SOME ASPECTS OF FUZZY SETS AND ROUGH SETS

THESIS

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By

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Dedicated to
my parents

CERTIFICATE

This to certify that the thesis entitled "**Some Aspects of Fuzzy Sets and Rough Sets**", which is being submitted by Sri Bubul Kumar Saikia for the award of degree of Doctor of Philosophy in Mathematics to the Tezpur University, Assam, India, is a record of bonafide research work carried out by him under our supervision and guidance. Sri Bubul Kumar Saikia, Senior Lecturer, Deptt. of Mathematics, Lakhimpur Girls' College has fulfilled the requirements prescribed for submitting the thesis for the award of Ph.D. Degree. The results embodied in this thesis have not been submitted to any other University or Institution for the award of any degree or diploma.

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DECLARATION

I, Sri Bubul Kumar Saikia, hereby declare that the contents of this thesis entitled "**Some Aspects of Fuzzy Sets and Rough Sets**" is the record of work done by me and this did not form basis of the award of any previous degree to me or, to the best of my knowledge, to anybody else and that the thesis has not been submitted by me for any research degree in any other university/institute.

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SYNOPSIS

In 1965, Prof.L.A. Zadeh[**100**] of the University of California, Los Angels(USA) laid the foundation of the Fuzzy Set theory by generalizing the classical notion of a set to accommodate the vagueness and inexactitude usually faced in a decision process. The fuzzy set theory and logic lay a form of mathematical precision to human thought process that in many ways are imprecise and ambiguous by the standard of classical Mathematics. This theory provides a strict mathematical frame work in which vague conceptual phenomenon can be precisely and rigorously studied. Out of several higher order fuzzy sets, intuitionistic fuzzy sets(IFS) introduced by Atanassov [**5,6**] is quite useful and applicable .IFS are not necessarily fuzzy sets, although these are defined with the help of membership functions[**5**]. But fuzzy sets are intuitionistic fuzzy sets. Research on the theory of fuzzy sets has been growing steadily since the inception of the theory in the mid- 1960s along with research on a board variety of applications of this comparatively new concept into different areas of study.

The most recent contemporary concern about knowledge and information systems has been another useful extension of elementary set theory (different from fuzzy set theory) is the Rough set theory by Z. Pawlak[**68**]. The underlying assumption behind rough set is that knowledge has granular structure which is caused by the situation when some objects of interest cannot be distinguished and they may appear to be identical. The indiscernibility relation thus generated is the mathematical basis of Pawlak's rough sets. It is an independent discipline to reason about vagueness and uncertainty. Rough sets, though different from fuzzy sets, are also suitable for modeling vague concepts, i.e., concepts without sharp boundaries. Rough set theory is emerging as a powerful theory dealing with incomplete data. It is an expanding research area which stimulates explorations on both real world applications and the theory itself, viz., decision analysis, machine learning, knowledge discovery, market research, conflict analysis etc. Recent theoretical developments on this theory and some of its applications are available in [**69**].

Yager[**93**] introduced the bag structure as a set like object in which repeated elements are significant. A set generally implies a collection into a whole, of definite, well distinguished objects where redundant objects are not counted. In fact there are many collections like collection of books in a library, collection of medicines in a medical shop, collection of zeros in an algebraic polynomial, etc, are not sets but bags. The application and usefulness of bag in real life situations is important, especially in relational database, decision making, etc.

Molodtsov[**60**] pointed out that classical methods cannot be successfully used to solve complicated problems in economics, engineering and environment because of various uncertainties typical of these problems. The important existing theories, viz., theory of probability, theory of fuzzy sets, theory of intuitionistic fuzzy sets, theory of vague sets, theory of interval mathematics, theory of rough sets, can be considered as mathematical tools for dealing with uncertainties. But all these theories have their own limitations as pointed out in [60]. The reason for this is possibly the inadequacy of the parametrization tool of the theories. Subsequently Molodtsov developed a new mathematical theory called 'Soft Set' for dealing with uncertainties and is free from the above limitations. The absence of any restrictions on the approximate description in soft set theory makes this theory very convenient and easily applicable. Zadeh's fuzzy set may be considered as a special case of the soft set.

The sole motivation behind this thesis is to study the notions of fuzzy sets and rough sets further, to explore as well as to formulate the fuzziness and roughness occurring within the frame work of a number of mathematical avenues and hence to promote some fuzzy and rough mathematical theories along with a few extensions to some existing frameworks.

The thesis mainly deals with the following:

1. The definition of union and intersection in [19] have been simplified to include union and intersection of more than two fuzzy sets in different universes and some existing propositions are proved using these generalized notions.

2. Cartesian product of fuzzy bags, bag relation and fuzzy bag relation are defined and some results are proved with examples as an extension of Yager's theory of bags and fuzzy bags and subsequently developed by Chakrabarty et al.[**19**].

3. A new concept of rough algebraic structures is introduced by defining rough Boolean algebra and rough subalgebra based on upper approximation of Pawlak's rough set and some results of rough Boolean algebras and rough subalgebras have been proved with examples.

4. Some properties of lower and upper approximations are studied with respect to congruence relation and fuzzy congruence relation in a lattice.

5. The concept of similarity measure has been used through intuitionistic fuzzy bag theory in deciding the best possible action out of 'n' alternatives involving 'm' criteria on the basis of judgments of 'p' judges where each criteria has its corresponding weight and an algorithm for the method is also presented along with a hypothetical case study.

6. Soft relation and fuzzy soft relation are introduced and then are applied in decision making problems.We have also studied the Sanchez's approach for medical diagnosis and extended this concept with the notion of fuzzy soft set theory and intuitionistic fuzzy soft set theory and exhibit the technique with a hypothetical case study for the two cases separately.

7. As an interdisciplinary application of fuzzy technology, a fuzzy rule based routine has been developed for estimating monthly discharge using the Jiadhal river (a tributary of river Brahmaputra) sub- basin covering parts of Arunachal Pradesh and Assam. This ensures the applicability of a fuzzy logic –based approach to modeling catchment process.

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Chapter 1

General Introduction

Fuzzy set theory provides a means for representing uncertainties and modeling the related concepts. Historically, probability theory has been the primary tool for representing uncertainty in mathematical models. But it deals with only random uncertainty. So nonrandom uncertainties are not suited to treatment or modeling by probability theory. In fact, an overwhelming amount of uncertainty associated with complex systems and issues, which humans address on a daily basis, is nonrandom in nature. Fuzzy set theory is a marvelous tool for modeling nonrandom uncertainty, i.e., uncertainty associated with vagueness, with imprecision, with lack of detailed information regarding the problem at hand.

The notion of fuzzy sets provide an alternative approach to the traditional notions of set membership and logic whose roots lie in ancient Greek philosophy. Some philosophers of that time, including Aristotle, proposed 'laws of thought' in their effort to formulate a theory of logic (foundations of Mathematics).It contained 'the law of excluded middle' meaning that every proposition must either be 'true' or 'false' .But Plato claimed that there exists a third region between 'true' and 'false'.A central idea in his philosophy is that, in the real world, elements are very closely associated with imperfection and hence, there exists no element that is perfectly round. "Perfect notions" or "exact concepts" are the sort of things envisaged in pure mathematics while "inexact structure" prevail in real life. Fuzzy sets deal with situations using truth values which are true or false or ranging between 'true' and 'false'. The membership function of a fuzzy set maps each element of the universe to a value in the unit interval. The primary feature of fuzzy sets is that their boundaries are not precise. This facilitates the assignment of a subjective membership value to the elements of the universe of interest without totally rejecting or accepting them. This approach of subjective memberships taking all relevant aspect of the situation makes the fuzzy set quite user friendly. Fuzzy set theory does not handle the value of non membership value of its members explicitly. It is automatically determined by the difference between the membership value and unity. For example,we often here statements like "sixty percent of the voters voted in favour of a party". It does not mean the remaining forty percent of the voters have not voted for the particular party because there may be voters who have failed to vote due some reasons.In other words, a voter who has not voted in an election is not same as he has not voted for a particular party. Thus non membership value is not always automatically the difference between unity and the membership value. Considering this fact that the membership value always does not determine the non-membership value of an element, an extended definition of fuzzy sets was initiated by Prof. Atanassov **[5,6]**. For a set, he defined a membership and a non membership function each of which maps each element of the set to a value in the unit interval with the constraint that sum of these two functions must be less than or equal to unity. The remaining part(the difference of the sum of the membership and non membership from unity), if non zero, is the indeterminacy part of the evaluator's conception of the particular element. This set with a membership and non membership (each of which may be partial) is called an intuitionistic fuzzy set which is in fact a generalization of fuzzy set.

Rough set theory is a comparatively recent mathematical approach to formulate sets with imprecise boundaries. It represents a different mathematical approach to vagueness and uncertainty. Here we associate some information (data, knowledge) with every object in the universe of discourse such that objects characterized by the same information are indiscernible (similar) in view of the available information about them. The indiscernibility relation generated in this way is the mathematical basis of the rough set theory. In recent years, there has been a rapid growth of interest worldwide in rough set theory and its application. It has been seen that fuzzy sets and rough sets are two different topics **[30]** and none generalizes the other.

A fundamental axiom in the theory of sets is the action of extensionality **[50]**.This action defines equality among sets. According to this axiom two sets A and B are equal, if for any $x \in A$ it follows that $x \in B$ and for any $y \in B$ implies $y \in A$. Two important properties that follow from this axiom are that ordering of elements in a set does not matter or repeated elements in a set are redundant. A list provides a structure in which ordering counts. In some situations we may want a structure which is a collection of objects in the same sense as a set but in which redundancy counts.

For example,a collection of objects corresponding to the ages of people in a company *do have a redundancy which we may desire to make explicit in the underlying set.* With this in mind, Yager [**93**] introduced the concept of a bag drawn from a set X and mentioned some possible applications of this bag structure in relational databases. He defined various operations on bags, introduced the concept of fuzzy bags and discussed some operations on them.

The theory of fuzzy sets developed by Zadeh [**100**] has established itself as quite an appropriate theory for dealing with uncertainties . But recently D. Molodtsov [**60**] showed in his paper 'Soft set theory – First Results' that fuzzy set theory have also some difficulties in handling uncertainties due to the inadequacy of the parametrization tool. Soft set theory is a new mathematical tool for dealing with uncertainties which is free from above difficulties. Also, Soft set theory has a rich potential for applications in several directions. Molodtsov has shown that fuzzy set might be considered as a special case of the soft set.

Before furnishing the summary of our results, we present a brief description of the theories of Fuzzy sets, Intuitionistic fuzzy sets, Rough sets, Bags, Fuzzy bags and Soft sets.

1.1 Fuzzy Sets

Fuzzy set theory proposed by Professor L.A.Zadeh at the University of California, Berkeley in 1965 is a generalization of classical or crisp sets. It makes possible to describe vague notions and deals with the concepts and techniques which lay in the form of mathematical precision to human thought processes that in many ways are imprecise and ambiguous within the ambit of classical mathematics .This theory reflects itself as a multi-dimensional field of inquiry, contributing to a wide spectrum of areas ranging from para-mathematical to human perception and judgment .It deals with situations using truth values ranging between the usual “true and false” .

When vague notions arise, it is sometimes difficult to determine the exact boundaries of the class and hence the decision that whether an element belongs to it or not is replaced by a measure from scale. Each element of the class is evaluated by a measure which expresses its place and role in the class. This measure is called the grade of membership in the given class. A set in which each element of the universal set is characterized by its membership grade is called a fuzzy set.

Let X be a classical set of objects called the universe and x be any arbitrary element of X. Membership in a classical subset A of X is defined through a characteristic function from X to $\{0,1\}$ such that

$$\mu_A(x)=\begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

where $\{0,1\}$ is called a valuation set. If the valuation set is taken to be the real interval $[0,1]$ then A is called a fuzzy set. In the year 1967, Goguen [33] proposed a purely mathematical definition of fuzzy set by taking a more general poset as a valuation set instead of $[0,1]$ as in Zadeh’s definition and discussed in detail the case when this poset is a complete lattice ordered semigroup .In a fuzzy set A, $\mu_A(x)$ is called the grade of the membership of x in A. The closer is the membership value $\mu_A(x)$ to 1, the more x belongs to A. Thus, we can view A as a subset of X that has no sharp boundary and is completely characterized by the set of pairs $\{(x, \mu_A(x) : x \in X\}$.The standard complement $\bar{A}$, of a fuzzy set A with respect to universal set X is defined by the characteristic function $\mu_{\bar{A}}(x)=1-\mu_A(x)$.

For fuzzy sets A, B, C and D in X, we have

$A=B$ iff $\mu_A(x)=\mu_B(x), \quad \forall x \in X$;

$A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x), \quad \forall x \in X$;

$B \supseteq A$ iff $A \subseteq B$;

$C=A \cup B$ iff $\mu_C(x)=\max\{\mu_A(x),\mu_B(x)\}, \quad \forall x \in X$;

$D=A \cap B$ iff $\mu_D(x)=\min\{\mu_A(x),\mu_B(x)\}, \quad \forall x \in X$.

Note that the operations fuzzy union, fuzzy intersection and fuzzy complement contrary to their counterparts in case of crisp sets , are not unique, i.e., different functions may be used to represent these operations in different contexts. Therefore, like membership value of fuzzy sets, the operations of fuzzy sets are context dependent. The law of contradiction and the law of excluded middle are not satisfied by fuzzy set and fuzzy logic.

1.1.1 Fuzzy Relations

Fuzzy relations also relate elements of one universe, say X, with elements of another universe, say, Y, through the Cartesian product of two universes. But the strength of the relationship is not measured with the characteristic function, rather with a membership function expressing various "degrees" of the relation on the interval [0,1]. Thus the degree or grade of membership of a member in the relation R is $\mu_R(x,y) \in [0,1], \forall\ x \in X$ and $\forall\ y \in Y$. Here R can be considered as a fuzzy set in the universe $X \times Y$.

Note that it is just a generalization of crisp relation. Another case of fuzzy relation that map Cartesian product of fuzzy sets $A \times B$ contained in the universal set $X \times Y$ into the unit interval $[0,1]$.

A fuzzy relation R on a single universe X is also a relation from X to X. It is called a fuzzy equivalence relation if it satisfies the following three properties:

i. Reflexivity

ii. Symmetry

iii. Transitivity

A fuzzy tolerance relation is a fuzzy relation that satisfies only the reflexive and symmetric properties.

1.1.2 Similarity Measures

A similarity measure is a matching function to measure the degree of similarity between two fuzzy sets. This degree of similarity indicates the closeness of two sets and associates a numerical value with the idea that a higher value indicates a greater similarity. A number of similarity measures have been proposed in the literature for measuring the degree of similarity between fuzzy sets.

Let F(X) be the set of all fuzzy sets drawn from the set X and

S: $F(X)\times F(X)\rightarrow[0,1]$.Then S(A,B) is said to be the degree of similarity between the fuzzy sets A and $B\in F(X)$ if S(A,B) satisfies the following properties:

i. $0\leq S(A,B)\leq 1$;

ii. S(A,B)= 1 if A=B;

iii. S(A,B)= S(B,A).

1.1.3 Fuzzy Numbers

Fuzzy numbers model imprecise quantities (numbers) like 'about 10', 'below 100' etc. A fuzzy number is a quantity whose value is imprecise, rather than exact as in the case with 'ordinary' (single-valued) numbers. Any fuzzy number can be thought of as a function whose domain is a specified set (usually the set of real numbers, and whose range is the span of non-negative real numbers between 0 and 1(both inclusive).Each numerical value in the domain is assigned a specific 'grade of membership', where 0 represents the smallest possible grade, and 1 is the largest possible grade. In many respects, fuzzy numbers depict the physical world more realistically than single-valued numbers. Fuzzy numbers are used in statistics, computer programming, engineering (especially communications), and experimental science. The concept takes into account the fact that all phenomena in the physical universe have a degree of inherent uncertainty.

1.1.4 Fuzzy Logic

Logic is the science of reasoning. Symbolic or mathematical logic has turned out to be a powerful computational paradigm. Not only does symbolic logic help in the description of events in the real world but also has turned out to be an effective tool for inferring or deducing information from a given set of facts. Just as mathematical sets have been classified into crisp sets and fuzzy sets, logic can also be broadly viewed as crisp and fuzzy logic. Also we have that crisps sets survive on a 2-state membership(0/1) and fuzzy sets on a multistate membership [0,1], similarly crisp logic is built on a 2-state truth value(true/false) and fuzzy logic on a multistate truth value(true/false/partly true/ partly false and so on). Fuzzy logic seems closer to the way our brains work. We aggregate data and form a number of partial truths which we aggregate further into higher truths. When certain thresholds are exceeded, these cause certain further results such as motor reaction. The ultimate goal of fuzzy logic is to form the theoretical foundation for reasoning about imprecise propositions and such reasoning has been referred to as approximate reasoning. Approximate reasoning is analogous to predicate logic for reasoning with precise propositions, and hence is an extension of classical predicate calculus for dealing with partial truths.

1.1.5 Fuzzification and Defuzzification

Fuzzification is the process of changing a crisp quantity into a fuzzy quantity. It can be done by simply recognizing that of the quantities which are considered to be crisp and deterministic are actually not deterministic at all. They carry considerable uncertainty. If the form of uncertainty happens to arise because of imprecision, ambiguity, or vagueness, then the variable is considered to be fuzzy and can be represented with the help of a membership function.

In many situations, where output is fuzzy, it is possible to take a crisp decision by converting the fuzzy output as a single scalar quantity. This conversion of a fuzzy set to a single crisp value is called defuzzification and involves the reverse process of fuzzification. There are several methods for defuzzification namely centroid method, centre of sums and mean of maxima etc.

1.2 Intuitionistic Fuzzy Sets (IFS)

Out of several higher order fuzzy sets, intuitionistic fuzzy sets(IFS) introduced by Atanassov[5,6] have a lot of potential for applications. IFS are not fuzzy sets, although these are defined with the help of membership functions. But fuzzy sets are intuitionistic fuzzy sets. Atanassov [5] himself gave one example showing that fuzzy sets are intuitionistic fuzzy sets, but the converse is not necessarily true. There are many situations where intuitionistic fuzzy sets are more useful to deal with. Let E be the set of all states of India with elective governments. Assume that we know for every state $x \in E$ the percentage of the electorate who have voted for the corresponding government. Let it be denoted by M(x) and let $\mu(x)=\frac{M(x)}{100}$. Let $v(x)=1-\mu(x)$. This number corresponds to that part of electorate who have not voted for the government. By means of the fuzzy set theory we cannot consider this value in more detail. However, if we define $v(x)$ as the number of votes given to parties or persons outside the government, then the part of electorate who have not voted at all will have membership value $1-\mu(x)-v(x)$. Thus the resulting set, denoted by $\{<x,\mu(x),v(x)>|x\in E\}$ is called an intuitionistic fuzzy set [5]. Let a (non fuzzy) set E be fixed .An intuitionistic fuzzy set (IFS) A in E is defined as $A=\{<x,\mu(x),v(x)>\}|x\in E\}$, where the functions $\mu_A:E\rightarrow[0,1]$ and $v_A:E\rightarrow[0,1]$ define the degree of membership and degree of non membership respectively of the element x to the set A and for every $x\in E, 0\le\mu_A(x)+v_A(x)\le 1$.

1.3 Rough Sets

Rough sets have been introduced by Z.Pawlak in 1982 [68] to provide a systematic framework for studying imprecise and insufficient knowledge. It is a strong mathematical tool to deal with vagueness. For a very long time, philosophers and logicians have been attracted by the concept which is related to the so called boundary line view. These objects can be classified neither to the concept nor to its complement and thus there are boundary line cases. The underlying assumption behind the concept of rough sets is that knowledge has granular structure which is caused by the situation when some objects of interest cannot be distinguished as they may appear to be identical. The indiscernibility relation thus generated is the mathematical basis of Pawlak's rough sets.This can be used to describe dependencies between attributes, to evaluate significance of attributes and to deal with inconsistent data. The main advantage of rough set theory is that it does not need any preliminary or additional information about the data. According to the approach of rough set theory we first of all assume that any vague concept is characterized by a pair of precise concepts called the lower and the upper approximations of the concerned vague concept. The lower approximation is the set of objects surely belonging to the concept and the upper approximation is the set consisting of all objects possibly belonging to the concept. The boundary region of the vague concept is merely the difference between the upper and the lower approximations of it. Suppose R is an equivalence relation defined over the universe U which, in turn, partitions U into disjoint equivalent classes. Then for any subset X of U, the sets $\underline{A}(X)=\{x:[x]_R \subseteq X\}$ and $\overline{A}(X)=\{x:[x]_R \cap X \neq \phi\}$ are respectively called the lower and the upper approximation of X. The pair A=(U,R) is called the approximation space and $A(X)=(\underline{A}(X),\overline{A}(X))$ is called the rough set of X in U. In the above, $[x]_R$ denotes the equivalence class with respect to R containing x. Further, for a fixed non empty subset X of U, the rough set of X i.e., A(X) is unique.

For any subset $X \subseteq U$ representing a concept of interest, the approximation space A=(U,R) can be characterized with three distinct regions of X : the so called positive region $\underline{A}$ (X), the boundary region $\overline{A}(X)$-$\underline{A}(X)$, and the negative region U-$\overline{A}(X)$.The characterization of objects in X by the indiscernibility relation R is not precise enough if the boundary region $\overline{A}(X)-\underline{A}(X)$ is not empty. In such a case it may be impossible to say whether an object belongs to X or not, so that the set X is said to be non definable in A, and X is a rough set. For simplicity, we denote a rough set $A(X)=(\underline{A}(X),\overline{A}(X))$ of X by $A(X)=(\underline{X},\overline{X})$. Let $A(X)=(\underline{X},\overline{X})$ and $A(Y)=(\underline{Y},\overline{Y})$ be any two rough sets in the approximation space A=(U,R). Then

(i) $A(X)\cup A(Y)=(\underline{X}\cup\underline{Y},\overline{X}\cup\overline{Y})$

(ii) $A(X)\cap A(Y)=(\underline{X}\cap\underline{Y},\overline{X}\cap\overline{Y})$

(iii) $A(X)\subset A(Y)$ iff $\underline{X}\subset\underline{Y},\overline{X}\subset\overline{Y}$

(iv) The rough complement of A(X) in (U,R) denoted by -A(X) and is defined by $-A(X)=(U-\underline{X},U-\overline{X})$

(v) $A(X)-A(Y)=(\underline{X}-\overline{Y},\overline{X}-\underline{Y})$.

1.4 Bags and Fuzzy Bags

Yager[93] introduced the bag structure as a set like object in which repeated elements are significant. A set generally implies a collection into a whole of definite well distinguished objects where redundant objects are not counted. In fact there are many collections like collection of books in a library, collection of medicine in a pharmacy, collection of zeros of an algebraic polynomial etc., which are not sets, but bags.The application and usefulness of bag in the real life situations is very important, especially in relational database[**93**], decision making[**28,73,74**]etc.

A bag (or crisp bag) B drawn from a set X is represented by a function $Count_B$ and is defined as C_B:X $\rightarrow$N, where N is the set of all non negative integers. The function C_B is called the count function of the bag and for each $x \in X$, the value $C_B(x)$ indicates the number of times (i.e., multiplicity) the element x appears in the bag B. The bag B is represented by $B=\{x/C_B(x):x \in X\}$. For example, the bag B drawn from the set $X=\{x_1,x_2,.....,x_m\}$ is represented as $B=\{x_1/n_1,x_2/n_2,.....,x_m/n_m\}$, where n_i is the number of occurrences of the element x_i in the bag B, i.e., $n_i=C_B(x_i),i=1,2,...,n$. In [**93**] Yager has proposed the operations of intersection, union, addition etc. on bags together with the operation of selection of elements from a bag and bag projection. In addition to these, he has also defined fuzzy bags (i.e., bags with fuzzy elements, in which an object(element) may appear with a number of different membership grades).Thus, a fuzzy bag F drawn from a set X is characterized by a function $CM_F : X \rightarrow B$, where B is the set of all bags drawn for the unit interval [0,1].Yager has also defined some operations on fuzzy bags such as the sum of fuzzy bags, removal of a fuzzy bag from another fuzzy bag, union and intersection of fuzzy bags etc.

1.5 Soft Sets

Molodtsov **[60]** pointed out that classical methods can not be successfully used to solve complicated problems in economics, engineering and environment because of various uncertainities typical of these problems. The important existing theories, viz., theory of probability, theory of fuzzy sets, theory of intuitionistic fuzzy sets, theory of vague sets, theory of interval mathematics, theory of rough sets etc.can be considered as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties as pointed out in **[60]**. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of these theories. Molodtsov **[60]** introduced a new mathematical theory called 'Soft set' for dealing with uncertainties which is free from the above difficulties. Let U be an universe set and let E be a set of parameters. A pair (F,E) is called a soft set over U if and only if F is a mapping of E into the set of all subsets of the set U, i.e., $F:E \to P(U)$, where P(U) is the power set of U. In other words, the soft set is a parametrized family of subsets of the set U. Every set F(e), for $e \in E$, from this family may be considered as the set of e–approximate elements of the soft set (F,E).For an illustration of soft set **[60]**, suppose U = the set of houses available for purchase, E= the set of parameters whose each parameter is a word or a sentence, say expensive houses, beautiful houses, and so on. It is worth nothing that the sets F(e) may be arbitrary, some of them may be empty, while some may have non empty intersection.

Zadeh's fuzzy set may be considered as a special case of the soft set. For this, let A be a fuzzy set, and μ_A be the membership function of the fuzzy set A, i.e., μ_A is a mapping of U into [0,1], where U is the universal set. Then $F(\alpha)=\{x \in U/\mu_A(x) \geq \alpha\}$, $\alpha \in [0,1]$ is a family of α-level sets of the function μ_A. If the family F is known, one can find the functions $\mu_A(x)$ by means of the following formulae :

$$\mu_A(x)=\sup_{\substack{\alpha \in [0,1] \\ x \in F(\alpha)}} \alpha$$

Thus, every Zadeh's fuzzy set A may be considered as the soft set (F,[0,1]).Again, let, (X,τ)be a topological space, that is, X is a set and τ is a topology(τ is a family of subsets of X, called the open sets of X). Then the family of open neighbourhoods T(x) of point x, where $T(x) = \{ V \in \tau : x \in V \}$, may be considered as the soft set $(T(x), \tau)$.

