1.1)-(2.1.3), we recall the space $H_0^1(\Omega) = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$. Then weak formulation of the problem (2.1.1)-(2.1.3) may be stated as: Find $u \in H_0^1(\Omega)$ such that u satisfies

$$A(u, v) = (f. v) \quad \forall\, v \in H_0^1(\Omega), \tag{2.1.4}$$

where $(\cdot, \cdot)$ denotes the inner product of the $L^2(\Omega)$ space.

The solution $u \in X \cap H_0^1(\Omega)$ satisfies the following a priori estimate (cf. [11])

$$\|u\|_X \leq C\|f\|_{L^2(\Omega)}. \tag{2.1.5}$$

The main objective of this chapter is to extend the results of quadrature based finite element method discussed in [16]. The main crucial technical tools used in our analysis are some Sobolev embedding inequality, approximations properties for linear interpolation operator, duality arguments, some known results on elliptic interface problems and some auxiliary projections. For the earlier works on finite element approximation to elliptic interface problems, we refer to Chapter 1.

The organization of this chapter is as follows. In section 2.2, we describe the finite element discretization and some known results for elliptic interface problems. Finally, in section 2.3 error estimates for linear elliptic interface problem are presented.

2.2 Finite Element Discretization

For the purpose of finite element approximation of the problems (2.1.1)-(2.1.3), we now describe the triangulation $\mathcal{T}_h$ of Ω as follows. We first approximate the domain Ω_1 by a domain Ω_1^h with the polygonal boundary Γ_h whose vertices all lie on the interface Γ. Let Ω_2^h be the approximation for the domain Ω_2 with polygonal exterior and interior boundaries as $\partial\Omega$ and Γ_h, respectively.

Triangulation $\mathcal{T}_h$ of the domain Ω satisfy the following conditions:

($\mathcal{A}1$) $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$.

($\mathcal{A}2$) If $K_1,\ K_2 \in \mathcal{T}_h$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is common vertex or edge of both triangles.

($\mathcal{A}3$) Each triangle $K \in \mathcal{T}_h$ is either in Ω_1^h or Ω_2^h and intersects Γ (interface) in at most two points.

($\mathcal{A}4$) For each triangle $K \in \mathcal{T}_h$, let h_K be the length of the largest side. Let $h = \max\{h_K : K \in \mathcal{T}_h\}$.

The triangles with one or two vertices on Γ are called the interface triangles, the set of all interface triangles is denoted by $\mathcal{T}_\Gamma^*$ and we write $\Omega_\Gamma^* = \cup_{K\in\mathcal{T}_\Gamma^*} K$.

Let V_h be a family of finite dimensional subspaces of $H_0^1(\Omega)$ defined on $\mathcal{T}_h$ consisting of piecewise linear functions vanishing on the boundary $\partial\Omega$. Examples of such finite element spaces can be found in [9] and [12].

For the coefficients $\beta(x)$, we define its approximation $\beta_h(x)$ as follows: For each triangle $K \in \mathcal{T}_h$, let $\beta_K(x) = \beta_i$ if $K \subset \Omega_i^h$, i=1 or 2. Then β_h is defined as

$$\beta_h(x) = \beta_K(x) \quad \forall K \in \mathcal{T}_h.$$

Then the finite element approximation to (2.1.4) is stated as follows: Find $u_h \in V_h$ such that

$$A_h(u_h, v_h) = (f, v_h) \quad \forall\, v_h \in V_h, \tag{2.2.1}$$

where $A_h(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ is defined as

$$A_h(w,v) = \sum_{K\in\mathcal{T}_h} \int_K \beta_K(x)\nabla w \cdot \nabla v dx \quad \forall w, v \in H^1(\Omega).$$

The following lemmas will be useful for our future analysis. For a proof, we refer to [45].

Lemma 2.2.1 *For $w_h, v_h \in V_h$, we have*

$$|A_h(w_h, v_h) - A(w_h, v_h)| \;\leq\; Ch \sum_{K\in\mathcal{T}_\Gamma^*} \|\nabla v_h\|_{L^2(K)} \|\nabla w_h\|_{L^2(K)}.$$

Lemma 2.2.2 *If Ω_Γ^* is the union of all interface triangles, then we have*

$$\|u\|_{H^1(\Omega_\Gamma^*)} \leq Ch^{\frac{1}{2}} \|u\|_X.$$

Let $\Pi_h : C(\overline{\Omega}) \rightarrow V_h$ be the Lagrange interpolation operator corresponding to the space V_h. As the solutions concerned are only in $H^1(\Omega)$ globally, one cannot apply

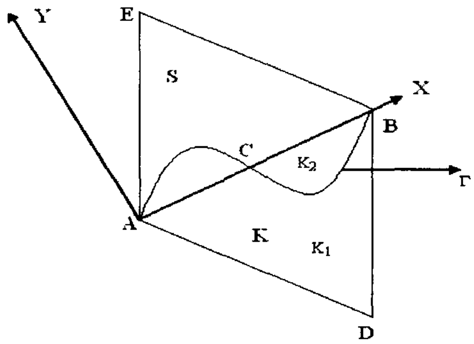

Figure 2.1: *Interface Triangles K, S, along with interface Γ and its approximation Γ_h.*

the standard interpolation theory directly. However, following the argument of [11] it is possible to obtain optimal error bounds for the interpolant Π_h (see Chapter 3, [15]). In [15], the authors have assumed that the solution $u \in X \cap W^{1,\infty}(\Omega_1 \cap \Omega_0) \cap W^{1,\infty}(\Omega_2 \cap \Omega_0)$, where Ω_0 is some neighborhood of the interface Γ. The following lemma shows that optimal approximation of Π_h can be derived for $u \in X$ with $[u] = 0$ along interface Γ.

Lemma 2.2.3 *Let $\Pi_h : C(\overline{\Omega}) \to V_h$ be the linear interpolation operator and u be the solution for the interface problem (2.1.1)-(2.1.3), then the following approximation properties*

$$\|u - \Pi_h u\|_{H^m(\Omega)} \leq Ch^{2-m}\|u\|_X, \; m = 0, 1,$$

hold true.

Proof. For any $v \in X$, let v_i be the restriction of v on Ω_i for $i = 1, 2$. As the interface is of class C^2, we can extend the function $v_i \in H^2(\Omega_i)$ on to the whole Ω and obtain the function $\tilde{v}_i \in H^2(\Omega)$ such that $\tilde{v}_i = v_i$ on Ω_i and

$$\|\tilde{v}_i\|_{H^2(\Omega)} \leq C\|v_i\|_{H^2(\Omega_i)}, i = 1, 2. \tag{2.2.2}$$

For the existence of such extensions, we refer to Stein [46]. Further, we have a C^2 function ϕ in $[C, B]$ such that (c.f. [24])

$$|\phi(x)| \leq Ch^2 \tag{2.2.3}$$

and hence

$$\text{meas}(K_2) = \int_C^B |\phi(x)|dx \leq Ch^2 \int_C^B dx \leq Ch^3.$$

Then, for $K \in \mathcal{T}_h$, we now define

$$\Pi_h u = \begin{cases} \Pi_h \tilde{u}_1 & \text{if } K \subseteq \Omega_1^h \\ \Pi_h \tilde{u}_2 & \text{if } K \subseteq \Omega_2^h. \end{cases}$$

Then it is easy to verify that $\Pi_h u \in V_h$ (cf. [17]).

Now, for any triangle $K \in \mathcal{T}_h \backslash \mathcal{T}_\Gamma^*$, the standard finite element interpolation theory (cf. [9, 12]) implies that

$$\|u - \Pi_h u\|_{H^m(K)} \le Ch^{2-m}\|u\|_{H^2(K)}, \ m = 0, 1. \tag{2.2.4}$$

For any element $K \in \mathcal{T}_\Gamma^*$, we write $K_i = K \cap \Omega_i, i = 1, 2$, for our convenience. Further, using the Hölder's inequality and the fact meas$(K_2) \le Ch^3$ we derive that for any $p > 2$, and $m = 0, 1$,

$$\begin{aligned} \|u - \Pi_h u\|_{H^m(K_2)} &\le Ch^{\frac{3(p-2)}{2p}}\|u - \Pi_h u\|_{W^{m,p}(K_2)} \\ &\le Ch^{\frac{3(p-2)}{2p}}\|u - \Pi_h u\|_{W^{m,p}(K)} \\ &\le Ch^{\frac{3(p-2)}{2p}+1-m}\|u\|_{W^{1,p}(K)}, \end{aligned} \tag{2.2.5}$$

in the last inequality, we used the standard interpolation theory (cf. [12]). On the other hand

$$\begin{aligned} \|u - \Pi_h u\|_{H^m(K_1)} &= \|\tilde{u}_1 - \Pi_h \tilde{u}_1\|_{H^m(K_1)} \\ &\le C\|\tilde{u}_1 - \Pi_h \tilde{u}_1\|_{H^m(K)} \\ &\le Ch^{2-m}\|\tilde{u}_1\|_{H^2(K)} \\ &\le Ch^{2-m}\|u\|_X, \end{aligned} \tag{2.2.6}$$

in the last inequality, we used (2.2.2). In view of (2.2.5)-(2.2.6), it now follows that

$$\begin{aligned} &\|u - \Pi_h u\|^2_{H^m(\Omega_\Gamma^*)} \\ &\le Ch^{4-2m}\|u\|_X^2 + C\sum_{K \in \mathcal{T}_\Gamma^*} h^{\frac{3(p-2)}{p}+2-2m}\|u\|^2_{W^{1,p}(K)} \\ &\le Ch^{4-2m}\|u\|_X^2 + C\sum_{K \in \mathcal{T}_\Gamma^*} h^{5-2m-\frac{6}{p}}\|u\|^2_{W^{1,p}(K)} \\ &\le Ch^{4-2m}\|u\|_X^2 + C\sum_{K \in \mathcal{T}_\Gamma^*} h^{5-2m-\frac{6}{p}}\{\|u\|^2_{W^{1,p}(K_1)} + \|u\|^2_{W^{1,p}(K_2)}\} \\ &\le Ch^{4-2m}\|u\|_X^2 + C\sum_{K \in \mathcal{T}_\Gamma^*} h^{5-2m-\frac{6}{p}}\{\|\tilde{u}_1\|^2_{W^{1,p}(K_1)} + \|\tilde{u}_2\|^2_{W^{1,p}(K_2)}\}. \end{aligned} \tag{2.2.7}$$

We now recall Sobolev embedding inequality for two dimensions (cf. Ren and Wei [42])

$$\|v\|_{L^p(\Omega)} \leq Cp^{\frac{1}{2}}\|v\|_{H^1(\Omega)} \;\; \forall v \in H^1(\Omega), \;\; p > 2. \tag{2.2.8}$$

Now, setting $p = 6$ in the Sobolev embedding inequality (2.2.8), we obtain

$$\begin{aligned} \|\tilde{u}_i\|_{L^6(K_i)} &\leq \|\tilde{u}_i\|_{L^6(\Omega_i)} \leq C\|\tilde{u}_i\|_{H^1(\Omega_i)}, \\ \|\nabla\tilde{u}_i\|_{L^6(K_i)} &\leq \|\nabla\tilde{u}_i\|_{L^6(\Omega_i)} \leq C\|\nabla\tilde{u}_i\|_{H^1(\Omega_i)}. \end{aligned}$$

In view of the above estimates, it now follows that

$$\|\tilde{u}_i\|_{W^{1,6}(K_i)} \;\; \leq \;\; C\|\tilde{u}_i\|_{H^2(\Omega_i)}.$$

This together with (2.2.7), we have

$$\|u - \Pi_h u\|^2_{H^m(\Omega^*_\Gamma)} \leq Ch^{4-2m}\|u\|^2_X, \;\; m = 0, 1. \tag{2.2.9}$$

Then Lemma 2.2.3 follows immediately from the estimates (2.2.4) and (2.2.9). □

2.3 Convergence Analysis for Elliptic Interface Problem

In this section, we will establish some new optimal error estimates for linear elliptic interface problem which will be useful in the subsequent error analysis of parabolic interface problems.

From (2.1.4) and (2.2.1), we note that

$$\begin{aligned} A(u_h - \Pi_h u, v_h) &= A(u - \Pi_h u, v_h) \\ &\quad + \{A(u_h, v_h) - A_h(u_h, v_h)\} \\ &\equiv: (I)_1 + (I)_2 \end{aligned} \tag{2.3.1}$$

By Lemma 2.2.3, we can bound the term $(I)_1$ by

$$\begin{aligned} |(I)_1| &\leq C\|u - \Pi_h u\|_{H^1(\Omega)}\|\nabla v_h\|_{L^2(\Omega)} \\ &\leq Ch\|u\|_X\|v_h\|_{H^1(\Omega)} \end{aligned} \tag{2.3.2}$$

For the term $(I)_2$, use Lemma 2.2.1 to have

$$
\begin{aligned}
|(I)_2| &\le Ch\|\nabla u_h\|_{L^2(\Omega)}\|\nabla v_h\|_{L^2(\Omega)} \\
&\le Ch\|\nabla u_h\|_{L^2(\Omega)}\|v_h\|_{H^1(\Omega)} \\
&\le Ch\|f\|_{L^2(\Omega)}\|v_h\|_{H^1(\Omega)} \qquad (2.3.3)
\end{aligned}
$$

where we have used the inequality

$$\|\nabla u_h\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$$

which follows directly from (2.2.1) by taking $v_h = u_h$ and using coercivity.

From the estimates (2.3.2)-(2.3.3), we conclude by taking $v_h = u_h - \Pi_h u$ in (2.3.1) that

$$\|u_h - \Pi_h u\|_{H^1(\Omega)} \le Ch(\|u\|_X + \|f\|_{L^2(\Omega)}). \qquad (2.3.4)$$

The above estimate (2.3.4) together with Lemma 2.2.3 and (2.1.5) leads to the following optimal order error estimate in H^1 norm.

Theorem 2.3.1 *Let u and u_h be the solutions of the problem (2.1.1)-(2.1.3) and (2.2.1), respectively. Then, for $f \in L^2(\Omega)$, the following H^1-norm error estimate holds*

$$\|u - u_h\|_{H^1(\Omega)} \le Ch\|f\|_{L^2(\Omega)}. \quad \square$$

For the L^2 norm error estimate we shall use the Nitsche's trick. We consider the following elliptic interface problem

$$-\nabla \cdot (\beta \nabla w) = u - u_h \quad \text{in } \Omega$$

with Dirichlet boundary condition

$$w(x) = 0 \quad \text{on } \partial\Omega$$

and interface conditions

$$[w] = 0, \quad \left[\beta \frac{\partial w}{\partial \mathbf{n}}\right] = 0 \quad \text{along } \Gamma.$$

Then clearly $w \in X \cap H_0^1(\Omega)$ and satisfies the weak form

$$A(w, v) = (u - u_h, v) \quad \forall v \in H_0^1(\Omega). \qquad (2.3.5)$$

Further, w satisfies the a priori estimate (cf. [11])

$$\|w\|_X \le C\|u - u_h\|_{L^2(\Omega)}. \tag{2.3.6}$$

We then define its finite element approximation to be the function $w_h \in V_h$ such that

$$A_h(w_h, v_h) = (u - u_h, v_h) \quad \forall v_h \in V_h. \tag{2.3.7}$$

Arguing as in the derivation of Theorem 2.3.1 and further using the a priori estimate (2.3.6), we have

$$\|w - w_h\|_{H^1(\Omega)} \le Ch\|u - u_h\|_{L^2(\Omega)}. \tag{2.3.8}$$

Setting $v = u - u_h \in H_0^1(\Omega)$ in (2.3.5) and using (2.1.4) and (2.2.1), we obtain

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega)}^2 &= A(w, u - u_h) \\ &= A(w - w_h, u - u_h) + A(w_h, u - u_h) \\ &= A(w - w_h, u - u_h) + \{A_h(u_h, w_h) - A(u_h, w_h)\} \\ &\equiv: (II)_1 + (II)_2 \end{aligned} \tag{2.3.9}$$

By Theorem 2.3.1 and (2.3.8) we immediately have

$$|(II)_1| \le Ch^2\|f\|_{L^2(\Omega)}\|u - u_h\|_{L^2(\Omega)} \tag{2.3.10}$$

Arguing as deriving (2.3.3) we can deduce

$$\begin{aligned} |(II)_2| &\le Ch \sum_{K \in \mathcal{T}_\Gamma^*} \|\nabla u_h\|_{L^2(K)}\|\nabla w_h\|_{L^2(K)} \\ &\le Ch\|\nabla u_h\|_{L^2(\Omega_\Gamma^*)}\|\nabla w_h\|_{L^2(\Omega_\Gamma^*)} \\ &\le Ch\|\nabla(u - u_h)\|_{L^2(\Omega_\Gamma^*)}\|\nabla w_h\|_{L^2(\Omega_\Gamma^*)} \\ &\quad + Ch\|\nabla u\|_{L^2(\Omega_\Gamma^*)}\|\nabla(w - w_h)\|_{L^2(\Omega_\Gamma^*)} \\ &\quad + Ch\|\nabla u\|_{L^2(\Omega_\Gamma^*)}\|\nabla w\|_{L^2(\Omega_\Gamma^*)} \\ &\le Ch^2\|f\|_{L^2(\Omega)}\|u - u_h\|_{L^2(\Omega)} \\ &\quad + Ch^{\frac{5}{2}}\|u\|_X\|u - u_h\|_{L^2(\Omega)} + Ch^2\|u\|_X\|w\|_X \end{aligned}$$

where we have used Theorem 2.3.1, Lemma 2.2.2 and (2.3.8), and the following inequality

$$\|\nabla w_h\|_{L^2(\Omega)} \le C\|u - u_h\|_{L^2(\Omega)}.$$

Thus, for the term $(II)_2$, we have

$$\begin{aligned} |(II)_2| &\le Ch^2\|f\|_{L^2(\Omega)}\|u-u_h\|_{L^2(\Omega)} \\ &\quad +Ch^2\|u\|_X\|u-u_h\|_{L^2(\Omega)} \\ &\le Ch^2\|f\|_{L^2(\Omega)}\|u-u_h\|_{L^2(\Omega)}. \end{aligned} \tag{2.3.11}$$

Finally using (2 3 10)-(2.3.11) in (2.3.9), we obtain

$$\|u-u_h\|^2_{L^2(\Omega)} \le Ch^2\|f\|_{L^2(\Omega)}\|u-u_h\|_{L^2(\Omega)}.$$

Thus, we have proved the following optimal order estimates in L^2 norm.

