CONTRIBUTIONS TO PARTITION IDENTITIES AND CONGRUENCES BY USING RAMANUJAN'S THETA FUNCTIONS, MODULAR EQUATIONS AND CONTINUED FRACTIONS

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN
MATHEMATICAL SCIENCES

By

Kanan Kumari Ojah

Registration No. 015 of 2002

Enrollment No. MSP08004 of Spring 2008

DEPARTMENT OF MATHEMATICAL SCIENCES
SCHOOL OF SCIENCES
TEZPUR UNIVERSITY, NAPAAM-784028
ASSAM, INDIA

MAY 2013

Dedicated to my Parents

Abstract

In this thesis, we study analogues of Ramanujan's partition identities and congruences by using his cubic continued fraction, theta function identities and modular equations. We also find several new partition identities and congruences for partitions with designated summands in which all parts are odd and overpartition pairs into odd parts.

DECLARATION BY THE CANDIDATE

I, Kanan Kumari Ojah, hereby declare that the subject matter in this thesis entitled, **"Contributions to Partition Identities and Congruences by using Ramanujan's Theta Functions, Modular Equations and Continued Fractions,"** is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/institute.

This thesis is being submitted to the Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

Place: Tezpur.

Date: 7.10.2013

Kanan Kumari Ojah

Signature of the candidate

TEZPUR UNIVERSITY

Professor Nayandeep Deka Baruah,
Deptartment of Mathematical Sciences
Napaam–784028, Assam, India

Phone: +91-3712 275506(Ex. 5506)
9954449469(M)
E-mail: nayan@tezu.ernet.in

CERTIFICATE OF THE SUPERVISOR

This is to certify that the thesis entitled **"Contributions to Partition Identities and Congruences by using Ramanujan's Theta Functions, Modular Equations and Continued Fractions,"** submitted to the School of Sciences of Tezpur University in partial fulfillment for the award of the degree of Doctor of Philosophy in Mathematical Sciences is a record of research work carried out by **Mrs. Kanan Kumari Ojah** under my supervision and guidance.

All help received by her from various sources have been duly acknowledged.

No part of this thesis have been submitted elsewhere for award of any other degree.

Place: Tezpur
Date: 07/10/2013

(Nayandeep Deka Baruah)

Acknowledgments

At the very outset, I take this opportunity to express my sincere gratitude and thanks to my supervisor Prof. Nayandeep Deka Baruah, for his invaluable suggestions, continuous encouragement and motivation.

It is my privilege to acknowledge the facilities provided at the Department of Mathematical Sciences, Tezpur University during my research time and I must appreciate the co-operation and support from the faculty members of this Department. I am also grateful to Assam Don Bosco University, Guwahati for giving me ample support in completion of my thesis work.

I thank my friends and fellow research scholars in the Department of Mathematical Sciences for their help in many ways. I also thank my hostel-mates and friends, specially, Surobhi, Mandakini, Padmakshi, Shreemoyee, Rasna, for their support and help.

I would like to thank my parents for their affection and support and a special thank to my spouse Kula for his encouragement and support during this entire period.

Kanan Kumari Ojah.

(Kanan Kumari Ojah)

Date: 7.10.2013

Place: Tezpur

Table of Contents

Chapter 1

Introduction

The thesis consists of six chapters including this introductory chapter. In the following few subsections we briefly introduce the basic concepts and terminology.

1.1 Partitions

A *partition* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of a natural number n is a finite sequence of non-increasing positive integer *parts* λ_i such that $n = \sum_{i=1}^{k} \lambda_i$. Let $p(n)$ denote the number of partitions of n. For example, $p(4) = 5$, since there are five partitions of 4, namely,

$$(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad \text{and } (1,1,1,1).$$

The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.1.1}$$

where, here and throughout the thesis, for $|q| < 1$, $(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n)$.

Ramanujan [80, 81], found nice congruence properties for $p(n)$ modulo 5, 7, and 11, namely, for any nonnegative integer n,

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1.2}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.1.3}$$

and

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.1.4}$$

In [80], Ramanujan deduced (1.1.2) by proving the identity

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}, \tag{1.1.5}$$

which G. H. Hardy [81, p. xxxv] described as Ramanujan's "Most Beautiful Identity". Many proofs of (1.1.5) are available in the literature. We refer to the commentary on Ramanujan's papers by B.C. Berndt in [81].

Ramanujan also offered a more general conjecture which states that if $\delta = 5^a 7^b 11^c$ and λ is an integer such that $24\lambda \equiv 1 \pmod{\delta}$, then

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta}.$$

Ramanujan sketched a proof of this conjecture for arbitrary a and $b = c = 0$. However, for arbitrary b and $a = c = 0$ the conjecture was corrected by Watson [93] as

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta'},$$

where $\delta' = 5^a 7^{b'} 11^c$ with $b' = b$ if $b = 0,\ 1,\ 2$ and $b' = [(b+2)/2]$ if $b > 2$.