The way of setting (or describing) any of object in the soft set theory principally differs from the way in which we use classical mathematics.In classical mathematics, we construct a mathematical model of an object and define the notion of the exact solution of the model. Usually the mathematical model is too complicated and we may not find the exact solution. So, in the second step we introduce the notion of the approximate solution and calculate that solution.In the soft set theory, we have the opposite approach to this problem. The initial description of the object has an approximate nature, and we do not need to introduce the notion of exact solution.

The absence of any restrictions of on the approximate description in soft set theory makes this theory very convenient and easily applicable in practice. We may use any parametrization we prefer: with the help of words and sentences, real numbers, functions, mappings and so on. It means that the problem of setting the membership function or any similar problem does not arise in the soft set theory.

1.6 Motivation of the thesis

The motivation of this thesis is to study the notions of fuzzy sets and rough sets further, to explore as well as to formulate the fuzziness and roughness occurring within the frame work of a number of mathematical avenues and hence to promote some fuzzy and rough mathematical theories along with a few extensions to some existing frameworks.

This thesis consists of nine chapters.

The first chapter is an introductory chapter. In this chapter, a brief introduction of fuzzy set, intuitionistic fuzzy set and rough set have been given along with a survey of the basic literature related to the proposed problems.

In chapter 2, the definition of union and intersection in [**19**] have been simplified to include union and intersection of more than two fuzzy sets in different universes and some existing propositions are proved using these generalized notions.

In chapter 3, Cartesian product of fuzzy bags, bag relation and fuzzy bag relation are defined and some results are proved with examples as an extension of Yager's theory of bags and fuzzy bags which were subsequently developed by Chakrabarty et al.[**18**].

In chapter 4, a concept of rough algebraic structure is introduced by defining rough Boolean algebra and rough subalgebra based on upper approximation of Pawlak's rough set and then some results of rough Boolean algebras and rough subalgebras have been proved.

In chapter 5, some properties of the lower and upper approximations with respect to the crisp congruence relation and fuzzy congruene relation on a lattice are studied .

In chapter6, intuitionistic fuzzy bags(IFB) concept is applied in multicriteria decision making problem and a hypothetical case study has been taken as an example.

In chapter 7, soft relation and fuzzy soft relation are introduced and then have applied in decision making problems. Also, fuzzy soft set and intuitionistic fuzzy soft set theory have been applied in medical diagnosis problems separately .

In chapter 8, a fuzzy rule based methodology is developed for estimating monthly discharge for the Jiadhal river basin in the upper Assam.

In chapter 9, an overall conclusion is made and some scopes for further research works are indicated.

Chapter 2

On Generalized Union and Intersection of Fuzzy Sets

In generalizing the Zadeh's notion of union and intersection of two fuzzy sets, Chakrabarty et al.[19]defined these concepts for two fuzzy sets in two different universes. In this chapter, we have simplified the above definition to include more than two fuzzy sets each coming from a different universe. Some of the existing results are also verified with the proposed generalization.

2.1 Introduction

Introducing fuzzy set in [**100**], Prof. L.A.Zadeh defined union and intersection of two fuzzy sets in the same universe .Chakrabarty et al.[**19**] extended the notion by defining union and intersection of two fuzzy sets in two different universes. Here the definition of union and intersection in [**19**] have been simplified to include union and intersection more than two fuzzy sets, each from different universes. It is observed that the present generalization include the notion of Zadeh[**100**] ,Dubois and Prade, Yager and Hamacher given in [**107**] .

[1]Contents of this chapter was presented in the International Conference on Recent Advances in Mathematical Sciences, December 20-22,2000,IIT,Kharagpur,India and have appeared as a paper entitled "On Generalized Union and Intersection of Fuzzy Sets" in the Proceedings (Vol I), pp.335-340,Narosa Publishing House,New Delhi.

2.2 Preliminaries

The union and intersection of a fuzzy set A in the universe X and another fuzzy set B in the universe Y, where X and Y are two different universes, in general, are defined in **[19]** as follows.

Definition 2.2.1

Let A be a fuzzy set of X with membership function μ_A and B be another fuzzy set of Y with membership function μ_B. Then the union of the fuzzy sets A and B, denoted by $A \tilde{\cup} B$, is a fuzzy set of $X \cup Y$ with membership function $\mu_{A\tilde{\cup}B}$ defined by

$$\mu_{A\tilde{\cup}B}(z) = \begin{cases} \mu_A(z) & \text{if} \quad z \in X\text{-}Y \\ \mu_B(z) & \text{if} \quad z \in Y\text{-}X \\ \max\{\mu_A(z), \mu_B(z)\} & \text{if} \quad z \in X \cap Y \end{cases}$$

Definition 2.2.2

For the fuzzy sets A and B in definition **2.2.1** above, the intersection of A and B, denoted by $A \tilde{\cap} B$, is a fuzzy set of $X \cup Y$ with membership function $\mu_{A\tilde{\cap}B}$ defined by

$$\mu_{A\tilde{\cap}B}(z) = \min\{\mu_A(z), \mu_B(z)\}, \quad \forall z \in X \cup Y$$

In the above, unlike Zadeh's notion, the definition of union of two fuzzy sets in two different universes is not straight to include more than two fuzzy sets. Also, the definition of intersection of two fuzzy sets contains inherent problem when $z \in X \Delta Y$.

Therefore, we need to simplify the above definitions and prove the results for 'n' fuzzy sets each of which is from a different universe.

2.3 Generalizations

In general we need to handle fuzzy sets in different universes so that the resulting union and intersection are fuzzy sets in a universe which is the union of universes or the elementary universes of the individual fuzzy sets. So, for the generalization of Zadeh's notion proposed by Chakrabarty et al.[19], it is required to fix up the membership grade for each element with reference to a particular fuzzy set which is not in the corresponding elementary universe. For this, we define

Definition 2.3.1

Let A_i be a fuzzy set in the (elementary) universe X_i with membership function μ_{A_i}, for i=1,2,.......n. Then the union of the fuzzy sets A_i's, denoted by $\tilde{\cup}_i A_i$, is a fuzzy set in the universe $\cup_i X_i$ whose membership function $\mu_{\tilde{\cup} A_i}(z)$ is defined by

$$\mu_{\tilde{\cup} A_i}(z) = \max_i\{\mu_{A_i}(z)\}, \forall z \in \cup_i X_i, \text{ where } \mu_{A_i}(z) = 0 \ \text{ if } z \notin X_i.$$

The graphical representation of the union of three fuzzy sets is given in Figure 1.

The above definition reduces to the definition proposed in [19] for n=2 and to that defined in [100] when X_1=X_2.

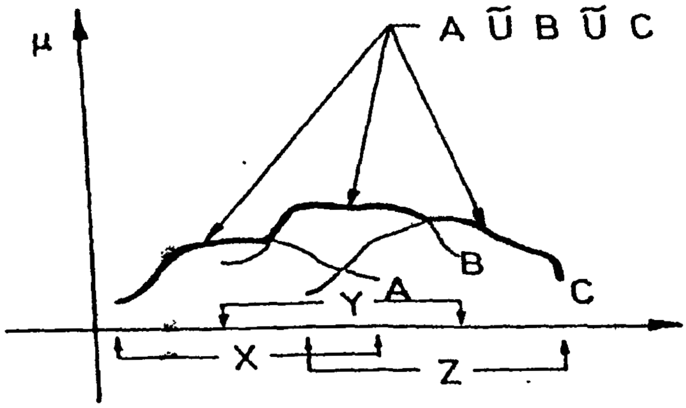

Figure 1. Union of three fuzzy sets.

Definition 2.3.2

Let A_i be a fuzzy set in (elementary)universe X_i with membership function μ_{A_i}, $i = 1,2,\ldots\ldots,n$. The intersection of the fuzzy sets A_i's , denoted by $\tilde{\cap}_i A_i$, is a fuzzy set in the universe $\cup_i X_i$ whose membership function $\mu_{\tilde{\cap} A_i}$ is defined by

$$\mu_{\tilde{\cap} A_i}(z) = \min_i\{\mu_{A_i}(z)\},\ \forall z \in \cup_i X_i,\ \text{ where } \mu_{A_i}(z) = 0 \ \text{ if } z \notin X_i$$

The graphical representation of the intersection of three fuzzy sets, given in Figure2.

The above definition reduces to the proposed in **[19]** for n=2 and then that defined in **[100]** when $X_1 = X_2$.

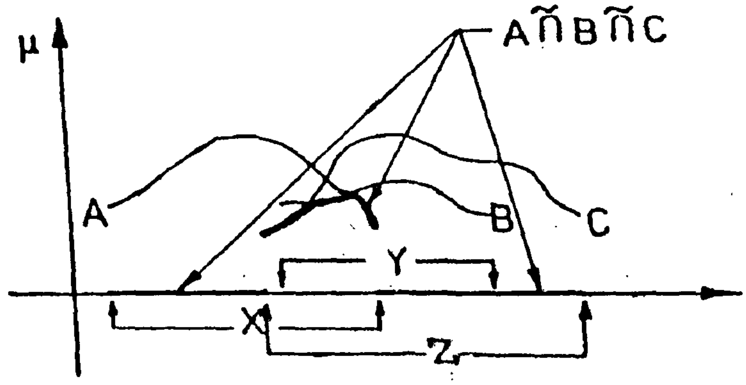

Figure 2.Intersection of three fuzzy sets.

Example 2.3.1

Let A_1={a/0.2,b/0.6, c/0.5,d/0.7}, A_2={ a/0.3,c/0.7,d/0.9, e/0.5,f/0.7, g/0.8} and A_3={ a/0.6,c/0.1, d/0.3 ,e/0.6, s/0.2,t/0.5}be fuzzy sets in the universes X_1={a, b, c, d}, X_2={ a, c, d, e, f, g} and X_3={ a, c, d, e, s, t} respectively. Then $A_1 \tilde{\cup} A_2 \tilde{\cup} A_3$={a/0.6,b/0.6,c/0.7,d/0.9,e/0.6,f/0.7,g/0.8,s/0.2,t/0.5} and $A_1 \tilde{\cap} A_2 \tilde{\cap} A_3$={a/0.2, c/0.1, d/0.3 }.

Proposition 2.3.1

Let A_i be a fuzzy set in the universe X_i with membership function μ_{A_i},

for i=1,2,….,n. Then

(i) $(\tilde{\bigcup_i} A_i)^c = \tilde{\bigcap_i} A_i^c$ and

(ii) $(\tilde{\bigcap_i} A_i)^c = \tilde{\bigcup_i} A_i^c$

where $(\tilde{\bigcup_i} A_i)^c$ is the complement of $\tilde{\bigcup_i} A_i$ in $\bigcup_i X_i$ and A_i^c is the complement of A_i in X_i

Proof: For $z \in \underset{i}{\cup} X_i$

$$\mu_{(\tilde{\cup} A_i)^c}(z) = 1 - \mu_{\tilde{\cup} A_i}(z)$$
$$= 1 - \max_i\{\mu_{A_i}(z)\}$$
$$= \min_i\{\mu_{A_i^c}(z)\}$$
$$= \mu_{\tilde{\cap} A_i^c}(z).$$

(ii) Similar to (i).

The above result reduces to the similar result in **[19]** when n=2.

The results involving α-cuts in two fuzzy sets in **[19]** also hold for n sets.

Proposition 2.3.2

If $(A_i)_\alpha$, $(\underset{i}{\tilde{\cup}} A_i)_\alpha$, $(\underset{i}{\tilde{\cap}} A_i)_\alpha$ denote α– cuts of the fuzzy sets A_i in the universe X_i, fuzzy sets $\underset{i}{\tilde{\cup}} A_i$ and $\underset{i}{\tilde{\cap}} A_i$ in the universe $\cup X_i$ respectively, for i=1,2,....n. Then

(i) $(\underset{i}{\tilde{\cup}} A_i)_\alpha = \underset{i}{\tilde{\cup}} (A_i)_\alpha$ and

(ii) $(\underset{i}{\tilde{\cap}} A_i)_\alpha = \underset{i}{\tilde{\cap}} (A_i)_\alpha$

Proof: (i) For $z \in (\underset{i}{\tilde{\cup}} A_i)_\alpha \Leftrightarrow \mu_{\tilde{\cup} A_i}(z) \geq \alpha$

$$\Leftrightarrow \max_i\{\mu_{A_i}(z)\} \geq \alpha$$
$$\Leftrightarrow \{\mu_{A_i}(z)\} \geq \alpha \text{ , for some i.}$$
$$\Leftrightarrow z \in \underset{i}{\tilde{\cup}} (A_i)_\alpha .$$

It is further observed that the above generalization of Zadeh's notion**[100]** of union and intersection of fuzzy sets each in a different universe also hold in case two fuzzy sets for the parameterized union and intersection operators given separately by Yager, Dubois and Prade and Hamacher**[107]**.To this end we note that Zadeh's notion is a particular case of Yager's notion when $p \to \infty$ and that of Dubois and

Prade when the parameter α =0. Now, for two fuzzy sets in different universes, the notion of their union and intersection also reduce, for specific value of parameters, to the max and min operator respectively according to the generalization suggested in this note. Also in case of Hamacher's definition, the union and intersection reduce, for the specific value of the parameters, to the algebraic sum and algebraic product respectively even if the two fuzzy sets are in two different universes.

2.4 Conclusion

The present generalization of Zadeh's notion **[100]** of union and intersection of fuzzy sets enhances the scope of its applications to many cases of real life problems involving more than two fuzzy sets, each in a different universe. A few results are proved using the generalized notion and it is examined that the existing results on union and intersection also hold in the case of two fuzzy sets which are either in two different universes or the same universe.

Chapter-3

Some Results on Yager's Theory of Bags and Fuzzy Bags

[1]In this chapter, the bag and fuzzy bag structure introduced by Yager and further developed by Chakrabarty et al.[**18**] have been considered. In continuation, Cartesian product of fuzzy bags, bag relation and fuzzy bag relation are defined and some results are proved.

3.1 Introduction

A bag or a multiset is a collection of elements in a universal set such that, unlike crisp set, an element can be repeated in a bag. It is well known that the data language SQL [17] for relational databases is based on multisets. Implementation of another useful version of database calculus have been suggested by Yager [**93**] with the help of bags or multisets. Also the fact that fuzzy relational databases constitute some important applications of fuzzy systems [**34**] has attracted considerable attention for the study of fuzzy bags.Fuzzy bags have been defined by Yager [**93**] and their applications have been discussed in [**28,51,73,74,75**]. Chakrabarty et al. [**18**] have

[1]Contents of this chapter was presented in the Fourth International Conference on Information Technology, December 20-23,2001,NIST,Palur Hills, Berhampur, India and have appeared as a paper entitled "Some Results on Yager's Theory of Bag and Fuzzy Bags"in the Proceedings, pp. 265-270, Tata McGraw-Hill Publishing Company Limited, New Delhi.

further studied Yager's theory of bags and fuzzy bags. In sequel to **[18,93]** some more results have been considered in the present chapter.

3.2. A brief review of Yager's theory of bags and fuzzy bags

In order to make our discussion self contained, we summarize below some of the basic definitions and results on the theory of bags and fuzzy bags introduced by Yager **[93]** and subsequently developed by Chakraborty et al **[18]**.

Definition 3. 2.1

A bag (or crisp bag) B drawn from a set X is represented by a function $count_B$ or C_B defined as $C_B : X \to N$, where N represents the set of nonnegative integers. Here $C_B(x)$ is the number of occurrences of the element $x \in X$ in the bag B. The bag B drawn from a set $X = \{x_1, x_2, .., x_m\}$ is represented as $B = \{ x_1/n_1, x_2/n_2 \ldots, x_m/n_m)$, where n_i is the number of occurrences of the element x_i $(i = 1, 2, .., m)$ in the bag B. It may be noted that the element of X with zero count in B are not included in B.

Example 3. 2.1

Let X = {a, b, c, d, e} be any set. Then B = {a, a, b, b, b, d, d} is a bag drawn from X and is represented by B = {a/2, b/3, d/2}.

Suppose B_1 and B_2 are two bags drawn from a set X. Then,

(1) $B_1 = B_2$ if $C_{B_1}(x) = C_{B_2}(x)$, $\forall\, x \in X$

(2) $B_1 \subseteq B_2$ if $C_{B_1}(x) \le C_{B_2}(x)$, $\forall\, x \in X$

(3) $B = B_1 \oplus B_2$ if $C_B(x) = C_{B_1}(x) + C_{B_2}(x)$, $\forall\ x \in X$

(4) $B = B_1 \ominus B_2$ if $C_B(x) = \max\{ C_{B_1}(x) - C_{B_2}(x) , 0\}, \forall\ x \in X$

(5) $B = B_1 \cup B_2$ if $C_B(x) = \max\{ C_{B_1}(x), C_{B_2}(x) \}, \forall\ x \in X$

(6) $B = B_1 \cap B_2$ if $C_B(x) = \min\{ C_{B_1}(x), C_{B_2}(x) \}, \forall\ x \in X$.

Definition 3. 2.2

Let B be a bag drawn from set X. Then

(i) the support of B denoted by B^*, is a subset of X with the characteristic function

given by $\phi_{B^*}(x) = \min\{C_B(x), 1\}$, $\forall\ x \in X$;

(ii) The cardinality of B, denoted by card B , is given by card B $= \sum_{x \in X} C_B(x)$

(iii) The insertion of an element $x \in X$ into the bag B results in a new bag B'

denoted by $B' = B \oplus x$, such that

$$C_{B'}(x) = C_B(x) + 1 ,$$

and $C_{B'}(y) = C_B(y)$, $\forall\ y\ (\neq x) \in X$;

(iv) the removal of x from the bag B results in a new bag B, denoted by $B' = B \ominus x$, such that,

$$C_{B'}(x) = C_B(x) - 1 ,$$

$$C_{B'}(y) = C_B(y) , \forall\ y\ (\neq x) \in X .$$

Definition 3. 2.3

Consider a universal bag U of an information system and X is a support set of U.

Then, for a sub bag B of U, the complement of B in U, denoted by B^c, is defined by

$C_B{}^c(x) = C_U(x) - C_B(x)$, $\forall x \in X$.

Example 3. 2.2

If $U = \{a/7, b/6, c/5, d/8\}$ is the universal bag for a fixed information system and $B = \{a/5, b/6, d/3\}$ is a sub bag of U, then $B^c = \{a/2, c/5, d/5\}$.

Proposition 3. 2.1

Suppose U is a universal bag of an information system and X is the support set of U . Then for subbag B and C of U drawn from X, the following hold:

(1) $(B^c)^c = B$

(2) $B^* \neq (B^c)^*$

(3) $B^* \cup (B^c)^* = U^* = X$

(4) $(B \oplus x)^c = B^c \ominus x$, $\forall x \in X$

(5) $(B \oplus C)^c = B^c \ominus C = C^c \ominus B$

(6) $(B \ominus x)^c = B^c \oplus x$, $\forall x \in X$

(7) $(B \ominus C)^c = B^c \oplus C$

(8) if $B = B_1 \oplus B_2$, then $B^* = B_1{}^* \cup B_2{}^*$

(9) $(B \cup C)^C = B^C \cap C^C$

(10) $(B \cap C)^C = B^C \cup C^C$

Definition 3. 2.4 (Cartesian product of bags)

Suppose A and B are two bags drawn from the sets X and Y respectively. Then their Cartesian product, denoted by $A \otimes B$, is a bag drawn from the set $X \times Y$ such that for all $(x, y) \in X \times Y$, $C_{A \otimes B}(x,y) = C_A(x) \,.\, C_B(y)$.

Note that the above two bags may be drawn from the same set.

Example 3.2.3

Let $X = \{a,b,c,d,e\}$ be any set and let $B_1=\{a/5, b/3, e/8\}$ and $B_2 = \{a/3, b/6, c/5, d/1\}$ be two bags drawn from X. Then , $B_1 \otimes B_2 = \{(a, a)/15, (b, b)/18, (a, b)/30, (b, a)/9, (a, c)/25, (a, d)/5, (b, c)/15, (b, d)/3, (e, a)/24, (e, b)/48, (e, c)/40, (e, d)/8\}$.

Note that $C_{B_1 \otimes B_2}(a,b) \neq C_{B_1 \otimes B_2}(b,a)$ in the above example. Thus, in general, for any two bags B_1 and B_2 drawn from the same set X, $C_{B_1 \otimes B_2}(x,y) \neq C_{B_1 \otimes B_2}(y,x)$.

Proposition 3.2.2

For any two bags A and B drawn from the sets X and Y respectively, then

(1) $A \otimes B \neq B \otimes A$

(2) For all $x \in X$ and all $y \in Y$, $C_{A \otimes B}{}^{(x, y)} = C_{B \otimes A}{}^{(y, x)}$

Proposition 3.2.3

For any three bags A, B & C drawn from the sets X, Y and Z respectively,

(1) $A \otimes (B \cup C) = (A \otimes B) \cup (A \otimes C)$

(2) $A \otimes (B \cap C) = (A \otimes B) \cap (A \otimes C)$

(3) $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$

(4) $A \otimes (B \ominus C) = (A \otimes B) \ominus (A \otimes C)$.

Definition 3.2.5

A fuzzy bag F drawn from a set X is characterized by a function $CM_F : X \to B$, where B is the set of all crisp bags drawn from the unit interval, I = [0, 1],i.e. , for every x, $CM_F(x)$ is a crisp bag drawn from I.

Again any crisp bag can be characterized by a count function over its set , so $CM_F(x)$ can be characterized by the count function $C_{CM_F^{(x)}} : I \to N$, where N is the set of non- negative integers. Here, for every $\alpha \in I$, $C_{CM_F^{(x)}}(\alpha)$ is a positive integer indicating the number of occurrences of x with membership value α in the fuzzy bag F.

Note3.2.1 For $x \in X, C_{CM_F}(x)$ is a crisp bag drawn from I and hence for any $\alpha \in [0,1]$, $C_{CM_F^{(x)}}(\alpha)$ is always a non-negative integer .

Note 3.2.2 For $x \in X$ and any fuzzy bag F drawn from X, $C_{CM_F^{(x)}}(0) = 0$

Example 3. 2.4

Let X = {a, b, c, d} be a set . A fuzzy bag F drawn from X is given by

$$CM_F^{(a)} = \{.2/5, .6/3, .1/7\}$$

$$CM_F^{(b)} = \{.1/2 , .3/5, .6/1\}$$

$$CM_F^{(c)} = \{.5/6\}$$

$$CM_F^{(d)} = \{.3/5, .1/2, .7/3, .8/4\}.$$

The fuzzy bag F is also represented as

F = { a/(.2/5, .6/3, .1/7), b/(.1/2, .3/6, .6/1), c/(.5/6), d/(.3/5, .1/2, .7/3, .8/4)}.

The other basic definitions namely union, intersection, addition, insertion of an element and removal of an element on fuzzy bags are on similar lines as of crisp bags.

Definition 3.2.6

Let X be any set and let U be the universal bag for some fixed information system. The universal fuzzy bag F(U) for this information system is a fuzzy bag where

$\forall\, x \in X, \forall\, \alpha \in [0, 1] = I$,

(i) $C_U(x) = \sum_{\alpha} C_{CM^x_{F(U)}}(\alpha)$

(ii) $C_{CM_F^x}(\alpha) \leq C_{CM^x_{F(U)}}(\alpha)$ for each fuzzy bag F drawn from X.

Definition 3.2.7

Let F(U) be the universal fuzzy bag drawn from the set X and let F be any fuzzy sub bag of F(U). Then the complement F^c of F is defined as

$$C_{CM^x_{F^c}}(\alpha) = C_{CM^x_{F(U)}}(\alpha) - C_{CM^x_F}(\alpha) \quad , \forall\, x \in X \; , \forall\, \alpha \in I .$$

Proposition 3.2.4

Suppose F be a fuzzy bag drawn from the set X, then

(i) $(F^c)^c = F$;

(ii) $F^* \neq (F^c)^*$;

(iii) $F^* \cup (F^c)^* = F(U)^*$ where F^* is the supporting set of F .

3.3 Bag Relations

Bag relation is defined in continuation to Cartesian product of two bags introduced by Chakrabarty et al. [**18**]

Definition 3. 3.1

Let A and B be two bags drawn from the sets X and Y respectively. A bag relation from A to B is defined as a subset of $A \otimes B$, denoted by R, if $\forall\, (x, y) \in X \times Y$

$$C_R{}^{(x, y)} \leq C_{A \otimes B}{}^{(x,y)}.$$

Definition 3.3.2

A bag relation R in a bag A drawn from X is said to be

(i) reflexive if $C_R\,(a, a) \leq C_{A\otimes A}\,(a, a),\ \forall\ a \in X$;

(ii) symmetric if $C_R\,(a, a) = C_R\,(b, a),\ \forall\ a, b \in X$;

(iii) transitive if $C_R\,(a, b) + C_R\,(b, c) \geq C_R\,(a, c),\ \forall\ a, b, c \in X$

.

Definition 3.3.3

A bag relation R *in a bag* A *drawn from a set* X *is called a bag equivalence relation if* it is simultaneously reflexive, symmetric and transitive.

Example 3.3.1

Suppose A= {a/6, b/10, c/5} is a bag drawn from X= {a, b, c} so that we have

$A\otimes A$={(a, a)/36, (b, b)/100, (c, c)/25,(a, b)/60, (a, c)/30, (b, c)/50,(b, a)/60, (c, a)/30, (c, b)/50}.

Then ,R={(a,a)/30,(b, b)/80, (c, c)/25, (a, b)/40,(b, a)/40, (b, c)/42, (c, b)/42,(a, c)/15, (c, a)/15} in a bag A is a bag equivalence relation.

It may be observed that for any bag equivalence relation R on a bag A drawn from X, $C_R(x, y)$ is greater than or equal to each of $\frac{1}{2} C_R(x, x)$ and $\frac{1}{2} C_R(y, y)$, $\forall$ x, y $\in$ X. Hence $C_R(x, y) = 0 \Rightarrow C_R(x, x) = 0$ and $C_R(y, y) = 0$, $\forall$ x, y $\in X$.