Theorem 2.3.2 *Let u and u_h be the solutions of the problem (2.1.1)-(2.1.3) and (2.2 1), respectively. Then, for $f \in L^2(\Omega)$, there exist a positive constant C independent of h such that*

$$\|u-u_h\|_{L^2(\Omega)} \le Ch^2\|f\|_{L^2(\Omega)}. \quad \square$$

Remark 2.3.1 *Under certain hypotheses. the error of approximation of solutions of certain nonlinear problems is basically the same as the error of approximation of solutions of related linear problems [10, 26]. Therefore an essential improvement of the results of [11] for the linear elliptic interface problems have been obtained in this work. Further, the results are also extended for the semilinear problems (cf. [17])*

$$A(u,v) = (f(u),v) \quad \forall\, v \in H^1_0(\Omega).$$

Chapter 3

$L^2(L^2)$ and $L^2(H^1)$ norms Error Estimates for Parabolic Interface Problems

In this chapter, we extend the finite element analysis of elliptic interface problems discussed in Chapter 2 to parabolic interface problems. Optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are derived for the linear parabolic interface problems.

3.1 Introduction

In this chapter, we consider a linear parabolic interface problem of the form

$$u_t + \mathcal{L}u = f(x,t) \quad \text{in } \Omega \times (0,T] \tag{3.1.1}$$

with initial and boundary conditions

$$u(x,0) = u_0 \ \text{ in } \Omega; \ \ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T] \tag{3.1.2}$$

and jump conditions on the interface

$$[u] = 0, \ \ \left[\beta\frac{\partial u}{\partial \mathbf{n}}\right] = g(x,t) \quad \text{along } \Gamma, \tag{3.1.3}$$

where, $f = f(x,t)$ and $g = g(x,t)$ are real valued functions in $\Omega \times (0,T]$, and $u_t = \frac{\partial u}{\partial t}$. Further, $u_0 = u_0(x)$ is real valued function in Ω. The domain Ω, operator $\mathcal{L}$, symbols $[v]$ and $\mathbf{n}$ are defined as in Chapter 1, and T $< \infty$.

To derive $O(h^m)$ $(m \geq 0)$ error estimates for non-interface parabolic problems in the literature generally require $u \in L^2(0,T;H^{m+1}(\Omega)) \cap H^1(0,T;H^{m-1}(\Omega))$, see, [47]. The purpose of the present chapter is to extend the convergence analysis of fitted finite element method for elliptic interface problems to parabolic interface problems. The convergence of finite element solution to the exact solution has been discussed in terms of $L^2(H^1)$ and $L^2(L^2)$ norms. The main crucial technical tools used in our analysis are Sobolev embedding inequality, approximation result for the linear interpolant and elliptic projection (see, Lemma 3.2.2), parabolic duality arguments and some known results on elliptic interface problems. The previous work on finite element analysis for parabolic interface problems can be found in Chapter 1.

The outline of this chapter is as follows. In section 3.2, the approximation properties related to the auxiliary projections ar presented and section 3.3 is devoted to the error analysis for the semidiscrete scheme. Finally, in section 3.4, a fully discrete scheme based on backward Euler method is proposed and optimal $L^2(H^1)$ norm is established.

3.2 Preliminaries

In this section, some approximation properties related to the auxiliary projection is obtained. Due to the presence of discontinuous coefficients the solution u, in general, does not belong to $H^2(\Omega)$. Regarding the regularity for the solution of the interface problem (3.1.1)-(3.1.3), we have the following result (cf. [11, 30, 44]).

Theorem 3.2.1 *Let $f \in H^1(0,T;L^2(\Omega))$, $g \in H^1(0,T;H^{\frac{1}{2}}(\Gamma))$ and $u_0 \in H_0^1(\Omega)$. Then the problem (3.1.1)-(3.1.3) has a unique solution $u \in L^2(0,T;X) \cap H^1(0,T;Y)$. Further, u satisfies the following a priori estimate*

$$\begin{aligned} \|u\|_{L^2(0,T;X)} \leq\ & C\Big\{\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{H^1(\Omega)} + \|g(x,0)\|_{H^{\frac{1}{2}}(\Gamma)} \\ & +\|g(x,T)\|_{H^{\frac{1}{2}}(\Gamma)} + \|g\|_{H^1(0,T;H^{\frac{1}{2}}(\Gamma))}\Big\}. \qquad (3.2.1) \end{aligned}$$

Now, we shall recall the finite element space $V_h \subset H_0^1(\Omega)$ consisting of piecewise linear polynomials vanishing on the boundary $\partial\Omega$ where interface triangles are straight triangles as discussed in Chapter 2. Further, we assume that V_h satisfy the inverse estimate

$$\|\phi\|_{H^1(\Omega)} \leq Ch^{-1}\|\phi\|_{L^2(\Omega)} \ \ \forall\, \phi \in V_h. \qquad (3.2.2)$$

Approximating the interface function $g(x)$ by its discrete specimen $g_h = \sum_{j=1}^{m_h} g(P_j)\Phi_j^h$, where $\{\Phi_j^h\}_{j=1}^{m_h}$ is the set of standard nodal basis functions corresponding to the nodes $\{P_j\}_{j=1}^{m_h}$ on the interface Γ, we have the following approximation result. For a proof, we refer to [11].

Lemma 3.2.1 *Let $g \in H^2(\Gamma)$. If Ω_Γ^* is the union of all interface triangles then we have*

$$\left| \int_\Gamma g v_h ds - \int_{\Gamma_h} g_h v_h ds \right| \le C h^{\frac{3}{2}} \|g\|_{H^2(\Gamma)} \|v_h\|_{H^1(\Omega_\Gamma^*)} \quad \forall v_h \in V_h.$$

We now define an operator $P_h : X \cap H_0^1(\Omega) \to V_h$ by

$$A_h(P_h v, \phi) = A(v, \phi) \quad \forall \phi \in V_h, \ v \in X \cap H_0^1(\Omega). \tag{3.2.3}$$

Earlier, in [11], the approximation results obtained for P_h in L^2 and H^1-norms are not optimal. However, the loss in accuracy for the H^1 norm is recovered in [45] under the assumption that the solution $u \in X \cap W^{1,\infty}(\Omega_1 \cap \Omega_0) \cap W^{1,\infty}(\Omega_2 \cap \Omega_0)$. The following lemma shows that optimal approximation of P_h in L^2 and H^1-norms can be derived for $u \in X \cap H_0^1(\Omega)$ only. This lemma is very crucial for our later analysis.

Lemma 3.2.2 *Having the projection P_h fixed in (3.2.3), there is a positive constant C independent of the mesh size parameter h such that*

$$\|u - P_h u\|_{L^2(\Omega)} + h\|u - P_h u\|_{H^1(\Omega)} \le Ch^2 \|u\|_X.$$

Proof. We first split $u - P_h u$ as

$$u - P_h u = (u - \Pi_h u) + (\Pi_h u - P_h u).$$

From Lemma 2.2.3 of Chapter 2 and (3.2.3), we note that

$$\begin{aligned}
&\|\Pi_h u - P_h u\|_{H^1(\Omega)}^2 \\
&\le A_h(\Pi_h u - u, \Pi_h u - P_h u) + A_h(u - P_h u, \Pi_h u - P_h u) \\
&\le Ch\|u\|_X \|\Pi_h u - P_h u\|_{H^1(\Omega)} + \{A_h(u, \Pi_h u - P_h u) - A(u, \Pi_h u - P_h u)\} \\
&= Ch\|u\|_X \|\Pi_h u - P_h u\|_{H^1(\Omega)} \\
&\quad + A_h(u - \Pi_h u, \Pi_h u - P_h u) - A(u - \Pi_h u, \Pi_h u - P_h u) \\
&\quad + \{A_h(\Pi_h u, \Pi_h u - P_h u) - A(\Pi_h u, \Pi_h u - P_h u)\} \\
&\le Ch\|u\|_X \|\Pi_h u - P_h u\|_{H^1(\Omega)} \\
&\quad + \{A_h(\Pi_h u, \Pi_h u - P_h u) - A(\Pi_h u, \Pi_h u - P_h u)\} \\
&\equiv: Ch\|u\|_X \|\Pi_h u - P_h u\|_{H^1(\Omega)} + (I).
\end{aligned} \tag{3.2.4}$$

Then using Lemma 2.2.1 of Chapter 2 for the term (I) to have

$$\begin{aligned}
|(I)| &\le Ch\|\Pi_h u\|_{H^1(\Omega)}\|\Pi_h u - P_h u\|_{H^1(\Omega)} \\
&\le Ch(\|\Pi_h u - u\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)})\|\Pi_h u - P_h u\|_{H^1(\Omega)} \\
&\le Ch\|u\|_X \|\Pi_h u - P_h u\|_{H^1(\Omega)}.
\end{aligned}$$

This in combination with (3.2.4) now leads to

$$\|\Pi_h u - P_h u\|_{H^1(\Omega)} \le Ch\|u\|_X.$$

By Lemma 2.2.3 and using triangle inequality, we obtain

$$\|u - P_h u\|_{H^1(\Omega)} \le Ch\|u\|_X. \tag{3.2.5}$$

For L^2-norm error estimate, we consider the following interface problem: Find $w \in H_0^1(\Omega)$ such that

$$A(w, v) = (u - P_h u, v) \;\; \forall v \in H_0^1(\Omega), \tag{3.2.6}$$

and let $w_h \in V_h$ be its finite element approximation such that

$$A_h(w_h, v_h) = (u - P_h u, v_h) \;\; \forall v_h \in V_h. \tag{3.2.7}$$

Note that $w \in H_0^1(\Omega)$ is the solution of (3.2.6) with jump conditions

$$[w] = 0 \quad \text{and} \quad \left[\beta \frac{\partial w}{\partial \mathbf{n}}\right] = 0 \;\; \text{along } \Gamma.$$

Then apply Theorem 2.3.2 for the above interface problem to have

$$\|w - w_h\|_{H^1(\Omega)} \le Ch\|w\|_X \le Ch\|u - P_h u\|_{L^2(\Omega)}. \tag{3.2.8}$$

In the last inequality, we have used regularity estimate for elliptic interface problem (3.2.6). Now, setting $v = u - P_h u$ in (3.2.6) and, using (3.2.5) and (3.2.8), we have

$$\begin{aligned}
\|u - P_h u\|^2_{L^2(\Omega)} &= A(w - w_h, u - P_h u) + A(w_h, u) - A(w_h, P_h u) \\
&= A(w - w_h, u - P_h u) + \{A(u, w_h) - A(P_h u, w_h)\} \\
&\equiv: A(w - w_h, u - P_h u) + (II) \\
&\le C\|w - w_h\|_{H^1(\Omega)}\|u - P_h u\|_{H^1(\Omega)} + (II) \\
&\le Ch^2\|u\|_X\|u - P_h u\|_{L^2(\Omega)} + (II).
\end{aligned} \tag{3.2.9}$$

For the term (II), we use (3.2.3) and Lemma 2.2.1 of Chapter 2 to have

$$\begin{aligned}|(II)| &= |A_h(P_h u, w_h) - A(P_h u, w_h)| \le Ch\|P_h u\|_{H^1(\Omega_\Gamma^*)}\|w_h\|_{H^1(\Omega_\Gamma^*)} \\ &\le Ch(\|P_h u - u\|_{H^1(\Omega_\Gamma^*)} + \|u\|_{H^1(\Omega_\Gamma^*)})(\|w_h - w\|_{H^1(\Omega_\Gamma^*)} + \|w\|_{H^1(\Omega_\Gamma^*)}) \\ &\le Chh^{\frac{1}{2}}\|u\|_X Ch^{\frac{1}{2}}\|w\|_X \le Ch^2\|u\|_X\|P_h u - u\|_{L^2(\Omega)}. \end{aligned} \tag{3.2.10}$$

In the last inequality, we have used Lemma 2.2.2 of Chapter 2. Then combining the estimates (3.2.9)-(3.2.10), we can conclude that

$$\|u - P_h u\|_{L^2(\Omega)} \le Ch^2\|u\|_X. \tag{3.2.11}$$

This completes the proof of Lemma 3.2.2. □

We need the standard L^2 projection $L_h : L^2(\Omega) \to V_h$ defined by

$$(L_h v, \phi) = (v, \phi) \quad \forall v \in L^2(\Omega), \quad \phi \in V_h, \tag{3.2.12}$$

satisfying the stability estimate

$$\|L_h v\|_{H^1(\Omega)} \le C\|v\|_{H^1(\Omega)} \quad \forall v \in H_0^1(\Omega). \tag{3.2.13}$$

It is well known that $L_h v \in V_h$ is the best approximation of $v \in L^2(\Omega)$ with respect to the L^2 norm. Thus Lemma 3.2.2 immediately implies

Lemma 3.2.3 *Let L_h be defined by (3.2.12). Then, for $m = 0, 1$, we have*

$$\|L_h v - v\|_{H^m(\Omega)} \le Ch^{2-m}\|v\|_X \quad \forall v \in H_0^1(\Omega) \cap X.$$

Proof. The L^2-norm estimate follows immediately from the fact that $L_h w \in V_h$ is the best approximation in the L^2 norm to $w \in L^2(\Omega)$ and Lemma 3.2.2. For H^1-norm estimate, we use the inverse inequality (3.2.2) and Lemma 3.2.2 to have

$$\begin{aligned}\|v - L_h v\|_{H^1(\Omega)} &\le \|v - P_h v\|_{H^1(\Omega)} + \|P_h v - L_h v\|_{H^1(\Omega)} \\ &\le Ch\|v\|_X + Ch^{-1}\|P_h v - L_h v\|_{L^2(\Omega)} \\ &\le Ch\|v\|_X + Ch^{-1}\{\|P_h v - v\|_{L^2(\Omega)} + \|v - L_h v\|_{L^2(\Omega)}\} \\ &\le Ch\|v\|_X.\end{aligned}$$

This completes the rest of the proof. □

3.3 Continuous time Galerkin Method

This section deals with the error analysis for the spatially discrete scheme for parabolic interface problems (3.1.1)-(3.1.3) and derive optimal error estimates in $L^2(0,T;H^1)$ and $L^2(0,T;L^2)$ norms.

The weak formulation of the problem (3.1.1)-(3.1.3) is stated as follows: Find $u \in H_0^1(\Omega)$ such that

$$(u_t, v) + A(u, v) = (f, v) + \langle g, v\rangle_\Gamma \quad \forall v \in H_0^1(\Omega), \quad t \in (0,T] \tag{3.3.1}$$

with $u(0) = u_0$. Here, $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_\Gamma$ are used to denote the inner products of $L^2(\Omega)$ and $L^2(\Gamma)$ spaces, respectively.

The continuous in time Galerkin finite element approximation to (3.3.1) which may be stated as follows: Find $u_h : [0,T] \to V_h$ such that $u_h(0) = L_h u_0$ and satisfies

$$(u_{ht}, v_h) + A_h(u_h, v_h) = (f, v_h) + \langle g_h, v_h\rangle_{\Gamma_h} \quad \forall v_h \in V_h, \quad t \in (0,T]. \tag{3.3.2}$$

We shall need the following Lemma for the semidiscrete solution u_h satisfying (3.3.2) for our future use. For a proof, we refer to [15].

Lemma 3.3.1 *Let $f \in L^2(\Omega)$ and $g \in H^2(\Gamma)$. Then we have*

$$\int_0^t \|u_h\|^2_{H^1(\Omega)} ds \le C\Big(\int_0^t \{\|f\|^2_{L^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\} ds + \|u_0\|^2_{L^2(\Omega)}\Big).$$

Now, we are in a position to discuss the main results of this section which is stated in the following theorems.

Theorem 3.3.1 *Let u and u_h be the solutions of (3.1.1)-(3.1.3) and (3.3.2), respectively. Then, for $u_0 \in H_0^1(\Omega), f \in L^2(\Omega)$ and $g \in H^2(\Gamma)$, there is a positive constant C independent of h such that*

$$\|u - u_h\|_{L^2(0,T,H^1(\Omega))} \le C(u_0, u, f, g)h.$$

Theorem 3.3.2 *Let u and u_h be the solutions of (3.1.1)-(3.1.3) and (3.3.2), respectively. Then, for $u_0 \in H_0^1(\Omega), f \in L^2(\Omega)$ and $g \in H^2(\Gamma)$, there is a positive constant C independent of h such that*

$$\|u - u_h\|_{L^2(0,T,L^2(\Omega))} \le C(u_0, u, f, g)h^2.$$

Proof of Theorem 3.3.1. Subtracting (3.3.2) from (3.3.1), for all $v_h \in V_h$, we have

$$\begin{aligned}(u_t - u_{ht}, v_h) + A(u - u_h, v_h) &= \langle g, v_h\rangle_\Gamma - \langle g_h, v_h\rangle_{\Gamma_h} \\ &\quad + A_h(u_h, v_h) - A(u_h, v_h). \end{aligned} \tag{3.3.3}$$

Define the error $e(t)$ as $e(t) = u(t) - u_h(t)$. Setting $v_h = L_h u$ in (3.3.3) and using (3.2.12), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|e(t)\|^2_{L^2(\Omega)} + A(e, e) \\ &= (I)_1 + (I)_2 + (I)_3 + \frac{1}{2}\frac{d}{dt}\|u - L_h u\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.3.4}$$

where the terms $(I)_i$, $i = 1, 2, 3$ are given by

$$\begin{aligned}(I)_1 &= \langle g, L_h u - u_h\rangle_\Gamma - \langle g_h, L_h u - u_h\rangle_{\Gamma_h}, \\ (I)_2 &= A_h(u_h, L_h u - u_h) - A(u_h, L_h u - u_h), \\ (I)_3 &= A(u_h - u, L_h u - u).\end{aligned}$$

Now, we estimate the terms $(I)_1$, $(I)_2$ and $(I)_3$ one by one. By Lemma 3.2.1, Lemma 3.2.3 and the triangle inequality, we obtain

$$\begin{aligned}|(I)_1| &\le Ch^{\frac{3}{2}}\|g\|_{H^2(\Gamma)}\|L_h u - u_h\|_{H^1(\Omega)} \\ &\le Ch^{\frac{5}{2}}\|g\|_{H^2(\Gamma)}\|u\|_X + Ch^{\frac{3}{2}}\|g\|_{H^2(\Gamma)}\|e(t)\|_{H^1(\Omega)} \\ &\le Ch^{\frac{5}{2}}\|g\|_{H^2(\Gamma)}\|u\|_X + Ch^3\|g\|^2_{H^2(\Gamma)} + \frac{1}{4}\|e(t)\|^2_{H^1(\Omega)} \\ &\le Ch^2(\|u\|^2_X + \|g\|^2_{H^2(\Gamma)}) + \frac{1}{4}\|e(t)\|^2_{H^1(\Omega)}.\end{aligned} \tag{3.3.5}$$

In the last inequality, we used Young's Inequality. Similarly, for $(I)_2$, using Lemma 2.2.1 and Lemma 3.2.3 to have

$$\begin{aligned}|(I)_2| &\le Ch\|u_h\|_{H^1(\Omega)}\|L_h u - u + u - u_h\|_{H^1(\Omega)} \\ &\le Ch\|u_h\|_{H^1(\Omega)}(\|L_h u - u\|_{H^1(\Omega)} + \|u - u_h\|_{H^1(\Omega)}) \\ &\le Ch^2\|u_h\|^2_{H^1(\Omega)} + C\|L_h u - u\|^2_{H^1(\Omega)} + \frac{1}{4}\|u - u_h\|^2_{H^1(\Omega)} \\ &\le Ch^2\|u_h\|^2_{H^1(\Omega)} + Ch^2\|u\|^2_X + \frac{1}{4}\|e(t)\|^2_{H^1(\Omega)}.\end{aligned} \tag{3.3.6}$$

Then, the last term $(I)_3$ can be bounded by using Lemma 3.2.3

$$\begin{aligned}|(I)_3| &\le Ch\|u\|_X\|e(t)\|_{H^1(\Omega)} \\ &\le Ch^2\|u\|^2_X + \frac{1}{4}\|e(t)\|^2_{H^1(\Omega)}.\end{aligned} \tag{3.3.7}$$

Integrating the identity (3.3.4) from 0 to t and using the estimates (3.3.5)-(3.3.7), we obtain

$$\begin{aligned}\int_0^t \|e\|^2_{H^1(\Omega)} ds &\leq Ch^2 \int_0^t \|u\|^2_X ds + Ch^2 \int_0^t \|u_h\|^2_{H^1(\Omega)} ds + \frac{3}{4} \int_0^t \|e\|^2_{H^1(\Omega)} ds \\ &+ \|u - L_h u\|^2_{L^2(\Omega)},\end{aligned}$$

This, together with Lemma 3.2.3 and Lemma 3.3.1 completes the rest of the proof of Theorem 3.3.1. □

Proof of Theorem 3.3.2. For the L^2 norm error estimate we shall use the parabolic duality trick. For any time $t > 0$ and $e = u - u_h$, let $w(s) \in H^1_0(\Omega)$ and $w_h(s) \in V_h$, respectively, be the solutions of the backward problems

$$\begin{aligned}(\phi, w_s) - A(\phi, w) &= (\phi, e) \quad \forall \phi \in H^1_0(\Omega), \quad s < t, \qquad (3.3.8)\\ w(t) &= 0;\\ (\phi_h, w_{hs}) - A_h(\phi_h, w_h) &= (\phi_h, e) \quad \forall \phi_h \in V_h, \quad s < t, \qquad (3.3.9)\\ w_h(t) &= 0\end{aligned}$$

with $[w] = 0$ and $g(x,t) = 0$ across the interface Γ. From (3.3.8) and (3.3.9), we obtain

$$(\phi_h, w_s - w_{hs}) - A(\phi_h, w - w_h) = A(\phi_h, w_h) - A_h(\phi_h, w_h) \qquad (3.3.10)$$

for all $\phi_h \in V_h$. Following the standard argument of [34], it is easy to show that

$$\int_0^t \|w_s - w_{hs}\|^2_{L^2(\Omega)} ds \leq C \int_0^t \|e\|^2_{L^2(\Omega)} ds. \qquad (3.3.11)$$

Further, we assume that the following identity

$$\int_0^t \left(h^{-2} \|w - w_h\|^2_{H^1(\Omega)} \right) ds \leq C \int_0^t \|e\|^2_{L^2(\Omega)} ds \qquad (3.3.12)$$

holds true. The estimate (3.3.12) is obtained by reversing time in the proof of Theorem 3.3.1 and further using Theorem 3.2.1 for the problem (3.3.8)-(3.3.9). Set $\phi = e$ in

(3.3.8). Then, using the identity (3.3.3), we obtain

$$
\begin{aligned}
\|e\|^2_{L^2(\Omega)} &= (e, w_s) - A(e, w) \\
&= (e, w_{hs}) + (e, w_s - w_{hs}) - A(e, w - w_h) - A(e, w_h) \\
&= \frac{d}{ds}(e, w_h) + (e, w_s - w_{hs}) - A(e, w - w_h) \\
&\quad -(e_s, w_h) - A(e, w_h) \\
&= \frac{d}{ds}(e, w_h) + (e, w_s - w_{hs}) - A(e, w - w_h) \\
&\quad +\{A(u_h, w_h) - A_h(u_h, w_h)\} + \{\langle g_h, w_h\rangle_{\Gamma_h} - \langle g, w_h\rangle_\Gamma\}.
\end{aligned}
$$

With an aid of (3.3.10), the above equation may be rewritten as

$$
\begin{aligned}
\|e\|^2_{L^2(\Omega)} &= \frac{d}{ds}(e, w_h) + (u - P_h u, w_s - w_{hs}) - A(u - P_h u, w - w_h) \\
&\quad +(P_h u - u_h, w_s - w_{hs}) - A(P_h u - u_h, w - w_h) \\
&\quad +\{A(u_h, w_h) - A_h(u_h, w_h)\} + \{\langle g_h, w_h\rangle_{\Gamma_h} - \langle g, w_h\rangle_\Gamma\} \\
&= \frac{d}{ds}(e, w_h) + (u - P_h u, w_s - w_{hs}) - A(u - P_h u, w - w_h) \\
&\quad +\{A(P_h u - u_h, w_h) - A_h(P_h u - u_h, w_h)\} \\
&\quad +\{A(u_h, w_h) - A_h(u_h, w_h)\} + \{\langle g_h, w_h\rangle_{\Gamma_h} - \langle g, w_h\rangle_\Gamma\} \\
&= \frac{d}{ds}(e, w_h) + (u - P_h u, w_s - w_{hs}) \\
&\quad -A(u - P_h u, w - w_h) + (II)_1 + (II)_2, \qquad (3.3.13)
\end{aligned}
$$

where $(II)_1 = A(P_h u, w_h) - A_h(P_h u, w_h)$ and $(II)_2 = \{\langle g_h, w_h\rangle_{\Gamma_h} - \langle g, w_h\rangle_\Gamma\}$.

We integrate (3.3.13) from 0 to t to obtain

$$
\begin{aligned}
\int_0^t \|e\|^2_{L^2(\Omega)} ds &= \int_0^t \{(u - P_h u, w_s - w_{hs}) - A(u - P_h u, w - w_h)\} ds \\
&\quad + \int_0^t (II)_1 ds + \int_0^t (II)_2 ds \\
&\le \int_0^t \|u - P_h u\|_{L^2(\Omega)} \|w_s - w_{hs}\|_{L^2(\Omega)} ds \\
&\quad +C \int_0^t \|u - P_h u\|_{H^1(\Omega)} \|w - w_h\|_{H^1(\Omega)} ds \\
&\quad + \int_0^t (II)_1 ds + \int_0^t (II)_2 ds.
\end{aligned}
$$

We, now use the Young's inequality to obtain

$$\begin{aligned}\int_0^t \|e\|^2_{L^2(\Omega)}ds &\leq \epsilon \int_0^t \{\|w_s - w_{hs}\|^2_{L^2(\Omega)} + h^{-2}\|w - w_h\|^2_{H^1(\Omega)}\}ds \\ &+\frac{C}{\epsilon}\int_0^t \{\|u - P_h u\|^2_{L^2(\Omega)} + h^2\|u - P_h u\|^2_{H^1(\Omega)}\}ds \\ &+ \int_0^t (II)_1 ds + \int_0^t (II)_2 ds.\end{aligned}$$

Apply (3.3.11) and (3.3.12) to have

$$\begin{aligned}\int_0^t \|e\|^2_{L^2(\Omega)}ds &\leq C\epsilon \int_0^t \|e(t)\|^2_{L^2(\Omega)}ds + \frac{C}{\epsilon}\int_0^t \{\|u - P_h u\|^2_{L^2(\Omega)} + h^2\|u - P_h u\|^2_{H^1(\Omega)}\}ds \\ &+ \int_0^t (II)_1 ds + \int_0^t (II)_2 ds. \qquad (3.3.14)\end{aligned}$$

The term $(II)_1$ can be bounded by using Lemma 2.2.1 and Lemma 2.2.2 of Chapter 2

$$\begin{aligned}|(II)_1| &\leq Ch\|P_h u\|_{H^1(\Omega^*_\Gamma)}\|w_h\|_{H^1(\Omega^*_\Gamma)} \\ &\leq Ch\|P_h u - u\|_{H^1(\Omega^*_\Gamma)}\|w_h\|_{H^1(\Omega^*_\Gamma)} + Ch\|u\|_{H^1(\Omega^*_\Gamma)}\|w_h\|_{H^1(\Omega^*_\Gamma)} \\ &\leq Ch\|u - P_h u\|_{H^1(\Omega^*_\Gamma)}\|w_h\|_{H^1(\Omega^*_\Gamma)} + Ch^{\frac{3}{2}}\|u\|_X\|w - w_h\|_{H^1(\Omega^*_\Gamma)} \\ &+Ch^{\frac{3}{2}}\|u\|_X\|w\|_{H^1(\Omega^*_\Gamma)} \\ &\leq Ch\|u - P_h u\|_{H^1(\Omega^*_\Gamma)}\|w - w_h\|_{H^1(\Omega^*_\Gamma)} + Ch\|u - P_h u\|_{H^1(\Omega^*_\Gamma)}\|w\|_{H^1(\Omega^*_\Gamma)} \\ &+Ch^{\frac{3}{2}}\|u\|_X\|w - w_h\|_{H^1(\Omega^*_\Gamma)} + Ch^2\|u\|_X\|w\|_X.\end{aligned}$$

Integrating this identity from 0 to t and using Young's inequality, we obtain

$$\begin{aligned}\int_0^t |(II)_1|ds &\leq Ch\int_0^t \|u - P_h u\|_{H^1(\Omega^*_\Gamma)}\|w - w_h\|_{H^1(\Omega^*_\Gamma)}ds \\ &+Ch^{\frac{3}{2}}\int_0^t \|u - P_h u\|_{H^1(\Omega^*_\Gamma)}\|w\|_X ds \\ &+Ch^{\frac{3}{2}}\int_0^t \|u\|_X\|w - w_h\|_{H^1(\Omega^*_\Gamma)}ds + Ch^2\int_0^t \|u\|_X\|w\|_X ds \\ &\leq \frac{C}{\epsilon}h^4\int_0^t \|u - P_h u\|^2_{H^1(\Omega)}ds + \frac{\epsilon}{2}h^{-2}\int_0^t \|w - w_h\|^2_{H^1(\Omega)}ds \\ &+\frac{C}{\epsilon}h^3\int_0^t \|u - P_h u\|^2_{H^1(\Omega)}ds + \frac{\epsilon}{2}\int_0^t \|w\|^2_X ds \\ &+\frac{C}{\epsilon}h^5\int_0^t \|u\|^2_X ds + \frac{\epsilon}{2}h^{-2}\int_0^t \|w - w_h\|^2_{H^1(\Omega)}ds \\ &+\frac{C}{\epsilon}h^4\int_0^t \|u\|^2_X ds + \frac{\epsilon}{2}\int_0^t \|w\|^2_X ds.\end{aligned}$$

Further, using the regularity result (cf. Theorem 3.2.1), (3.3.8) and (3.3.12), we obtain

$$\begin{aligned}\int_0^t |(II)_1| ds \;&\le\; \frac{C}{\epsilon} h^3 \int_0^t \|u - P_h u\|^2_{H^1(\Omega)} ds + C\epsilon \int_0^t \|e\|^2_{L^2(\Omega)} ds \\ &\quad + \frac{C}{\epsilon} h^4 \int_0^t \|u\|^2_X ds. \qquad (3.3.15)\end{aligned}$$

Finally, Lemma 3.2.1 and similar argument leads to

$$\int_0^t |(II)_2| ds \le \frac{C}{\epsilon} h^4 \int_0^t \|g\|^2_{H^2(\Gamma)} ds + C\epsilon \int_0^t \|e\|^2_{L^2(\Omega)} ds. \qquad (3.3.16)$$

Thus, combining the estimates (3.3.15)-(3.3.16), together with (3.3.14) and Lemma 3.2.2 completes the rest of the proof of Theorem 3.3.2. □

Remark 3.3.1 *The convergence results for the linear parabolic interface problems are also extended for the semilinear problems (cf. [18]) into the Brezzi-Rappaz-Raviart ([10]) framework.*

3.4 Error Analysis for Fully Discrete Scheme

A fully discrete scheme based on backward Euler method is proposed and analyzed in this section. Optimal $L^2(0,T;H^1(\Omega))$ norm error estimate is obtained for fully discrete scheme. For the simplicity, we have assumed $g(x,t) = 0$.

We first partition the interval [0, T] into M equally spaced subintervals by the following points

$$0 = t_0 < t_1 < \ldots < t_M = T$$

with $t_n = nk,\ k = \frac{T}{M}$, be the time step. Let $I_n = (t_{n-1}, t_n]$ be the n-th subinterval. Now we introduce the backward difference quotient

$$\Delta_k \phi^n = \frac{\phi^n - \phi^{n-1}}{k},$$

for a given sequence $\{\phi^n\}_{n=0}^M \subset L^2(\Omega)$.

The fully discrete finite element approximation to the problem (3.3.2) is defined as follows: For $n = 1, \ldots, M$, find $U^n \in V_h$ such that

$$(\Delta_k U^n, v_h) + A_h(U^n, v_h) = (f^n, v_h) \ \ \forall v_h \in V_h \qquad (3.4.1)$$

with $U^0 = L_h u_0$. For each $n = 1, \ldots, M$, the existence of a unique solution to (3.4.1) can be found in [11]. We then define the fully discrete solution to be a piecewise constant function $U_h(x, t)$ in time and is given by

$$U_h(x,t) = U^n(x) \ \ \forall t \in I_n, \ \ 1 \leq n \leq M.$$

We now prove the main result of this section in the following theorem.

Theorem 3.4.1 *Let u and U be the solutions of the problem (4.1.1)-(4.1.3) and (4.5.1), respectively. Assume that $U^0 = L_h u_0$ and u_0 is sufficiently smooth. Then there exists a constant C independent of h and k such that*

$$\begin{aligned}
&\|U(t_n) - u(t_n)\|_{L^2(\Omega)} \\
&\leq C(h^2 + k) \sum_{i=1}^{2} \left\{ \|u^0\|_{H^2(\Omega_i)} + \|u_t\|_{L^2(0,T,H^2(\Omega_i))} + \|u_{tt}\|_{L^2(0,T;L^2(\Omega_i))} \right\}
\end{aligned}$$

Proof. For simplicity of the exposition, we write $u^n = u(x, nk)$, $e^n = u^n - U^n$ and $w^n = u^n - P_h u^n$. Using (3.3.1) and (3.4.1), it follows that

$$\begin{aligned}
(\Delta_k e^n, e^n) + A(e^n, e^n) &= (\Delta_k e^n, w^n) + A(e^n, w^n) + (\Delta_k u^n - u_t^n, P_h u^n - U^n) \\
&\quad + \{A_h(U^n, P_h u^n - U^n) - A(U^n, P_h u^n - U^n)\} \\
&= C \sum_{j=1}^{4} I_j. \qquad (3.4.2)
\end{aligned}$$

where

$$\begin{aligned}
&I_1 = (\Delta_k e^n, w^n), \quad I_2 = A(e^n, w^n), \quad I_3 = (\Delta_k u^n - u_t^n, P_h u^n - U^n) \\
&I_4 = \{A_h(U^n, P_h u^n - U^n) - A(U^n, P_h u^n - U^n)\}.
\end{aligned}$$

Summing (3.4.2) over n from $n = 0$ to $n = M$, we have

$$\begin{aligned}
\frac{1}{2}\|e^M\|^2_{L^2(\Omega)} + k\sum_{n=0}^{M} A(e^n, e^n) + \frac{1}{2}\sum_{n=0}^{M} \|\Delta_k e^n\|^2_{L^2(\Omega)} &\leq \frac{1}{2}\|e^0\|^2_{L^2(\Omega)} \\
&\quad + k\sum_{n=0}^{M}(I_1 + I_2 + I_3 + I_4). \qquad (3.4.3)
\end{aligned}$$

Using Lemma 3.2.2 and Young's inequality, we obtain

$$k\sum_{n=0}^{M} I_1 \leq Ch^2 k \sum_{n=0}^{M} \|u^n\|_X^2 + \frac{k}{4}\sum_{n=0}^{M} \|\Delta_k e^n\|_{L^2(\Omega)}^2. \tag{3.4.4}$$

Similarly,

$$k\sum_{n=0}^{M} I_2 \leq C(\epsilon) h^2 k \sum_{n=0}^{M} \|u^n\|_X^2 + \epsilon k \sum_{n=0}^{M} \|e^n\|_{H^1(\Omega)}^2. \tag{3.4.5}$$

To estimate $k\sum_{n=0}^{M} I_3$, we first note that

$$\Delta_k u^n - \frac{\partial u^n}{\partial t} = -\frac{1}{k}\int_{t_{n-1}}^{t_n} (s - t_{n-1}) u_{ss}(s) ds.$$

and hence using Lemma 3.2.2, we obtain

$$k\sum_{n=0}^{M} I_3 \leq Ck^2 \|u_{tt}\|_{L^2(0,T;L^2(\Omega))} + Ch^2 k \sum_{n=0}^{M} \|u^n\|_X^2 + k\sum_{n=0}^{M} \|e^n\|_{L^2(\Omega)}^2. \tag{3.4.6}$$

Using Lemma 2.2.1, we obtain

$$\begin{aligned} k\sum_{n=0}^{M} I_4 &\leq Chk\sum_{n=0}^{M} \Big\{ \|U^n\|_{H^1(\Omega)} \|P_h u^n - U^n\|_{H^1(\Omega)} \Big\} \\ &\leq Chk\sum_{n=0}^{M} \Big\{ \|U^n\|_{H^1(\Omega)}^2 + \frac{\epsilon}{4}\|P_h u^n - U^n\|_{H^1(\Omega)}^2 \Big\} \\ &\leq Chk\sum_{n=0}^{M} \|e^n\|_{H^1(\Omega)}^2 + Chk\sum_{n=0}^{M} \|u^n\|_X^2. \end{aligned} \tag{3.4.7}$$

In the last inequality, we have used Lemma 3.2.2. Combining (3.4.3)-(3.4.7) and using the standard kickback argument, we arrive at

$$\begin{aligned} \|e^M\|_{L^2(\Omega)}^2 + k\sum_{n=0}^{M} \|e^n\|_{H^1(\Omega)}^2 &\leq Ck^2 \|u_{tt}\|_{L^2(0,T;L^2(\Omega))}^2 + Ch\Big(k\sum_{n=0}^{M} \|u^n\|_X^2\Big) \\ &+ Ck\sum_{n=0}^{M} \|e^n\|_{L^2(\Omega)}^2. \end{aligned}$$

For sufficiently small k, we obtain

$$\begin{aligned} \|e^M\|_{L^2(\Omega)}^2 + k\sum_{n=0}^{M} \|e^n\|_{H^1(\Omega)}^2 &\leq Ck^2 \|u_{tt}\|_{L^2(0,T;L^2(\Omega))}^2 + Ch\Big(k\sum_{n=0}^{M} \|u^n\|_X^2\Big) \\ &+ Ck\sum_{n=0}^{M-1} \|e^n\|_{L^2(\Omega)}^2. \end{aligned}$$

An application of discrete version of Gronwall's lemma leads to

$$\|e^M\|^2_{L^2(\Omega)} + k\sum_{n=0}^{M}\|e^n\|^2_{H^1(\Omega)} \le Ck^2\|u_{tt}\|^2_{L^2(0,T;L^2(\Omega))} + Ch\Big(k\sum_{n=0}^{M}\|u^n\|^2_X\Big). \tag{3.4.8}$$

Finally, by a simple calculation we have

$$\|u - U_h\|_{L^2(0,T;H^1(\Omega))} \le Ck\|u_t\|_{L^2(0,T;Y)} + C(k\sum_{n=0}^{M}\|u^{n+1} - U^{n+1}\|^2_{H^1(\Omega)})^{\frac{1}{2}}. \tag{3.4.9}$$

Since $k = O(h)$, (3.4.9) combine with (3.4.8) leads to the desired result. □

Chapter 4

$L^{\infty}(L^2)$ and $L^{\infty}(H^1)$ norms Error Estimates for Parabolic Interface Problems

The purpose of this chapter is to establish some new a priori error estimates in finite element method for parabolic interface problems. Optimal $L^{\infty}(H^1)$ and $L^{\infty}(L^2)$ norms error estimates have been derived under practical regularity assumption of the true solution for fitted finite element method with straight interface triangles.