Definition 3.3.4

Suppose R is a bag relation from bag A to bag B drawn from X and Y respectively. Then R^{-1}, defined by $C_{R^{-1}}(y, x) = C_R(x, y)$, $\forall (x, y) \in X \times Y$, is also a bag relation from B to A. This R^{-1} is called inverse of the bag relation R.

Note that R^{-1} is also a bag equivalence relation whenever R is one.

Definition 3.3.5

Suppose R and R' be two bag relations on bags A and B drawn from the sets X and Y respectively. Then the union of the bag relations R and R' denoted by $R \cup R'$ and their intersection denoted by $R \cap R'$ are defined through the function count

$$C_{R \cup R'}(z) = \max \{ C_R(z), C_{R'}(z) \}$$

$$C_{R \cap R'}(z) = \min \{ C_R(z), C_{R'}(z) \}, \forall z = (x,y) \in X \times Y.$$

Proposition 3.3.1

Let R and R' be two bag equivalence relations on a bag A drawn from the set X. Then $R \cup R'$ is also a bag equivalence relations on A.

Proof: (i) We have for each $a \in X$,

$C_{R \cup R'}(a, a) = \max \{C_R(a, a), C_{R'}(a, a)\} \leq C_{A \otimes A}(a, a)$

(since both R and R' are reflexive)

This implies $R \cup R'$ is reflexive.

(ii) For $a, b \in X$, $C_{R \cup R'}(a, b) = \max \{C_R(a, b), C_{R'}(a, b)\}$

$= \max \{C_R(b, a), C_{R'}(b, a)\}$

(since R and R' are both symmetric)

$= C_{R \cup R'}(b, a)$.

This implies $R \cup R'$ is symmetric.

(iii) For $a, b, c \in X$, $C_{R \cup R'}(a, b) + C_{R \cup R'}(b, c)$

$= \max \{C_R(a, b), C_{R'}(a, b)\} + \max \{C_R(b, c), C_{R'}(b, c)\}$

$= \max \{C_R(a, b) + C_R(b, c), C_{R'}(a, b) + C_R(b, c), C_R(b, c) + C_{R'}(b, c),$

$C_{R'}(a, b) + C_{R'}(b, c)\}$

$\geq \max \{C_R(a, b) + C_R(b, c), C_{R'}(a, b) + C_{R'}(b, c)\}$

$\geq \max \{C_R(a, c), C_{R'}(a, c)\}$

$= C_{R \cup R'}(a, c)$.

Hence $R \cup R'$ is a bag equivalence relation.

Remark. $R \cap R'$ is not necessarily a bag equivalence relation.

Example3.3.2

Consider a bag A= { a/5, b/7, c/4} drawn from the set X={a, b, c}.

Then $A \otimes A$ = { (a, a)/25, (b, b)/49,(c, c)/16, (a, b)/35, (b, a)/35, (b, c)/28, (c, b)/28, (c, a)/20, (a, c)/20}.

Consider two bag equivalence relations

R={(a,a)/10, (b,b)/15, (c,c)/8, (a,b)/10, (b,a)/10, (b,c)/8, (c,b)/8, (a,c)/18, (c,a)/18)}

and R'={(a,a)/15, (b,b)/12, (c,c)/5, (a,b)/8, (b,a)/8, (b,c)/10,(c,b)/10,(a,c)/17,(c, a)/17}

defined on A. Then the bag relation

$R \cap R'$ ={(a,a)/10, (b,b)/12, (c,c)/5, (a,b)/8, (b,a)/8, (b,c)/8, (c,b)/8,(a,c)/17, (c,a)/17}

is not a bag equivalence relation since

$C_{R \cap R'}(a, b) + C_{R \cap R'}(b, c) = 8+8 = 16 \ngtr C_{R \cap R'}(a, c) = 17.$

3.4. Cartesian product of fuzzy bags

Definition 3.4.1

Let F_1 & F_2 be two fuzzy bags drawn from the sets X and Y respectively. Then their Cartesian product, denoted by $F_1 \otimes F_2$, is a fuzzy bag drawn from $X \times Y$ such that for all $(x, y) \in X \times Y$

$$C_{CM_{F_1 \otimes F_2}^{(x,y)}}(\alpha) = \max \{ C_{CM_{F_1}^{x}}(\alpha_1) . C_{CM_{F_2}^{y}}(\alpha_2)\}, \text{ where } \alpha = \min\{\alpha_1, \alpha_2\}$$

The notations $C_{CM_{F_1}^{x}}(\alpha_1)$ and $C_{CM_{F_2}^{x}}(\alpha_2)$ in the above have been defined at **3.2.5.**

Note that the two fuzzy bags above may also be drawn from the same set and in that case F_1 and F_2 may or may not be equal . .

Example 3.4.1

Suppose F_1 = {a/(.3/2, .5/6), b/(.6/5, .7/2)} and F_2 = {l/(.4/5), m/(.6/5, .2/4)} are two fuzzy bags drawn from the sets X = {a, b} & Y = {l, m} respectively. Then

$F_1 \otimes F_2$={(a,l)/(.3/10,.4/30),(a,m)/(.3/10,.2/24, .5/30),(b,l)/(.4/25),(b,m)/(.6/25, .2/20)}

and $F_1 \otimes F_1$ = {(a, a)/(.3/12, .5/36), (b, b)/(.6/25, .7/10), (a, b)/(.3/10, .5/30),

(b, a)/(.3/10,.5/30)}.

Example 3.4.2

Let X = {a, b, c, d} be a set and F_1={a/(.2/3, .4/5), b/(.5/6), d/(.3/3, .6/7)} and

F_2 = {a/(.6/8, .1/5), b/(.3/4, .2/5), c/(.4/5)} be two fuzzy bags drawn from X.

Then, $F_1 \otimes F_2$={(a,a)/(.2/24, .1/25, .4/40), (b,a)/(.5/48, .1/30),(d,a)/(.3/24, .1/35, .6/56),

(a, b)/(.2/25,.3/20), (b, b)/(.3/24, .2/30), (d, b)/(.3/28, .2/35), (a, c)/(.2/25, .4/25),

(b, c)/(.4/30), (d, c)/(.3/15,.4/35)} .

It may be observed from the above example that

$$C_{CM_{F_1 \otimes F_2}(x,y)}(\alpha) \neq C_{CM_{F_1 \otimes F_2}(y,x)}(\alpha) \text{ for } \alpha \in [0, 1] \text{, in general.}$$

We denote $F_1 \otimes F_1$ by F_1^2, $F_1^2 \otimes F_1$ by F_1^3 and so on.

Proposition 3.4.1

For any two fuzzy bags F_1 and F_2 drawn from the sets X and Y respectively,

(i) $F_1 \otimes F_2 \neq F_2 \otimes F_1$, in general ;

(ii) $C_{CM_{F_1 \otimes F_2}(x,y)}(\alpha) = C_{CM_{F_2 \otimes F_1}(y,x)}(\alpha)$, $\forall\, \alpha \in [0,1]$, $x \in X$, $y \in Y$.

Proposition 3.4.2

If three fuzzy bags A, B and C are drawn from the sets X, Y and Z respectively, then the following hold :

(i) $A \otimes (B \cup C) = (A \otimes B) \cup (A \otimes C)$

(ii) $A\otimes (B \cap C) = (A\otimes B) \cap (A\otimes C)$

Proof . (i) For any $x \in X$ and $y \in Y$ and $\alpha \in [0,1]$

$$C_{C M^{(x,y)}_{(A \otimes B) \cup (A \otimes C)}}(\alpha) = \max \{C_{C M^{(x,y)}_{(A\otimes B)}}(\alpha) , C_{CM^{(x,y)}_{(A\otimes C)}}(\alpha)\}$$

$$= \max \{\max \{ C_{C M_A^x} (\alpha_1). C_{CM_B^y} (\alpha_2)\},$$

$$\max \{ C_{CM_A^x} (\alpha_1). \ C_{CM_C}(\alpha_2)\}\}, \text{ where } \alpha = \min\{\alpha_1, \alpha_2\}.$$

$$= \max \{C_{CM_A^x} (\alpha_1). \max \{ C_{CM_B^y} (\alpha_2), \ C_{CM_C^x} (\alpha_2)\}\}$$

$$= \max \{C_{CM_A^x} (\alpha_1). C_{CM^y_{B\cup C}} (\alpha_2)\}$$

$$= C_{CM^{(x,y)}_{A\otimes (B\cup C)}}(\alpha) \text{ where } \alpha = \min \{\alpha_1, \alpha_2\}$$

i.e. $A\otimes (B\cup C) = (A\otimes B)\cup(A\otimes C)$

(ii) Similar to (i) .

Remark. Cartesian product of fuzzy bags is not distributive with respect to addition and subtraction of fuzzy bags.

Example 3.4.3

Let $X = \{a, b\}$, $Y = \{ l, m\}$ and $Z = \{m\}$ be any three sets .

Let A= {a/(.2/7, .4/5), b/(.3/8, .4/7)}, B = {l/(.4/5), m/(.4/2, .6/5)} and

C ={m/(.4/9, .2/3)} be the fuzzy bags drawn from X, Y & Z respectively. Then

$(A\otimes B) \oplus (A\otimes C)$ = {(a, l)/(.2/35, .4/25), (a, m)/(.2/98, .4/70), (b, m)/(.3/112, .2/24, .4/98), (b, l)/(.3/40, .4/35)}.

$A\otimes (B \oplus C)$ = {(a, l)/(.2/35, .4/25), (a, m)/(.2/77, .4/55), (b, l)/(.3/40, .4/35), (b, m)/(.3/88,.4/77, .2/24)}

Therefore, $A\otimes(B \oplus C) \neq (A\otimes B) \oplus (A\otimes C)$.

Similarly it can be checked that $A\otimes(B \ominus C) \neq (A\otimes B)\ominus(A\otimes C)$.

3.5. Fuzzy bag relations

Definition 3.5.1

Consider two fuzzy bags A and B drawn from the sets X & Y respectively. Then a fuzzy bag relation from A to B is a subset R of A⊗B whose function count is given by

$$C_{CM_R(x,y)}(\alpha) \leq. C_{CM_{A\otimes B}(x,y)}(\alpha), \qquad \forall (x,y) \in X \times Y \;\&\; \alpha \in [0,1].$$

It may be noted that a fuzzy bag relation R is also a sub bag of A⊗ B.

Definition 3.5.2

A fuzzy bag relation R in a fuzzy bag A drawn from a set X is said to be

(i) reflexive if $C_{CM_R(x,x)}(\alpha) \leq \max\{C_{CM_A x}(\alpha_1), C_{CM_A x}(\alpha_2)\}$, for every $x \in X$, where $\alpha = \min\{\alpha_1, \alpha_2\}$;

(ii) symmetric if $C_{CM_R(x,y)}(\alpha) = C_{CM_R(y,x)}(\alpha), \forall\, x, y \in X$;

(iii) transitive if $\{C_{CM_R(x,y)}(\alpha) + C_{CM_R(y,z)}(\alpha)\} \geq C_{CM_R(x,z)}(\alpha) \;\; \forall\, x, y, z \in X$.

Definition 3.5.3

A fuzzy bag relation R on a fuzzy bag A drawn from X is called a fuzzy bag equivalence relation if it is reflexive, symmetric and transitive.

Example 3.5.1

Consider a fuzzy bag A={a/(.2/3,.5/6, .7/1),b/(.3/1, .4/5, .7/3),c/(.1/5, .4/3, .5/2, .7/5)} drawn from the set X = {a, b, c} . Then,

A⊗A ={(a,a)/(.2/18, .5/6, .7/1), (b,b)/(.3/5, .4/25, .7/9), (c, c)/(.1/25, .4/9, .5/4, .7/25), (a,b)/(.2/25, .3/6, .3/18, .7/3, .4/30), (b,a)/ (.2/15, .3/6, .4/30, .5/18, .7/3), (b,c)/(.3/3, .4/15, .1/25, .5/6, .7/15), (c, b)/(.1/25, .3/3, .5/6, .4/15, .7/15) (c, a)/(.1/30, .2/9, .4/18, .5/12, .7/5), (a, c)/(.1/30, .2/9, .4/18, .5/12, .7/5)}

and the fuzzy bag relation, R= {(a, a)/(.2/10, .5/6), (b, b)/(.3/5, .4/20, .7/5), (c, c) /(.1/22, .4/5, .5/4, .7/20), (a, b)/(.2/6, .5/5, .3/4,.4/12,.7/3), (b, a)/(.2/6, .3/4, .4/12,.5/5,.7/3), (b, c)/(.1/15, .3/3, .4/10, .5/5, .7/12), (c, b)/(.1/5, .3/3, .4/10, .5/5, .7/12), (c, a)/(.1/12, .2/5, .4/4,.5/3,.7/10) (a, c)/(.1/12, .2/5,.4/4,.5/3,7/10)} is a fuzzy bag equivalence relation.

It may be observed that for any fuzzy bag A drawn from a set X, A⊗A is not always a fuzzy bag equivalence relation . However, A⊗A is a fuzzy bag relation if

1. $C_{CM_A^x}(\alpha) \neq 0$, $\forall x \in X$, where $\alpha = \max\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ and $C_{CM_A^x}(\alpha_i) \neq 0$;
2. $C_{CM_R^{(x,y)}}(\alpha) \geq$ each of $\frac{1}{2} C_{CM_R^{(x,x)}}(\alpha)$ and $\frac{1}{2} C_{CM_R^{(y,y)}}(\alpha)$, $\forall\, x, y \in X$;

   and hence $C_{CM_R^{(x,y)}}(\alpha) = 0 \Rightarrow C_{CM_R^{(x,x)}}(\alpha) = 0$ and $C_{CM_R^{(y,y)}}(\alpha) = 0$.

Proposition 3.5.1

If R and R' be two fuzzy bag equivalence relations on a fuzzy bag A drawn from X, then $R \cup R'$ is also a fuzzy bag equivalence relation on A .

Proof : Suppose R & R' are two fuzzy bag equivalence relations on a fuzzy bag A drawn from X. Then

(i) For all a, b $\in$ X, $C_{CM_{R\cup R'}}(a, a)(\alpha) = \max\{C_{CM_R}(a, a)(\alpha), C_{CM_{R'}}(a, a)(\alpha)\}$

$$\leq \max\{\max\{C_{CM_A}a(\alpha_1).\ C_{CM_A}a(\alpha_2)\}\},$$

$$\max\{C_{CM_A}a(\alpha_1).\ C_{CM_A}a(\alpha_2)\}\}$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$. (since R and R' are both reflexive) .

$$\leq \max\{C_{CM_A}a(\alpha_1).\ C_{CM_A}a(\alpha_2)\}$$

This implies that $R\cup R'$ is reflexive.

(ii) For all a, b $\in$ X, $C_{CM_{R\cup R'}}(a, b)(\alpha) = \max\{C_{CM_R}(a, b)(\alpha), C_{CM_{R'}}(a, b)(\alpha)\}$

$$= \max\{C_{CM_R}(b, a)(\alpha), C_{CM_{R'}}(b, a)(\alpha)\}$$

$$= C_{CM_{R\cup R'}}(b, a)(\alpha)$$ (since R and R' are both symmetric)

This implies that $R\cup R'$ is symmetric.

(iii) For all a, b, c $\in$ X,

$$C_{CM_{R\cup R'}}(a, b)(\alpha) + C_{CM_{R\cup R'}}(b, c)(\alpha)$$

$$= \max\{C_{CM_R}(a, b)(\alpha), C_{CM_{R'}}(a, b)(\alpha)\} + \max\{C_{CM_R}(b, c)(\alpha), C_{CM_{R'}}(b, c)(\alpha)\}$$

$$= \max\{C_{CM_R}(a, b)(\alpha) + C_{CM_R}(b, c)(\alpha)\}, C_{CM_R}(a, b)(\alpha) + C_{CM_{R'}}(a, b)(\alpha)\},$$

$$C_{CM_{R'}}(a, b)(\alpha) + C_{CM_R}(b, c)(\alpha), C_{CM_{R'}}(a, b)(\alpha) + C_{CM_{R'}}(b, c)(\alpha)\}$$

$$\geq \max\{C_{CM_R}(a, b)(\alpha) + C_{CM_R}(b, c)(\alpha), C_{CM_{R'}}(a, b)(\alpha) + C_{CM_{R'}}(b, c)(\alpha)\}$$

$\geq \max \{ C_{CM_R(a,c)}(\alpha), C_{CM_{R'}(a,c)}(\alpha)\}$ (since both R and R' are transitive)

$= C_{CM_{R \cup R'}(a,c)}(\alpha)$

This implies that $R \cup R'$ is transitive.

Thus $R \cup R'$ is a fuzzy bag equivalence relation.

Remark: $R \cap R'$ is not necessarily a fuzzy bag equivalence relation .

Example 3.5.2

In addition to the fuzzy bag equivalence relation R in example **3.5.1**, consider another fuzzy bag equivalence relation R' = {(a, a)/(.2/12, .5/6), (b, b)/(.3/4, .4/22, .7/7), (c, c)/(.1/20, .4/5, .5/3, .7/22), (a, b)/(.2/7, .5/4, .3/3, .4/14, .7/5), (b, a) /(.2/7, .5/4, .3/3, .4/14, .7/5), (b, c)/(.1/17, .3/2, .4/15, .5/5, .7/14), (c, b)/(.1/17, .3/2, .4/15, .5/5, .7/14), (c, a)/(.1/14, .2/10, .4/4, .5/3, .7/15), (a, c)/(.1/14, .2/10, .4/4, .5/3, .7/15)}

Then, $R \cap R'$={(a,a)/(.2/10, .5/6), (b,b)/(.3/4, .4/20, .7/5), (c,c)/(.1/20, .4/4, .5/3, .7/22), (a, b)/(.2/7, .3/3, .5/4, .4/14, .7/3), (b, a)/(.2/7, .3/3, .5/4, .4/14, .7/3), (b, c)/(.1/17, .3/2, .4/15, .5/5, .7/14), (c, b))/(.1/17, .3/2, .4/15, .5/5, .7/14), (c, a)/(.1/14, .2/9, .4/4, .5/3, .7/3), (a, c))/(.1/14, .2/9, .4/4, .5/3, .7/3)} .

Here $C_{CM_{R \cap R}(b,c)}(.3) + C_{CM_{R \cap R'}(c,a)}(.3) = 2 + 0 \not> C_{CM_{R \cap R'}(b,a)}(.3) = 3$.

Therefore, $R \cap R'$ is not a fuzzy bag equivalence relation.

3.6. Complement of a fuzzy bag

In continuation to the work of Chakrabarty et al. **[18]** regarding complement of a fuzzy bag we add the following definitions and results.

Definition 3.6.1

Let F_1 and F_2 be two fuzzy bag drawn from a set X and let F(U) be the universal fuzzy bag for the some fixed information system. Then $\forall\, x \in X, \alpha \in I = [0, 1]$;

(i) $C_{CM^x_{F_1 \oplus F_2}}(\alpha) = \min\{C_{CM^x_{F_1}}(\alpha) + C_{CM^x_{F_2}}(\alpha), C_{CM^x_{F(U)}}(\alpha)\}$

(ii) $C_{CM^x_{F_1 \ominus F_2}}(\alpha) = \max\{C_{CM^x_{F_1}}(\alpha) - C_{CM^x_{F_2}}(\alpha), 0\}$

(iii) $C_{CM^x_{F_1 \oplus x}}(\alpha) = \min\{C_{CM^x_{F_1}}(\alpha) + 1, C_{CM^x_{F(U)}}(\alpha)\}$

(iv) $C_{CM^x_{F_1 \ominus x}}(\alpha) = \max\{C_{CM^x_{F_1}}(\alpha) - 1, 0\}$.

Proposition 3.6.1

For any $x \in X$ and any fuzzy bag F drawn from X, the following holds.

(1) $(F \oplus x)^c = F^c \ominus x$

(2) $(F \ominus x)^c = F^c \oplus x$

Proposition 3.6.2

If F_1 and F_2 be to fuzzy sub bags of the universal fuzzy bag F(U) drawn from the set X, then

(1) $(F_1 \oplus F_2)^c = F_1^c \ominus F_2 = F_2^c \ominus F_1$

(2) $(F_1 \ominus F_2)^c = F_1^c \oplus F_2$

(3) $(F_1 \cup F_2)^c = F_1^c \cap F_2^c$

(4) $(F_1 \cap F_2)^c = F_1^c \cup F_2^c$

3.7 Conclusion

The bag structure introduced by Yager [93] where 'bags' provide a natural structure for representing 'set-like' objects in which a count of the number of elements is of relevance. He has also pointed out that the bag and fuzzy bag structure would be very useful for the development of an advanced version of a database calculus. This impetus has led to the study of bags and fuzzy bags. With this in mind, bags and fuzzy bags are further studied and some results are proved. In the process it is established that union of two bags (fuzzy bags) equivalence relations is a bag (fuzzy bag) relation but their intersection is not always one.

Chapter 4

Rough Boolean Algebras

[1]The concept of rough Boolean algebra and rough sub algebra are introduced based upon Pawlak's notion of indiscernibility relation between elements in a set. Some characterizations of rough Boolean algebras and rough subalgebras are given.

4.1 Introduction

The algebraic approach of rough sets was studied by some authors, for example, Z. Bonikowaski[**15**],T.B. Iwinski[**39**] and J. Pomykala and J.A.Pomykala[**71**].Recently Biswas and Nanda [**10**] introduced the notion of rough subgroups based on upper approximation of rough set. Boolean algebra has the potential applications in different fields. In view of this, a concept of rough algebraic structures is introduced by defining rough Boolean algebra and rough subalgebra based on upper approximation of Pawlak's rough set and then some results of rough Boolean algebras and rough subalgebras have been proved and also checked through examples.

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[1]Parts of this chapter entitled " Rough Boolean Algebras" has been accepted for publication in Journal of Fuzzy Mathematics, Los Angels.

4.2 Preliminaries

Definition of rough set and some basic operations on rough sets necessary to introduce rough Boolean algebras are given.

Definition 4.2.1

Suppose R is an equivalence relation defined over an universe U .Then for any subset X of U, the sets

$$\underline{A}(X)=\{x:[x]_R \in X\} \text{ and } \bar{A}(X)=\{x:[x]_R \cap X \neq \phi\}$$

are called lower and upper approximations of X respectively.

Furthermore, $A(X)=(\underline{A}(X),\bar{A}(X))$ is called the rough set of X in the approximation space S=(U,R).

In the above $[x]_R$ denotes the equivalence class of R containing x . Further, for a fixed non empty subset X of U, the rough set of X, i.e., A(X) is unique.

Example 4.2.1

Given an approximation space S=(U,R), where R is an equivalence relation on $U=\{x_1,x_2,\ldots,x_8\}$ with the following equivalence classes:

$E_1=\{x_1,x_4,x_5\}$

$E_2=\{x_2,x_5,x_7\}$

$E_3=\{x_3\}$

$E_4=\{x_6\}$.

Let $X=\{x_3,x_5\}$, then $\underline{A}(X)=\{x_3\}$ and $\bar{A}(X)=\{x_2,x_3,x_5,x_7\}$ and so

$A(X)=\{\{x_3\},\{x_2,x_3,x_5,x_7\}\}$ is a rough set of X.

Definition 4.2.2

A subset X of U is called definable if $\underline{A}(X)=\bar{A}(X)$. If $X\subseteq U$ is given by a predicate P and $x\in U$, then

1. $x\in\underline{A}(X)$ means that x certainly has property P,
2. $x\in\bar{A}(X)$ means that x possibly has property P,
3. $x\in U\setminus\bar{A}(X)$ means that x definitely does not have property P.

Definition 4.2.3

If $A=(\underline{A},\bar{A})$ and $B=(\underline{B},\bar{B})$ are two rough sets in the approximation space $S=(U,R)$, then $A\cup B$ (union of two rough sets), $A\cap B$ (intersection of two rough sets), $A\subseteq B$ (rough set inclusion) and -A(rough complement of A) are defined by

$$\text{(i) } A\cup B = (\underline{A}\cup\underline{B},\bar{A}\cup\bar{B})$$

$$\text{(ii) } A\cap B = (\underline{A}\cap\underline{B},\bar{A}\cap\bar{B})$$

$$\text{(iii) } A\subseteq B \Leftrightarrow (\underline{A}\subseteq\underline{B},\bar{A}\subseteq\bar{B})$$

$$\text{(iv) } -A = (U\text{-}\bar{A},U\text{-}\underline{A})$$

$$\text{(v) } A\text{-}B = A\cap(-B)=(\underline{A}-\bar{B},\bar{A}-\underline{B})$$

Definition 4. 2.4

Let R be an equivalence relation, defined over a set U. Then the relation $\dot{R}$ on $U\times U$ is defined by $(x,y)\dot{R}(p,q) \Leftrightarrow xRp$ and yRq.

Corollary: The relation $\dot{R}$ is an equivalence relation in $U\times U$.

4.3 Rough Boolean Algebra

Definition 4.3.1

Suppose S=(B,R) is an approximation space, where $B=(B,\vee,\wedge,')$ is a Boolean algebra and R is an equivalence relation on B. Then the rough set $A(X)=(\underline{A}(X),\bar{A}(X))$ of a non empty subset X of B is a rough Boolean algebra in S with respect to the operations in B if

(i) $x\vee y \& x\wedge y\in\bar{A}(X), \forall x,y\in X$; and

(ii) for every $x\in X, \exists x'\in\bar{A}(X)$ s.t. $x\vee x'=u$ and $x\wedge x'=0$, where 0 and u are $\inf B$ and $\sup B$ respectively.

In the above ($A(X),\vee,\wedge,'$) or simply $A(X)$ (when the operations are understood) denotes a rough Boolean algebra.

Remarks : Suppose $\bar{A}(X)$ is a replaced by an arbitrary super set Y of X, then either $X\subset Y\subset\bar{A}(X)$ or $X\subset\bar{A}(X)\subset Y$ .The latter is outside the rough structure whereas former does not affect the definition of a rough Boolean algebra. Moreover, the lower and upper approximation operations are dual of each other and so, the upper approximation is considered.

It can be easily seen that the following properties of a Boolean algebra also hold in case of a rough Boolean algebra.