4.1 Introduction

In $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, we shall again recall the following parabolic interface problem

$$u_t + \mathcal{L}u = f(x,t) \quad \text{in } \Omega \times (0,T] \tag{4.1.1}$$

with initial and boundary conditions

$$u(x,0) = u_0 \ \text{ in } \Omega; \ \ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T] \tag{4.1.2}$$

and jump conditions on the interface

$$[u] = 0, \ \ \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = 0 \ \ \text{along } \Gamma, \tag{4.1.3}$$

where, $f = f(x,t)$ is real valued functions in $\Omega \times (0,T]$, and $u_t = \frac{\partial u}{\partial t}$. Further, $u_0 = u_0(x)$ is real valued function in Ω. The operator $\mathcal{L}$, symbols $[v]$ and $\mathbf{n}$ are defined as in Chapter 1, and $T < \infty$.

Due to low global regularity of the solutions, it is difficult to achieve optimal $L^\infty(L^2)$ and $L^\infty(H^1)$ error estimates for parabolic interface problems. More recently, Deka and Sinha ([19]) have studied the pointwise-in-time convergence in finite element method for parabolic interface problems. They have shown optimal error estimates in $L^\infty(H^1)$ and $L^\infty(L^2)$ norms under the assumption that grid line exactly follow the actual interface. This may causes some technical difficulties in practice for the evaluation of the integrals over those curved elements near the interface. Therefore, in present work an attempt has been made to extend the results obtained in [19] for a more practical finite element discretization discussed in [11]. In this chapter, we are able to show that the standard energy technique of finite element method can be extended to parabolic interface problems under the assumptions that solution as well as its normal derivative along interface are continuous. Optimal order pointwise-in-time error estimates in the L^2 and H^1 norms are established for the semidiscrete scheme. In addition, a fully discrete method based on backward Euler time-stepping scheme is analyzed and related optimal pointwise-in-time error bounds are derived. To the best of our knowledge, optimal pointwise in time error estimates for a finite element discretization based on [11] have not been established earlier for the parabolic interface problem.

A brief outline of this chapter is as follows. In section 4.2, we introduce some standard notations, recall some basic results from the literature and obtain the a priori estimate for the solution. In section 4.3, we describe a finite element discretization for the problem (4.1.1)-(4.1.3) and prove some approximation properties related to the auxiliary projection used in our analysis. While Section 4.4 is devoted to the error analysis for the semidiscrete finite element approximation, error estimates for the fully discrete backward Euler time stepping scheme are derived in section 4.5.

4.2 Preliminaries

The purpose of this chapter is to introduce some new a priori estimates for the solutions of parabolic interface problems.

In order to introduce the weak formulation of the problem, we now define the local bilinear form $A^l(.,.) : H^1(\Omega_l) \times H^1(\Omega_l) \to \mathbb{R}$ by

$$A^l(w,v) = \int_{\Omega_l} \beta_l \nabla w \cdot \nabla v dx, \quad l = 1, 2.$$

Then the global bilinear form $A(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is defined by

$$\begin{aligned} A(w,v) &= \int_\Omega \beta(x)\nabla w \cdot \nabla v dx \\ &= A^1(w,v) + A^2(w,v) \ \ \forall\, w,\ v \in H_0^1(\Omega). \end{aligned}$$

Then weak form for the problem (4.1.1)-(4.1.3) is defined as follows: Find $u : (0,T] \to H_0^1(\Omega)$ such that

$$(u_t, v) + A(u,v) = (f,v) \ \ \forall v \in H_0^1(\Omega),\ a.e.\ t \in (0,T] \tag{4.2.1}$$

with $u(x,0) = u_0(x)$.

Remark 4.2.1 *Let $f(x,0) = f_0(x)$. Then it is clear from (1.1.1) that $u_t(0) \in H^2(\Omega)$ provided $u_0 \in H_0^1(\Omega) \cap H^4(\Omega)$ and $f_0 \in H^2(\Omega)$. From therein, we assume that $u_0 \in H_0^1(\Omega) \cap H^4(\Omega)$, $f \in H^1(0,T;L^2(\Omega))$ and $f_0 \in H^2(\Omega)$.*

Under the assumption $f \in H^1(0,T;L^2(\Omega))$, we have

$$u_{tt} - \nabla \cdot (\beta(x)\nabla u_t) = f_t \quad \text{in } \Omega_i,\ i = 1,2. \tag{4.2.2}$$

Further u_t, satisfies the following initial and boundary condition

$$u_t(x,0) = u_t(0) \ \text{ and } \ u_t(x,t) = 0 \ \text{ on } \partial\Omega \times (0,T] \tag{4.2.3}$$

along with the jump conditions

$$[u_t] = 0 \quad \text{and} \quad \left[\beta \frac{\partial u_t}{\partial \mathbf{n}}\right] = 0 \ \text{ along } \Gamma. \tag{4.2.4}$$

Thus $v = u_t \in \Omega_i,\ i = 1,2$ satisfies a parabolic interface problem (4.2.2)-(4.2.4). Therefore, for $f_t \in H^1(0,T;L^2(\Omega))$ and $u_t(0) \in H^2(\Omega)$, apply Theorem 3.2.1 to have the following result.

Lemma 4.2.1 *Let $f \in H^2(0,T;L^2(\Omega))$, $f_0 \in H^2(\Omega)$ and $u_0 \in H_0^1(\Omega) \cap H^4(\Omega)$. Then the problem (1.1.1)-(1.1.3) has a unique solution $u \in H^1(0,T;H^2(\Omega_1) \cap H^2(\Omega_2)) \cap H_0^1(\Omega) \cap H^2(0,T;L^2(\Omega))$. Further, u_t satisfies the following a priori estimate*

$$\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)} \le C\{\|f_t\|_{L^2(\Omega)} + \|u_{tt}\|_{L^2(\Omega)}\}.$$

Proof. The proof of the existence of unique solution $u \in H^1(0,T;H^2(\Omega_1)\cap H^2(\Omega_2))\cap H_0^1(\Omega)\cap H^2(0,T;L^2(\Omega))$ follows from the assumptions and Theorem 3.2.1.

Next, to obtain the a priori estimate we consider the following elliptic interface problem: For a.e $t \in (0,T]$, find $w = w(x,t) \in H_0^1(\Omega)\cap X$ satisfying

$$
\begin{aligned}
-\nabla\cdot(\beta(x)\nabla w(x,t)) &= f_t(x,t) - u_{tt}(x,t) \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial\Omega, \\
[w]=0, \quad \left[\beta\frac{\partial w}{\partial \mathbf{n}}\right] &= 0 \ \text{along } \Gamma.
\end{aligned}
\tag{4.2.5}
$$

From the elliptic regularity estimate for elliptic interface problem (cf. [11]), it follows that

$$\|w\|_{H^2(\Omega_1)} + \|w\|_{H^2(\Omega_2)} \le C\{\|f_t\|_{L^2(\Omega)} + \|u_{tt}\|_{L^2(\Omega)}\}. \tag{4.2.6}$$

Now, multiplying (4.2.5) by $\phi \in L^2(\Omega)\cap H^1(\Omega_1)\cap H^1(\Omega_2)\cap\{\psi \in L^2(\Omega):\ \psi = 0 \text{ on } \partial\Omega,\ [w]=0 \text{ on } \Gamma\}$ and then integrating over Ω_1 and Ω_2, we get

$$A^1(w,\phi) + A^2(w,\phi) = (f_t - u_{tt}, \phi). \tag{4.2.7}$$

Similarly from (4.2.2), we get

$$A^1(u_t,\phi) + A^2(u_t,\phi) = (f_t - u_{tt}, \phi). \tag{4.2.8}$$

Thus, for all such ϕ, we have

$$A^1(w-u_t,\phi) + A^2(w-u_t,\phi) = 0.$$

Again, $u_t \in L^2(\Omega)\cap H^1(\Omega_1)\cap H^1(\Omega_2)\cap\{\psi\in L^2(\Omega):\ \psi = 0 \text{ on } \partial\Omega,\ [w]=0 \text{ on } \Gamma\}$. Finally, setting $\phi = w - u_t$ in the above equation and using the coercivity of each local bilinear map, we have $w = u_t$ in Ω_i, $i = 1,\ 2$. Then the desire estimate follows from (4.2.6). □

4.3 Some Auxiliary Projections

In this chapter, we introduce linear interpolant and some auxiliary projections. Further, the convergence of such operators are obtained under global minimum regularity assumption of the true solutions.

Since the global regularity of the true solution is low, it is not favorable to work on $H^1(\Omega)$ in estimating pointwise-in-time error estimates. Therefore, we introduce $X^\star$ be the collection of all $v \in L^2(\Omega)$ with the property that $v \in H^2(\Omega_1) \cap H^2(\Omega_2) \cap \{\psi : \psi = 0 \text{ on } \partial\Omega\}$ and $[v] = 0$ along Γ. Let Π_h be the Lagrange's interpolation operator defined in Chapter 2. Then, for $K \in \mathcal{T}_h$ and $v \in X^\star$, we now define

$$v_I = \begin{cases} \Pi_h \tilde{v}_1 & \text{if } K \subseteq \Omega_1^h \\ \Pi_h \tilde{v}_2 & \text{if } K \subseteq \Omega_2^h. \end{cases} \tag{4.3.1}$$

For a finite dimensional space $V_h \subset H_0^1(\Omega)$ discussed in Chapter 2, it is easy to verify that $v_I \in V_h$.

Following the lines of proof for Lemma 2.2.3, it is possible to obtain the following optimal error bounds for linear interpolant v_I in $X^\star$. We include the proof for the completeness of this work.

Lemma 4.3.1 *For any $v \in X^\star$, we have*

$$\|v - v_I\|_{H^1(\Omega_1)} + \|v - v_I\|_{H^1(\Omega_2)} \leq Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

Proof. For H^1 norm estimate, we have

$$\begin{aligned} &\|v - v_I\|_{H^1(\Omega_1)} + \|v - v_I\|_{H^1(\Omega_2)} \\ &\leq \sum_{K \in \mathcal{T}_h \setminus \mathcal{T}_\Gamma^*} \|v - v_I\|_{H^1(K)} + \sum_{K \in \mathcal{T}_\Gamma^*} \{\|v - v_I\|_{H^1(K_1)} + \|v - v_I\|_{H^1(K_2)}\} \\ &\leq Ch\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\} \\ &\quad + \sum_{K \in \mathcal{T}_\Gamma^*} \{\|v - v_I\|_{H^1(K_1)} + \|v - v_I\|_{H^1(K_2)}\}. \end{aligned} \tag{4.3.2}$$

Here, $K_1 = K \cap \Omega_1$ and $K_2 = K \cap \Omega_2$. Again, for any $K \in \mathcal{T}_h$ either $K \subseteq \Omega_1^h$ or $K \subseteq \Omega_2^h$. Let $K \subseteq \Omega_1^h$, then $v_I = \Pi_h \tilde{v}_1$ and hence, we have

$$\begin{aligned} \|v - v_I\|_{H^1(K_1)} &= \|\tilde{v}_1 - \Pi_h \tilde{v}_1\|_{H^1(K_1)} \leq \|\tilde{v}_1 - \Pi_h \tilde{v}_1\|_{H^1(K)} \\ &\leq Ch\|\tilde{v}_1\|_{H^2(K)} \leq Ch\|v_1\|_{H^2(\Omega_1)}. \end{aligned} \tag{4.3.3}$$

Again, since $v \in H^2(\Omega_2)$ and $K_2 \subseteq \Omega_2$ with $\text{meas}(K_2) \leq Ch^3$, we have

$$
\begin{aligned}
\|v - v_I\|_{H^1(K_2)} &\leq Ch^{\frac{3(p-2)}{2p}} \|v - v_I\|_{W^{1,p}(K_2)} \;\; \forall p > 2 \\
&= Ch\|v - v_I\|_{W^{1,6}(K_2)} = Ch\|v_2 - \Pi_h \tilde{v}_1\|_{W^{1,6}(K_2)} \\
&\leq Ch\|\tilde{v}_2 - \tilde{v}_1\|_{W^{1,6}(K_2)} + Ch\|\tilde{v}_1 - \Pi_h \tilde{v}_1\|_{W^{1,6}(K_2)} \\
&\leq Ch\|\tilde{v}_2 - \tilde{v}_1\|_{W^{1,6}(K)} + Ch\|\tilde{v}_1 - \Pi_h \tilde{v}_1\|_{W^{1,6}(K)} \\
&\leq Ch\|\tilde{v}_2 - \tilde{v}_1\|_{H^2(\Omega)} + Ch\|\tilde{v}_1\|_{H^2(K)} \\
&\leq Ch\|\tilde{v}_1\|_{H^2(\Omega)} + Ch\|\tilde{v}_2\|_{H^2(\Omega)} \\
&\leq Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}). \qquad (4.3.4)
\end{aligned}
$$

Then Lemma 4.3.1 follows immediately from the estimates (4.3.2)-(4.3.4). □

In the error analysis of parabolic problems the term $\rho = u - P_h u$ and $\rho_t = u_t - P_h u_t$ plays very crucial role, where P_h is the standard elliptic projection (c.f. [47]). But in our present case solution $u \in H^1(\Omega)$ and $u_t \in L^2(\Omega)$, and therefore the standard elliptic projection P_h at u_t is not defined in usual manner. Therefore a modification in the definition of elliptic projection has been proposed and analyzed in this work. For any $v \in X^\star$ with $[\beta \partial v / \partial \mathbf{n}] = 0$ along Γ, we define

$$
f^* = \begin{cases} -\nabla \cdot (\beta_1 \nabla v) & \text{in } \Omega_1 \\ -\nabla \cdot (\beta_2 \nabla v) & \text{in } \Omega_2. \end{cases}
$$

Clearly $f^* \in L^2(\Omega)$. We denote X^{**} to be the collection of all such $v \in X^*$. Then define $R_h : X^* \to V_h$ by

$$
A_h(R_h v, v_h) = (f^*, v_h) \;\; \forall v_h \in V_h. \qquad (4.3.5)
$$

The existence and uniqueness of such $R_h v$ can be verified by setting $R_h v = \sum c_i \Phi_i$ in (4.3.5) and then applying the coercivity of $A_h(.,.)$. Here, Φ_i represents basis function corresponding to the ith grid. Again,

$$
\begin{aligned}
(f^*, v_h) &= -\int_{\Omega_1} \nabla \cdot (\beta_1 \nabla v) v_h dx - \int_{\Omega_2} \nabla \cdot (\beta_2 \nabla v) v_h dx \\
&= -\int_\Gamma \beta_1 \frac{\partial v}{\partial \eta} v_h ds + \int_{\Omega_1} \beta_1 \nabla v \cdot \nabla v_h dx \\
&\quad + \int_\Gamma \beta_2 \frac{\partial v}{\partial \eta} v_h ds + \int_{\Omega_2} \beta_2 \nabla v \cdot \nabla v_h dx \\
&= \int_{\Omega_1} \beta_1 \nabla v \cdot \nabla v_h dx + \int_{\Omega_2} \beta_2 \nabla v \cdot \nabla v_h dx + \int_\Gamma \left[\beta \frac{\partial v}{\partial \eta}\right] v_h ds \\
&= A^1(v, v_h) + A^2(v, v_h). \qquad (4.3.6)
\end{aligned}
$$

In the last equality, we have used the fact that $\left[\beta\frac{\partial v}{\partial \eta}\right] = 0$ along Γ. Combining (4.3.5) and (4.3.6), we have

$$A_h(R_h v, v_h) = A^1(v, v_h) + A^2(v, v_h) \quad \forall v_h \in V_h. \tag{4.3.7}$$

Regarding the approximation properties of R_h operator defined by (4.3.7), we have the following results

Lemma 4.3.2 *Let R_h be defined by (4.3.7). then for any $v \in X^{**}$ there is a positive constant C independent of the mesh parameter h such that*

$$\|R_h v - v\|_{H^1(\Omega_1)} + \|R_h v - v\|_{H^1(\Omega_2)} \leq Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

Proof. Coercivity of each local bilinear map and the definition of R_h projection leads to

$$\begin{aligned}
&\|v - R_h v\|^2_{H^1(\Omega_1)} + \|v - R_h v\|^2_{H^1(\Omega_2)} \\
&\leq C\{A^1(v - R_h v, v - v_h) + A^2(v - R_h v, v - v_h)\} \\
&\quad + CA^1(v, v_h - R_h v) - CA^1(R_h v, v_h - R_h v) \\
&\quad + CA^2(v, v_h - R_h v) - CA^2(R_h v, v_h - R_h v) \\
&= C\{A^1(v - R_h v, v - v_h) + A^2(v - R_h v, v - v_h)\} \\
&\quad + C\{A_h^1(R_h v, v_h - R_h v) - A^1(R_h v, v_h - R_h v)\} \\
&\quad + C\{A_h^2(R_h v, v_h - R_h v) - A^2(R_h v, v_h - R_h v)\} \\
&= C\{A^1(v - R_h v, v - v_h) + A^2(v - R_h v, v - v_h)\} \\
&\quad + C\{A_h(R_h v, v_h - R_h v) - A(R_h v, v_h - R_h v)\}.
\end{aligned}$$

Then it follows from Lemma 2.2.1 of Chapter 2 and Young's inequality that

$$\begin{aligned}
&\|v - R_h v\|^2_{H^1(\Omega_1)} + \|v - R_h v\|^2_{H^1(\Omega_2)} \\
&\leq C\|v - R_h v\|_{H^1(\Omega_1)}\|v - v_h\|_{H^1(\Omega_1)} + C\|v - R_h v\|_{H^1(\Omega_2)}\|v - v_h\|_{H^1(\Omega_2)} \\
&\quad + Ch\|R_h v\|_{H^1(\Omega)}\|v_h - R_h v\|_{H^1(\Omega)} \\
&\leq \epsilon\|v - R_h v\|^2_{H^1(\Omega_1)} + \frac{C}{\epsilon}\|v - v_h\|^2_{H^1(\Omega_1)} + \epsilon\|v - R_h v\|^2_{H^1(\Omega_2)} \\
&\quad + \frac{C}{\epsilon}\|v - v_h\|^2_{H^1(\Omega_2)} + \frac{Ch^2}{\epsilon}\|R_h v\|^2_{H^1(\Omega)} + \epsilon\|v_h - R_h v\|^2_{H^1(\Omega)}.
\end{aligned}$$

Again applying the fact $\|R_h v\|_{H^1(\Omega)} \leq C(\|v\|_{H^1(\Omega_1)} + \|v\|_{H^1(\Omega_2)})$ and for suitable $\epsilon > 0$, we have

$$\begin{aligned}\|v - R_h v\|^2_{H^1(\Omega_1)} + \|v - R_h v\|^2_{H^1(\Omega_2)} &\leq C\|v - v_h\|^2_{H^1(\Omega_1)} + C\|v - v_h\|^2_{H^1(\Omega_2)} \\ &\quad + Ch^2\{\|v\|^2_{H^1(\Omega_1)} + \|v\|^2_{H^1(\Omega_2)}\}.\end{aligned}$$

Now, setting $v_h = v_I$ and then using Lemma 4.3.1, we have

$$\|v - R_h v\|_{H^1(\Omega_1)} + \|v - R_h v\|_{H^1(\Omega_2)} \leq Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

This completes the proof of Lemma 4.3.2. $\square$

Corollary 4.3.1 *Let u be the exact solution of the interface problem (4.1.1)-(4.1.3), then*

$$\|u - R_h u\|_{H^1(\Omega_1)} + \|u - R_h u\|_{H^1(\Omega_2)} \leq Ch(\|u\|_{H^2(\Omega_1)} + \|u\|_{H^2(\Omega_2)}).$$

Proof. Since the solution $u \in X \cap H^1_0(\Omega)$ with $[u] = 0$ and $[\beta \frac{\partial u}{\partial \eta}] = 0$, thus $u \in X^{**}$ and hence the result follows from the previous result.