(i) $x \vee x = x,\ x \wedge x = x$ (idempotence)

(ii) $x \vee y = y \vee x,\ x \wedge y = y \wedge x$ (commutative)

(iii) $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (associative)

(iv) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ (absorption)

(v) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (distributive)

(vi) $(x \vee y)' = x' \wedge y', (x \wedge y)' = x' \vee y'$ (De Morgan's laws)

(vii) $x \leq y \Leftrightarrow x \wedge y' = 0 \Leftrightarrow x' \vee y = 1$.

Example 4.3.1

Consider B={1,2,3,5,6,10,15,30}=the set of all factors of 30.

Define operations $\vee$ and $\wedge$ in B as $x \vee y = \text{lcm}\{x,y\}$ and $x \wedge y = \gcd\{x,y\}$.

Then $(B, \vee, \wedge, ')$ is a Boolean algebra in which the complements of 1,3,5,6,10, 15 and 30 are respectively 30,15,10,6,5,3,2 and 1.Also define, for all

$x,y \in B, xRy \Leftrightarrow x \equiv y (\text{mod}\, 2)$.

Then R is an equivalence relation in B and the resulting equivalence classes are

[1]={1,3,5,15] and [2]={2,6,10,30}

Now, (i) for $X = \{1,2\} \subset B, \underline{A}(X) = \phi$, $\bar{A}(X)=B$ and $(A(X), \vee, \wedge, ')$ is a rough Boolean algebra , but (ii) for $X = \{2,6\} \subset B, \underline{A}(X) = \phi, \bar{A}(X) = [2] = \{2,6,10,30\}$ and $(A(X), \vee, \wedge, ')$ is not a rough Boolean algebra since $2' = 15 \notin \bar{A}(X)$.

Proposition 4. 3.1

For any subalgebra X of a Boolean algebra B, A(X) is a rough Boolean algebra.

Proof: For any subalgebra X, $x,y \in X \Rightarrow x \vee y, x \wedge y, x'$ and $y' \in X \subseteq \overline{A}(X)$.

Corollary: For any subalgebra X of a Boolean algebra B, the homomorphic image of $A(X)$ is also a rough Boolean algebra.

Example 4. 3.2

Let B be the set of all ordered triples with entries 0 or 1

i.e.$B=\{(0,0,0),(1,1,1),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1)\}$.

For $(a_1,b_1,c_1),(a_2,b_2,c_2) \in B$,

the $\vee, \wedge$ and $'$ are defined as $(a_1,b_1,c_1) \vee (a_2,b_2,c_2) = (a_1 \vee a_2, b_1 \vee b_2, c_1 \vee c_2)$ where

$a_1 \vee a_2 = \max\{a_1,a_2\}$ etc.

$(a_1,b_1,c_1) \wedge (a_2,b_2,c_2) = (a_1 \wedge a_2, b_1 \wedge b_2, c_1 \wedge c_2)$ where $a_1 \wedge a_2 = \min\{a_1,a_2\}$ etc,

and $(a_1,b_1,c_1)' = (1\text{-}a_1, 1\text{-}b_1, 1\text{-}c_1)$ .Then $(B,\vee,\wedge,')$ is a Boolean algebra.

Also S=(B,R) is an approximation space with the equivalence relation R defined by

$xRy \Leftrightarrow$ x and y have the same number of 0's(1's), $\forall x,y \in B$.

The resulting equivalence classes are

$E_1 = \{(0,0,0)\}$
$E_2 = \{(1,1,1)\}$
$E_3 = \{(1,0,0),(0,1,0),(0,0,1)\}$
$E_4 = \{(1,1,0),(1,0,1),(0,1,1)\}$

Then $X = \{(0,0,0),(1,1,1),(1,0,0),(0,1,1)\}$ is a subalgebra where $\underline{A}(X)=\{E_1,E_2\}$,

$\overline{A}(X)=B$ and $A(X)$ is a rough Boolean algebra.

Definition 4.3.2

If the rough set $A(X)$,for $\phi \neq X \subseteq B$,is a rough Boolean algebra in an approximation space $S=(B,R)$ with operations in B then a rough subset $A(Y)$of $A(X)$ with $Y \subseteq X$,is a rough subalgebra of $A(X)$ provided $A(Y)$ forms a rough Boolean algebra with operations in B.

Example 4.3.3

In example 4.3.1, if we take $X=\{1,2,3\}$and $Y=\{1,2\}$,then $A(X)$ is a rough Boolean algebra and $A(Y)$ is a rough subalgebra.

The following results are straight forward.

Proposition 4.3.2

If $A(X)$ is a rough Boolean algebra in an approximation space $S=(B,R)$ such that $\overline{A}(X)=X$, then $\overline{A}(X)$ is a subalgebra of B .

Proposition 4.3.3

Let $S=(B,R)$ be an approximation space, where B is a Boolean algebra and $A(X)=(\underline{A}(X),\overline{A}(X))$ be the rough set of X in S. If $\overline{A}(X)$ is a subalgebra then $A(X)$ is a rough Boolean algebra.

Proof: Since $\overline{A}(X)$ is a subalgebra of$(B,\vee,\wedge,')$,

$x \vee y$ and $x \wedge y \in \bar{A}(X), \forall x,y \in \bar{A}(X)$. Also $x' \in \bar{A}(X)$ for any $x \in \bar{A}(X)$.

But $X \subseteq \bar{A}(X)$, so $x,y \in X \Rightarrow x \vee y$ and $x \wedge y \in \bar{A}(X)$ and $x' \in \bar{A}(X)$ for any $x \in X$.

The converse of the above proposition is not necessarily true.

Definition 4.3.3

A rough Boolean algebra $A(X)$ in an approximation space $S=(B,R)$ is a complete rough Boolean algebra if every non empty subset of X has a least upper bound and a greatest lower bound in $\bar{A}(X)$.

Example 4.3.3

The rough Boolean algebra of example **4.3.1(i)** and **4.3.2** are also complete rough Boolean algebra.

Proposition 4.3.4

If for any rough Boolean algebra $A(X)$ in an approximation space $S=(B,R)$,

$\bar{A}(X)$ is a complete sub algebra of B then $A(X)$ is a complete rough Boolean algebra.

Proof: Since $X \subseteq \bar{A}(X)$ and $\bar{A}(X)$ is a complete subalgebra, every non empty subset of X has a lub and glb in $\bar{A}(X)$ and so, $A(X)$ is a complete rough Boolean algebra.

Proposition 4.3.5

If $A(X)$ is a complete rough Boolean algebra in $S=(B,R)$ and $\bar{A}(X)=X$, then X is complete subalgebra of B.

Proof: $A(X)$ is a complete rough Boolean algebra implies $A(X)$ is a rough Boolean algebra and every subset of X has a lub and a glb in $\overline{A}(X)=X \Rightarrow X$ is a complete sub algebra.

Lemma 4.3.1

Let $S=(U,R)$ be an approximation space.

Then $\overline{A}(X)\times\overline{A}(Y) \subseteq \overline{A}(X\times Y)$, for any subsets X,Y of U.

Proof: Let $(x,y) \in \overline{A}(X)\times\overline{A}(Y) \Rightarrow x \in \overline{A}(X) \& y \in \overline{A}(Y) \Rightarrow [x]_R \cap X \neq \phi$

$\& [y]_R \cap Y \neq \phi \Rightarrow ([x]_R \times [y]_R) \cap (X\times Y) \neq \phi \Rightarrow [(x,y)]_R \cap (X\times Y) \neq \phi \Rightarrow (x,y) \in \overline{A}(X\times Y)$.

Prposition 4.3.6

Suppose $S=(B,R)$ is an approximation space where B is a Boolean algebra. If $A(X)$ and $A(Y)$, for $X,Y \subseteq B$, are rough Boolean algebra then $A(X\times Y)$ is also a rough Boolean algebra.

Proof: $A(X)$ and $A(Y)$ are two rough Boolean algebras in S, therefore

$\forall x_1,x_2 \in X$, $x_1 \vee x_2, x_1 \wedge x_2 \in \overline{A}(X)$, and for every $x \in X$, $\exists x' \in \overline{A}(X)$ such that

$x \vee x' = u$ and $x \wedge x' = 0$ and $\forall y_1,y_2 \in Y$, $y_1 \vee y_2$, $y_1 \wedge y_2 \in \overline{A}(Y)$

and for every $y \in Y$, $\exists y' \in \overline{A}(Y)$ such that $y \vee y'=u$ and $y \wedge y' = 0$, where 0 and u are inf B and supB respectively.

Now take $(x_1,y_1), (x_2,y_2) \in X \times Y$.

Then $(x_1,y_1) \vee (x_2,y_2) = (x_1 \vee x_2, y_1 \vee y_2) \in \bar{A}(X) \times \bar{A}(Y) \subseteq \bar{A}(X \times Y)$ and

$(x_1,y_1) \wedge (x_2,y_2) = (x_1 \wedge x_2, y_1 \wedge y_2) \in \bar{A}(X) \times \bar{A}(Y) \subseteq \bar{A}(X \times Y)$.

Also $(x,y) \vee (x',y') = (x \vee x', y \vee y') = (u,u)$ and

$(x,y) \wedge (x',y') = (x \wedge x', y \wedge y') = (0,0)$

Therefore $A(X \times Y)$ is a rough Boolean algebra.

4.4 Conclusion

Pawlak's rough sets are generalization of sets which are associated with impreciseness in the form of indiscernibility between the elements of the set. So rough algebraic structures could be useful in dealing with some knowledge representation problems involving uncertainty in the form of indscernibility. In this work, we have introduced rough Boolean algebra based on Pawlak's rough set. Rough Boolean algebra consists of a rough set, two binary operations and one unary operation whereas a Boolean algebra consists of a set and the same operations. Further, we propose to carry forward the work regarding the relationship between the rough structure and algebraic structure.

Chapter 5

Some Results of the Lower and Upper Approximations in Lattices

In this chapter, we present some properties of lower and upper approximations with respect to congruence relation and fuzzy congruence relation in a lattice.

5.1 Introduction

Kuroki et al.[**48**] introduced the concept of a rough ideal in a semigroup and proved few results on the lower and upper approximations with respect to the congruences and the fuzzy congruences on a semigroup. Davvaz, [**26**] defined the notion of rough subring and rough ideal with respect to an ideal of a ring and studied certain properties of the lower and the upper approximation in a ring. In sequel, some properties of the lower and upper approximations with respect to the congruence relation in a lattice, sublattice and ideal and also fuzzy congruence relation in a lattice have been examined here.

5.2 Rough subsets in a lattice

A binary relation θ in a lattice L is a congruence relation if

(i) θ is an equivalence relation;

(ii) For all a, b, c$\in$ L, aθb implies $(a\vee c)\theta(b\vee c)$ and $(a\wedge c)\theta(b\wedge c)$.

Being an equivalence relation θ partitions the lattice L into a set of disjoint θ-classes. The θ-class containing an element a$\in$ L, will be denoted by $a\theta$, i.e., $a\theta=\{x\in L: x\theta a\}$. For any two elements a, b $\in L$, the join and meet operations of θ-classes are defined as $(a\theta)\vee(b\theta)=(a\vee b)\theta$ and $(a\theta)\wedge(b\theta)=(a\wedge b)\theta$.

Definition 5.2.1

Let A be a nonempty subset of a lattice L and θ is a congruence relation on L. Then the sets $\theta_{-}(A)=\{x\in L: x\theta\subseteq A\}$ and $\theta^{-}(A)=\{x\in L: x\theta\cap A\neq\phi\}$ are called lower and upper approximations of A respectively with respect to θ. For a nonempty subset A of L, $\theta(A)=(\theta_{-}(A),\theta^{-}(A))$ is called a rough set with respect to θ.

Example 5.2.1

Let $(L,\vee,\wedge)$ be a lattice, where L is a the set of all positive integers where

$x\vee y$=L.C.M of x and y

$x\wedge y$=G.C.D of x and y.

Define a relation θ such that $\forall$ x, y$\in$L, $x\theta y \Leftrightarrow x\equiv y$ (mod2)

Then θ is clearly an equivalence relation and $\forall$ x, y, z$\in$L, and $x\theta y$ implies $(x\vee z)\theta$ $(y\vee z)$ and $(x\wedge z)\theta(y\wedge z)$. Hence θ is a congruence relation over L. Take a subset A={2,3,4,5,6} of L. Then 1θ=the set of odd positive integers and 2θ=the set of even positive integers. Also $\theta_{-}(A)=\phi$ and $\theta^{-}(A)$=L, i.e., $\theta(A)$= $(\phi$, L) is a rough set with respect to θ over L.

Proposition 5.2.1

Let θ and π be congruence relations on a lattice L. If A and B are non empty subsets of L, then

(i) $\theta_{-}(A) \subseteq A \subseteq \theta^{-}(A)$

(ii) $\theta^{-}(A \cup B) = \theta^{-}(A) \cup \theta^{-}(B)$

(iii) $\theta_{-}(A \cap B) = \theta_{-}(A) \cap \theta_{-}(B)$

(iv) $A \subseteq B \Rightarrow \theta^{-}(A) \subseteq \theta^{-}(B)$

(v) $A \subseteq B \Rightarrow \theta_{-}(A) \subseteq \theta_{-}(B)$

(vi) $\theta_{-}(A \cup B) \supseteq \theta_{-}(A) \cup \theta_{-}(B)$

(vii) $\theta^{-}(A \cap B) \subseteq \theta^{-}(A) \cap \theta^{-}(B)$

(viii) $\theta \subseteq \pi \Rightarrow \theta_{-}(A) \subseteq \pi_{-}(A)$

(ix) $\theta \subseteq \pi \Rightarrow \theta^{-}(A) \subseteq \pi^{-}(A)$.

Proof: (i) Let $a \in \theta_{-}(A)$, then $a \in a\theta \subseteq A$, which implies $\theta_{-}(A) \subseteq A$. Again if $a \in A$, then $a\theta \cap A \neq \phi$ ($\because a \in a\theta$) and so $a \in \theta^{-}(A)$, i.e., $A \subseteq \theta^{-}(A)$.

(ii) Now $a \in \theta^{-}(A \cup B) \Leftrightarrow a\theta \cap (A \cup B) \neq \phi$

$\Leftrightarrow (a\theta \cap A) \cup (a\theta \cap B) \neq \phi$

$\Leftrightarrow$ either $a\theta \cap A \neq \phi$ or $a\theta \cap B \neq \phi$

$\Leftrightarrow$ either $a \in \theta^{-}(A)$ or $a \in \theta^{-}(B)$

$\Leftrightarrow a \in \theta^{-}(A) \cup \theta^{-}(B)$.

(iii) Now $a \in \theta_{-}(A \cap B) \Leftrightarrow a\theta \subseteq A \cap B$

$\Leftrightarrow a\theta \subseteq A$ and $a\theta \subseteq B$

$\Leftrightarrow a\in \theta_{-}(A)$ and $a\in \theta_{-}(B)$

$\Leftrightarrow a\in \theta_{-}(A) \cap \theta_{-}(B)$.

Therefore, $\theta_{-}(A\cap B) = \theta_{-}(A)\cap \theta_{-}(B)$.

(iv) From (ii), we have $\theta^{-}(B)=\theta^{-}(A\cup B)$ ($\because A\subset B$ iff $A\cup B=B$)

i.e., $\theta^{-}(A)\subseteq \theta^{-}(B)$.

(v) Follows from (iii).

(vi) From (v), we have $\theta_{-}(A)\subseteq \theta_{-}(A\cup B)$ and $\theta_{-}(B)\subseteq \theta_{-}(A\cup B)$.

This implies $\theta_{-}(A)\cup \theta_{-}(B)\subseteq \theta_{-}(A\cup B)$.

(vii) Follows from (iv).

(viii) Let $a\in\theta^{-}(A) \Rightarrow a\theta\cap A\neq\phi$

$\Rightarrow a\pi\cap A\neq\phi$ ($\because$ $a\theta\subseteq a\pi$ as $\theta\subseteq\pi$)

$\Rightarrow a\in\pi^{-}(A)$

i.e., $\theta^{-}(A) \subseteq \pi^{-}(A)$

(ix) Let a be any element of $\theta^{-}(A)$.Then there exists $x\in a\theta\cap A$.

$\Rightarrow x\in a\pi\cap A$ ($\because a\theta\subseteq a\pi$ as $\theta\subseteq\pi$)

$\Rightarrow a\in \pi^{-}(A)$.

i.e., $\theta\subseteq\pi$ implies $\theta^{-}(A)\subseteq\pi^{-}(A)$.

The following example shows that the converse of (vi) and (vii) in Proposition **5.2.1** is not true.

Example 5.2.2

Let $(L,\vee,\wedge)$ be a lattice, where L={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14} whose

$x \vee y = x +_3 y = (x+y)(\text{mod}3)$

$x \wedge y = x \times_3 y = xy(\text{mod}3)$.

Define a relation θ such that $\forall$ x, y $\in$ L, $x\theta y \Leftrightarrow x \equiv y$ (mod3)

Then θ is clearly an equivalence relation and $\forall$ x, y, z $\in$ L, and $x\theta y$ implies $(x \vee z)\theta\ (y \vee z)$ and $(x \wedge z)\theta(y \wedge z)$. Hence θ is a congruence relation over L. The θ-congruence classes are

$E_1=\{0,3,6,9,12\}$, $E_2=\{1,4,7,10,13\}$ and $E_3=\{2,5,8,11,14\}$.

(i)Take A,B$\subseteq$L, where A={0,3,6,9} and B={1,2,4,7,10,12,13}.Then $\theta_-(A)= \phi$ and $\theta_-(B)= E_2$ and $\theta_-(A \cup B)=E_1 \cup E_2$ and so, $\theta_-(A \cup B) \not\subseteq \theta_-(A) \cup \theta_-(B)$.

(ii)Again, take A,B$\subseteq$L, where A={1,3,6,9} and B={0,3,4,7}.Then

$\theta^-(A)= \theta^-(B)= E_1 \cup E_2$ and $\theta^-(A \cap B)= E_1$ where $\theta^-(A) \cap \theta^-(B)= E_1 \cup E_2$ and therefore $\theta^-(A) \cap \theta^-(B) \not\subseteq \theta^-(A \cap B)$.

Proposition 5.2.2

Let θ and ψ be congruence relations on a lattice L. If A be non empty subset of L then $(\theta \cap \psi)^-(A) \subseteq \theta^-(A) \cap \psi^-(A)$.

Proof: We know that $\theta \cap \psi$ is also a congruence relation on L.

Let $x \in (\theta \cap \psi)^-(A)$, then $x(\theta \cap \psi) \cap A \neq \phi$.

$\Rightarrow$ there exists an element a$\in$L such that $a \in x(\theta \cap \psi) \cap A$.

$\Rightarrow$ $(a,x) \in (\theta \cap \psi) \Rightarrow (a,x) \in \theta$ and $(a,x) \in \psi$

$\Rightarrow a \in x\theta$ and $a \in x\psi$

Now, $a \in x\theta$ and $a \in A$ $\Rightarrow x\theta \cap A \neq \phi \Rightarrow x \in \theta^-(A)$

and $a \in x\psi$ and $a \in A \Rightarrow x\psi \cap A \neq \phi \Rightarrow x \in \psi^{-}(A)$

This implies, $x \in \theta^{-}(A) \cap \psi^{-}(A)$

i.e., $(\theta \cap \psi)^{-}(A) \subseteq \theta^{-}(A) \cap \psi^{-}(A)$.

Proposition 5.2.3

Let θ and ψ be congruence relations on a lattice L.If A be non empty subset of L then $(\theta \cap \psi)_{-}(A) = \theta_{-}(A) \cap \psi_{-}(A)$.

Proof: $x \in (\theta \cap \psi)_{-}(A) \Leftrightarrow x(\theta \cap \psi) \subseteq A$

$\Leftrightarrow x\theta \subseteq A$ and $x\psi \subseteq A$

$\Leftrightarrow x \in \theta_{-}(A)$ and $x \in \psi_{-}(A)$

$\Leftrightarrow x \in \theta_{-}(A) \cap \psi_{-}(A)$

i.e., $(\theta \cap \psi)_{-}(A) = \theta_{-}(A) \cap \psi_{-}(A)$.

Note1: Each congruence class is a sublattice.

Note2: If $x\theta$ and $y\theta$ are two congruence classes of a lattice L with respect to the congruence relation θ then $(x,y)\theta$ is also a congruence class of the lattice $L \times L$ and $x\theta \times y\theta = (x,y)\theta$.

Proposition 5.2.4

Let θ be a congruence relation on a lattice L. If A and B are non empty subsets of L then $\theta^{-}(A) \times \theta^{-}(B) \subseteq \theta^{-}(A \times B)$.

Proof: Let $(x,y) \in \theta^{-}(A) \times \theta^{-}(B)$

$\Rightarrow x \in \theta^{-}(A)$ and $y \in \theta^{-}(B)$

$\Rightarrow x\theta \cap A \neq \phi$ and $y\theta \cap B \neq \phi$

$\Rightarrow \exists$ m,n $\in$ L such that $m \in x\theta \cap A$ and $n \in y\theta \cap B$

$\Rightarrow m \in x\theta$, $n \in y\theta$ and $m \in A$, $n \in B$

$\Rightarrow (m,n) \in x\theta \times y\theta$ and $(m,n) \in A \times B$

$\Rightarrow (m,n) \in (x, y)\theta$ and $(m,n) \in A \times B$

$\Rightarrow (x, y)\theta \cap (A \times B) \neq \phi$

$\Rightarrow (x, y) \in \theta^{-}(A \times B)$

i.e., $\theta^{-}(A) \times \theta^{-}(B) \subseteq \theta^{-}(A \times B)$.

Proposition 5.2.5

Let θ be a congruence relation on a lattice L. If A and B are non empty subsets of L then $\theta_{-}(A) \times \theta_{-}(B) \subseteq \theta_{-}(A \times B)$.

Proof: Let $(x,y) \in \theta_{-}(A) \times \theta_{-}(B)$

$\Rightarrow x \in \theta_{-}(A)$ and $y \in \theta_{-}(B)$

$\Rightarrow x\theta \subseteq A$ and $y\theta \subseteq B$

$\Rightarrow x\theta \times y\theta \subseteq A \times B$

$\Rightarrow (x, y)\theta \subseteq A \times B$

$\Rightarrow$ $(x, y) \in \theta_-(A \times B)$

i.e., $\theta_-(A) \times \theta_-(B) \subseteq \theta_-(A \times B)$.

5.3 Upper and Lower Approximations in sublattices and ideals

An example is considered below to exhibit the existence of lower and upper approximation in rough sublattices and rough ideals before examining the some results on them.

Example 5.3.1

In example **5.2.2,** take A={0,1,2,3, 6, 9,12} then $\theta_-(A)$=E_1

$\theta^-(A)$= L, $\forall x, y \in \theta_-(A) \Rightarrow x \vee y \in \theta_-(A)$ and $x \wedge y \in \theta_-(A)$.

And $\forall x, y \in \theta^-(A) \Rightarrow x \vee y \in \theta^-(A)$ and $x \wedge y \in \theta^-(A)$.

Thus $\theta_-(A)$ and $\theta^-(A)$ are sublattices of L. Similarly it can be shown that $\theta_-(A)$ and $\theta^-(A)$ are ideals of L.

Proposition 5.3.1

Suppose θ be a congruence relation on a lattice L, and $\phi \neq A, B \subseteq L$.

If $\theta^-(A)$ and $\theta^-(B)$ are sublattices of L, then $\theta^-(A \times B)$ is also a sublattice of $L \times L$.

Proof: Let (x_1,y_1), $(x_2,y_2) \in \theta^-(A \times B)$ $\quad \forall x_1,x_2,y_1,y_2 \in L$

Now, $(x_1,y_1) \vee (x_2,y_2) = (x_1 \vee x_2, y_1 \vee y_2) \in \theta^-(A) \times \theta^-(B) \subseteq \theta^-(A \times B)$

Similarly, $(x_1,y_1) \wedge (x_2,y_2) \in \theta^-(A \times B)$.

Therefore $\theta^-(A \times B)$ is a sublattice of $L \times L$.

Proposition 5.3.2

Suppose θ be a congruence relation on a lattice L, and $\phi \neq A,B \subseteq L$. If $\theta_-(A)$ and $\theta_-(B)$ are sublattices of L, then $\theta_-(A \times B)$ is also a sublattice of $L \times L$.

Proof: The proof is similar to the theorem **5.3.1.**

Proposition 5.3.3

Suppose θ is a congruence relation on a lattice L, and $\phi \neq A,B \subseteq L$.

If $\theta^-(A)$ and $\theta^-(B)$ are ideals of L, then $\theta^-(A \times B)$ is also an ideal of $L \times L$.

Proof: Let $(x_1,y_1),(x_2,y_2) \in \theta^-(A \times B) \quad \forall\, x_1,x_2,y_1,y_2 \in L$

Now, $(x_1,y_1) \vee (x_2,y_2) = (x_1 \vee x_2, y_1 \vee y_2) \in \theta^-(A) \times \theta^-(B) \subseteq \theta^-(A \times B)$

Again $(x_1,y_1) \in \theta^-(A \times B)$, $(m,n) \in L \times L$

This implies $(x_1,y_1) \wedge (m,n) = (x_1 \wedge m, y_1 \wedge n) \in \theta^-(A) \times \theta^-(B) \subseteq \theta^-(A \times B)$

Therefore, $\theta^-(A \times B)$ is an ideal of $L \times L$.

Definition 5.3.1

Suppose (L,θ) be an approximation space, where θ is a congruence relation on lattice L. Also for $\phi \neq A \subseteq L$ and $\theta(A)=(\theta_-(A),\theta^-(A))$ is a rough set in the approximation space (L,θ). If $\theta_-(A)$ and $\theta^-(A)$ are ideals (or sublattices)of L, then we call $\theta(A)$ is a rough ideal (or rough sublattice) in (L,θ).

Proposition 5.3.4

Intersection of two rough ideals is again a rough ideal with respect to same congruence relation.

Proof: Follows from the intersection of two rough sets.

This completes the proof.

5.4 Lower and upper approximation with respect to fuzzy congruences

A fuzzy relation θ on L is a mapping θ:L×L→I, where I is the unit interval [0,1].Let θ and ψ be two fuzzy relations on L. Then the composition of two fuzzy relations on L is defined by $(\theta \circ \psi)(a, b)= \bigvee_{x\in L} \{\theta (a, x) \wedge \psi (x, b) \}$, $\forall a,b\in L$.