Corollary 4.3.2 *Let u be the exact solution of the interface problem (4.1.1)-(4.1.3), then*

$$\|u_t - R_h u_t\|_{H^1(\Omega_1)} + \|u_t - R_h u_t\|_{H^1(\Omega_2)} \leq Ch(\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)}).$$

Proof. Again $u_1 = u_2$ and $\beta_1 \frac{\partial u_1}{\partial \eta} = \beta_2 \frac{\partial u_2}{\partial \eta}$ along Γ, therefore taking time derivative, we have

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial t} \ \textit{and}\ \beta_1 \frac{\partial u_{1t}}{\partial \eta} = \beta_2 \frac{\partial u_{2t}}{\partial \eta} \Rightarrow [u_t] = 0 \ \textit{and}\ \left[\beta \frac{\partial u_t}{\partial \eta}\right] = 0 \ \textit{along}\ \Gamma.$$

Therefore, $u_t \in X^{**}$ and hence an application of Lemma 4.3.2 leads to the desired result.

Lemma 4.3.3 *Let R_h be defined fixed in (4.3.7), then for any $v \in X^{**}$ there is a positive constant C independent of the mesh size parameter h such that*

$$\|R_h v - v\|_{L^2(\Omega)} \leq Ch^2(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

Proof. For L^2 norm error estimate, we will use the duality argument. For this purpose, we consider the following interface problem

$$-\nabla \cdot (\beta \nabla \phi) = v - R_h v$$

with the boundary condition $\phi = 0$ on $\partial\Omega$ and interface conditions $[\phi] = 0, \ \ [\beta \frac{\partial \phi}{\partial \eta}] = 0$ along Γ.

Now multiply the above equation by w with $w \in L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2) \cap \{\psi : \psi = 0 \text{ on } \partial\Omega\}$ and $[w] = 0$ along Γ, and then integrate over Ω to have

$$\begin{aligned}
(v - R_h v, w) &= \int_\Omega -\nabla \cdot (\beta \nabla \phi) w dx \\
&= -\int_{\Omega_1} \nabla \cdot (\beta_1 \nabla \phi) w dx - \int_{\Omega_2} \nabla \cdot (\beta_2 \nabla \phi) w dx \\
&= \int_{\Omega_1} \beta_1 \nabla \phi . \nabla w dx - \int_\Gamma \beta_1 \frac{\partial \phi}{\partial \eta} w ds + \int_{\Omega_2} \beta_2 \nabla \phi . \nabla w dx \\
&\quad + \int_\Gamma \beta_2 \frac{\partial \phi}{\partial \eta} w ds \\
&= A^1(\phi, w) + A^2(\phi, w) + \int_\Gamma \left[\beta w \frac{\partial \phi}{\partial \eta} \right] ds.
\end{aligned}$$

Again $w_1 = w_2$ and $\beta_1 \partial \phi_1 / \partial \eta = \beta_2 \partial \phi_2 / \partial \eta$ along Γ implies $\left[\beta w \partial \phi / \partial \eta \right] = 0$ along Γ. Thus, the above equation reduces to

$$A^1(\phi, w) + A^2(\phi, w) = (v - R_h v, w). \tag{4.3.8}$$

Let $\phi_h \in V_h$ be the finite element approximation to ϕ defined as: Find $\phi_h \in V_h$ such that

$$A_h(\phi_h, w_h) = (v - R_h v, w_h) \ \ \forall w_h \in V_h. \tag{4.3.9}$$

Arguing as deriving Lemma 4.3.2, it can be concluded that

$$\begin{aligned}
\|\phi - \phi_h\|_{H^1(\Omega_1)} &+ \|\phi - \phi_h\|_{H^1(\Omega_2)} \\
&\leq C(\|\phi - w_h\|_{H^1(\Omega_1)} + \|\phi - w_h\|_{H^1(\Omega_2)}) \\
&\quad + Ch(\|\phi\|_{H^2(\Omega_1)} + \|\phi\|_{H^2(\Omega_2)}) \ \ \forall w_h \in V_h.
\end{aligned}$$

Let ϕ_I be defined as in (4.3.1) and then set $w_h = \phi_I$ to have

$$\begin{aligned}
\|\phi - \phi_h\|_{H^1(\Omega_1)} + \|\phi - \phi_h\|_{H^1(\Omega_2)} &\leq Ch(\|\phi\|_{H^2(\Omega_1)} + \|\phi\|_{H^2(\Omega_2)}) \\
&\leq Ch\|v - R_h v\|_{L^2(\Omega)}.
\end{aligned}$$

In the last inequality, we used the elliptic regularity estimate $\|\phi\|_X \le C\|v - R_h v\|_{L^2(\Omega)}$ (cf. [11]). Thus, we have

$$\|\phi - \phi_h\|_{H^1(\Omega)} \le Ch\|v - R_h v\|_{L^2(\Omega)}. \tag{4.3.10}$$

Since $[v - R_h v] = 0$ along Γ and $v - R_h v \in L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2) \cap \{\psi : \psi = 0 \text{ on } \partial\Omega\}$, therefore we can set $w = v - R_h v$ in (4.3.8) to have

$$\begin{aligned}
\|v - R_h v\|^2_{L^2(\Omega)} &= A^1(\phi, v - R_h v) + A^2(\phi, v - R_h v) \\
&= A^1(\phi - \phi_h, v - R_h v) + A^2(\phi - \phi_h, v - R_h v) \\
&\quad + \{A^1(\phi_h, v - R_h v) + A^2(\phi_h, v - R_h v)\} \\
&\le C\|\phi - \phi_h\|_{H^1(\Omega_1)}\|v - R_h v\|_{H^1(\Omega_1)} \\
&\quad + C\|\phi - \phi_h\|_{H^1(\Omega_2)}\|v - R_h v\|_{H^1(\Omega_2)} \\
&\quad + \{A^1(\phi_h, v) + A^2(\phi_h, v)\} - \{A^1(\phi_h, R_h v) + A^2(\phi_h, R_h v)\} \\
&\le Ch\|v - R_h v\|_{L^2(\Omega)} \cdot Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}) \\
&\quad + A_h(R_h v, \phi_h) - A(R_h v, \phi_h) \\
&= Ch^2\|v - R_h v\|_{L^2(\Omega)}(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}) \\
&\quad + \{A_h(R_h v, \phi_h) - A(R_h v, \phi_h)\} \\
&= Ch^2\|v - R_h v\|_{L^2(\Omega)}(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}) + (J).
\end{aligned} \tag{4.3.11}$$

Now, we apply Lemma 2.2.1 to have

$$\begin{aligned}
|(J)| &\le Ch \sum_{K \in \mathcal{T}_\Gamma^*} \|R_h v\|_{H^1(K)}\|\phi_h\|_{H^1(K)} \\
&\le Ch \sum_{K \in \mathcal{T}_\Gamma^*} \|R_h v\|_{H^1(K_1)}\|\phi_h\|_{H^1(K_1)} \\
&\quad + Ch \sum_{K \in \mathcal{T}_\Gamma^*} \|R_h v\|_{H^1(K_2)}\|\phi_h\|_{H^1(K_2)} \\
&= (J)_1 + (J)_2.
\end{aligned} \tag{4.3.12}$$

Again, using Corollary 4.3.1 and estimate (4.3.10), we have

$$\begin{aligned}
&\|R_h v\|_{H^1(K_2)}\|\phi_h\|_{H^1(K_2)} \\
&\le \{\|R_h v - v\|_{H^1(K_2)} + \|v\|_{H^1(K_2)}\}\{\|\phi_h - \phi\|_{H^1(K_2)} + \|\phi\|_{H^1(K_2)}\} \\
&\le \{\|R_h v - v\|_{H^1(\Omega_2)} + \|\tilde{v}_2\|_{H^1(K_2)}\}\{\|\phi_h - \phi\|_{H^1(\Omega_2)} + \|\phi\|_{H^1(K_2)}\} \\
&\le C\{h\|v\|_{H^2(\Omega_1)} + h\|v\|_{H^2(\Omega_2)} + \|\tilde{v}_2\|_{H^1(K)}\} \\
&\quad \times \{h\|v - R_h v\|_{L^2(\Omega)} + \|\phi\|_{H^1(K)}\}. \qquad (4.3.13)
\end{aligned}$$

Setting $p = 4$ in the Sobolev embedding inequality (2.2.8), we obtain

$$\begin{aligned}
\|\tilde{v}_2\|_{H^1(K)} &= \|\tilde{v}_2\|_{L^2(K)} + \|\nabla\tilde{v}_2\|_{L^2(K)} \\
&\le Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{L^4(K)} + Ch^{\frac{1}{2}}\|\nabla\tilde{v}_2\|_{L^4(K)} \\
&\le Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{H^1(K)} + Ch^{\frac{1}{2}}\|\nabla\tilde{v}_2\|_{H^1(K)} \\
&\le Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{H^2(K)} \le Ch^{\frac{1}{2}}\|v_2\|_{H^2(\Omega_2)}, \qquad (4.3.14)
\end{aligned}$$

where we have used the fact that $\text{meas}(K) \le Ch^2$. Similarly, for $\|\phi\|_{H^1(K)}$, we have

$$\|\phi\|_{H^1(K)} \le Ch^{\frac{1}{2}}\|\phi\|_X \le Ch^{\frac{1}{2}}\|v - R_h v\|_{L^2(\Omega)}. \qquad (4.3.15)$$

Combining (4.3.13)-(4.3.15), we have

$$\begin{aligned}
&\|R_h v\|_{H^1(K_2)}\|\phi_h\|_{H^1(K_2)} \\
&\le Ch\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\}\|v - R_h v\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore, for $(J)_2$, we have

$$(J)_2 \le Ch^2\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\}\|v - R_h v\|_{L^2(\Omega)}. \qquad (4.3.16)$$

Similarly, for $(J)_1$, we have

$$(J)_1 \le Ch^2\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\}\|v - R_h v\|_{L^2(\Omega)}. \qquad (4.3.17)$$

Then, using the estimates (4.3.16) and (4.3.17) in (4.3.12), we have

$$|(J)| \le Ch^2\|v - R_h v\|_{L^2(\Omega)}(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}). \qquad (4.3.18)$$

Finally, (4.3.11) and (4.3.18) leads to the following optimal L^2 norm estimate

$$\|v - R_h v\|_{L^2(\Omega)} \le Ch^2(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

This completes the rest of the proof.

Corollary 4.3.3 *Let u be the exact solution of the interface problem (4.1.1)-(4.1.3), then*

$$\|u - R_h u\|_{L^2(\Omega)} \leq Ch^2 \|u\|_X,$$

$$\|u_t - R_h u_t\|_{L^2(\Omega)} \leq Ch^2 (\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)}).$$

4.4 Error Analysis for the Semidiscrete Scheme

In this section, we discuss the semidiscrete finite element method for the problem (4.1.1)-(4.1.3) and derive optimal error estimates in L^2 and H^1 norms.

The continuous-time Galerkin finite element approximation to (4.2.1) is stated as follows: Find $u_h(t) \in V_h$ such that $u_h(0) = R_h u_0$ and

$$(u_{ht}, v_h) + A_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h, \;\; t \in (0, T]. \tag{4.4.1}$$

Subtracting (4.4.1) from (4.2.1), we have

$$(u_t - u_{ht}, v_h) + A(u, v_h) - A_h(u_h, v_h) = 0. \tag{4.4.2}$$

Define the error $e(t) = u - u_h = u - R_h u + R_h u - u_h = \rho + \theta$, with $\rho = u - R_h u$ and $\theta = R_h u - u_h$. Again, using (4.3.7) for $v = u \in X^{**}$ and further differentiating with respect to t, we have

$$A_h((R_h u)_t, v_h) = A^1(u_t, v_h) + A^2(u_t, v_h).$$

Also,

$$A_h(R_h u_t, v_h) = A^1(u_t, v_h) + A^2(u_t, v_h).$$

From the above two equations, we have

$$A_h((R_h u)_t - R_h u_t, v_h) = 0 \;\; \forall v_h \in V_h.$$

Setting $v_h = (R_h u)_t - R_h u_t$ in the above equation, we obtain $(R_h u)_t = R_h u_t$.

Now, by the definition of R_h operator and (4.4.2), we obtain

$$\begin{aligned}
(\theta_t, v_h) + A_h(\theta, v_h) &= ((R_h u)_t - u_{ht}, v_h) + A_h(R_h u - u_h, v_h) \\
&= (R_h u_t, v_h) - (u_{ht}, v_h) + A_h(R_h u, v_h) - A_h(u_h, v_h) \\
&= (u_t - \rho_t, v_h) - (u_{ht}, v_h) + A(u, v_h) - A_h(u_h, v_h) \\
&= (-\rho_t, v_h) + (u_t - u_{ht}, v_h) - (u_t - u_{ht}, v_h) \\
&= (-\rho_t, v_h).
\end{aligned}$$

For $v_h = \theta$, we have

$$\begin{aligned}(\theta_t, \theta) + C\|\theta\|^2_{H^1(\Omega)} &\le \|\rho_t\|_{L^2(\Omega)}\|\theta\|_{L^2(\Omega)} \\ &\le C_\epsilon\|\rho_t\|^2_{L^2(\Omega)} + \frac{\epsilon}{2}\|\theta\|^2_{H^1(\Omega)}.\end{aligned}$$

Integrating the above equation form 0 to t and using Corollary 4.3.3, we obtain

$$\begin{aligned}\|\theta(t)\|^2_{L^2(\Omega)} &\le C\int_0^t \|\rho_t\|^2_{L^2(\Omega)}ds + \|\theta(0)\|^2_{L^2(\Omega)} \\ &\le C\int_0^t \|\rho_t\|^2_{L^2(\Omega)}ds \\ &\le Ch^4\int_0^t (\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)})^2. \qquad (4.4.3)\end{aligned}$$

Now, combining Corollary 4.3.3 and (4.4.3), we have the following optimal pointwise-in-time L^2-norm error estimates.

Theorem 4.4.1 *Let u and u_h be the solution of the problem (4.1.1)-(4.1.3) and (4.4.1), respectively. Assume that $u_h(0) = R_h u_0$. Then there exists a constant C independent of h such that*

$$\|e(t)\|_{L^2(\Omega)} \le Ch^2\Big\{\|u\|_X + \Big(\int_0^t (\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)})^2 ds\Big)^{\frac{1}{2}}\Big\}.$$

For H^1-norm estimate, we first use Corollary 4.3.1 to have

$$\sum_{i=1}^{2}\|\rho(t)\|_{H^1(\Omega_i)} \le Ch\sum_{i=1}^{2}\|u\|_{H^2(\Omega_i)}. \qquad (4.4.4)$$

Applying inverse estimate 2.2 of Chapter 3, we obtain

$$\begin{aligned}\sum_{i=1}^{2}\|\theta(t)\|_{H^1(\Omega_i)} &\le Ch^{-1}\sum_{i=1}^{2}\|\theta(t)\|_{L^2(\Omega_i)} \\ &\le Ch^{-1}.h^2\Big\{\int_0^t (\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)})^2\Big\}^{\frac{1}{2}} \\ &= Ch\Big\{\int_0^t (\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)})^2\Big\}^{\frac{1}{2}}. \qquad (4.4.5)\end{aligned}$$

Combining (4.4.4) and (4.4.5), we have the following optimal pointwise-in-time H^1-norm error estimates.

Theorem 4.4.2 *Let u and u_h be the solution of the problem (4.1.1)-(4.1.3) and (4.4.1), respectively. Assume that $u_h(0) = R_h u_0$. Then there exists a constant C independent of h such that*

$$\|e(t)\|_{H^1(\Omega)} \leq Ch\Big\{\sum_{i=1}^{2}\|u\|_X + \Big(\int_0^t (\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)})^2\Big)^{\frac{1}{2}}\Big\}.$$

4.5 Error Analysis for the Fully Discrete Scheme

A fully discrete scheme based on backward Euler method is proposed and analyzed in this section. Optimal L^2 norm error estimate is obtained for fully discrete scheme.

We first partition the interval [0, T] into M equally spaced subintervals by the following points

$$0 = t_0 < t_1 < \ldots < t_M = T$$

with $t_n = nk,\ k = \frac{T}{M}$, be the time step. Let $I_n = (t_{n-1}, t_n]$ be the n-th subinterval. Now we introduce the backward difference quotient

$$\Delta_k \phi^n = \frac{\phi^n - \phi^{n-1}}{k},$$

for a given sequence $\{\phi^n\}_{n=0}^M \subset L^2(\Omega)$.