A fuzzy relation θ on L is called fuzzy congruence relation on L if $\forall a, b, c\in L$, the following hold:

(i) θ is an equivalence relation on L;

(ii) $\theta (a, b) \leq \theta (a\vee c, b\vee c) \wedge \theta (a\wedge c, b\wedge c)$, $\forall a,b,c \in L$.

Let θ be a fuzzy congruence relation on L. For each $a\in L$, we define a fuzzy subset θ_a as follows:

$\theta_a (x)= \theta (a, x)$ for all $x\in L$. This fuzzy subset θ_a is called a fuzzy congruence class containing $a\in L$. We write $\frac{S}{\theta}=\{\theta_a : a\in L\}$

Then $\frac{S}{\theta}$ is a lattice under the binary operations $\vee$ and $\wedge$ defined by

$\theta_a \vee \theta_b = \theta_{a\vee b}$ and $\theta_a \wedge \theta_b = \theta_{a\wedge b}$ for all $a, b\in L$.

Lemma 5.4.1

Let θ be a fuzzy congruence relation on a lattice L. Then

$$\theta'(1)=\{(a, b)\in L\times L : \theta (a, b)=1\} \text{ is a congruence relation on L.}$$

Proof: It is obvious that $\theta'(1)$ is reflexive and symmetric. Therefore ,to prove $\theta'(1)$ is transitive, let (a,b), (b,c) $\in \theta'(1)$.This implies that $\theta(a, b)= \theta(b, c)=1$

Again since θ is a fuzzy congruence relation on L, we have

$$\begin{aligned}\theta(a,c) &\geq (\theta o\, \theta)(a,c)\\ &= \bigvee_{x\in L} \{\theta (a, x) \wedge \theta (x,c) \}\\ &\geq \theta(a,b) \wedge \theta(b,c)\\ &= 1 \wedge 1\\ &= 1,\end{aligned}$$

and so $\theta(a, c)=1$. Thus $(a, c) \in \theta'(1)$.

Again let (a, b) $\in \theta'(1)$ and $c\in L$ .Then, since θ is a fuzzy congruence relation on L, we have $\theta (a\vee c, b\vee c) \wedge \theta (a\wedge c, b\wedge c) \geq \theta(a, b)=1$.

$$\Rightarrow \theta(a\vee c,b\vee c) \wedge \theta(a\wedge c,b\wedge c) \geq 1$$

$$\Rightarrow \theta (a\vee c, b\vee c) \geq 1 \text{ and } \theta (a\wedge c, b\wedge c) \geq 1$$

$$\Rightarrow \theta (a\vee c, b\vee c) =1 \text{ and } \theta (a\wedge c, b\wedge c) =1$$

$$\Rightarrow (a\vee c, b\vee c) \in \theta'(1) \text{ and } (a\wedge c, b\wedge c) \in \theta'(1)$$

Thus $\theta'(1)$ is a congruence relation on L.

Proposition 5.4.1

Let θ and ψ be fuzzy congruence relations on a lattice L. Then $\theta \cap \psi$ is a fuzzy congruence relation on L and $(\theta \cap \psi)'(1) = \theta'(1) \cap \psi'(1)$.

Proof: It is easy to prove that $\theta \cap \psi$ is a fuzzy congruence relation on L.

Let (a, b) $\in (\theta \cap \psi)'(1)$

$\Rightarrow (\theta \cap \psi)$(a,b)=1

$\Rightarrow \min\{\theta$ (a, b) , ψ (a, b)$\}$= 1

$\Rightarrow \theta(a,b) = \psi(a,b) = 1$

$\Rightarrow (a,b) \in \theta'(1)$ and $(a,b) \in \psi'(1)$

$\Rightarrow (a,b) \in \theta'(1) \cap \psi'(1)$

$\Rightarrow (\theta \cap \psi)'(1) \subseteq \theta'(1) \cap \psi'(1)$..

Again, let $(a,b) \in \theta'(1) \cap \psi'(1)$

$\Rightarrow (a,b) \in \theta'(1)$ and $(a,b) \in \psi'(1)$

$\Rightarrow \theta(a,b) = \psi(a,b) = 1$

Then we have $(\theta \cap \psi)$(a,b)= $\min\{\theta$ (a, b) , θ (a, b)$\}$=$\min\{1,1\}$=1

and so, (a,b) $\in (\theta \cap \psi)'(1)$

$\Rightarrow \theta'(1) \cap \psi'(1) \subseteq (\theta \cap \psi)'(1)$.

This completes the proof.

Proposition 5.4.2

Let θ and ψ be fuzzy congruence relations on a lattice L and if A be non empty subset of L$\times$L. Then

(i) $(\theta \cap \psi)'(1)_{-}(A) = (\theta'(1) \cap \psi'(1))_{-}(A)$

$$= \theta'(1)_{-}(A) \cap \psi'(1)_{-}(A).$$

(ii) $(\theta \cap \psi)'(1)^{-}(A) = (\psi'(1) \cap \psi'(1))^{-}(A)$

$$= \theta'(1)^{-}(A) \cap \psi'(1)^{-}(A).$$

Proof: Suppose (a, b) $(\theta \cap \psi)'(1)_{-}(A)$

$\Leftrightarrow (a,b) \in (\theta \cap \psi)'(1)$ and $[(a,b)]_{(\theta \cap \psi)'(1)} \subseteq A$

$\Leftrightarrow (\theta \cap \psi)(a,b)=1$ and$(a,b) \in A$

$\Leftrightarrow \min\{\theta(a,b)=1, \psi(a,b)=1\}$ and$(a,b) \in A$

$\Leftrightarrow \theta(a,b)=\psi(a,b)=1$ and $(a,b) \in A$

$\Leftrightarrow (a,b) \in \theta'(1)$ and $(a,b) \in \psi'(1)$ and $(a,b) \in A$

$\Leftrightarrow (a,b) \in \theta'(1) \cap \psi'(1)$ and $(a,b) \in A$

$\Leftrightarrow [(a,b)]_{\theta'(1) \cap \psi'(1)} \subseteq A$

$\Leftrightarrow (a,b) \in (\theta'(1) \cap \psi'(1))_{-}(A)$.

Again suppose, (a, b) $(\theta \cap \psi)'(1)_{-}(A)$

$\Leftrightarrow [(a,b)]_{(\theta \cap \psi)'(1)} \subseteq A$

$\Leftrightarrow [(a,b)]_{\theta'(1) \cap \psi'(1)} \subseteq A$

$\Leftrightarrow [(a,b)]_{\theta'(1)} \subseteq A$ and $[(a,b)]_{\psi'(1)} \subseteq A$

$\Leftrightarrow (a,b) \in \theta'(1)_{-}(A)$ and $(a,b) \in \psi'(1)_{-}(A)$

$\Leftrightarrow (a,b) \in \theta'(1)_{-}(A) \cap \psi'(1)_{-}(A)$.

This completes the proof.

(ii) Similar to (i).

5.5 Conclusion

Some results on lower and upper approximations of subsets in a lattice, sublattice and ideal with respect to congruence relation have been established including some results with respect to fuzzy congruence relation in lattice.

Chapter 6

Multicriteria Decision Making using Intuitionistic Fuzzy Bag Theory

In this chapter[1], intuitionistic fuzzy bags(IFB) concept is applied in multicriteria decision making problem and a hypothetical case study has been taken as an example.

6.1 Introduction

In real life, we often come across some collection of objects, i.e., set-like structure in which redundancy is significant ,for example, collection of books in a library or collection of marks scored by students in a school final examination. Crisp set representation of these collections fail to give information like presence of multiple copies of books in a library or number of students with equal marks. To overcome such difficulties Yager[93] introduced the bag structure as a set-like objects in which repeated elements are significant . The notion of bag and specially fuzzy bag are useful tools for the development of an advance version of a database calculus[93] and decision making problems[28,73,74].Intuitionistic fuzzy set(IFS)

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[1]Contents of this chapter was presented in the International Conference on Analysis and Discrete Structure, December 20-22,2002,IIT,Kharagpur, India and have appeared as a paper entitled "An Application of Intuitionistic Fuzzy Bag in Multicriteria Decision-Making"in the Proceedings (Combinatorial and Computational Mathematics) pp. 335-340,Narosa Publishing House, New Delhi.

theory introduced by Attanassov [5] is not necessarily a fuzzy set but a fuzzy set in an intuitionistic fuzzy set. Therefore, IFS is an alternative theory to deal with vagueness. The problems which are dealt with fuzzy set theory can also be well dealt with IFS theory , but there are some situations where IFS theory is more appropriate than fuzzy set theory. Chakrabarty et al.[20] introduced intuitionistic fuzzy bag (IFB) theory which can be used in some decision making problems where bag or fuzzy bag theory are not applicable. In this chapter, we apply the IFB concept in multi criteria decision making problems.

6.2 Preliminaries

Definition 6.2.1

Let E be fixed crisp set. An intuitionistic fuzzy set (or IFS) A in E is an object of the form $A = \{ <x, \mu_A(x), \nu_A(x)>: x \in E \}$, where the function $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ represent the degree of membership and degree of non membership respectively of the element $x \in E$ to the set A . It is clear that

$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$

6.2.1 Some Operations

Definition 6.2.2

If A and B are two IFSs of the set E, then

$A \subset B$ iff $\forall x \in E, [\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)]$ and $A \subset B$ iff $B \supset A$.

$A = B$ iff $\forall x \in E, [\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)]$.

$\bar{A}$={< x, $\nu_A(x), \mu_A(x)$ >} ,complement of the intuitionistic fuzzy set A.

$A \cap B = \{< x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)> | x \in E\}$.

$A \cup B = \{< x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)> | x \in E\}$.

$A+B=\{< x, \mu_A(x) + \mu_B(x) - \mu_A(x) . \mu_B(x), \nu_A(x) . \nu_B(x)> | x \in E\}$.

$A.B=\{< x, \mu_A(x) . \mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x) . \nu_B(x)> | x \in E\}$.

$\square A = \{< x, \mu_A(x), 1 - \mu_A(x)> | x \in E\}$.

$\lozenge A = \{< x, \nu_A(x), 1 - \nu_A(x)> | x \in E\}$.

$C(A)=\{< x,k,l> | x \in E\}$, where $k = \max_{x \in E} \mu_A(x)$, $l = \min_{x \in E} \nu_A(x)$

$I(A)=\{x,k',l'> | x \in E\}$, where $k' = \min_{x \in E} \mu_A(x)$, $l' = \max_{x \in E} \nu_A(x)$

Clearly every fuzzy set can be written in the form $\{< x, \mu_A(x), \mu_{A^c}(x)> | x \in E\}$ and hence is also an IFS.

Definition 6.2.3

An intuitionistic fuzzy bag(IFB) F drawn from a non empty set X is characterized by a count function CM_F:X→B, where B is the set of all crisp bags drawn from the Cartesian product I×I of the unit interval I=[0,1].Thus for any $x \in X$, $CM_F(x)$ is a crisp bag drawn from I×I and $C_{CM_F(x)}$:I×I→N which is the characterizing count function for the bag $CM_F(x)$.Here, for each $(\alpha,\beta) \in$ I×I, and $0 \leq \alpha+\beta \leq 1$, $C_{CM_F^x}(\alpha,\beta)$

is a non negative integer which indicates the number of occurrences of x with membership value α and non membership value β in the IFB F.

Note 1: For all F and x, $C_{CM_F^x}(0,0)=0$.

Note2: An IFB F reduces to a fuzzy bag F if for each $x\in X$, $CM_F(x)$ is a crisp bag in which $(\alpha,\beta)\in CM_F(x), \alpha+\beta=1$.

Note3: An IFB ,denoted by ϕ is called null IFB, if for each $x\in X$, $CM\phi$ (x) is an empty bag, i.e., $C_{CM_F^x}(\alpha,\beta)=0$.

6.2.2 Some Operations

Definition 6.2.4

Suppose F_1 and F_2 are two IFBs drawn from a set X. Then for all $x\in X$ and $(\alpha,\beta)\in I\times I$ with $0\leq\alpha+\beta\leq 1$,

(i) $F_1=F_2$ if $C_{CM_{F_1}^x}(\alpha,\beta)=C_{CM_{F_2}^x}(\alpha,\beta)$

(ii) $F_1\subseteq F_2$ if $C_{CM_{F_1}^x}(\alpha,\beta)\leq C_{CM_{F_2}^x}(\alpha,\beta)$

(iii) $F=F_1\oplus F_2$ if $C_{CM_F^x}(\alpha,\beta)=C_{CM_{F_1}^x}(\alpha,\beta)+C_{CM_{F_2}^x}(\alpha,\beta)$

(iv) $F=F_1\ominus F_2$ if $C_{CM_F^x}(\alpha,\beta)=\max\{C_{CM_{F_1}^x}(\alpha,\beta)-C_{CM_{F_2}^x}(\alpha,\beta), 0\}$

(v) $F=F_1\cup F_2$ if $C_{CM_F^x}(\alpha,\beta)=\max\{C_{CM_{F_1}^x}(\alpha,\beta), C_{CM_{F_2}^x}(\alpha,\beta)\}$

(vi) $F=F_1\cap F_2$ if $C_{CM_F^x}(\alpha,\beta)=\min\{C_{CM_{F_1}^x}(\alpha,\beta), C_{CM_{F_2}^x}(\alpha,\beta)\}$

Example 6.2.1

Let $F_1=\{a/\{(.3,.5)/5,(.5,.2)/2\}, b/\{(.1,.7)/6,(.6,.4)/5,(.8,.1)/12,(.9,.1)/7\},$

$c/\{(.2,.7)/2\}$ } and $F_2=\{ a/\{(.5,.2)/3\}, b/\{(.6,.4)/4,(.9,.1)/8,(.3,.4)/4\}$

be two IFBs drawn from a set $X=\{a, b, c\}$. Then

$F_1 \cup F_2=\{a/ \{(.3,.5)/5,(.5,.2)/3\}, b/\{(.1,.7)/6,(.6,.4)/5,(.8,.1)/12, (.9,.1)/8,(.3,.4)/4\},$

$c/\{(.2,.7)/2\}\}$

$F_1 \cap F_2 =\{a/\{ (.5,.2)/2\}, b/\{ (.6,.4)/4,(.9,.1)/7\} \}$

$F_1 \oplus F_2=\{a/\{(.3,.5)/5,(.5,.2)/5\}, b/\{(.1,.7)/6,(.6,.4)/9,(.8,.1)/12,(.9,.1)/15,(.3,.4)/4\},$

$c/\{(.2,.7)/2\}\}$

$F_1 \ominus F_2 =\{ a/ \{ (.3,.5)/5\}, b/ \{(.1,.7)/6, (.6,.4)/1, (.8,.1)/12 \}\}$.

Proposition 6.2.1

Let A,B and C be IFBs drawn from the set X. Then

(i) $A\cup B= B\cup A$, $A\cap B= B\cap A$

(ii) $A\cup(B\cup C)=(A\cup B)\cup C$, $A\cap(B\cap C)=(A\cap B)\cap C$

(iii) $A\cup A=A$, $A\cap A=A$

(iv) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$, $A\cap(B\cup C) =(A\cap B)\cup(A\cap C)$

(v) $A\oplus(B\cup C)=(A\oplus B)\cup(A\oplus C)$, $A\oplus (B\cap C)=(A\oplus B) \cap (A\oplus C)$,

(vi) $A\oplus B=B\oplus A$

(vii) $A\oplus(B\oplus C)=(A\oplus B) \oplus C$

Definition 6.2.5

Let A and B be two non empty fuzzy bags drawn from the set X. Then a similarity measure between A and B denoted by S(A,B) is defined by

$$S(A,B)=\frac{1}{\#(X)}\sum_{x\in X}\left(1-\frac{d(A(x),B(x))}{M(x)}\right), \quad \text{where } \#(X) = \text{cardinality of } X$$

and $d(A(x),B(x))=\left|\sum_{\theta}\theta.C_{CM_A^x}(\theta)-\sum_{\theta}\theta.C_{CM_B^x}(\theta)\right|$, $0\leq\theta\leq1$ and

$M(x)=\max\{\sum_{\theta}C_{CM_A^x}(\theta),\sum_{\theta}C_{CM_B^x}(\theta)\}$.

Clearly ,$0\leq S(A,B)\leq1$, $S(A,A)=1$ and $S(A,B)=S(B,A)$,for any non empty bags A and B.

Definition 6.2.6

Let A and B be two non empty fuzzy bags drawn from the set X. Suppose $w(x)$ is the weight of the element x in X such that $w(x)\in[0,1]$, then the degree of similarity between two fuzzy bags A and B can be defined by

$$WS(A,B)=\frac{1}{\sum_{x\in X}w(x)}\sum_{x\in X}[1-\frac{d(A(x),B(x))}{M(x)}],\text{where } w(x)=\text{weight of the element x in X.}$$

Clearly ,$0\leq WS(A,B)\leq1$,$WS(A,A)=1$ and $WS(A,B)=WS(B,A)$,for any two non empty fuzzy bags A and B.

6.3 Application of similarity measure in decision analysis

Definition 6.3.1

The standard IFB of an IFB in a set X is defined as

$$S(F)=\left\{\frac{x}{\left\{\frac{(.5,.5)}{\sum_{\alpha,\beta\in I}C_{CM_F^x}(\alpha,\beta)}\right\}}:x\in X\right\}$$

Consider the problem of deciding the best possible action out of 'n' alternatives involving m criteria on the basis of judgment of 'p' judges where each criteria has its corresponding weight. Let the 'n' alternatives be respectively $A_1,A_2,A_3,\dots A_n$ each of which depends upon all of the m factors or criteria $X_1,X_2,\dots,X_m$ with weights w_i, $i=1,2,\dots,m$ respectively and these m criteria values are evaluated by p judges $E_1,E_2,\dots,E_p$. These criteria values clearly form an IFB drawn from the set X of all criteria , corresponding to the action A_i. Let us form a criteria matrix P by the criteria values given by p judges, as below

P=

	X_1 X_2 ....X_m
A_1	p_{11} p_{12} ....p_{1m}
A_2	p_{21} p_{22} ....p_{2m}
....	....
A_n	p_{n1} p_{n2} ....p_{nm}

where $p_{ij}=(\mu_{ij},\nu_{ij})$ consists of the membership value μ_{ij} and the non membership value ν_{ij} with respect to i^{th} alternative and j^{th} criterion respectively. Here, corresponding to action A_i, the criteria values $\{p_{i1}, p_{i2}, \dots p_{im}\}$ form an IFB F_i drawn from the set $X=\{X_1,X_2,\dots X_m\}$ where each X_i has its weight w_i, $i=1,2,..m$. For each IFB F_i , the standard IFB S(F) is defined as $S(F)=\{X_1/\{(.5,.5)/p\}, X_2/\{(.5,.5)/p\},\dots\dots X_m/\{(.5,.5)/p\}\}$, which is independent of i. Now each F_i is compared with the standard IFB S(F) by the similarity measure with respect to μ_i and υ_i separately. Suppose $S_i = S_{\mu_i} - S_{\nu_i}$ is the difference of similarity measure for each i. Then arranging the Si's in descending order of magnitude, we obtain a ranking of the actions in order of merit.

An algorithm for the above method is presented below.

6.3.1 Algorithm

1.Construct the criteria matrix P from the available information supplied by the m judges.

2. Take the standard IFB S(F).

3. Calculate the n weighted similarity measures WS_i= WS[S(F),F_i], i=1,2,..n with respect to membership value and non membership value separately .

4.For each i, take $S_i = S_{\mu_i} - S_{\nu_i}$, the difference of similarity measure with respect to membership value and non membership value and arrange S_i's in descending order of magnitude , say $S_{i_1} \geq S_{i_2} \geq ...... \geq S_{i_3}$.

5. Choose A_{r_i} corresponding to S_i as the best action out of n alternatives.

6. Stop.

6.3.2 Case- Study

Consider the problem of selection of an office assistant in a company that a perspective candidate has to satisfy some characteristic such as Hand writing, Typing speed ,Working under tension and English vocabulary, etc. However, we consider here only three criteria for easy handling of the problem which are Hand writing, Typing and working under tension denoted by X_1, X_2 and X_3 .Thus $X=\{X_1,X_2,X_3\}$denote the set of criteria such that the weight of X_1,X_2 and X_3 are .5,.8, and .9 respectively. Suppose there are four candidates A_1,A_2,A_3 and A_4 for the post of office assistant and there are five judges to select the best candidate for the particular post. The evaluation by the five judges based on the three criteria X_1, X_2, X_3 is given in following criteria matrix.

	X_1	X_2	X_3
A_1	{(.6,.3)/2, (.7,.2)/2,(.8,.1)/1}	{(.6,.4)/2,(.8,.1)/2,(.9,.1)/1}	{(.3,.6)/3,(.5,.4)/1,(.6,.2)/1}
A_2	{(.8,.1)/5}	{(.3,.5)/1,(.4,.5)/2,(.5,.3)/2}	{ (.6,.2)/4, (.7,.2)/1}
A_3	{(.3,.5)/2,(.4,.5)/2,(.5,.3)/1}	{(.3,.1)/1,(.5,.4)/3,(.6,.2)/1}	{(.7,.2)/1,(.8,.1)/3,(.9,.1)/1}
A_4	{(.6,.3)/4,(.7,.2)/1}	{(.6,.2)/2,(.7,.2)/2,(.9,.1)/1}	{(.4,.5)/3,(.5,.2)/2}

Therefore $F=\{X_1/\{(.6,.3)/2,(.7,.2)/2,(.8,.1)/1\},X_2/\{(.6,.4)/2,(.8,.1)/2,(.9,.1)/1\},$

$X_3/\{(.3,.6)/3,(.5,.4)/1,(.6,.2)/1\}\}$

$F_2=\{X_1/\{(.8,.1)/5\},X_2/\{(.3,.5)/1,(.4,.5)/2,(.5,.3)/2\},X_3/\{(.6,.2)/4, (.7,.2)/1\}\}$

$F_3=\{X_1/\{(.3,.5)/2,(.4,.5)/2,(.5,.3)/1\},X_2/\{(.3,.1)/1,(.5,.4)/3,(.6,.2)/1\},$

$X_3/\{(.7,.2)/1,(.8,.1)/3,(.9,.1)/1\}\}$

$F_4=\{X_1/\{(.6,.3)/4,(.7,.2)/1\},X_2/\{(.6,.2)/2,(.7,.2)/2,(.9,.1)/1\},X_3/\{(.4,.5)/3, (.5,.2)/2\}\}$

Now $S(F)=\{X_1/\{(.5,.5)/5\}, X_2/\{(.5,.5)/5\}, X_3/\{(.5,.5)/5\}\}$

Then $S\mu_1=WS(S(F),F_1)=$ .9036 , $S\nu_1=WS(S(F),F_1)=$ .8309

$S\mu_2=WS(S(F),F_2)=$ .8523 , $S\nu_2=WS(S(F),F_2)=$.7573

$S\mu_3$=WS(S(F),F_3)= .8427 , $S\nu_3$=WS(S(F),F_3)= .7627

$S\mu_4$=WS(S(F),F_4)= .8755 , $S\nu_4$=WS(S(F),F_4)= .7845

Therefore S_1=.0727, S_2=.095, S_3=.08 and S_4=.091 and hence

$S_2 \geq S_4 \geq S_3 \geq S_1$ i.e., A_2(2^{nd} candidate) dominates all the others and so 2^{nd} candidate is selected .

6.4 Conclusion

The concept of IFB is reviewed and then applied in decision analysis in selecting the most suitable action out of n alternatives on the basis m weighted criteria depending on the information of p judges.

Chapter 7

On Soft Sets, Fuzzy Soft Sets and Intuitionistic Fuzzy Soft Sets

7.1 Introduction

Most of our real life problems in medical sciences, engineering, management, environment and social sciences often involve data which are not always all crisp, precise and deterministic in character because of various uncertainties typical for these problems. Such uncertainties are usually being handled with the help of the topics like probability, fuzzy sets, intuitionistic fuzzy sets, interval mathematics and rough sets etc. However, Molodtsov [**60**] has shown that each of the above topics suffers from some inherent difficulties due to inadequacy of their parametrization tools and introduced a concept called 'Soft Set Theory' having parametrization tools for successfully dealing with various types of uncertainties. The absence of any restrictions on the approximate description in soft set theory makes this theory very convenient and easily applicable in practice. Subsequently Maji et al. [**54,55,56**] extended their works by studying the theoretical aspects of the ' Soft Set Theory, 'Fuzzy Soft sets' and 'Intuitionistic Fuzzy Soft sets'.

In the first section of this chapter, some well known definitions and results of soft set, fuzzy soft set and intuitionistic fuzzy soft set are listed. In the next section, soft relation and fuzzy soft relation are introduced and then have applied in decision making problems. In the last two sections fuzzy soft set theory and intuitionistic fuzzy soft set theory have been applied in medical diagnosis problems separately .

7.2 A brief survey of soft sets, fuzzy soft sets and intuitionistic fuzzy soft sets

7.2.1 Soft Sets

Definition 7.2.1.1

Let X be a universal set, E a set of parameters and $A \subset E$. Then a pair (F,A) is called soft set over X, where F is a mapping from A to 2^X , the power set of X.

Definition 7.2.1.2

Let (F,A) and (G,B) be two soft sets over a common universe X, then

(i) (F,A) $\tilde{\subset}$ (G,B) , if $A \subset B$ and $\forall e \in A$, F(e)=G(e),

(ii) (F,A) = (G,B) , if (F,A) $\tilde{\subset}$ (G,B) and (G,B) $\tilde{\subset}$ (F,A) ,

(iii) Let $E=\{e_1,e_2,\ldots..,e_n\}$ be a set of parameters. The NOT set of E, denoted by $\rceil E$,

is defined by $\rceil E=\{\rceil e_1, \rceil e_2, \rceil e_3, \ldots.,\rceil e_n\}$, where $\rceil e_i$=not e_i, $\forall$ i.

It may be noted that $\rceil$ and $\rceil$ are different operators.

(iv) The complement of a soft set (F,A), denoted by $(F,A)^c$, is defined by

$(F,A)^c=(F^c,\rceil A)$,where $F^c{:}\rceil A \rightarrow 2^X$ is a mapping such that $F^c(e)=2^X-F(\rceil e)$, $\forall e \in \rceil A$.