The fully discrete finite element approximation to the problem (4.2.1) is defined as follows: For $n = 1, \ldots, M$, find $U^n \in V_h$ such that

$$(\Delta_k U^n, v_h) + A_h(U^n, v_h) = (f^n, v_h)\ \ \forall v_h \in V_h \tag{4.5.1}$$

with $U^0 = R_h u_0$. For each $n = 1, \ldots, M$, the existence of a unique solution to (4.5.1) can be found in [11]. We then define the fully discrete solution to be a piecewise constant function $U_h(x,t)$ in time and is given by

$$U_h(x,t) = U^n(x)\ \ \forall t \in I_n,\ \ 1 \leq n \leq M.$$

We now prove the main result of this section in the following theorem.

Theorem 4.5.1 *Let u and U be the solutions of the problem (4.1.1)-(4.1.3) and (4.5.1), respectively. Assume that $U^0 = R_h u_0$. Then there exists a constant C independent of h*

and k such that

$$\|U(t_n) - u(t_n)\|_{L^2(\Omega)}$$
$$\leq C(h^2 + k) \sum_{i=1}^{2} \left\{ \|u^0\|_{H^2(\Omega_i)} + \|u_t\|_{L^2(0,T,H^2(\Omega_i))} + \|u_{tt}\|_{L^2(0,T,L^2(\Omega_i))} \right\}$$

Proof. We write the error $U^n - u^n$ at time t_n as

$$U^n - u^n = (U^n - R_h u^n) + (R_h u^n - u^n) \equiv: \theta^n + \rho^n$$

where $\theta^n = U^n - R_h u^n$ and $\rho^n = R_h u^n - u^n$.

For θ^n, we have the following error equation

$$\begin{aligned}
&(\Delta_k \theta^n, v_h) + A_h(\theta^n, v_h) \\
&= (-\Delta_k R_h u^n + \Delta_k U^n, v_h) + A_h(-R_h u^n + U^n, v_h) \\
&= (\Delta_k U^n, v_h) + A_h(U^n. v_h) - (\Delta_k R_h u^n, v_h) - A_h(R_h u^n, v_h) \\
&= (f^n, v_h) - (\Delta_k R_h u^n, v_h) - A(u^n, v_h) \\
&= (f^n, v_h) - (\Delta_k R_h u^n, v_h) + (u_t^n, v_h) - (f^n, v_h) \\
&\equiv: -(w^n, v_h)
\end{aligned} \tag{4.5.2}$$

where $w^n = \Delta_k R_h u^n - u_t^n$. For simplicity of the exposition, we write $w^n = w_1^n + w_2^n$, where $w_1^n = R_h \Delta_k u^n - \Delta_k u^n$ and $w_2^n = \Delta_k u^n - u_t^n$.

Now, setting $v_h = \theta^n$ in (4.5.2), we have

$$(\Delta_k \theta^n, \theta^n) + A_h(\theta^n, \theta^n) = -(w^n. \theta^n) \tag{4.5.3}$$

Since $A_h(\theta^n, \theta^n) \geq 0$, we have

$$\begin{aligned}
\|\theta^n\|_{L^2(\Omega)} &\leq k\|w^n\|_{L^2(\Omega)} + \|\theta^{n-1}\|_{L^2(\Omega)} \\
&\leq \|\theta^0\|_{L^2(\Omega)} + k\sum_{j=1}^{n} \|w_1^j\|_{L^2(\Omega)} + k\sum_{j=1}^{n} \|w_2^j\|_{L^2(\Omega)}.
\end{aligned} \tag{4.5.4}$$

In Ω_1, the term w_1^j can be expressed as

$$\begin{aligned}
w_1^j &= R_h \Delta_k u_1^j - \Delta_k u_1^j = (R_h - I)(\Delta_k u_1^j) \\
&= (R_h - I)\frac{1}{k}\int_{t_{j-1}}^{t^j} u_{1,t} dt = \frac{1}{k}\int_{t_{j-1}}^{t^j} (R_h u_{1,t} - u_{1\,t}) dt,
\end{aligned}$$

where u_i, $i = 1, 2$ is the restriction of u in Ω_i and $u_{i,t} = \frac{\partial u_i}{\partial t}$.
An application of Corollary 4.3.3 leads to

$$k\|w_1^j\|_{L^2(\Omega_1)} \leq Ch^2 \int_{t_{j-1}}^{t^j} \Big\{ \sum_{i=1}^{2} \|u_t\|_{H^2(\Omega_i)} \Big\} dt$$

Similarly, we obtain

$$k\|w_1^j\|_{L^2(\Omega_2)} \leq Ch^2 \int_{t_{j-1}}^{t^j} \Big\{ \sum_{i=1}^{2} \|u_t\|_{H^2(\Omega_i)} \Big\} dt.$$

Using above two estimates, we have

$$k \sum_{j=1}^{n} \|w_1^j\|_{L^2(\Omega)} \leq Ch^2 \int_0^{t_n} \Big\{ \sum_{i=1}^{2} \|u_t\|_{H^2(\Omega_i)} \Big\} dt. \tag{4.5.5}$$

Similarly, for the term w_2^n, we have

$$kw_2^j = u^j - u^{j-1} - ku_t^j = -\int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt} ds$$

and hence

$$k\|w_2^j\|_{L^2(\Omega_i)} \leq k \int_{t_{j-1}}^{t_j} \|u_{tt}\|_{L^2(\Omega_i)} ds.$$

Summing over j from $j = 1$ to $j = n$, we obtain

$$k \sum_{j=1}^{n} \|w_2^j\|_{L^2(\Omega)} \leq Ck \int_0^{t_n} \Big\{ \sum_{i=1}^{2} \|u_{tt}\|_{L^2(\Omega_i)} \Big\} dt. \tag{4.5.6}$$

Combining (4.5.4), (4.5.5) and (4.5.6), and further using the fact that $\theta^0 = 0$, we obtain

$$\begin{aligned} \|\theta^n\|_{L^2(\Omega)} &\leq C(h^2 + k) \sum_{i=1}^{2} \int_0^{t_n} \Big\{ \|u_t\|_{H^2(\Omega_i)} + \|u_{tt}\|_{L^2(\Omega_i)} \Big\} dt \\ &\leq C(h^2 + k) \sum_{i=1}^{2} \Big\{ \|u_t\|_{L^2(0,T;H^2(\Omega_i))} + \|u_{tt}\|_{L^2(0,T;L^2(\Omega_i))} \Big\} \end{aligned} \tag{4.5.7}$$

An application of Corollary 4.3.3 for ρ^n yields

$$\|\rho^n\|_{L^2(\Omega)} \leq Ch^2 \sum_{i=1}^{2} \|u^n\|_{H^2(\Omega_i)}.$$

Again, it is easy to verify that

$$\|u^n\|_{H^2(\Omega_i)} \leq \|u^0\|_{H^2(\Omega_i)} + \int_0^{t_n} \|u_t\|_{H^2(\Omega_i)} dt$$

Thus, we have

$$\|\rho^n\|_{L^2(\Omega)} \leq Ch^2 \sum_{i=1}^{2} \left\{ \|u^0\|_{H^2(\Omega_i)} + \|u_t\|_{L^2(0,T,H^2(\Omega_i))} \right\} \tag{4.5.8}$$

Combining (4.5 7) and (4.5.8) the desired estimate is easily obtained. □

Remark 4.5.1 *Although the error analysis of Sections 4.4-4.5 depends on standard ρ and θ argument given in Thomee's monograph ([47]) for non interface problem, the novelty of this chapter are contained in Section 4.3, where we have introduced modified elliptic projection and approximation properties of such projection under minimum regularity assumption of the solution. Due to low global regularity of the solution the classical analysis is difficult to apply for the convergence analysis of the interface problems. Section 4.3 bridges the gap between standard finite element technique for non interface problems and interface problems.*

Chapter 5

Finite Element Method for Hyperbolic Interface Problems

A finite element method is proposed and analyzed for hyperbolic problems with discontinuous coefficients. The main emphasize is given on the convergence of such method. For a finite element discretization discussed in Chapter 2, optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$ norms are established for continuous time discretization. Further, a fully discrete scheme based on a symmetric difference approximation is considered and optimal order convergence in $L^\infty(H^1)$ norm is established.

5.1 Introduction

In $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, we consider the following hyperbolic interface problem

$$u_{tt} - \nabla \cdot (\beta(x)\nabla u) = 0 \quad \text{in } \Omega \times (0, T] \tag{5.1.1}$$

with initial and boundary conditions

$$u(x, 0) = u_0 \;\;\&\;\; u_t(x, 0) = v_0 \;\text{ in } \Omega; \;\; u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T] \tag{5.1.2}$$

and jump conditions on the interface

$$[u] = 0, \;\; \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = 0 \quad \text{along } \Gamma. \tag{5.1.3}$$

Here, $u_0 = u_0(x)$ & $v_0 = v_0(x)$ are real valued functions in Ω. The domain Ω, symbols $[v]$ and $\mathbf{n}$ are defined as in Chapter 1, and $\mathrm{T} < \infty$.

The main objective of this chapter is to extend the results obtained in previous chapter to hyperbolic interface problems. More precisely, we are able to prove optimal order pointwise-in-time error estimates in L^2 and H^1 norms for the hyperbolic interface problem (5.1.1)-(5.1.3) with semidiscrete scheme. Fully discrete scheme based on a symmetric difference approximation is also analyzed and optimal H^1 norm error is obtained. To the best of our knowledge there is hardly any literature concerning the convergence of finite element solutions to the true solutions of hyperbolic interface problems.

The rest of the chapter is organized as follows. In section 5.2, *we recall some basic* results from the literature. Further, we define some auxiliary projections and discuss their approximation properties. Section 5.3 is devoted to the error analysis for the semidiscrete finite element approximation. Finally, error estimates for the fully discrete scheme are derived in section 5.4.

5.2 Preliminaries

Due to the presence of discontinuous coefficients the solution u of the interface problem (5.1.1)-(5.1.3), in general, does not belong to $H^2(\Omega)$. However, the solution is assumed to be smooth in each individual subdomain Ω_i, $i = 1, 2$. More precisely, the problem (5.1.1)-(5.1.3) has a unique solution $u \in L^2(0, T; X \cap H_0^1(\Omega)) \cap H^1(0, T; H^2(\Omega_1) \cap H^2(\Omega_2)) \cap H^2(0, T; Y)$ (cf. [13, 30]).

As a first step towards the finite element approximation, the weak form for the problem (5.1.1)-(5.1.3) is defined as follows: Find $u : (0, T] \to H_0^1(\Omega)$ such that

$$(u_{tt}, v) + A(u, v) = 0 \ \ \forall v \in H_0^1(\Omega), \ \ a.e. \ t \in (0, T] \tag{5.2.1}$$

with $u(0) = u_0$ and $u_t(0) = v_0$.

Let $\Pi_h : C(\overline{\Omega}) \to V_h$ be the Lagrange interpolation operator corresponding to the space V_h. As the solutions concerned are only on $H^1(\Omega)$ globally, one can not apply the standard interpolation theory directly. However, working in the space

$$X^\star = \{v \in L^2(\Omega) : v \in H^2(\Omega_1) \cup H^2(\Omega_2)\} \cap \{\psi : \psi = 0 \ \text{ on } \partial\Omega \ \ \& \ [v] = 0 \text{ along } \Gamma\},$$

we have derived the optimal error bounds for the interpolant Π_h in the previous chapter. Further, the results are also extended for elliptic projection R_h defined by (4.3.7) in the space $X^{\star\star} = \{v \in X^\star : \ [\beta \partial v / \partial \mathbf{n}] = 0 \text{ along } \Gamma\}$. The following results for the linear interpolant and elliptic projection are recalled for our convenience.

Lemma 5.2.1 *For any $v \in X^\star$, we have*

$$\|v - \Pi_h v\|_{H^1(\Omega_1)} + \|v - \Pi_h v\|_{H^1(\Omega_2)} \le Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

Lemma 5.2.2 *Let R_h be defined by (4.3.7), then for any $v \in X^{\star\star}$ there is a positive constant C independent of the mesh parameter h such that*

$$\|R_h v - v\|_{H^1(\Omega_1)} + \|R_h v - v\|_{H^1(\Omega_2)} \le Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}),$$
$$\|R_h v - v\|_{L^2(\Omega)} \le Ch^2(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

Remark 5.2.1 *Let u be the solution for the problem (5.1.1)-(5.1.3). Then clearly $u,\ u_t \in X^{\star\star}$ and hence above results are also holds true for $v = u,\ u_t$.*

Then, the following result for L^2 projection, which is an extension of Lemma 3.2.3, is an immediate consequence of previous Lemma 5.2.2.

Lemma 5.2.3 *Let L_h be defined by (3.2.12). Then, for $v \in X^{\star\star}$, there is a positive constant C independent of the mesh size parameter h such that*

$$(a) \qquad \|v - L_h v\|_{L^2(\Omega)} \le Ch^2\Big(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\Big),$$
$$(b) \qquad \|v - L_h v\|_{H^1(\Omega_1)} + \|v - L_h v\|_{H^1(\Omega_2)} \le Ch\Big(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\Big).$$

Proof. Part (a) follows from the fact that $L_h v$ is the best approximation to v in V_h with respect to L^2 norm and Lemma 5.2.2.

For H^1 norm estimate, we use inverse inequality (3.2.2) to have

$$\begin{aligned}
\sum_{l=1}^{2} \|v - L_h v\|_{H^1(\Omega_l)} &\le \sum_{l=1}^{2} \|v - R_h v\|_{H^1(\Omega_l)} + \|R_h v - L_h v\|_{H^1(\Omega)} \\
&\le \sum_{l=1}^{2} \|v - R_h v\|_{H^1(\Omega_l)} + Ch^{-1}\|R_h v - L_h v\|_{L^2(\Omega)} \\
&\le \sum_{l=1}^{2} \|v - R_h v\|_{H^1(\Omega_l)} + Ch^{-1}\Big(\|R_h v - v\|_{L^2(\Omega)} \\
&\quad + \|v - L_h v\|_{L^2(\Omega)}\Big)
\end{aligned}$$

which together with Lemma 5.2.2 leads to Part (b) of Lemma 5.2.3.

5.3 Error analysis for the Semidiscrete Scheme

This section deals with the pointwise-in-time error analysis for the spatially discrete scheme. Optimal order of convergence in $L^\infty(L^2)$ and $L^\infty(H^1)$ are established.

The continuous time Galerkin finite element approximation to (5.2.1) is stated as follows: Find $u_h(t) : [0,T] \to V_h$ such that

$$(u_{htt}, v_h) + A_h(u_h, v_h) = 0 \ \ \forall v_h \in V_h, \ \ t \in (0,T] \tag{5.3.1}$$

with $u_h(0) = R_h u_0$ and $u_{ht}(0) = L_h v_0$. We assume that $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ & $v_0 \in H_0^1(\Omega)$.

Regarding the stability for u_h, we have the following result. The proofs involve standard energy arguments and therefore the proof is omitted.

Lemma 5.3.1 *Let u_h satisfy (5.3.1). then, for $i = 1,2,3,4$, we have*

$$\left\|\frac{\partial^i}{\partial t^i}u_h(t)\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial^{i-1}}{\partial t^{i-1}}u_h(t)\right\|^2_{H^1(\Omega)} \le C\sum_{l=1}^{2}\{\|u_0\|^2_{H^i(\Omega_l)} + \|v_0\|^2_{H^{i-1}(\Omega_l)}\}.$$

Now, subtracting (5.3.1) from (5.2.1), we have

$$(u_{tt} - u_{htt}, v_h) + A(u - u_h. v_h) = A_h(u_h, v_h) - A(u_h, v_h) \ \ \forall v_h \in V_h. \tag{5.3.2}$$

Define the error $e(t)$ as $e(t) = u(t) - u_h(t)$. Then we have the following error equation

$$(e_{tt}, v_h) + A(e, v_h) = A_h(u_h, v_h) - A(u_h, v_h) \ \ \forall v_h \in V_h. \tag{5.3.3}$$

Setting $v_h = L_h u_t$ in (5.3.3) and using (3.2.12), we obtain the following error equation

$$\begin{aligned}
(e_{tt}, e_t) + A(e, e_t) &= \{A_h(u_h, L_h u_t) - A(u_h, L_h u_t)\} \\
&\quad + (u_{tt} - u_{htt}, u_t - L_h u_t) + A(u - u_h, u_t - L_h u_t) \\
&\quad - \{(u_{tt} - u_{htt}, u_{ht}) + A(u - u_h, u_{ht})\} \\
&= \{A_h(u_h, L_h u_t) - A(u_h, L_h u_t)\} \\
&\quad + (u_{tt} - L_h u_{tt}, u_t - L_h u_t) + (L_h u_{tt} - u_{htt}, u_t - L_h u_t) \\
&\quad + A(u - u_h, u_t - L_h u_t) - \{A_h(u_h, u_{ht}) - A(u_h, u_{ht})\} \\
&= \{A_h(u_h, L_h u_t) - A(u_h. L_h u_t)\} \\
&\quad + \frac{1}{2}\frac{d}{dt}(u_t - L_h u_t, u_t - L_h u_t) + A(u - u_h, u_t - L_h u_t) \\
&\quad - \frac{1}{2}\frac{d}{dt}\{A_h(u_h. u_h) - A(u_h, u_h)\}.
\end{aligned}$$

Integrate the above equation from 0 to t, we get

$$
\begin{aligned}
&\frac{1}{2}\|e_t\|^2_{L^2(\Omega)} + C\|e\|^2_{H^1(\Omega)} \\
&\le \frac{1}{2}\|e_t(0)\|^2_{L^2(\Omega)} + C\|e(0)\|^2_{H^1(\Omega)} + \int_0^t |A_h(u_h, L_h u_s) - A(u_h, L_h u_s)| ds \\
&+\frac{1}{2}\|u_t - L_h u_t\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_t(0) - L_h u_t(0)\|^2_{L^2(\Omega)} + \int_0^t A(u - u_h, u_s - L_h u_s) ds \\
&+\frac{1}{2}\{A_h(u_h, u_h) - A(u_h, u_h)\} - \frac{1}{2}\{A_h(u_h(0), u_h(0)) - A(u_h(0), u_h(0))\}.
\end{aligned}
$$

With $u_h(0) = R_h u_0$, $u_{ht}(0) = L_h v_0$ and the fact that $\|e_t\|_{L^2(\Omega)} \ge \|u_t - L_h u_t\|_{L^2(\Omega)}$, and further using Lemma 5.2.3 we obtain

$$
\begin{aligned}
\|e\|^2_{H^1(\Omega)} \le\ & Ch^2(\|u_0\|^2_{H^2(\Omega_1)} + \|u_0\|^2_{H^2(\Omega_2)}) + C\sum_{l=1}^{2} \|v_0 - L_h v_0\|^2_{L^2(\Omega_l)} \\
&+ \int_0^t \{A_h(u_h, L_h u_t) - A(u_h, L_h u_t)\} ds \\
&+C\int_0^t \{\|u_s - L_h u_s\|^2_{H^1(\Omega_1)} + \|u_s - L_h u_s\|^2_{H^1(\Omega_2)}\} ds \\
&+C\int_0^t \|e\|^2_{H^1(\Omega)} ds + C\{A_h(u_h, u_h) - A(u_h, u_h)\} \\
&+C|A_h(u_h(0), u_h(0)) - A(u_h(0), u_h(0))|. \qquad (5.3.4)
\end{aligned}
$$

Using Lemma 2.2.1 and Lemma 2.2.2, we obtain

$$
\begin{aligned}
&|A_h(u_h, L_h u_t) - A(u_h, L_h u_t)| \le Ch\|u_h\|_{H^1(\Omega^*_\Gamma)}\|L_h u_t\|_{H^1(\Omega^*_\Gamma)} \\
&\le Ch\|u - u_h\|_{H^1(\Omega^*_\Gamma)}\|L_h u_t\|_{H^1(\Omega^*_\Gamma)} + Ch\|u\|_{H^1(\Omega^*_\Gamma)}\|L_h u_t\|_{H^1(\Omega^*_\Gamma)} \\
&\le Ch^2\|L_h u_t\|^2_{H^1(\Omega^*_\Gamma)} + C_\epsilon\|e\|^2_{H^1(\Omega^*_\Gamma)} \\
&\quad +Ch^{\frac{3}{2}}\|u\|_X \sum_{K\in\mathcal{T}^*_\Gamma} \{\|L_h u_t\|_{H^1(K_1)} + \|L_h u_t\|_{H^1(K_2)}\} \\
&\le Ch^2\|L_h u_t\|^2_{H^1(\Omega^*_\Gamma)} + C_\epsilon\|e\|^2_{H^1(\Omega^*_\Gamma)} \\
&\quad +Ch^{\frac{3}{2}}\|u\|_X \sum_{K\in\mathcal{T}^*_\Gamma} \{\|L_h u_t - u_t\|_{H^1(K_1)} + \|u_t\|_{H^1(K_1)}\} \\
&\quad +Ch^{\frac{3}{2}}\|u\|_X \sum_{K\in\mathcal{T}^*_\Gamma} \{\|L_h u_t - u_t\|_{H^1(K_2)} + \|u_t\|_{H^1(K_2)}\}.
\end{aligned}
$$

We now recall the extension $\tilde{u}_{tl} \in H^2(\Omega),\ l = 1,2$ of $u_{tl} = u_t|_{\Omega_l}$ satisfying (2.2.2) to have

$$
\begin{aligned}
&|A_h(u_h, L_h u_t) - A(u_h, L_h u_t)| \\
&\le Ch^2 \|L_h u_t\|^2_{H^1(\Omega^*_\Gamma)} + C_\epsilon \|e\|^2_{H^1(\Omega^*_\Gamma)} + Ch^{\frac{3}{2}} \|u\|_X \sum_{l=1}^{2} \|L_h u_t - u_t\|_{H^1(\Omega_l)} \\
&\quad + Ch^{\frac{3}{2}} \|u\|_X \sum_{K \in \mathcal{T}^*_\Gamma} \{\|\tilde{u}_{t1}\|_{H^1(K_1)} + \|\tilde{u}_{t2}\|_{H^1(K_2)}\} \\
&\le Ch^2 \|L_h u_t\|^2_{H^1(\Omega^*_\Gamma)} + C_\epsilon \|e\|^2_{H^1(\Omega^*_\Gamma)} + Ch^{\frac{5}{2}} \|u\|_X \sum_{l=1}^{2} \|u_t\|_{H^2(\Omega_l)} \\
&\quad + Ch^{\frac{3}{2}} \|u\|_X \sum_{l=1}^{2} \|\tilde{u}_{tl}\|_{H^1(\Omega^*_\Gamma)} \\
&\le Ch^2 \|L_h u_t\|^2_{H^1(\Omega^*_\Gamma)} + C_\epsilon \|e\|^2_{H^1(\Omega^*_\Gamma)} + Ch^{\frac{5}{2}} \|u\|_X \sum_{l=1}^{2} \|u_t\|_{H^2(\Omega_l)} \\
&\quad + Ch^2 \|u\|_X \sum_{l=1}^{2} \|\tilde{u}_{tl}\|_{H^2(\Omega)} \\
&\le Ch^2 \|L_h u_t\|^2_{H^1(\Omega^*_\Gamma)} + C_\epsilon \|e\|^2_{H^1(\Omega^*_\Gamma)} + Ch^{\frac{5}{2}} \|u\|_X \sum_{l=1}^{2} \|u_t\|_{H^2(\Omega_l)} \\
&\quad + Ch^2 \|u\|_X \sum_{l=1}^{2} \|u_t\|_{H^2(\Omega_l)} \\
&\le Ch^2 (\|u\|^2_X + \sum_{l=1}^{2} \|u_t\|^2_{H^2(\Omega_l)}) + C_\epsilon \|e\|^2_{H^1(\Omega)}.
\end{aligned} \tag{5.3.5}
$$

Similarly arguing as in (5.3.5), we obtain

$$
\begin{aligned}
&|A_h(u_h, u_h) - A(u_h, u_h)| \\
&\le |A_h(u_h, u_h - L_h u) - A(u_h, u_h - L_h u)| + |A_h(u_h, L_h u) - A(u_h, L_h u)| \\
&\le Ch \|u_h\|_{H^1(\Omega^*_\Gamma)} \|u_h - L_h u\|_{H^1(\Omega^*_\Gamma)} + Ch \|u_h\|_{H^1(\Omega^*_\Gamma)} \|L_h u\|_{H^1(\Omega^*_\Gamma)} \\
&\le Ch \|u_h\|_{H^1(\Omega^*_\Gamma)} \{\|u - u_h\|_{H^1(\Omega^*_\Gamma)} + \|u - L_h u\|_{H^1(\Omega^*_\Gamma)}\} \\
&\quad + Ch \{\|u_h - u\|_{H^1(\Omega^*_\Gamma)} + \|u\|_{H^1(\Omega^*_\Gamma)}\} \|L_h u\|_{H^1(\Omega^*_\Gamma)} \\
&\le Ch (\|u_h\|_{H^1(\Omega^*_\Gamma)} \|e\|_{H^1(\Omega^*_\Gamma)} + \|u_h\|_{H^1(\Omega^*_\Gamma)} \|u - L_h u\|_{H^1(\Omega^*_\Gamma)}) \\
&\quad + Ch \|e\|_{H^1(\Omega^*_\Gamma)} \|L_h u\|_{H^1(\Omega^*_\Gamma)} + Ch \|u\|_{H^1(\Omega^*_\Gamma)} \|L_h u\|_{H^1(\Omega^*_\Gamma)}.
\end{aligned}
$$

Then apply Young's inequality to have

$$
\begin{aligned}
&|A_h(u_h,u_h) - A(u_h,u_h)| \\
&\le Ch^2\|u_h\|^2_{H^1(\Omega^*_\Gamma)} + C_\epsilon\|e\|^2_{H^1(\Omega^*_\Gamma)} + Ch^2\|u_h\|_{H^1(\Omega^*_\Gamma)}\|u\|_X \\
&\quad + Ch^2\|L_h u\|^2_{H^1(\Omega^*_\Gamma)} + C_\epsilon\|e\|^2_{H^1(\Omega^*_\Gamma)} + Ch\cdot h^{\frac{1}{2}}\|u\|_X\cdot h^{\frac{1}{2}}\|u\|_X \\
&\le Ch^2\|u_h\|^2_{H^1(\Omega)} + C_\epsilon\|e\|^2_{H^1(\Omega)} + Ch^2\|u_h\|^2_{H^1(\Omega)} + Ch^2\|u\|^2_X \\
&\quad + Ch^2\|u\|^2_X + Ch^2\|u\|^2_X \\
&\le Ch^2\|u_h\|^2_{H^1(\Omega)} + Ch^2\|u\|^2_X + C_\epsilon\|e\|^2_{H^1(\Omega)} \\
&\le Ch^2(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{L^2(\Omega)}) + Ch^2\|u\|^2_X + C_\epsilon\|e\|^2_{H^1(\Omega)}. \qquad (5.3.6)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&|A_h(u_h(0),u_h(0)) - A(u_h(0),u_h(0))| \\
&\le |A_h(u_h(0),u_h(0)-L_hu_0) - A(u_h(0),u_h(0)-L_hu_0)| \\
&+|A_h(u_h(0). L_hu_0) - A(u_h(0),L_hu_0)| \\
&\le Ch\|u_h(0)\|_{H^1(\Omega^*_\Gamma)}\|u_h(0)-L_hu_0\|_{H^1(\Omega^*_\Gamma)} + Ch\|u_h(0)\|_{H^1(\Omega^*_\Gamma)}\|L_hu_0\|_{H^1(\Omega^*_\Gamma)} \\
&\le Ch\|u_h(0)\|_{H^1(\Omega^*_\Gamma)}\{\|R_hu_0-u_0\|_{H^1(\Omega^*_\Gamma)} + \|u_0-L_hu_0\|_{H^1(\Omega^*_\Gamma)}\} \\
&+Ch\{\|R_hu_0-u_0\|_{H^1(\Omega^*_\Gamma)} + \|u_0\|_{H^1(\Omega^*_\Gamma)}\}\|L_hu_0\|_{H^1(\Omega^*_\Gamma)} \\
&\le Ch\|R_hu_0\|_{H^1(\Omega^*_\Gamma)}\{Ch\|u_0\|_X + Ch\|u_0\|_X\} + Ch\{Ch\|u_0\|_X + \|u_0\|_{H^1(\Omega^*_\Gamma)}\}h^{\frac{1}{2}}\|u_0\|_X \\
&\le Ch^2\cdot h^{\frac{1}{2}}\|u_0\|^2_X + Ch^2\cdot h^{\frac{1}{2}}\|u_0\|^2_X + Ch\cdot h^{\frac{1}{2}}\|u_0\|_X\cdot h^{\frac{1}{2}}\|u_0\|_X \\
&\le Ch^2\|u_0\|^2_X. \qquad (5.3.7)
\end{aligned}
$$

Using (5.3.5)-(5.3.7) in (5.3.4), we obtain

$$
\begin{aligned}
\|e\|^2_{H^1(\Omega)} \le\; & Ch^2(\|u_0\|^2_X + \sum_{l=1}^{2}\|v_0\|_{H^1(\Omega_l)}) + Ch^2\int_0^t \|u\|^2_X ds \\
&+Ch^2\int_0^t\sum_{l=1}^{2}\|u_t\|_{H^2(\Omega_l)}ds + C_\epsilon\int_0^t \|e\|^2_{H^1(\Omega)}ds \\
&+Ch^2\sum_{l=1}^{2}\|u_t\|_{L^2(0,T,H^2(\Omega_l))} + Ch^2(\|u_0\|^2_X + \|u\|^2_X) \\
&+C_\epsilon\|e\|^2_{H^1(\Omega)} + C\int_0^t \|e\|^2_{H^1(\Omega)}ds.
\end{aligned}
$$

An application of Gronwall's lemma leads to the following optimal H^1-norm error estimate

Theorem 5.3.1 *Let u and u_h be the solution of the problem (5.1.1)-(5.1.3) and (5.3.1), respectively. Assume that $u_h(0) = R_h u_0$ and $u_{ht}(0) = L_h v_0$. Then, for sufficiently smooth u_0, v_0 in Ω_i, $i = 1, 2$, we have*

$$\begin{aligned} \|e(t)\|_{H^1(\Omega)} &\leq Ch\Big\{\|u_0\|_X + \|u\|_X \\ &\quad + \sum_{i=1}^{2}(\|v_0\|_{H^1(\Omega_i)} + \|u_t\|_{L^2(0,T;H^2(\Omega_i))}) + \|u\|_{L^2(0,T;X)}\Big\}. \quad \square \end{aligned}$$

For any function ψ in $[0, T]$, we define $\hat{\psi}(t)$ as

$$\hat{\psi}(t) = \int_0^t \psi(s) ds.$$

Clearly $\hat{\psi}_t = \psi$. For L^2 norm error estimate, we integrate (5.3.1) from 0 to t to have

$$(\hat{u}_{htt}, v_h) + A_h(\hat{u}_h, v_h) = (L_h v_0, v_h) \;\; \forall v_h \in V_h, \;\; t \in (0, T] \tag{5.3.8}$$

with $\hat{u}_{htt}(0) = u_{ht}(0) = L_h v_0$. Similarly, integrating (5.2.1) from 0 to t, to obtain

$$(\hat{u}_{tt}, v_h) + A(\hat{u}, v_h) = (v_0, v_h) \;\; \forall v_h \in V_h, \;\; t \in (0, T]. \tag{5.3.9}$$

Subtracting (5.3.9) from (5.3.8), we obtain

$$(\hat{u}_{htt} - \hat{u}_{tt}, v_h) + A_h(\hat{u}_h, v_h) - A(\hat{u}, v_h) = (L_h v_0 - v_0, v_h) \;\; \forall v_h \in V_h, \;\; t \in (0, T].$$

For optimal error estimate, we split the error $e = u_h - u$ as

$$e = u_h - R_h u + R_h u - u = \theta + \rho.$$

Then, for θ, we have the following error equation

$$\begin{aligned} (\hat{\theta}_{tt}, v_h) + A_h(\hat{\theta}, v_h) &= (\hat{u}_{htt} - R_h \hat{u}_{tt}, v_h) + A_h(\hat{u}_h - R_h \hat{u}, v_h) \\ &= (\hat{u}_{htt} - \hat{u}_{tt} + \hat{u}_{tt} - R_h \hat{u}_{tt}, v_h) + A_h(\hat{u}_h, v_h) - A(\hat{u}, v_h) \\ &= -(\hat{\rho}_{tt}, v_h) + (\hat{u}_{htt} - \hat{u}_{tt}, v_h) + A_h(\hat{u}_h, v_h) - A(\hat{u}, v_h) \\ &= -(\rho_t, v_h). \end{aligned} \tag{5.3.10}$$

Here, we have used the fact that $(L_h v_0 - v_0, v_h) = 0$. Setting $v_h = \hat{\theta}_t$ in the above equation, we have

$$\frac{1}{2}\frac{d}{dt}\|\hat{\theta}_t\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}A_h(\hat{\theta}, \hat{\theta}) \leq C\|\rho_t\|^2_{L^2(\Omega)} + C\|\hat{\theta}_t\|^2_{L^2(\Omega)}.$$

Integrating from 0 to t and further applying Lemma 4.3.3 of Chapter 4, we obtain

$$\begin{aligned}\|\theta\|^2_{L^2(\Omega)} &+ A_h(\hat{\theta},\hat{\theta}) \\ &\leq Ch^4\int_0^t \{\|u_s\|^2_{H^2(\Omega_1)} + \|u_s\|^2_{H^2(\Omega_2)}\}ds + C\int_0^t \|\theta\|^2_{H^1(\Omega)}ds.\end{aligned}$$

Here, we have used the fact that $u_h(0) = R_h u_0$ and $\hat{\theta}(0) = 0$. Further, a simple application of Gronwall's lemma leads to

$$\|\theta\|_{L^2(\Omega)} \leq Ch^2 \sum_{i=1}^{2} \|u_t\|_{L^2(0,T;H^2(\Omega_i))}. \tag{5.3.11}$$

This together with Lemma 5.2.1 gives the following optimal L^2 norm error estimate

Theorem 5.3.2 *Let u and u_h be the solution of the problem (5.1.1)-(5.1.3) and (5.3.1), respectively. Assume that $u_h(0) = R_h u_0$ and $u_{ht}(0) = L_h v_0$. Then, for sufficiently smooth u_0, v_0 in Ω_i, $i = 1, 2$, we have*

$$\|e(t)\|_{L^2(\Omega)} \leq Ch^2 \sum_{i=1}^{2} (\|u\|_{H^2(\Omega_i)} + \|u_t\|_{L^2(0,T;H^2(\Omega_i))}). \quad \square$$

5.4 Error Analysis for the Fully Discrete Scheme

A discrete-in-time scheme based on a symmetric difference approximation around the nodal points is considered and analyzed in this section.