(v) A soft set (F,A) is said to be a null soft set , denoted by Φ ,if $\forall\, e \in A$, $F(e)= \Phi$ (null set) of X.

(vi) A soft set (F,A) is said to be absolute set over X, denoted by $\tilde{A}$, if $\forall\, e \in A$, $F(e)=X$.

(vii) AND operation of two soft sets : If (F,A) and (G,B) be two soft sets then "(F,A) AND (G,B)", denoted by $(F,A) \wedge (G,B)=(H,A\times B)$, where

$H(\alpha,\beta)=F(\alpha) \cap G(\beta), \forall\, (\alpha,\beta) \in A\times B$.

(viii) OR operations of two soft sets : If (F,A) and (G,B) be two soft sets then "(F,A) OR(G,B)" denoted by $(F,A) \vee (G,B)=(O,A\times B)$,where

$O(\alpha,\beta)=F(\alpha) \cup G(\beta), \forall\, (\alpha,\beta) \in A\times B$.

(ix)Union of two soft sets of (F,A) and (G,B) over the common universe X is the soft set $(F,A)\ \tilde{\cup}\ (G,B)=(H,C)$., where $C=A\cup B$, such that $\forall\, e \in C$,

$$
\begin{aligned}
H(e) &= F(e), && \text{if } e \in A-B, \\
&= G(e), && \text{if } e \in B-A, \\
&= F(e) \cup G(e), && \text{if } e \in A \cap B.
\end{aligned}
$$

(x) Intersection of two soft sets of (F,A) and (G,B) over the common universe X is the soft set $(F,A)\ \tilde{\cap}\ (G,B)=(O,C)$, where $C=A\cap B$, and $\forall\, e \in C$, $O(e)=F(e)$ or $G(e)$.

Example 7.2.1.1

Let $X=\{c_1,c_2,c_3\}$ be the set of three cars and $E =\{$costly(e_1), metallic colour (e_2)

cheap (e_3)} be the set of parameters ,where $A=\{e_1,e_2\} \subset E$. Then

$(F,A)=\{F(e_1)=\{c_1,c_2,c_3\},F(e_2)=\{c_1,c_3\}\}$ is the crisp soft set over X which describes the " attractiveness of the cars" which Mr. S(say) is going to buy .

7.2.2 Fuzzy Soft Sets

Definition 7.2.2.1

Let X be a universal set, E a set of parameters and $A \subset E$. Let $\mathbb{F}(X)$ denotes the set of all fuzzy subsets of X. Then a pair (F,A) is called fuzzy soft set over X, where F is a mapping from A to $\mathbb{F}(X)$.

Definition 7.2.2.2

Let (F,A) and (G,B) be two fuzzy soft sets over a common universe X, then

(i) $(F,A) \tilde{\subset} (G,B)$, if $A \subset B$ and $\forall\ e \in A$, F(e) is a fuzzy subset of G(e).

(ii) (F,A)=(G,B) , if $(F,A) \tilde{\subset} (G,B)$ and $(G,B) \tilde{\supset} (F,A)$,

(iii) The complement of a fuzzy soft set (F,A) denoted by

$(F,A)^c$, is defined by $(F,A)^c = (F^c, \rceil A)$, where $F^c\colon \rceil A \to \mathbb{F}(x)$ is a mapping given by $F^c(\alpha)$= fuzzy complement of $F(\rceil\alpha)$, $\forall\ \alpha \in \rceil A$.

(iv) (F,A) is said to be a null fuzzy soft set, denoted by ϕ, if $\forall\ e \in A$, F(e)=null fuzzy set of X.

(v) A fuzzy soft set (F,A) is said to be absolute fuzzy soft set over X , denoted by $\tilde{A}$, if $\forall\ e \in A$, F(e)=X.

(vi)Union of two fuzzy soft sets (F,A) and (G,B) is a fuzzy soft set, denoted by

$(H,C)=(F,A)\tilde{\cup}(G,B)$, if $C=A\cup B$ and $\forall\ e\in C$,

$$H(e) = F(e), \quad \text{if } e\in A-B,$$
$$=G(e), \quad \text{if } e\in B-A,$$
$$= F(e)\tilde{\cup}G(e), \text{ if } e\in A\cap B.$$

(vii) Intersection of two fuzzy soft sets (F,A) and (G,B) is a fuzzy soft set, denoted by $(H,C)=(F,A)\tilde{\cap}(G,B)$, if $C=A\cap B$ and $\forall\ e\in C, H(e)=F(e)\cap G(e)$.

(viii) AND($\wedge$) operation of two fuzzy soft sets : If (F,A) and (G,B) are two fuzzy soft sets then (F,A) AND (G,B),denoted by

$(H,A\times B)=(F,A)\wedge(G,B)$, where $H(\alpha,\beta)=F(\alpha)\tilde{\cap}G(\beta), \forall\alpha\in A$ and $\forall\beta\in B$.

(ix) OR($\vee$) operation of two fuzzy soft sets: If (F,A) and (G,B) are two fuzzy soft sets, then (F,A) OR(G,B), denoted by

$(O,A\times B)=(F,A)\vee(G,B)$, where $O(\alpha,\beta)=F(\alpha)\tilde{\cup}G(\beta), \forall\alpha\in A$ and $\forall\beta\in B$.

Example 7.2.2.1

Let $X=\{c_1,c_2,c_3\}$ be the set of three cars and E ={costly(e_1), metallic colour(e_2) cheap(e_3)} be the set of parameters ,where $A=\{e_1,e_2\}\subset E$. Then $(G,A)=\{G(e_1)=\{c_1/.6,c_2/.4,c_3/.3\},G(e_2)=\{c_1/.5,c_2/.7,c_3/.8\}\}$ is the fuzzy soft set over X describes the " attractiveness of the cars" which Mr. S(say) is going to buy .

7.2.3 Intuitionistic Fuzzy Soft Sets(IFSSs)

Definition 7.2.3.1

Let X be a universal set, E a set of parameters and $A\subset E$. Let $\mathcal{I}(X)$ denote the set of all intuitionistic fuzzy subsets of X. Then a pair (F,A) is called an intuitionistic fuzzy soft set(IFSS) over X, where F is a mapping from A to $\mathcal{I}(X)$.

Definition 7. 2.3.2

Let (F,A) and (G,B) be two IFSSs over a common universe X, then

(i) $(F,A) \tilde{\subseteq} (G,B)$, if $A \subset B$ and $\forall\ e \in A$, F(e) is an intuitionistic fuzzy subset of G(e).

(ii) (F,A)=(G,B), if $(F,A) \tilde{\subseteq} (G,B)$ and $(G,B) \tilde{\supseteq} (F,A)$,

(iii) The complement of an IFSS (F,A) denoted by $(F,A)^c$, is defined by

$(F,A)^c = (F^c, \rceil A)$, where $F^c: \rceil A \rightarrow \mathcal{I}(X)$ is a mapping given by

$F^c(\alpha)$= intuitionistic fuzzy complement of $F(\rceil\alpha)$, $\forall\ \alpha \in \rceil A$.

(iv) (F,A) is said to be a null IFSS, denoted by ϕ if $\forall\ e \in A$, F(e)=null intuitionistic fuzzy set of X.

(v) An IFSS set (F,A) is said to be absolute IFSS over X, denoted by $\tilde{A}$, if $\forall\ e \in A$, F(e)=X.

(vi) Union of two IFSSs (F,A) and (G,B) is an IFSS, denoted by

$(H,C)=(F,A) \tilde{\cup} (G,B)$, if $C=A \cup B$ and $\forall\ e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A-B, \\ G(e), & \text{if } e \in B-A, \\ F(e) \tilde{\cup} G(e), & \text{if } e \in A \cap B. \end{cases}$$

(vii) Intersection of two IFSSs (F,A) and (G,B) is an IFSS, denoted by

$(H,C)=(F,A) \tilde{\cap} (G,B)$, if $C=A \cap B$ and $\forall\ e \in C$, $H(e)= F(e) \cap G(e)$.

(viii) AND($\wedge$) operation of two IFSSs : If (F,A) and (G,B) are two IFSSs then

(F,A) AND (G,B) is an IFSS, denoted by

$(H, A\times B) = (F,A) \wedge (G,B)$, where $H(\alpha, \beta) = F(\alpha) \tilde{\cap} G(\beta), \forall \alpha \in A$ and $\forall \beta \in B$.

(ix) OR($\vee$) operation of two IFSSs : If (F,A) and (G,B) are two IFSSs then

(F,A)OR(G,B) is an IFSS, denoted by $(O, A\times B) = (F,A) \vee (G,B)$,

where $O(\alpha, \beta) = F(\alpha) \tilde{\cup} G(\beta), \forall \alpha \in A$ and $\forall \beta \in B$.

Example 7.2.3.1

Let $X=\{c_1,c_2,c_3\}$ be the set of three cars and E={costly(e_1), metallic colour(e_2), cheap(e_3),}be the set of parameters. Consider two intuitionistic fuzzy soft sets (F,A) and (G,B),where $A=\{e_1,e_2\} \subset E$ and $B=\{e_1,e_2,e_3\} \subset E$ and

$(F,A)=\{F(e_1)=\{c_1/(.6,.3),c_2/(.4,.4),c_3/(.3,.6)\}, F(e_2)=\{c_1/(.6,.2),c_2/(.7,.2),c_3/(.5,.4)\}\}$

and $(G,B)=\{G(e_1)=\{c_1/(.6,.3),c_2/(.4,.5),c_3/(.3,.6)\}, G(e_2)=\{c_1/(.6,.4),c_2/(.7,.2), c_3/(.5,.5)\}, G(e_3)=\{c_1/(.2,.6),c_2/(.4,.4),c_3/(.5,.3)\}\}$.Then

(i) $(F,A)^c=\{ F(\urcorner e_1)=\{c_1/(.3,.6),c_2/(.4,.4),c_3/(.6,.3)\},F(\urcorner e_2)=\{c_1/(.2,.6),c_2/(.2,.7),c_3/(.4,.5)\}$

(i i) $(F,A) \tilde{\subset} (G,B)$.

(iii) $(H,C)=(F,A)\tilde{\cup}(G,B)=\{H(e_1)=\{c_1/(.6,.3),c_2/(.4,.5),c_3/(.3,.6)\},$

$H(e_2)=\{ c_1/(.6,.2),c_2/(.7,.2),c_3/(.5,.4) \}, H(e_3)=\{ c_1/(.2,.6),c_2/(.4,.4)c_3/(.5,.3)\}\}$

(iv) $(H,C)=(F,A) \tilde{\cap} (G,B)=\{H(e_1)=\{c_1/(6,.3),c_2/(.4.4),c_3/(.3,.6)\},$

$H(e_2)=\{ c_1/(.6,.2),c_2/(.7,.2),c_3/(.5,.4)\}\}$.

Proposition 7.2.3.1

If (F,A) and (G,B) are two soft sets (or fuzzy soft sets or intuitionistic fuzzy soft sets) then (i) $(F,A)\ \tilde{\cup}\ (F,A) = (F,A)$

(ii) $(F,A)\ \tilde{\cap}\ (F,A) = (F,A)$

(iii) $(F,A)\ \tilde{\cup}\ \phi = \phi$

(iv) $(F,A)\ \tilde{\cap}\ \phi = \phi$

(v) $(F,A)\ \tilde{\cup}\ \tilde{A} = \tilde{A}$,where $\tilde{A}$ is the absolute soft set(or absolute fuzzy soft set / IFSS).

(vi) $(F,A)\ \tilde{\cap}\ \tilde{A} = \tilde{A}$

(vii) $((F,A)\ \tilde{\cup}\ (G,B))^c = (F,A)^c\ \tilde{\cup}\ (G,B)^c$

(viii) $((F,A)\ \tilde{\cap}\ (G,B))^c = (F,A)^c\ \tilde{\cap}\ (G,B)^c$

(ix) $((F,A) \vee (G,B))^c = (F,A)^c \wedge (G,B)^c$

(x) $((F,A) \wedge (G,B))^c = (F,A)^c \vee (G,B)^c$.

7.3 Soft Relations and Fuzzy Soft Relations

7.3.1 Soft Relations

[1]The concept of soft set is one of the recent topics developed for dealing with the uncertainties present in most of our real life situations. Further the availibity of the parametrization tools in soft set theory enhances the flexibility of its applications. Here soft and fuzzy soft relations are introduced and then applied in a decision making problem.

Definition 7.3.1.1

Let U and V be two initial universe sets and let E be a set of parameters and (F,E) and (G,E) be soft sets over U and V respectively, then (H,E) is a soft relation between (F,E) and (G,E) over U×V if

$$H:E\to 2^{U\times V} \text{ is a mapping such that } H(e)=\{(u_i,v_j):u_i\in F(e) \text{ and } v_j\in G(e),\forall e\in E\}$$
$$=\phi, \text{ otherwise.}$$

Example 7. 3.1.1

Let $U=\{u_1,u_2,u_3,u_4\}$ be the set of four houses and $V=\{v_1,v_2,v_3\}$ be the set of three farm houses. Also let, $E=\{e_1,e_2,e_3\}$ be the set of parameters namely e_1=expensive, e_2=wooden and e_3=prime location. Suppose that $F(e_1)=\{u_1,u_3\}$,$F(e_2)=\{u_2,u_4\}$and $F(e_3)=\{u_1,u_2\}$.Then the soft set (F,E) is a parametrized family $\{F(e_i),i=1,2,3\}$of

[1]Parts of this chapter will appear as a paper entitled " Application of Soft and Fuzzy Soft Relations in Decision Making Problems" in 'Bulletin of Pure and Applied Sciences'.

subsets of U, giving a collection of approximate description of houses. Similarly the soft set (G,E) is also a parametrized family $\{G(e_i), i=1,2,3\}$ of subsets of V, where $G(e_1)=\{v_1,v_3\}$, $G(e_2)=\{v_1,v_2\}$, $G(e_3)=\{v_2,v_3\}$, giving another collection of approximate description of farm houses. Then (H,E) is a parametrized family $\{H(e_i), i=1,2,3\}$ of subsets $U\times V$, where $H(e_1)=\{(u_1,v_1),(u_1,v_3),(u_3,v_1),(u_3,v_3)\}$, $H(e_2)=\{(u_2,v_1),(u_2,v_2),(u_4,v_1),(u_4,v_2)\}$ and $H(e_3)=\{(u_1,v_2),(u_1,v_3),(u_2,v_2),(u_2,v_3)\}$ and (H,E) is a soft relation between (F,E) and (G,E).

The tabular representation of soft sets (F,E) ,(G,E) and soft relation (H,E) are given below

(F,E)

U	e_1	e_2	e_3
u_1	1	0	1
u_2	0	1	1
u_3	1	0	0
u_4	0	1	0

(G,E)

V	e_1	e_2	e_3
v_1	1	1	0
v_2	0	1	1
v_3	1	0	1

(H,E)

U×V	e_1	e_2	e_3
(u_1,v_1)	1	0	0
(u_1,v_2)	0	0	1
(u_1,v_3)	1	0	1
(u_2,v_1)	0	1	0
(u_2,v_2)	0	1	1
(u_2,v_3)	0	0	1
(u_3,v_1)	1	0	0
(u_3,v_3)	1	0	0
(u_4,v_1)	0	1	0
(u_4,v_2)	0	1	0

Definition 7.3.1.2

Let (H_1,E) and (H_2,E) be two soft relations between (F,E) and (G,E) over U×V. Then

(i) the union of (H_1,E) and (H_2,E) is also soft relation, denoted by

$(H,E)= (H_1,E) \tilde{\cup} (H_2,E)$, such that $H(e) = H_1(e) \cup H_2(e), \forall\ e \in E$ and

(ii) the intersection of (H_1,E) and (H_2,E) is also soft relation, denoted by

$(K,E) = (H_1,E) \tilde{\cap} (H_2,E)$, such that $K(e) = H_1(e) \cap H_2(e)$, $\forall\ e \in E$.

Example 7. 3.1.2

From example **7.3.1.1**, let

(H_1,E)

$U\times V$	e_1	e_2	e_3
(u_1,v_1)	1	0	0
(u_1,v_2)	0	0	1
(u_1,v_3)	1	0	1
(u_2,v_1)	0	1	0
(u_2,v_2)	0	1	1
(u_4,v_1)	0	1	0

(H_2,E)

$U\times V$	e_1	e_2	e_3
(u_1,v_1)	1	0	0
(u_2,v_1)	0	1	1
(u_2,v_3)	0	0	1
(u_4,v_2)	0	1	0

$(H,E)= (H_1,E)\ \tilde{\cup}\ (H_2,E)$

$U\times V$	e_1	e_2	e_3
(u_1,v_1)	1	0	0
(u_1,v_2)	0	0	1
(u_1,v_3)	1	0	1
(u_2,v_1)	0	1	1
(u_2,v_2)	0	1	1
(u_2,v_3)	0	0	1
(u_4,v_1)	0	1	0
(u_4,v_2)	0	1	0

and

$(K,E) = (H_1,E)\ \tilde{\cap}\ (H_2,E)$

$U\times V$	e_1	e_2	e_3
(u_1,v_1)	1	0	0
(u_2,v_1)	0	1	0

Definition 7.3.1.3(Complement of a soft relation)

Suppose $(F,E)^c$ and $(G,E)^c$ are the complements of the soft sets (F,E) and (G,E) over U and V respectively. Then the complement of the soft relation (H,E) over U×V is also a soft relation $(H,E)^c$ over U×V if

$H^c: \rceil E \rightarrow 2^{U\times V}$ is a mapping such that

$H^c(\rceil e) = \{(u_i,v_j) : u_i \in F(\rceil e) \text{ and } v_j \in G(\rceil e), \forall e \in E\}$

Example 7.3.1.3

From example **7.3.1.1**

$(F,E)^c$ (G,E)^c and $(H,E)^c$

U	e_1	e_2	e_3
u_1	0	1	0
u_2	1	0	0
u_3	0	1	1
u_4	1	0	1

V	e_1	e_2	e_3
v_1	0	0	1
v_2	1	0	0
v_3	0	1	0

U×V	e_1	e_2	e_3
(u_1,v_3)	0	1	0
(u_2,v_2)	1	0	0
(u_3,v_1)	0	0	1
(u_3,v_3)	0	1	0
(u_4,v_1)	0	0	1
(u_4,v_2)	1	0	0

Definition 7.3.1.4

Suppose (F,E), (G,E) and (H,E) are three soft sets over U and V and W respectively. Further, (K_1,E) and (K_2,E) are soft relations between the pair of soft sets (F,E) and (G,E) over U×V and the pair of soft sets (G,E) and (H,E) over V×W respectively. Then the composition of (K_1,E) and (K_2,E) is also a soft relation (K,E) over U×W if

$K : E \rightarrow 2^{U\times V}$ such that

$K(e)= \{ (u,w) : \exists\, v \in V \text{ s.t.} (u,v) \in K_1(e) \text{ and } (v,w) \in K_2(e), \forall\, u \in U, w \in W\}$

Definition 7.3.1.5

Let (H,E) be a soft relation between the soft sets (F,E) and (G,E) over $U \times U$, then (H,E) is called,

i) reflexive iff $(u_i,u_i) \in H(e)$, $\forall\ e \in E$ and $\forall\ u_i \in U$.

ii) symmetric iff $(u_i,u_j) \in H(e) \Rightarrow (u_j,u_i) \in H(e)$, $\forall\ e \in E$, $u_i,u_j \in U$

iii) transitive iff (u_i,u_j) and $(u_j,u_k) \in H(e) \Rightarrow (u_i,u_k) \in H(e)$, $\forall\ e \in E$ and $\forall u_i,u_j,u_k \in U$

Definition 7.3.1.6

A soft relation is said to be a soft tolerance relation if it is reflexive and symmetric.

Definition 7.3.1.7

A soft relation is said to be soft equivalence relation if it is reflexive, symmetric and transitive.

Example 7.3.1.4

The soft relation (H,E) between the soft sets (F,E) and (G,E) is a soft equivalence relation where

(F,E)

U	e_1	e_2	e_3
u_1	1	1	1
u_2	1	1	1
u_3	1	1	1
u_4	1	1	1

(G,E)

V	e_1	e_2	e_3
v_1	1	1	1
v_2	1	1	1
v_3	1	1	1

(H,E)

$U\times V$	e_1	e_2	e_3
(u_1,v_1)	1	1	1
(u_1,v_2)	1	1	1
(u_1,v_3)	1	1	1
(u_2,v_1)	1	1	1
(u_2,v_2)	1	1	1
(u_2,v_3)	1	1	1
(u_3,v_1)	1	1	1
(u_3,v_2)	1	1	1
(u_3,v_3)	1	1	1
(u_4,v_1)	1	1	1
(u_4,v_2)	1	1	1
(u_4,v_3)	1	1	1

7.3.2 Fuzzy Soft Relations

Definition 7.3.2.1

Let U and V be two initial universe sets and let E be a set of parameters and (F,E) and (G,E) be two fuzzy soft sets over U and V respectively and $\mathcal{F}(U\times V)$be the set of all fuzzy subsets of $U\times V$, then (H,E) is a fuzzy soft relation between (F,E) and (G,E) over $U\times V$ if

$H:E\rightarrow\mathcal{F}(U\times V)$is a mapping such that ,$\forall e\in E$,$H(e)=\{(u_i,v_j)/\mu_{ij} : \mu_{ij} =\min\{\mu_i ,\mu_j \}\}$,

where $(u_i,\mu_i)\in F(e)$ and $(v_j , \mu_j) \in G(e)$.

Example 7.3.2.1

Suppose $U=\{u_1,u_2, u_3,u_4\}$is a set of four houses and $V=\{v_1,v_2,v_3\}$ is another set of three farm houses. Further, suppose, $E=\{e_1,e_2, e_3\}$is the set of parameters, namely e_1=expensive,e_2=wooden and e_3=prime location such that $F(e_1)=\{u_1/.4, u_2/1.0, u_3/.5, u_4/.2\}$,$F(e_2)=\{u_1/.4,u_2/.6,u_3/.8,u_4/.3\}$and $F(e_3)=\{u_1/.2,u_2/.3,u_3/.7,u_4/.3\}$.The fuzzy soft set (F,E) is a parametrized family $\{F(e_i), i=1,2,3\}$of fuzzy subsets of U, giving a collection of an approximate description of houses and (G,E) is also a parametrized family$\{G(e_i),i=1,2,3\}$of fuzzy subsets of V, where $G(e_1)=\{v_1/.3,v_2/.7,v_3/.5\}$, $G(e_2)=\{v_1/.7, v_2/.4,v_3/.6\}$ and $G(e_3)=\{v_1/.3,v_2/.8,v_3/.9\}$, giving another collection of approximate description of farm houses. Then (H,E) is a parametrized family $\{H(e_i),i=1,2,3\}$of fuzzy subsets of $U\times V$ where $H(e_1)=\{(u_1,v_1)/.3,(u_1,v_2)/.7, (u_1,v_3)/.4, (u_2,v_1)/.3,(u_2,v_2)/.7,(u_2,v_3)/.5,(u_3,v_1)/.3,(u_3,v_2)/.5,(u_3,v_3)/.5,(u_4,v_1)/.2,(u_4,v_2)/.2, (u_4,v_3)/.2\}$, $H(e_2)=\{(u_1,v_1)/.5, (u_1,v_2)/.4, (u_1,v_3)/.5, (u_2,v_1)/.6, (u_2,v_2)/.4, (u_2,v_3)/.6, (u_3,v_1)/.7, (u_3,v_2)/.4, (u_3,v_3)/.6, (u_4,v_1)/.3, (u_4,v_2)/.3, (u_4,v_3)/.3 \}$ and $H(e_3)=\{(u_1,v_1)/.2, (u_1,v_2)/.2,(u_1,v_3)/.2,(u_2,v_1)/.3,(u_2,v_2)/.3,(u_2,v_3)/.3,(u_3,v_1)/.3,(u_3,v_2)/.7,(u_3,v_3)/.7, (u_4,v_1)/.3, (u_4,v_2)/.3, (u_4,v_3)/.3 \}$, and

(H,E) represents a fuzzy soft relation between (F,E) and (G,E).

The tabular representation of fuzzy soft sets (F,E) ,(G,E) and fuzzy soft relation (H,E) are given below.

(F,E)

U	e_1	e_2	e_3
u_1	.4	.5	.2
u_2	1	.6	3
u_3	.5	.8	.7
u_4	.2	.3	.3

(G,E)

V	e_1	e_2	e_3
v_1	.3	.7	.3
v_2	.7	.4	.8
v_3	.5	.6	.9

(H,E)

U×V	e_1	e_2	e_3
(u_1,v_1)	.3	.5	.2
(u_1,v_2)	.7	.4	.2
(u_1,v_3)	.4	.5	.2
(u_2,v_1)	.3	.6	.3
(u_2,v_2)	.7	.4	.3
(u_2,v_3)	.5	.6	.3
(u_3,v_1)	.3	.7	.3
(u_3,v_2)	.5	.4	.7
(u_3,v_3)	.5	.6	.7
(u_4,v_1)	.2	.3	.3
(u_4,v_2)	.2	.3	.3
(u_4,v_3)	.2	.3	.3

Definition 7.3.2.2

Let (H_1,E) and (H_2,E) be two fuzzy soft relations between (F,E) and (G,E) over U×V.

Then

(i) the union of (H_1,E) and (H_2,E) is also fuzzy soft relation denoted by

$(H,E) = (H_1,E) \tilde{\cup} (H_2,E)$ such that $H(e) = H_1(e) \tilde{\cup} H_2(e)$, $\forall\ e \in E$

(where $\tilde{\cup}$ is the operation of fuzzy union of two fuzzy sets) and

(ii) the intersection of (H_1,E) and (H_2,E) is also fuzzy soft relation denoted by

$(K,E) = (H_1,E) \tilde{\cap} (H_2,E)$ such that $K(e) = H_1(e) \tilde{\cap} H_2(e)$, $\forall\ e \in E$, (where $\tilde{\cap}$ is the operation of fuzzy intersection of two fuzzy sets) .

Example 7.3.2.2

Suppose (H_1,E) and (H_2,E) are two fuzzy soft relations between the fuzzy soft sets (F,E) and (G,E) over $U \times V$.