We first divide the interval [0.T] into M equally spaced subintervals by the points

$$0 = t_0 < t_1 < \ldots < t_M = T$$

with $t_n = nk$, $k = T/M$ being the time step. Let $U^n = U(t_n)$ be an approximation of $u(t_n)$. Then the fully discrete finite element approximation to the problem (5.3.1) is defined as follows: For given U^0 and U^1, seek a function $U^n = U(t_n)$ such that

$$(\partial_t\bar{\partial}_t U^n, v_h) + A_h(\tilde{U}^n, v_h) = 0, \quad n \geq 1, \; v_h \in V_h \tag{5.4.1}$$

with

$$\begin{aligned}&\partial_t U^n = k^{-1}(U^{n+1} - U^n), \; \bar{\partial}_t U^n = k^{-1}(U^n - U^{n-1}) \text{ and} \\ &\tilde{U}^n = (U^{n+1} + 2U^n + U^{n-1})/4 = (U^{n+1/2} + U^{n-1/2})/2.\end{aligned}$$

We write $\xi^n = U^n - u_h^n$. Then (5.3.1) and (5.4.1) leads to the following error equation in ξ^n

$$(\partial_t\overline{\partial}_t\xi^n, v_h) + A_h(\widetilde{\xi}^n, v_h) = A_h(u_h^n - \widetilde{u}_h^n, v_h) + (\tau^n, v_h) \quad v_h \in V_h \tag{5.4.2}$$

where $\tau^n = u_{htt}^n - \partial_t\overline{\partial}_t u_h^n$. Setting $v_h = \overline{\partial}_t\xi^{n+\frac{1}{2}}$ in the above equation, we have

$$(\partial_t\overline{\partial}_t\xi^n, \overline{\partial}_t\xi^{n+\frac{1}{2}}) + A_h(\widetilde{\xi}^n, \overline{\partial}_t\xi^{n+\frac{1}{2}}) = A_h(u_h^n - \widetilde{u}_h^n, \overline{\partial}_t\xi^{n+\frac{1}{2}}) + (\tau^n, \overline{\partial}_t\xi^{n+\frac{1}{2}}). \tag{5.4.3}$$

Again, it is easy to verify that

$$(\partial_t\overline{\partial}_t\xi^n, \overline{\partial}_t\xi^{n+\frac{1}{2}}) = (\partial_t\overline{\partial}_t\xi^n, \frac{1}{2}(\partial_t\xi^n + \overline{\partial}_t\xi^n)) = \frac{1}{2}\overline{\partial}_t\|\partial_t\xi^n\|^2_{L^2(\Omega)} \text{ and}$$

$$A_h(\widetilde{\xi}^n, \overline{\partial}_t\xi^{n+\frac{1}{2}}) = \frac{1}{2}\overline{\partial}_t A_h(\xi^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}) - \frac{1}{2}\overline{\partial}_t A(\xi^{n-\frac{1}{2}}, \xi^{n-\frac{1}{2}}).$$

Substituting these expressions in (5.4.3), we obtain

$$\begin{aligned}\frac{1}{2}\overline{\partial}_t\|\partial_t\xi^n\|^2_{L^2(\Omega)} + \frac{1}{2}\overline{\partial}_t A_h(\xi^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}) &= A_h(u_h^n - \widetilde{u}_h^n, \overline{\partial}_t\xi^{n+\frac{1}{2}}) \\ &\quad + (\tau^n, \overline{\partial}_t\xi^{n+\frac{1}{2}}) \\ &\quad + \frac{1}{2}\overline{\partial}_t A(\xi^{n-\frac{1}{2}}, \xi^{n-\frac{1}{2}}) \\ &\equiv: I_1^n + I_2^n + \frac{1}{2}\overline{\partial}_t A(\xi^{n-\frac{1}{2}}, \xi^{n-\frac{1}{2}}).\end{aligned}$$

Further, applying the coercivity of $A_h(\ldots)$, we have

$$\begin{aligned}\frac{1}{2k}\Big\{\|\partial_t\xi^n\|^2_{L^2(\Omega)} - \|\partial_t\xi^{n-1}\|^2_{L^2(\Omega)}\Big\} \;+\; &\frac{C}{2k}\Big\{\|\xi^{n+\frac{1}{2}}\|^2_{H^1(\Omega)} - \|\xi^{n-\frac{1}{2}}\|^2_{H^1(\Omega)}\Big\} \\ &\le I_1^n + I_2^n + \frac{1}{2}\overline{\partial}_t A(\xi^{n-\frac{1}{2}}, \xi^{n-\frac{1}{2}}).\end{aligned}$$

Summing over n from $n = 1$ to $n = l$, we obtain

$$\begin{aligned}&\frac{1}{2}\Big\{\|\partial_t\xi^l\|^2_{L^2(\Omega)} - \|\partial_t\xi^0\|^2_{L^2(\Omega)}\Big\} + \frac{C}{2}\Big\{\|\xi^{l+\frac{1}{2}}\|^2_{H^1(\Omega)} - \|\xi^{\frac{1}{2}}\|^2_{H^1(\Omega)}\Big\} \\ &\le k\sum_{n=1}^{l}(I_1^n + I_2^n) + \frac{1}{2}k\sum_{n=1}^{l}\overline{\partial}_t A(\xi^{n-\frac{1}{2}}, \xi^{n-\frac{1}{2}}).\end{aligned} \tag{5.4.4}$$

For I_1^n, we use Taylor's expansion to obtain

$$\widetilde{u}_h^n - u_h^n = \frac{1}{2}\int_{t_{n-\frac{1}{2}}}^{t_n}(t_n - s)\frac{\partial^2 u_h}{\partial t^2}ds + \frac{1}{2}\int_{t_n}^{t_{n+\frac{1}{2}}}(t_{n+\frac{1}{2}} - s)\frac{\partial^2 u_h}{\partial t^2}ds$$

which immediately implies

$$\|\widetilde{u}_h^n - u_h^n\|_{H^1(\Omega)} \leq Ck^2\|u_{htt}\|_{L^\infty(H^1(\Omega))}. \tag{5.4.5}$$

Thus,

$$\begin{aligned} kI_1^n &\leq Ck\|\overline{\partial}_t\xi^{n+\frac{1}{2}}\|_{H^1(\Omega)}\|\widetilde{u}_h^n - u_h^n\|_{H^1(\Omega)} \\ &\leq C\{\|\xi^{n+\frac{1}{2}}\|_{H^1(\Omega)} + \|\xi^{n-\frac{1}{2}}\|_{H^1(\Omega)}\}k^2\|u_{htt}\|_{L^\infty(H^1(\Omega))}. \end{aligned} \tag{5.4.6}$$

Here, we have used the estimate (5.4.5).

Define $A_l = max_{0\leq n\leq l}|||\xi^{n+\frac{1}{2}}|||$, where $|||\xi^{n+\frac{1}{2}}|||^2 = \|\partial_t\xi^n\|^2_{L^2(\Omega)} + \|\xi^{n+\frac{1}{2}}\|^2_{H^1(\Omega)}$. Then summing (5.4.6) over n from $n = 1$ to $n = l$, we obtain

$$k\sum_{n=1}^{l} I_1^n \leq CA_lk^2\|u_{htt}\|_{L^\infty(H^1(\Omega))}. \tag{5.4.7}$$

Next, for I_2^n, we note that

$$kI_2^n \leq Ck\|\tau^n\|_{L^2(\Omega)}\{\|\partial_t\xi^n\|_{L^2(\Omega)} + \|\partial_t\xi^{n-1}\|_{L^2(\Omega)}\}. \tag{5.4.8}$$

For τ^n, we have the following expression

$$\|\tau^n\|_{L^2(\Omega)} \leq Ck^{\frac{3}{2}}\|u_{htttt}\|_{L^\infty(L^2(\Omega))}. \tag{5.4.9}$$

Summing (5.4.8) over n from $n = 1$ to $n = l$ and further applying (5.4.9), we have

$$k\sum_{n=1}^{l} I_2^n \leq CA_lk^2\|u_{htttt}\|_{L^\infty(L^2(\Omega))}. \tag{5.4.10}$$

Then applying (5.4.7) and (5.4.10) in the equation (5.4.4), we obtain

$$\begin{aligned} \|\partial_t\xi^l\|^2_{L^2(\Omega)} + \|\xi^{l+\frac{1}{2}}\|^2_{H^1(\Omega)} &\leq C\|\partial_t\xi^0\|^2_{L^2(\Omega)} + C\|\xi^{\frac{1}{2}}\|^2_{H^1(\Omega)} \\ &\quad + CA_lk^2(\|u_{htttt}\|_{L^\infty(L^2(\Omega))} + \|u_{htt}\|_{L^\infty(H^1(\Omega))}) \\ &\quad + k\sum_{n=1}^{l-1}|||\xi^{n+\frac{1}{2}}|||^2. \end{aligned}$$

Further, applying Young's inequality for $\epsilon > 0$, we have

$$\begin{aligned} |||\xi^{l+\frac{1}{2}}|||^2 &\leq C\|\partial_t\xi^0\|^2_{L^2(\Omega)} + C\|\xi^{\frac{1}{2}}\|^2_{H^1(\Omega)} \\ &\quad + \epsilon CA_l^2 + C(\epsilon)k^4(\|u_{htttt}\|_{L^\infty(L^2(\Omega))} + \|u_{htt}\|_{L^\infty(H^1(\Omega))})^2 \\ &\quad + k\sum_{n=1}^{l-1}|||\xi^{n+\frac{1}{2}}|||^2. \end{aligned}$$

The above relation hold true from $l \geq 1$. Thus, for a suitable ϵ, we obtain

$$\begin{aligned} A_n^2 &\leq C\|\partial_t \xi^0\|^2_{L^2(\Omega)} + C\|\xi^{\frac{1}{2}}\|^2_{H^1(\Omega)} \\ &+C(\epsilon)k^4\Big\{\|u_{htttt}\|_{L^\infty(L^2(\Omega))} + \|u_{htt}\|_{L^\infty(H^1(\Omega))}\Big\}^2 \\ &+k\sum_{n=1}^{l-1}|||\xi^{n+\frac{1}{2}}|||^2, \end{aligned}$$

and hence

$$\begin{aligned} A_n &\leq C\Big\{\|\partial_t \xi^0\|_{L^2(\Omega)} + \|\xi^{\frac{1}{2}}\|_{H^1(\Omega)}\Big\} + C(\epsilon)k^2\sum_{l=1}^{2}\Big\{\|u_0\|_{H^4(\Omega_l)} + \|v_0\|_{H^3(\Omega_l)}\Big\} \\ &+Ck\sum_{n=1}^{l-1}|||\xi^{n+\frac{1}{2}}|||. \end{aligned} \tag{5.4.11}$$

Now, replacing $|||\xi^{n+\frac{1}{2}}|||$ in the sum on the right by A_n and applying discrete Gronwall's lemma, we obtain the following estimate which is crucial for our error analysis.

Lemma 5.4.1 *Let ξ^n satisfy (5.4.2). Then, there exists a positive constant C independent of h and k such that*

$$\begin{aligned} &\|\partial_t \xi^n\|_{L^2(\Omega)} + \|\xi^{n+\frac{1}{2}}\|_{H^1(\Omega)} \\ \leq\ & C\{\|\partial_t \xi^0\|_{L^2(\Omega)} + \|\xi^{\frac{1}{2}}\|_{H^1(\Omega)}\} + Ck^2\sum_{l=1}^{2}\{\|u_0\|_{H^4(\Omega_l)} + \|v_0\|_{H^3(\Omega_l)}\}. \end{aligned}$$

For the convergence analysis, we need to fix U^0 and U^1. Let U^0, $\widetilde{P}_1$ and $\widetilde{P}_2$ be appropriate projections of $u_0 = u(0)$, $v_0 = u_t(0)$ and $w_1 = u_{tt}(0)$, respectively. Now, we set $U^1 = U^0 + k\widetilde{P}_1 + \frac{k^2}{2}\widetilde{P}_2$ with $U^0 = R_h u_0$, $\widetilde{P}_1 = L_h v_0$ and $\widetilde{P}_2 = L_h w_1$. We now have the following theorem:

Theorem 5.4.1 *Let u and U^n be the solution of (5.1.1)-(5.1.3) and (5.4.1), respectively. Let $u_0 \in H^4(\Omega_1) \cap H^4(\Omega_2) \cap H^1_0(\Omega)$ and $v_0 \in H^3(\Omega_1) \cap H^3(\Omega_2) \cap L^2(\Omega) \cap \{\psi : \psi = 0$ on $\partial\Omega\}$, and $k = O(h)$. Then there exist a constant C such that*

$$\begin{aligned} &\|U^{n+\frac{1}{2}} - u(t_{n+\frac{1}{2}})\|_{H^1(\Omega)} \\ &\leq C(h+k^2)\Big(\sum_{l=1}^{2}\{\|u_0\|_{H^4(\Omega_l)} + \|v_0\|_{H^3(\Omega_l)} + \|u_t\|_{L^2(0,t_{n+\frac{1}{2}};H^2(\Omega_l))}\} + \|u(t_{n+\frac{1}{2}})\|_X\Big). \end{aligned}$$

Proof. Clearly, $\xi^0 = U^0 - u_h(0)$ and $\partial_t \xi^0 = (\xi^1 - \xi^0)/k = (U^1 - u_h(t_1))/k$. Using Taylor's expansion, we have

$$u_h(t_1) = u_h(0) + k u_{ht}(0) + \frac{k^2}{2} u_{htt}(0) + \frac{1}{2}\int_0^{t_1} (t_1 - s)^2 \frac{\partial^3 u_h}{\partial t^3} ds.$$

Now

$$\|U^1 - u_h(t_1)\|_{H^1(\Omega)} \le Ck^2 \|L_h u_{tt}(0) - u_{htt}(0)\|_{H^1(\Omega)} + k^3 \|u_{httt}\|_{L^\infty(H^1(\Omega))}.$$

Using (5.2.1), (5.3.1) and the definition of L^2 projection, we note as $t \to 0$ that

$$\begin{aligned}
(L_h u_{tt}(0) - u_{htt}(0), v_h) &= (L_h u_{tt}(0), v_h) - (u_{htt}(0), v_h) \\
&= (u_{tt}(0), v_h) - (u_{htt}(0), v_h) \\
&= A_h(u_h(0), v_h) - A(u(0), v_h) \\
&= A_h(u_h(0), v_h) - A(u_h(0), v_h) + A(u_h(0), v_h) \\
&\quad - A(u(0), v_h) \\
&= A_h(R_h u_0, v_h) - A(R_h u_0, v_h) + A(R_h u_0 - u_0, v_h) \\
&\le Ch\|R_h u_0\|_{H^1(\Omega^*)} \|v_h\|_{H^1(\Omega^*)} \\
&\quad + C\|R_h u_0 - u_0\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)} \\
&\le Ch\|u_0\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)} \\
&\quad + Ch^2 \sum_{l=1}^{2} \|u_0\|_{H^3(\Omega_l)} \|v_h\|_{H^1(\Omega)} \\
&\le Ch^2 \sum_{l=1}^{2} \|u_0\|_{H^3(\Omega_l)} \|v_h\|_{H^1(\Omega)}.
\end{aligned}$$

Applying inverse inequality and setting $v_h = L_h u_{tt}(0) - u_{htt}(0)$, we have

$$\|L_h u_{tt}(0) - u_{htt}(0)\|_{H^m(\Omega)} \le Ch^{1-m} \sum_{l=1}^{2} \|u_0\|_{H^3(\Omega_l)}, \quad m = 0, 1 \tag{5.4.12}$$

and hence,

$$\begin{aligned}
\|\xi^{\frac{1}{2}}\|_{H^1(\Omega)} &= \|\xi^1\|_{H^1(\Omega)} = \|U^1 - u_h(t_1)\|_{H^1(\Omega)} \\
&\le Ck^2 \sum_{l=1}^{2} (\|u_0\|_{H^4(\Omega_l)} + \|v_0\|_{H^3(\Omega_l)}).
\end{aligned} \tag{5.4.13}$$

In the last inequality, we have used Lemma 5.3.1. Similarly, we write

$$\begin{aligned}\|\partial_t \xi^0\|_{L^2(\Omega)} &= \frac{1}{k}\|U^1 - u_h(t_1)\|_{L^2(\Omega)} \\ &\leq Ck^2 \sum_{l=1}^{2}(\|u_0\|_{H^3(\Omega_l)} + \|v_0\|_{H^2(\Omega_l)}). \qquad (5.4.14)\end{aligned}$$

Finally, a simple application of Lemma 5.4.1, Theorem 5.3.1 and the triangle inequality

$$\|U^{n+\frac{1}{2}} - u(t_{n+\frac{1}{2}})\|_{L^2(\Omega)} \leq \|\xi^{n+\frac{1}{2}}\|_{L^2(\Omega)} + \|e(t_{n+\frac{1}{2}})\|_{L^2(\Omega)}$$

leads to the desired result. □

Chapter 6

Numerical Results

In this chapter, we shall present some numerical experiment of a two-dimensional problems to illustrate our theoretical findings. All computations have been carried out using the software MATLAB-6.

For each example, we compute the error between the exact solution and the finite element solution in L^2 and H^1 norms. Numerical results for fitted finite element method is presented in this chapter.

6.1 Example 1

We consider the following two point boundary value problem in Ω

$$-\nabla \cdot (\beta_i \nabla u_i) + u_i = f_i \quad \text{in} \ \ \Omega_i, \ i = 1, 2, \tag{6.1.1}$$

$$u_i = 0 \quad \text{on} \ \ \partial\Omega \cap \bar{\Omega}_i, \ \ i = 1, 2, \tag{6.1.2}$$

$$u_1|_\Gamma = u_2|_\Gamma, \quad (\beta_1 \nabla u_1 \cdot \mathbf{n}_1)|_\Gamma + (\beta_2 \nabla u_2 \cdot \mathbf{n}_2)|_\Gamma = 0, \tag{6.1.3}$$

where $\mathbf{n}_i$ denotes the unit outer normal vector on Ω_i, $i = 1, 2$. Here, the domain is the rectangle $\Omega = (0, 2) \times (0, 1)$. The interface occurs at $x = 1$ so that $\Omega_1 = (0, 1) \times (0, 1)$, $\Omega_2 = (1, 2) \times (0, 1)$ and the interface $\Gamma = \bar{\Omega}_1 \cap \bar{\Omega}_2$.

For the exact solution, we choose

$$u_1(x, y) = \sin(\pi x) \sin(\pi y), \quad (x, y) \in \Omega_1$$

and

$$u_2(x, y) = -\sin(2\pi x) \sin(\pi y), \quad (x, y) \in \Omega_2.$$

The right-hand sides f_1 and f_2 in (6.1.1) are determined from the choice for u_1 and u_2, respectively with $\beta_1 = 1$ and $\beta_2 = \frac{1}{2}$.

Let h_x and h_y be the discretization parameters along x and y axes, respectively. Then we choose our mesh parameter h such that $h^2 = {h_x}^2 + {h_y}^2$. From Table 6.1, we see the convergence of the finite element solution to the exact solution in L^2 and H^1 norms.

h^2	(h_x, h_y)	$\|\|u - u_h\|\|_{L^2(\Omega)}$	$\|\|u - u_h\|\|_{H^1(\Omega)}$
1/8	(1/2, 1/2)	.056165	.158861
1/32	(1/4, 1/4)	.014041	.079430
1/128	(1/8, 1/8)	.003510	.039709
1/512	(1/16, 1/16)	.000877	.019854

Table 6.1: *Numerical results for the test problem (6.1.1)-(6.1.3).*

6.2 Example 2

We consider the following parabolic interface problem in Ω

$$u_t - \nabla \cdot (\beta \nabla u) = f \quad \text{in} \ \ \Omega \times (0, 1], \ i = 1, 2, \tag{6.2.4}$$

$$u(x, y, 0) = u_0(x, y) \ \ \text{in} \ \Omega, \ \ u(x, y, t) = 0 \ \ \text{on} \ \ \partial\Omega \times (0, 1] \tag{6.2.5}$$

$$u_1|_\Gamma = u_2|_\Gamma, \quad (\beta_1 \nabla u_1 \cdot \mathbf{n}_1)|_\Gamma + (\beta_2 \nabla u_2 \cdot \mathbf{n}_2)|_\Gamma = 0, \tag{6.2.6}$$

where $\mathbf{n}_i$ denotes the unit outer normal vector on Ω_i, $i = 1, 2$. For the exact solution, we choose

$$u_1(x, y) = e^{\sin t} \sin(\pi x) \sin(\pi y) \quad \text{in} \ \Omega_1 \times (0, 1]$$

and

$$u_2(x, y) = -e^{\sin t} \sin(2\pi x) \sin(\pi y) \quad \text{in} \ \Omega_2 \times (0, 1].$$

Then the source function f and the initial data u_0 are determined from the choice for u_1 and u_2 with $\beta_1 = 1$ and $\beta_2 = \frac{1}{2}$.

The L^2-norm and H^1-norm errors at $t = 1/130$ for various step size h are presented in Table 6.2 for the fully discrete solution. The convergence rates are found to be within our expectation.

h	$\|u - U_h\|_{L^2(\Omega)}$	$\|u - U_h\|_{H^1(\Omega)}$
1/8	2.06247×10^{-3}	5.17359×10^{-2}
1/16	5.28838×10^{-4}	2.72294×10^{-2}
1/32	1.36298×10^{-4}	1.36831×10^{-2}
1/64	3.47701×10^{-5}	6.94573×10^{-3}

Table 6.2: *Numerical results for the test problem (6.2.4)-(6.2.6).*

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