(H_1,E)

$U \times V$	e_1	e_2	e_3
(u_1,v_1)	.5	.4	.2
(u_1,v_2)	.7	0	.3
(u_2,v_2)	.4	1	.8
(u_3,v_2)	.3	.1	.8
(u_4,v_1)	.7	.4	0

and

(H_2,E)

$U \times V$	e_1	e_2	e_3
(u_1,v_1)	.4	.6	.5
(u_1,v_3)	.3	.2	.4
(u_2,v_3)	0	1	.8
(u_3,v_2)	.2	.6	.3
(u_4,v_2)	.6	.2	.1

(H,E)

$U\times V$	e_1	e_2	e_3
(u_1,v_1)	.5	.6	.5
(u_1,v_2)	.7	0	.3
(u_1,v_3)	.3	.2	.4
(u_2,v_3)	0	1	.8
(u_2,v_2)	.4	1	.8
(u_3,v_2)	.3	.6	.8
(u_4,v_1)	.7	.4	0
(u_4,v_2)	.6	.2	.1

and

(K,E)

$U\times V$	e_1	e_2	e_3
(u_1,v_1)	.4	.4	.2
(u_2,v_1)	.2	.6	.3

Definition 7. 3.2.3(Complement of a fuzzy soft relation)

Suppose $(F,E)^c$ and $(G,E)^c$ are the complements of the fuzzy soft sets (F,E) and (G,E) over U and V respectively. Then the complement of the fuzzy soft relation (H,E) between(F,E) and (G,E) over $U\times V$ is also a soft relation $(H,E)^c$ over $U\times V$ if

$H^c: \rceil E \rightarrow \mathcal{F}(U\times V)$ is a mapping such that

$H^c(\rceil e) = \{(u_i,v_j)/\mu_{ij} : \mu_{ij} = \min\{\mu_i , \mu_j \}, (u_i,\mu_i)\in F(\rceil e) \text{ and } (v_j, \mu_j)\in G(\rceil e),\forall e\in E\}$.

Example 7.3.2.3

From example **7.3.2.1,**

$(F,E)^c$

U	e_1	e_2	e_3
u_1	.6	.5	.8
u_2	0	.4	.7
u_3	.5	.2	.3
u_4	.8	.7	.7

and

$(G,E)^c$

V	e_1	e_2	e_3
v_1	.7	.3	.7
v_2	.3	.6	.2
v_3	.5	.4	.1

$(H,E)^c$

U×V	e_1	e_2	e_3
(u_1,v_1)	.6	.3	.7
(u_1,v_2)	.3	.5	.2
(u_1,v_3)	.5	.4	.1
(u_2,v_1)	0	.3	.7
(u_2,v_2)	0	.4	.2
(u_2,v_3)	0	.4	.1
(u_3,v_1)	.5	.2	.3
(u_3,v_2)	.3	.2	.2
(u_3,v_3)	.5	.2	.1
(u_4,v_1)	.7	.3	.7
(u_4,v_2)	.3	.6	.2
(u_4,v_3)	.5	.4	.1

Definition 7. 3.2.4

Let (H,E) be a fuzzy soft relation between (F,E) and (G,E) over U×U, then

(i) (H,E) is called reflexive if $\mu_{H(e)}(h_i,h_i)=1$, $\forall e\in E$ and $\forall h_i\in U$.

(ii) (H,E) is called symmetric if $\mu_{H(e)}(h_i,h_j)=\mu_{H(e)}(h_j,h_i)$, $\forall e\in E$ and $h_i,h_j\in U$.

(iii) (H,E) is called transitive if $\mu_{H(e)}(h_i,h_j)=\lambda_1$, $\mu_{H(e)}(h_j,h_k)=\lambda_2$ implies

$\mu_{H(e)}(h_i,h_k)=\lambda$, where $\lambda \geq \min\{\lambda_1,\lambda_2\}$, $\forall e\in E$ and $h_i,h_j,h_k\in U$.

Definition 7.3.2.5

A fuzzy soft relation is said to be fuzzy tolerance relation if it is reflexive and symmetric.

Definition 7.3.2.6

A fuzzy soft relation is called fuzzy soft equivalence relation if it is reflexive, symmetric and transitive.

Example 7.3.2.4

The fuzzy soft relation (H,E) between the fuzzy soft sets (F,E) and (G,E) is a fuzzy soft equivalence relation where

(F,E)

U	e_1	e_2	e_3
u_1	1	1	1
u_2	1	1	1
u_3	1	1	1
u_4	1	1	1

(G,E)

V	E_1	e_2	e_3
v_1	1	1	1
v_2	1	1	1
v_3	1	1	1

(H,E)

U×V	e_1	e_2	e_3
(u_1,v_1)	1	1	1
(u_1,v_2)	1	1	1
(u_1,v_3)	1	1	1
(u_2,v_1)	1	1	1
(u_2,v_2)	1	1	1
(u_2,v_3)	1	1	1
(u_3,v_1)	1	1	1
(u_3,v_2)	1	1	1
(u_3,v_3)	1	1	1
(u_4,v_1)	1	1	1
(u_4,v_2)	1	1	1
(u_4,v_3)	1	1	1

7.3.3 Applications of soft relations and fuzzy soft relations

Molodtsov **[60]** introduced soft set theory and showed its applications in several different directions like game theory, operation research, smoothness of functions, Riemann integration, Perron integration, probability, theory of measurement etc. Later, Maji et al. **[55,56,57]** presented applications of soft set and fuzzy soft set in decision making problems. In this section, we present applications of soft relation in both crisp and fuzzy setting. Suppose $U=\{m_1,m_2,m_3,m_4\}$ is the set of male and $V=\{f_1,f_2,f_3\}$ is the set of female tennis players having ranking in the mixed double version of the game seeking sponsorship. Also suppose the set

$E=\{e_1$=height,e_2=adaptability to climate change,e_3= stamina for playing long game, e_4= physical fitness and e_5=high rank holder$\}$represent the attributes required to judge the merit of a tennis player. The set of attributes is taken as the set of parameters in the language of soft set.

Suppose one Mr. S is interested to sponsor a mixed pair of tennis players on the basis of his choice of parameters e_1=height,e_2=adaptability to climate change and e_3=stamina for long game forming a subset A of E.

(i) In crisp soft setting

We consider the soft sets (F,A) over U and another soft set (G,A) over V given by

$(F,A)=\{F(e_1)=\{m_1,m_2\},F(e_2)=\{m_1,m_2,m_4\},F(e_3)=\{m_1,m_3\}\}$and
$(G,A)=\{G(e_1)=\{f_1,f_3\},G(e_2)=\{f_1,f_2\},G(e_3)=\{f_1,f_2,f_3\}\}$respectively. The tabular representation of these soft sets are

(F,A)

U	e_1	e_2	e_3
m_1	1	1	1
m_2	1	1	0
m_3	0	0	1
m_4	0	1	0

(G,A)

V	e_1	e_2	e_3
f_1	1	1	1
f_2	0	1	1
f_3	1	0	1

Definition 7.3.3.1

Choice value of a pair of players $(m_i,f_j) \in U\times V$ is r_{ij}, given by

$$r_{ij} = \sum_{e\in A}(m_i,f_j),$$

where $(m_i,f_j)$are the entries in the table of soft relation. The suffixes i,j of r_{ij} represent the suffixes of 'm' and 'f' respectively of the ordered pair (m_i,f_j).

Algorithm for selection of the mixed double players.

1. input the soft sets (F,A) and (G,A) and the soft relation (H,A) w.r.t. the choice of the parameters of Mr. S.

2. input the choice value $r_{ij} = \sum_{e \in A} (m_i, f_j)$ of each pair (m_i,f_j) of (H,A).

3.find m =max r_{ij}

If m has more than one value, then any one of them can be chosen by Mr. S by using his option.

From the above two tables of soft set, we formulate the tabular representation of the soft relations (H,A) between the soft sets (F,A) and (G,A) together with the respective choice value.

U×V	e_1	e_2	e_3	Total choice value
(m_1,f_1)	1	1	1	$r_{11}=3$
(m_1,f_2)	0	1	1	$r_{12}=2$
(m_1,f_3)	1	0	1	$r_{13}=2$
(m_2,f_1)	1	1	0	$r_{21}=2$
(m_2,f_2)	0	1	0	$r_{22}=1$
(m_2,f_3)	1	0	0	$r_{23}=1$
(m_3,f_1)	0	0	1	$r_{31}=1$
(m_3,f_2)	0	0	1	$r_{32}=1$
(m_3,f_3)	0	0	1	$r_{33}=1$
(m_4,f_1)	0	1	0	$r_{41}=1$
(m_4,f_2)	0	1	0	$r_{42}=1$
(m_4,f_3)	0	0	0	$r_{43}=0$

Here max $r_{ij}=r_{11}$.Thus the choice for Mr. S is the pair (m_1,f_1).

(ii) In fuzzy soft setting

We consider the fuzzy soft sets (F,A) over U and another fuzzy soft set (G,A) over V given by (F,A)= { $F(e_1)=\{m_1/1.0, m_2/.4, m_3/.3, m_4/.7\}$, $F(e_2)=\{m_1/.5, m_2/.6, m_3/.5, m_4/1.0\}$, $F(e_3)=\{m_1/.8, m_2/.9, m_3/1.0, m_4/.6\}$} and (G,A)= {$G(e_1)=\{s_1/.9, s_2/.5, s_3/.6\}$, $G(e_2)=\{s_1/.6, s_2/.8, s_3/.1.0\}$, $G(e_3)=\{s_1/.7, s_2/.6, s_3/.5\}$}.

The tabular representation of these fuzzy soft sets (F,A) and (G,A) are respectively

(F,A)

U	e_1	e_2	e_3
m_1	1	.5	.8
m_2	.4	.6	.9
m_3	.3	.5	1.0
m_4	.7	1.0	.6

(G,A)

V	e_1	e_2	e_3
f_1	.9	.6	.7
f_2	.5	.8	.6
f_3	.6	1.0	.5

Definition 7. 3.3.2 Comparison table of a fuzzy soft relation (H,A) :

The fuzzy soft relation is a square table having equal number of rows and columns and rows and columns are labelled by the pairs (m_i,f_j) of the cartesian product of both universes and the entries are r_{ij} i,j=1,2,...,n given by r_{ij}=the number of parameters for which the membership value of (m_i,f_j) exceeds or equal to the membership value of (m_i,f_j),i.e. $r_{ij}=p$, $\forall$ where p is the number of parameters in E.

Definition 7.3.3.3 Row sum and column sum of a pair (m_i,f_j):

Row sum of a pair (m_i,f_j) is denoted by s_{ij} and is calculated by the formula

$s_{ij}=\sum_{e\in A} r_{ij}$ and the column sum of a pair (m_i,f_j) is denoted by p_{ij} and is calculated by

the formula $p_{ij}=\sum_{e\in A} r_{ij}$

Definition 7.3.3.4

Score of pair (m_i,f_j) is S_{ij} and is calculated by the formula $S_{ij}= s_{ij}- p_{ij}$

Finally algorithm similar to that given in case of crisp soft setting can also be written in case of fuzzy soft setting .

From the above two tables of fuzzy soft sets in the application of fuzzy soft relations, the tabular representation of the fuzzy soft relations (H,A) between the soft sets (F,A) and (G,A) is formulated and then the comparison table is prepared together with row and column sums score of each pair (m_i,f_j).

(H,A)

U×V	e_1	e_2	e_3
(m_1,f_1)	.9	.5	.7
(m_1,f_2)	.5	.5	.6
(m_1,f_3)	.6	.5	.5
(m_2,f_1)	.4	.6	.7
(m_2,f_2)	.4	.6	.6
(m_2,f_3)	.4	.6	.5
(m_3,f_1)	.3	.5	.7
(m_3,f_2)	.3	.5	.6
(m_3,f_3)	.3	.5	.5
(m_4,f_1)	.7	.6	.6
(m_4,f_2)	.5	.8	.6
(m_4,f_3)	.6	1.0	.5

Comparison Table

	(m_1,f_1)	(m_1,f_2)	(m_1,f_3)	(m_2,f_1)	(m_2,f_2)	(m_2,f_3)	(m_3,f_1)	(m_3,f_2)	(m_3,f_3)	(m_4,f_1)	(m_4,f_2)	(m_4,f_3)	Row sum	Score
(m_1,f_1)	3	3	3	2	2	2	3	3	3	2	2	2	30	14
(m_1,f_2)	1	3	2	1	2	2	2	3	3	1	2	1	23	-3
(m_1,f_3)	1	2	3	1	1	2	2	2	3	0	1	2	20	-8
$(m_2,f_1$	2	2	2	3	3	3	3	3	3	2	1	1	28	10
(m_2,f_2)	1	2	2	2	3	3	2	3	3	2	1	1	25	2
(m_2,f_3)	1	1	2	2	2	3	2	3	3	1	0	1	20	-7
(m_3,f_1)	2	2	2	1	1	1	3	3	3	1	1	1	21	-6
(m_3,f_2)	1	2	2	0	1	1	2	3	3	1	1	1	18	-13
(m_3,f_3)	1	1	2	0	0	1	2	2	3	0	0	1	13	-23
(m_4,f_1)	1	3	3	2	3	3	2	3	3	3	2	2	30	14
(m_4,f_2)	1	3	2	2	3	3	2	2	3	2	3	1	27	11
(m_4,f_3)	1	2	3	2	2	3	2	2	3	1	2	3	26	9
Column sum	16	26	28	18	23	27	27	31	36	16	16	17		

Clearly the maximum score is 14 secured by the pairs (m_1,f_1) and (m_4,f_1). Therefore Mr.S can sponsor any pair of these two.

7.3.4 Conclusion

The concept of the soft relations and fuzzy soft relations are introduced and applied in decision making problem with separate examples.

7.4 An Application of Fuzzy Soft sets in Medical Diagnosis

[1]The concept of fuzzy soft set is applied here to extend Sanchez's approach for medical diagnosis with a hypothetical case study.

7.4.1 Application of fuzzy soft sets in medical diagnosis problem

Here we present an application of fuzzy soft set theory in medical diagnosis following Sanchez's approach[87].For this, suppose S is a set of symptoms, D is a set of diseases and P is a set of patients. Then a fuzzy soft set (F,D) is constructed over S, where F is a mapping $F: D \rightarrow I^S$ .This fuzzy soft set gives a relation matrix, say R_1,called symptom- disease matrix. Its complement $(F,D)^c$ gives another relation matrix, say R_2, called non symptom-disease matrix. Analogous to Sanchez's notion of 'Medical knowledge' we refer to the matrices R_1 and R_2 as 'Soft Medical Knowledge'.

Again we construct another soft set (F_1,S) over P, where F_1 is a mapping given by $F_1: S \rightarrow I^P$ .This fuzzy soft set gives a relation matrix Q called patient- symptom matrix. Then we obtain two new relation matrices $T_1 = Q_o R_1$ and $T_2 = Q_o R_2$, called

[1]Contents of this chapter was presented in the Conference on Fuzzy Set Theory and its Mathematical Aspects and Applications., December 26-28,2002,Deptt. of Mathematics,BHU,Varanasi,India and have appeared as a paper entitled "Application of Fuzzy Soft Sets in Medical Diagonosis"in the Proceedings, pp. 172-175,Allied Publishers Pvt. Limited, New Delhi.

symptom -patient matrix and non symptom–patient matrix respectively, in which the membership values are given by

$$\mu_{T_1}(p_i,d_j)=\vee[\mu_Q(p_i,e_k)\wedge\mu_{R_1}(e_k,d_j)] \text{ and}$$

$$\mu_{T_2}(p_i,\ d_j)=\vee[\mu_Q(p_i,e_k)\wedge\mu_{R_2}(e_k,\ d_j)] \text{ ,where, } \vee=\max \text{ and } \wedge=\min.$$

Now if $\max\{\mu_{T_1}(p_i,d_j)-\mu_{T_2}(p_i,\ d_j)\}$ occurs for exactly (p_i, d_k) only then we conclude that the acceptable diagnostic hypothesis for patient p_i is the disease d_k. In case there is a tie the process has to be repeated for patient p_i by reassessing the symptoms.

7.4.2 Case study

Suppose there are three patients p_1,p_2, and p_3 in a hospital with symptoms temperature, headache, cough and stomach problem. Let the possible diseases relating to the above symptoms be viral fever and malaria. We consider the set $S=\{e_1,e_2,e_3,e_4\}$ as universal set where e_1,e_2,e_3 and e_4 represent symptoms temperature, headache, cough and stomach problem respectively and the set $D=\{d_1,d_2\}$ where d_1 and d_2 represent the parameters viral fever and malaria respectively. Suppose that $F(d_1)=\{e_1/.9,e_2/.4,e_3/.5,e_4/.2\}$,$F(d_2)=\{e_1/.6,e_2/.5,e_3/.2,e_4/.8\}$. The fuzzy soft set (F,D) is a parametrized family { $F(d_1)$, $F(d_2)$}of all fuzzy sets over the set S and are determined from expert medical documentation. Thus the fuzzy soft set (F, D) gives an approximate description of the soft medical knowledge of the two diseases and their symptoms. This fuzzy soft set (F,D) and its complement $(F,D)^c$ are represented by two relation matrices $R_1$and R_2 ,called symptom-disease matrix and non symptom disease matrix respectively, given by

$$R_1=\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} .9 & .6 \\ .4 & .5 \\ .5 & .2 \\ .2 & .8 \end{bmatrix} \end{array} \quad \text{and} \quad R_2=\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} .1 & .4 \\ .6 & .5 \\ .5 & .8 \\ .8 & .2 \end{bmatrix} \end{array}$$

Again we take $P=\{p_1,p_2,p_3\}$ as the universal set where p_1,p_2 and p_3 represent the patients under consideration and $S=\{e_1,e_2,e_3,e_4\}$as the set of parameters detailed above. Suppose that, $F_1(e_1)=\{p_1/.8,p_2/.7,p_3/.4\}$,$F_1(e_2)=\{p_1/.4,p_2/.3,p_3/.5\}$ $F_1(e_3)=\{p_1/.6,p_2/.4,p_3/.4\}$and $F_1(e_4)=\{p_1/.3,p_2/.6,p_3/.7\}$. The fuzzy soft set is another parametrized family of all fuzzy sets over P and gives a collection of approximate description of the symptoms in the hospital. This fuzzy soft set (F_1,S) represents a relation matrix Q called patient –symptom matrix given by

$$Q=\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} .8 & .4 & .6 & .3 \\ .7 & .3 & .4 & .6 \\ .4 & .5 & .4 & .7 \end{bmatrix} \end{array}$$

Then combining the relation matricesR_1 and R_2 separately with Q we get two matricesT_1andT_2 called patient-disease and patient–non disease matrices respectively given by

$$T_1=Q{\circ}R_1=\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} .8 & .6 \\ .7 & .6 \\ .4 & .7 \end{bmatrix} \end{array}, \quad T_2=Q{\circ}R_2=\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} .5 & .6 \\ .6 & .4 \\ .7 & .5 \end{bmatrix} \end{array} \quad \text{and } T_1-T_2=\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} .3 & .0 \\ .1 & .2 \\ -.3 & .2 \end{bmatrix} \end{array}$$

Now, it clear from matrix T_1-T_2 that the patient p_1 is suffering from disease d_1 and patient p_2 and p_3 are both suffering from disease d_2.

7.4.3 Conclusion

Here we applied the notion of fuzzy soft sets in Sanchez's method of medical diagnosis. A case study has been taken to exhibit the simplicity of the technique.

7.5 An Application of Intuitionistic Fuzzy Soft Sets in Medical Diagnosis

[1]In this part we extend Sanchez's approach for medical diagnosis using intuitionistic fuzzy soft sets and exhibit the technique with a hypothetical case study.

7.5.1 Introduction

Out of several generalizations of fuzzy set theory for various objectives, the notion introduced by Atanassov[5,6] in defining intuitionistic fuzzy sets (IFSs) is interesting and useful. But Molodtsov [60] has shown that this topic suffers from some inherent difficulties due to inadequacy of parametriation tools and introduced a concept called 'Soft Set Theory' having parametrization tools for successfully dealing with various types of uncertainties. Maji et al.[56] have developed a theoretical study of the 'Intuitionistic Fuzzy Soft Set'(IFSS).The combination of Intuitionistic Fuzzy Set and Soft Set will be more useful in the field of applications wherever uncertainty appear. De et al.[27] have studied Sanchez's[87] method of medical diagnosis using intuitionistic fuzzy set. Our proposed method is an attempt to improve the results in [27]using the complement concept of IFSS to formulate a pair of medical knowledge, hereafter called soft medical knowledges.

In section 7.5.2, we present a new method for medical diagnosis through IFSS and section 7.5.3 contains an algorithm of the method. Then in the next section a hypothetical case study is discussed using the proposed method .

[1]Parts of this chapter has appeared as a paper entitled 'An Application of Intuitionistic Fuzzy Soft Sets in Medical Diagnosis' in ' Bio Science Research Bulletin', Vol.2 (2003) 121-127.

7.5.2. Application of intuitionistic fuzzy soft set in medical diagnosis problem

Suppose S is a set of symptoms related to sickness, D is a set of diseases and P is a set of patients. Construct an intuitionistic fuzzy soft set (F,D) over S, where F is a mapping $F:D \rightarrow \mathbb{P}(S)$. A relation matrix say, R_1 is constructed from the intuitionistic fuzzy soft set (F,D) and name it symptom-disease matrix. Similarly its complement $(F,D)^c$ gives another relation matrix, say R_2, called non symptom-disease matrix. Analogous to Sanchez's notion of 'Medical knowledge' we refer to each of the matrices R_1 and R_2 as 'Intuitionistic Soft Medical Knowledge'. Again we construct another intuitionistic soft set (F_1,S) over P, where F_1 is a mapping given by $F_1:S \rightarrow \mathbb{P}(P)$.This intuitionistic fuzzy soft set gives another relation matrix Q called patient-symptom matrix. Then we obtain two new relation matrices $T_1=Q \circ R_1$ and $T_2=Q \circ R_2$, called symptom-patient matrix and non symptom–patient matrix respectively, in which the membership values are given by

$$\mu_{T_1}(p_i,d_k) = \vee\{\mu_Q(p_i,e_j) \wedge \mu_{R_1}(e_j,d_k)\}$$

$$\mu_{T_2}(p_i,d_k) = \vee\{\mu_Q(p_i,e_j) \wedge \mu_{R_2}(e_j,d_k)\}$$

and the non-membership function given by

$$\nu_{T_1}(P_i, d_k) = \wedge\{\nu_Q(P_i,e_j) \vee \nu_{R_1}(e_j,d_k)\},$$

$$\nu_{T_2}(p_i,d_k) = \wedge\{\nu_Q(p_i,e_j) \vee \nu_{R_2}(e_j,d_k)\},$$

where $\vee = \max$ and $\wedge = \min$.

If π_{T_1}=(1- $\mu_{T_1} - \nu_{T_1}$) and π_{T_2}=(1- $\mu_{T_2} - \nu_{T_2}$) are the hesitation parts with respect to. T_1 and T_2 respectively then we calculate $S_{T_1} = \mu_{T_1} - \nu_{T_1}\pi_{T_1}$ and $S_{T_2} = \mu_{T_2} - \nu_{T_2}\pi_{T_2}$, which we call as diagnosis score for and against the disease respectively.

Now, if $\max_j \{ s_{T_1}(p_l, d_j) - s_{T_2}(p_l, d_j)\}$ occurs for exactly (p_l, d_k) only ,then we conclude that the acceptable diagnostic hypothesis for patient p_l is the disease d_k.

In case there is a tie, the process has to be repeated for patient p_l by reassessing the symptoms.

7.5.3 Algorithm

1.input the IFSSs (F,D) and $(F,D)^c$ over the sets S of symptoms, where D is the set of diseases. Also write the soft medical knowledge R_1 and R_2 representing the relation matrices of the IFSS (F,D) and $(F,D)^c$ respectively.

2. input the IFSS (F_1,S) over the set P of patients and write its relation matrix Q.

3. compute the relation matrices $T_1=Q \circ R_1$ and $T_2=Q \circ R_2$.

4. compute the diagnosis scores S_{T_1} and S_{T_2} .

5. find $S_k = \max_j \{ s_{T_1}(p_l, d_j) - s_{T_2}(p_l, d_j)\}$. Then conclude that the patient p_l is suffering from the disease d_k

6. if S_k has more than one value then go to step one and repeat the process by reassessing the symptoms for the patients.

7.5.4 Case Study

Suppose there are three patients p_1,p_2 and p_3 in a hospital with symptoms temperature, headache, cough and stomach problem. Let the possible diseases relating to the above symptoms be viral fever and malaria .We consider the set S={ e_1,e_2,e_3,e_4} as universal set, where e_1,e_2,e_3 and e_4 represent the symptoms temperature, headache,cough and stomach problem respectively and the set D={d_1,d_2}where d_1 and d_2 represent the parameters viral fever and malaria respectively. Suppose that F(d_1)={e_1/(.9,.1), e_2/(.4,.5),e_3/(.5,.3),e_4/(.2,.7)},F(d_2)={e_1/(.6,.2),e_2/(.5,.3),e_3/(.2,.6),e_4/(.8,.1)}. The intuitionistic fuzzy soft set (F,D) is a parametrized family {F(d_1), F(d_2)} of all intuitionistic fuzzy sets over the set S and are determined from expert medical documentation. Thus the fuzzy soft set (F,D) gives an approximate description of the intuitionistic soft medical knowledge of the two diseases and their symptoms. This intuitionistic fuzzy soft set (F,D) and its complement$(F,D)^c$ are represented by two relation matrices R_1 and R_2, called symptom-disease matrix and non symptom-disease matrix respectively, given by

$$R_1 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (.9,.1) & (.6,.2) \\ (.4,.5) & (.5,.3) \\ (.5,.3) & (.2,.6) \\ (.2,.7) & (.8,.1) \end{bmatrix} \end{array} \quad \text{and} \quad R_2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (.1,.9) & (.2,.6) \\ (.5,.4) & (.3,.5) \\ (.3,.5) & (.6,.2) \\ (.7,.2) & (.1,.8) \end{bmatrix} \end{array}.$$

Again ,we take $P=\{ p_1, p_2, p_3\}$ as the universal set where p_1,p_2 and p_3 represent patients respectively and $S= \{ e_1, e_2, e_3, e_4\}$ as the set of parameters. Suppose that, $F_1 (e_1)= \{p_1/(.8,.2),p_2/(.7,.1),p_3/(.4,.5)\},F_1 (e_2)= \{p_1/(.4,.5),p_2/(.3,.6) p_3/(.5,.5) \}$, $F_1 (e_3)= \{p_1/(.6,.3),p_2/(.4,.5), p_3/(4,.6)\}$and $F_1 (e_4)= \{p_1/(3,.4), p_2/(.6,.3), p_3/(.7,.2)\}$. The intuitionistic fuzzy soft set (F_1,S) is another parametrized family of all intuitionistic fuzzy sets and gives a collection of approximate description of the patient-symptoms in the hospital. This intuitionistic fuzzy soft sets (F_1,S) represents a relation matrix Q called patient–symptom matrix given by

$$Q= \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} (.8,.2) & (.4,.5) & (.6,.3) & (..3,.4) \\ (.7,.1) & (.3,.6) & (.4,.5) & (.6,.3) \\ (..4,.5) & (.5,.5) & (.4,.6) & (.7,.2) \end{bmatrix} \end{array}$$

Then combining the relation matrices R_1 and R_2 separately with Q we get two matrices T_1 and T_2 called patient-disease and patient-non disease matrices respectively, given by

$$T_1=Q\circ R_1= \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (.8,.2) & (.6,.2) \\ (.7,.1) & (.6,.2) \\ (.4,.5) & (.7,.2) \end{bmatrix} \end{array} , T_2=Q\circ R_2= \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (.4,.4) & (.6,.3) \\ (.6,.3) & (.4,.5) \\ (.7,.2) & (.4,.5) \end{bmatrix} \end{array}$$

Now we calculate

$S_{T_1} - S_{T_2}$	d_1	d_2
p_1	.48	−.01
p_2	.11	.21
p_3	−.33	.33

Now, it is clear that the patient p_1 is suffering from the disease d_1 and patients p_2 and p_3 are both suffering from disease d_2.

7.5. 5 Conclusion

We have applied the notion of intuitionistic fuzzy soft sets in Sanchez's method of medical diagnosis. A case study has been taken to exhibit the simplicity of the technique.

Chapter 8

Application of fuzzy logic in modeling river catchment

In this chapter, a fuzzy rule based methodology is developed for estimating monthly discharge using the Jiadhal river basin in the upper Assam.

8.1 Introduction

The state of Assam in the North East India has a chronic history of devastating flood and the situation has not changed much over the years.Despite the importance of rain water discharge there is no universally accepted method for the purpose and hence reasonable rain water discharge estimates are critical for developing accurate models of river basins.These models in spite of the inaccuracy are often used for flood forecasting and reducing the damages in a river basin. The fuzzy rule based approach presented here simplifies the model input by using fewer and easily quantifiable parameters like rainfall and past river discharge. Moreover, fuzzy set theory developed by Zadeh[**100**] are more suitable for handling the implicit vagueness in the data.

8.2 Study site

The Jiadhal river is a north bank tributary of river Brahmaputra and lies between latitudes $27°08'$N and $27°45'$N and longitudes $94°15'$E and $94°38'$E The Jiadhal sub-basin falls in the west Siang district of Arunachal Pradesh and Dhemaji district of Assam. It is bounded by the Subansiri sub basin on its west and north and by Maridhal sub basin on its east. The southern side of the subbasin is bounded by the Kherkatia Suti, a channel of Brahmaputra. Total catchment area of the sub basin is about 1346 sq. km. out of which 306 sq. km. lies in the hills of Arunachal Pradesh and 1040 sq. km. lies in the plains of Assam. The hill catchmant comprises of nearly 23 % of the overall catchment of the sub basin. The geographic location of the Jiadhal river basin and data collection centers are indicated in Figure 1.

The Jiadhal river is primarily fed by groundwater covering mostly hilly regions. The river has minimal flow during off season but it creates havoc during the monsoon. This necessitates flood forecast during the monsoon to save life and property in the lower part of the basin. The data used in this study are obtained from the branch at Lakhimpur of the Brahamaputra Board.

8.3 Fuzzy Methodology

All quantitative rules pertaining to physical science are normally described by mathematical functions which, for every element in the domain, assign a unique

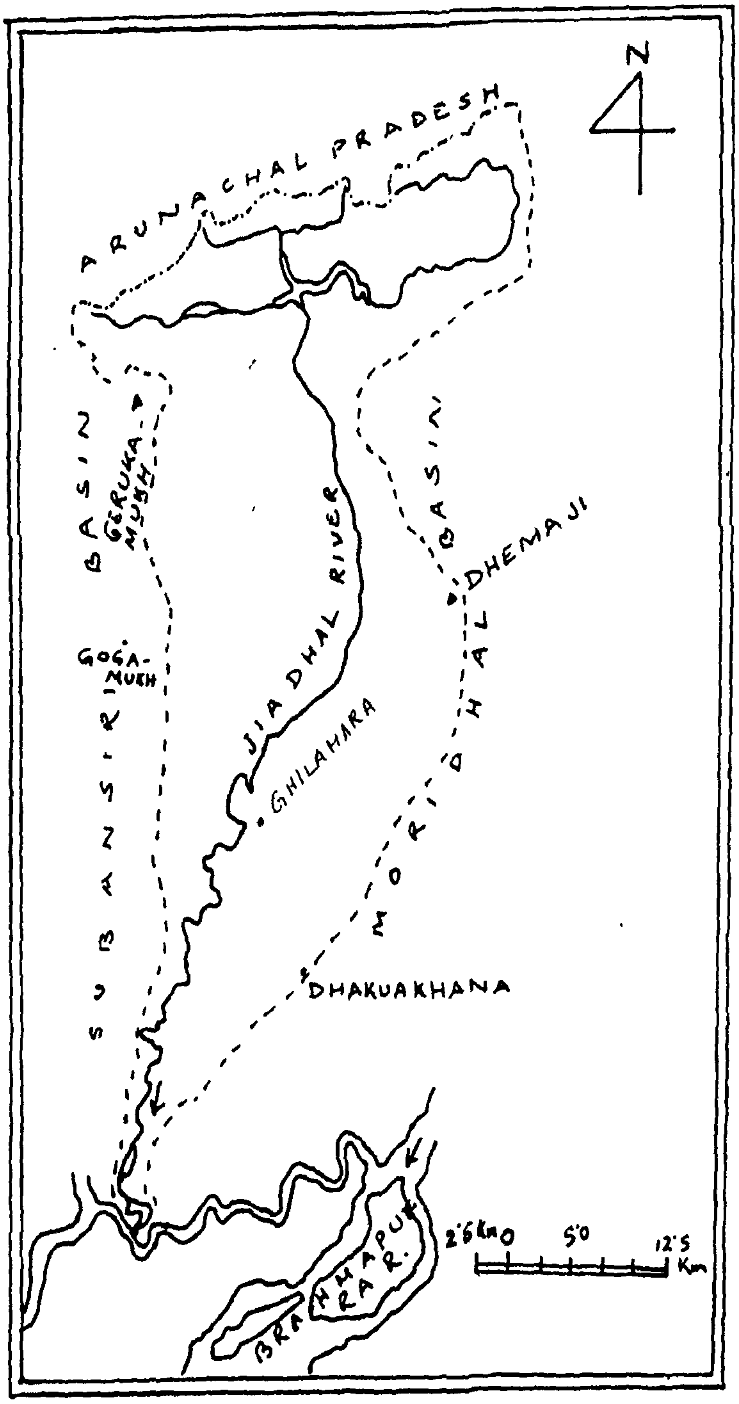


Figure1. Jiadhal River Basin study site

output value. There are also certain classes of rules applied to linguistic variables, which do not have unique numerical values. For example, suppose the quantitative rule for the linguistic variable 'low temperature' refers to the temperature 'around 5^0 Celsius'. Here the term 'low temperature' cannot be a definite numerical value. It can have value within an arbitrarily chosen range and all temperature values within the defined range may not be considered equally low temperature. A fuzzy logic based modeling approach enables one to establish a one to one relationship between 'low temperature' and 'around 5^0 Celsius' in a way that is quite different from a conventional functional form .

Fuzzy logic modeling is based on the theory of fuzzy sets in which, unlike an ordinary binary set, the boundary is not clearly defined. A fuzzy set is a generalization of ordinary (crisp or classical) set in the sense that the former includes partial memberships along with full and no membership. This concept of partial membership is responsible for the unclear boundary of a fuzzy set. Again the unclear boundary or the transitional region of a fuzzy set entertains gradual transition from full membership to no membership and allows modeling of concepts containing linguistic variables. In the above example, the temperature of 12^0 Celsius may have zero membership whereas the temperature of 8^0Celsius may have partial membership, say 0.2 and temperature of 5^0Celsius may have membership1.(Figure2).

Figure 2. Example of membership function

Thus each element in a fuzzy set is assigned a membership value which can be between 0 and 1 inclusively whose membership value 0 represents no membership and membership value 1 represents full membership and the values between 0 and 1 represents partial membership. The function μ that assigns a value to each member of a fuzzy A is referred to as the membership function associated with the fuzzy set and is denoted by μ_A.

Fuzzy numbers are special cases of fuzzy sets and are defined by having a non increasing part, non decreasing part (quasi convexity assumption) and at least one value such that its membership function is 1(normality assumption). Of the different fuzzy numbers representation, triangular fuzzy numbers are often used to define membership functions for different classes. The base of the triangular fuzzy numbers defines the range over which full or partial membership exists and is known as the support of the number. A fuzzy number can be expressed as $(a_1, a_2, a_3)_T$ such that $a_1 \leq a_2 \leq a_3$.

A fuzzy rule based model is based on an 'if ....then' principle where 'if' corresponds to a vector of input variables and 'then' corresponds to consequences. Fuzzy rules consist of a vector of arguments in the form of fuzzy sets $A_{i,k}$ with membership functions $\mu_{A_{i,k}}$, and a consequence in the form of a fuzzy set B_i, where i signifies rule number and k the input variable index .Fuzzy rules describe a function with domain as the Cartesian product of the fuzzy classes defined on each of the input variables and whose range is the fuzzy set defined on the output variable. The total possible number of rules is the product of the number of classes in each variable. Again training data consists of representative example of input vectors and their corresponding consequences. A rule based model serves as a good representation of the physical situation provided the rule outcomes reflect actual outcomes and it is achieved through 'degree of fulfillment calculation' and 'adjustment of membership functions to the training data'. There are a number of methods for constructing fuzzy rules utilizing training data. In the present discharge model, the three components of input vector are total monthly rainfall, mean monthly temperature and the pervious month's discharge and the corresponding consequence is total monthly discharge. The input vector consisting of the element $a_k(s)$, obtained at time s, is matched with the corresponding consequence denoted by b(s) for all months in the training set. Thus the "training set" T derived from the data is usually expressed as $T=\{ a_1(s), a_2(s), a_3(s), b(s):s=1,2,....,S\}$.

Model calibration consists of deriving fuzzy rules from an algorithm that uses the membership function of the input premises in conjunction with the training data.

Algorithm

The algorithm consists of the following steps:

I. Classify each of the inputs(for example, mean monthly temperature) into classes (for example, low, medium etc.) with membership functions. Then the rule premises arise through combination of the input classes, one from each type of input.

II. Calculate the degree of fulfillment (DOF), denoted by v_i (s), for each input vector corresponding to the training set T and each rule i following the classification in step I above. The DOF is defined as the product of the values of the membership function for the inputs as

$$v_i(A_1, A_2, A_3) = \mu_{i1}(a_1) \times \mu_{i2}(a_2) \times \mu_{i3}(a_3)$$

III. Select a number $\varepsilon > 0$ such that only computed DOF's with value at least equal to ε are considered in the construction of input fuzzy numbers and the corresponding response for each rule.

IV. Both the input premises $A_{i,k}$ and the corresponding response B_i are assumed to be triangular fuzzy numbers for all rules i. The triangular fuzzy number response for rule i is denoted by $(\beta_i-, \beta_i, \beta_i+)_T$, where β_i- is the minimal answer with a DOF at least ε, β_i is the mean answer with a DOF at least ε and β_i+ is the maximal answer with a DOF at least ε. The triangular fuzzy numbers $(\alpha_{i,k}-, \alpha_{i,k}, \alpha_{i,k}+)_T$ are similarly determined for the k inputs for rule i.

Defuzzification

For any input vector, all rules are checked to see if they fulfill some minimum DOF. For rules that do, the process of defuzzification combines the fuzzy consequences/responses of all such rules to produce a 'crisp' or a single numerical output value. Of various methods of defuzzification, a common one is the weighted fuzzy mean method which combines the weighted sum of the DOFs, v_i, with their corresponding fuzzy mean responses, $M(B_i)$, to produce the corresponding crisp value $M(B)$. For a fuzzy response in case of the i-th rule, $B_i = (\beta_i-, \beta_i, \beta_i+)_T = (\beta_1, \beta_2, \beta_3)_T$, where β_i- = the minimal answer satisfying the required DOF , β_i = the mean answer satisfying the required DOF and β_i+ = the maximal answer satisfying the required DOF, the formulae of fuzzy mean response and the weighted fuzzy mean are $M(B_i) = (\beta_1+\beta_2+\beta_3)/3$ ...(1) and $M(B) = \sum_{i=1}^{I} v_i \left(M(B_i) \Big/ \sum_{i=1}^{I} v_i \right)$ ....(2) respectively.

8.4 Fuzzy Rule- Based Discharge Methodology

In this study, all fuzzy rule based discharge models use three input premises- total monthly rainfall (R), mean monthly temperature (T) and the previous months discharge(PMD) to derive the single output of total monthly discharge. The mean monthly temperature is used to measure of potential evaporative and transpiration losses. The total previous month's discharge is used as measure of the potential runoff during precipitation events. The total monthly rainfall together with the above two inputs decides the total discharge during the month.

The characteristics of fuzzy rule based models are

(i) the number of rules used
(ii) the manner in which the support for the fuzzy numbers are obtained
(iii) the DOF for rule activation
(iv) computation of the mean fuzzy responses for each rule.

The number of rules in a fuzzy rule based model is a function of both the number of input premises and the number of classes of each input variable. Here each of the three input variables has been classified into three classes, say low, medium and high giving $27(=3^3)$ rules. The number of classes of each variable may also be increased to four or five classes giving a total of 64 or 125 rules respectively. Such increase in the number of classes will enhance the sensitivity of the model performances.

8.5 Example of model implementation:

As an example, the details of the translation of the linguistic rule for the month of August is presented here for 27 rule case. The membership functions for the input premises are shown in the Figures 3, 4 and 5.

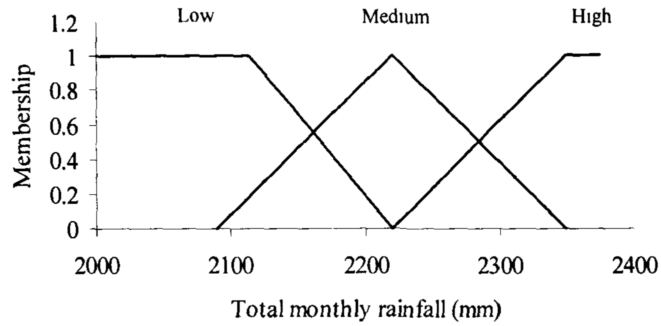


Figure 3. Rainfall membership functions for 27 rule case

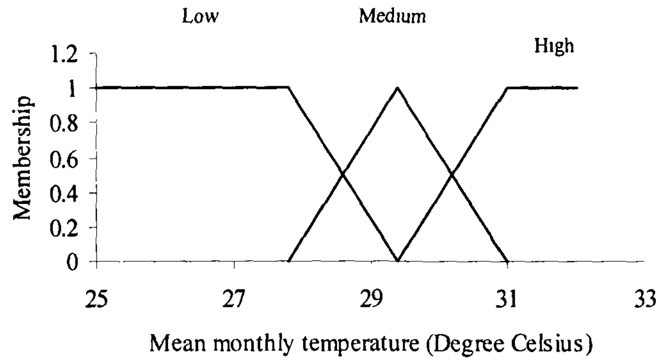


Figure 4.Temperature membership functions for 27 rule case

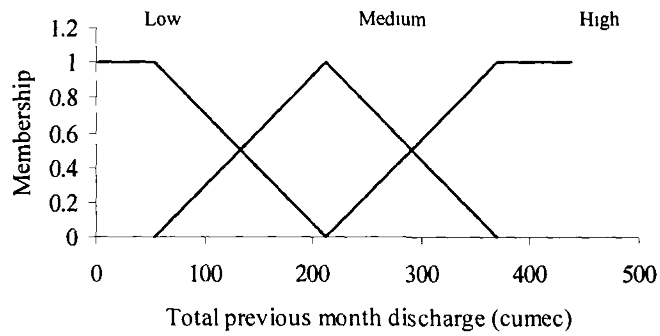


Figure 5. Previous month discharge for 27 rule case

The membership function values are computed for each of the three input variables for the same 'August' month from 1976 to 1998. During training only 4 rules were identified that fulfill the DOF(ε) = 0.3

Table 1.Fuzzy classes of total monthly rainfall

Amount of rainfall	Fuzzy number representation (mm)
Low	$(-\infty, 2090, 2220)_T$
Medium	$(2090, 2220, 2350)_T$
High	$(2220, 2350, \infty+)$

Table 2.Fuzzy classes of mean monthly temperature

Class of temperature	Fuzzy number representation (^0C)
Low	$(-\infty, 27.8, 29.4)_T$
Medium	$(27.8, 29.4, 31)_T$
High	$(29.4, 31, \infty+)_T$

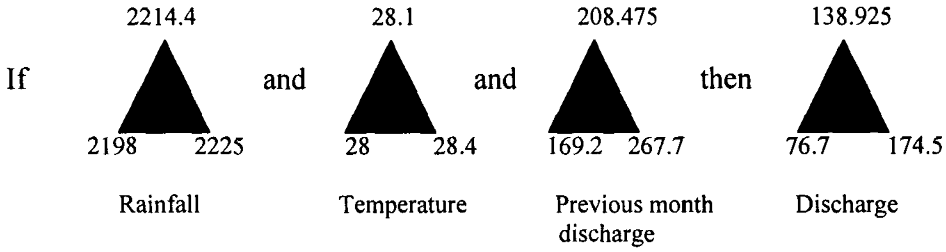
Table 3.Fuzzy classes of previous month discharge

Previous month discharge	Fuzzy number representation (Cumec)
Low	$(-\infty, 54, 212)_T$
Medium	$(54, 212, 370)_T$
High	$(212, 370, \infty+)_T$

The rules are

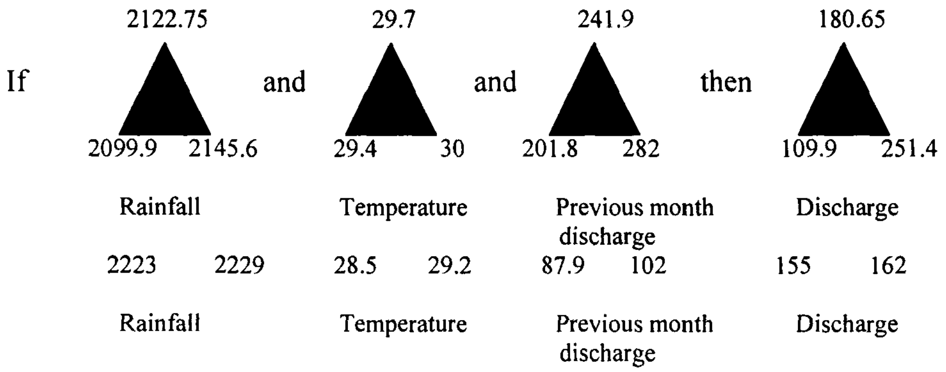
1.Rule5:

If total monthly rainfall is medium and mean monthly temperature is low and total previous month discharge is medium then total monthly discharge is some triangular fuzzy number.



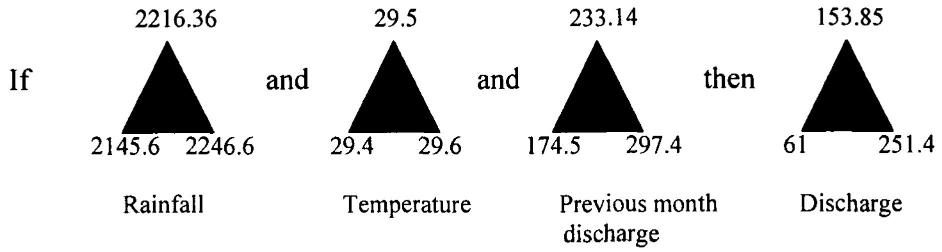
2. Rule11:

If total monthly rainfall is low and mean monthly temperature is medium and total previous month discharge is medium then total monthly discharge is some triangular fuzzy number.



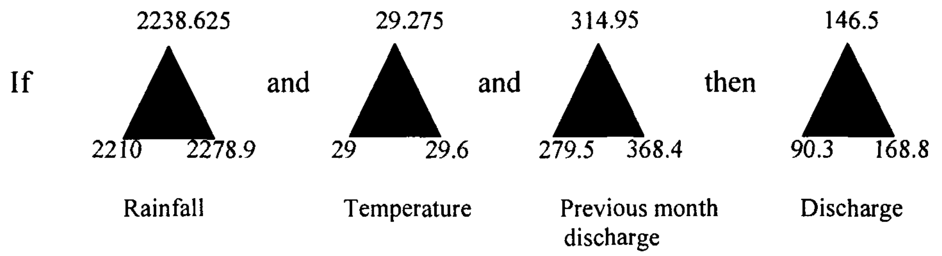
3.Rule14:

If total monthly rainfall is medium and mean monthly temperature is medium and total previous month discharge is medium then total monthly discharge is some triangular fuzzy number.



4.Rule15:

If total monthly rainfall is medium and mean monthly temperature is medium and total previous month discharge is medium then total monthly discharge is some triangular fuzzy number.



Defuzzification:

We take validation year so that all above five rules are satisfied .

Validation year: Rainfall(R)=2150 mm, Temperature(T)=28 ^{0}C and Previous month discharge(PMD)=300 cumec.

Rule	μ_R	μ_T	μ_{PMD}	Degree of freedom	Mean discharge
5	.46	.875	.44	0.177	130.04
11	.769	.125	.44	0.042	180.65
14	.46	.125	.44	0.0253	155.3 15
15	.46	.125	.56	0.032	135.2

Crisp discharge: 140.72 cumec. (applying (2)).

The above discharge of 140.72 cumec in August is less than that during July. This is quite reasonable looking at the rainfall difference during the two month.

Table4: Result summary for four months

Month	Validation data Rainfall/MMT/PMD mm / ^{0}C / cumec	No. of effective rules	Discharge on cumec
June	2200/29/40	03	87.82
July	3100/29/120	04	221.92
August	2150/28/300	04	140.72
September	1620/27.5/130	05	142.13

Discussion

The above table reflects the water discharge pattern of the river basin using the same DOF 0.3 for all the four months considered. It may be noted here that a reduction in the value of DOF is likely to increase the number of effective rules and this, in turn, will improve the final output for the corresponding month. Further, an increase in the number of classes for each input variable will improve the performance of the model.

8.6 Conclusion

The above fuzzy rule based model with more number of classes for the input variable is likely to serve as a model for flood forecasting of the underlying river basin.

Chapter 9

General conclusion and Future Scope of research

9.1 Conclusion

The theory of sets has been the base for the foundation of mathematics and so is considered as one of the most significant branches in mathematics. The fact that any mathematical concept can be interpreted with the help of set theory has not only increased its versatility but has established this theory to be the universal language of mathematics. In the recent past, a relook to the concept of uncertainty in science and mathematics has brought in paradigmatic changes. Prof. Zadeh, through his classical paper[**100**], introduced the concept of modified set called fuzzy set to be used a mathematical tool to handle different types of uncertainty with the help of linguistic variable. In continuation, other modified sets like intuitionistic fuzzy set, rough set and soft set have also been introduced. All these consider the boarder line objects in different ways to handle the uncertainty or vagueness in different areas.

Presently, all the sets are being so widely applied in different branches that one can hardly find any branch into which at least one of these sets and its methods cannot be applied. In the present work, we have studied some aspects of these comparatively recent concepts.

In this thesis, we have discussed a discrete variety of fuzzy and rough mathematical problems as stated below:

The first chapter is an introductory chapter. Besides a brief discussion on fuzzy set, intuitionistic fuzzy set, rough set and soft set, it includes the summary of the theories and results discussed throughout this thesis chapter wise one by one.

In Chapter 2 of this thesis , the generalized notions of union and intersection of fuzzy sets coming from different universes are proposed and some existing results are proved.

The investigation reported in Chapter 3 may looked upon as extension of Yager's theory of bags and fuzzy bags and subsequent together with development by Chakrabarty et al. Here we have defined Cartesian product of fuzzy bags, bag relation and fuzzy bag relation and some results are proved with examples.

In Chapter 4, the concept of rough Boolean algebra and rough sub algebra are introduced based upon Pawlak's notion of indiscernibility relation between elements in a set. Some characterizations of rough Boolean algebras and rough subalgebras are given.

Chapter 5 contains some results of lower and upper approximations of Pawlak's rough set with respect to congruence relation and fuzzy congruence relation in lattices, sub-lattices and ideals in a lattice.

In Chapter 6 of this thesis, intuitionistic fuzzy bags(IFB) concept is applied in multicriteria decision making problem and a hypothetical case study has been taken as an example. Here similarity measurement method is applied with respect to membership value and non membership value separately of intuitionistic fuzzy bag theory.

In Chapter 7, a new generalization of fuzzy sets called soft sets are extended by defining soft relations and fuzzy soft relations and then have been applied in decision making problems. Also, fuzzy soft set theory and intuitionistic fuzzy soft set theory have been applied in medical diagnosis problems separately .

In Chapter 8, an application of fuzzy rule based methodology is developed for estimating monthly discharge using the Jiadhal river basin in the upper Assam.

9.2 Future Scope of Research

The fuzzy set theory and rough set theory are two different approaches to extend the scope of the classical set theory. But both these obtained their pitching grounds from the boarder line objects. So, a closer look and comparison around the boarder line objects in both approaches is likely to provide finer concept to handle the so called uncertainty. Soft set being a very recent concept, there is scope to examine the applicability of this set in different branches of study.

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