Next, we define the general partition function $p_r(n)$ by

$$\sum_{n=0}^{\infty} p_r(n)q^n := \frac{1}{(q;q)_\infty^r}. \tag{1.1.6}$$

The function $p_r(n)$ has been studied by various mathematicians. For example, Atkin [9], Garvan [42], Boylan [27], Gordon [44], Kimming and Olsson [56], Newman [69]-[75], Ramanathan [77], Ramanujan [82, p. 182] (Berndt, Gugg and Kim [25] have proved and discussed Ramanujan's claims, and established further results depending on his ideas), Serre [89], and Sinick [90]. In particular, Boylan [27], Kimming and Olsson [56], and Sinick [90] addressed the characterization of Ramanujan-type congruences, i.e., congruences of the form $c(\ell n + a) \equiv 0 \pmod{\ell}$ for all $n \in \mathbb{Z}$ with ℓ prime, for the function $c(n)$ defined by

$$\sum_{n=0}^{\infty} c(n)q^n = \prod_{i=1}^{r} \frac{1}{(q^{a_i};q^{a_i})_\infty},$$

where a_i's are positive integers, not necessarily distinct. Note that when $a_i = 1$ for each i, then $c(n) = p_r(n)$.

1.2 Frobenius partitions

G. E. Andrews [5] introduced the idea of generalized Frobenius partitions (or simply F-partitions) of n which is a notation of the form

$$\begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix}$$

of non-negative integers a_i's, b_i's with

$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i,$$

where each row is of the same length and each is arranged in non-increasing order.

In particular, Andrews [5] introduced $c\phi_m(n)$, the number of F-partitions of n with m colors and strict decrease in each row. He gave the generating function for $c\phi_m(n)$ and obtained the q-product representations of the generating functions for $c\phi_1(n)$, $c\phi_2(n)$, and $c\phi_3(n)$. Furthermore, Andrews proved the congruences

$$c\phi_2(5n+3) \equiv 0 \pmod{5},$$
$$c\phi_m(n) \equiv 0 \pmod{m^2}, \quad \text{if } m \text{ is prime and does not divide } n.$$

Recently, q-product representations for the generating functions for $c\phi_4(n)$ and $c\phi_5(n)$ are given by Baruah and Sarmah [19]. They also deduced some congruences for $c\phi_4(n)$.

Again, L. Kolitsch [57, 58] introduced the partition function $\overline{c\phi_m}(n)$, which denotes the number of F-partitions of n with m colors whose order is m under cyclic permutation of the m colors. For example, the F-partitions enumerated by $\overline{c\phi_2}(2)$ are $\begin{pmatrix} 1_r \\ 0_r \end{pmatrix}$, $\begin{pmatrix} 1_g \\ 0_r \end{pmatrix}$, $\begin{pmatrix} 1_r \\ 0_g \end{pmatrix}$, $\begin{pmatrix} 1_g \\ 0_g \end{pmatrix}$, $\begin{pmatrix} 0_r \\ 1_r \end{pmatrix}$, $\begin{pmatrix} 0_r \\ 1_g \end{pmatrix}$, $\begin{pmatrix} 0_g \\ 1_r \end{pmatrix}$, and $\begin{pmatrix} 0_g \\ 1_g \end{pmatrix}$, where the subscripts represent the two colors red and green of the non negative integers. The generating function for $\overline{c\phi}_m(n)$ is given by [58],

$$\sum_{n=0}^{\infty} \overline{c\phi}_m(n) q^n = \frac{m \sum q^{Q(\mathbf{k})}}{(q;q)_\infty^m}, \tag{1.2.1}$$

where the sum on the right extends over all vectors $\mathbf{k} = (k_1, k_2, \cdots, k_m)$ with $\mathbf{k}\cdot\bar{1} = 1$ and $Q(\mathbf{k}) = \frac{1}{2}\sum_{i=1}^{m}(k_i - k_{i+1})^2$ wherein $\bar{1} = (1, 1, 1, \cdots, 1)$ and $k_{m+1} = k_1$

Next, Kolitsch proved that, for all $n \geq 1$ and for any $m \geq 2$, $\overline{c\phi_m}(n) \equiv 0 \pmod{m^2}$. In particular, Kolitsch [57] found that

$$\sum_{n=0}^{\infty}\overline{c\phi_3}(n)q^n = \frac{9q(q^9;q^9)_\infty^3}{(q;q)_\infty^3(q^3;q^3)_\infty}, \tag{1.2.2}$$

which readily implies that $\overline{c\phi_3}(n) \equiv 0 \pmod{3^2}$. In a short note, J. Sellers [86], found that, for all $n \geq 1$,

$$\overline{c\phi_5}(5n) \equiv 0 \pmod{5^3},$$
$$\overline{c\phi_7}(7n) \equiv 0 \pmod{7^3},$$

and

$$\overline{c\phi_5}(11n) \equiv 0 \pmod{11^3}.$$

Furthermore, by employing a well-known result of Jacobi in (1.2.2), Sellers [88] proved an analogous result involving $\overline{c\phi_3}(3n)$ modulo 3^4. Recently, Baruah and Sarmah [19] have found an expression for the generating function for $\overline{c\phi_4}(n)$ and also deduced some related congruences. For example,

$$\overline{c\phi_4}(2n) \equiv 0 \pmod{4^3},$$
$$\overline{c\phi_4}(4n+3) \equiv 0 \pmod{4^4},$$
$$\overline{c\phi_4}(4n) \equiv 0 \pmod{4^4}.$$

1.3 Partitions with designated parts

The notion of partition with designated summands goes back to MacMahon [66]. He considered partitions with designated summands in his work on generalized divisor sums. Indeed MacMahon's $A_{n,k}$ is the number of partition of n with designated summands wherein exactly k different magnitudes occur among all the parts. MacMahon [66, Section 17] is able to connect $A_{n,k}$ with numerous divisor sum identities due to Glaisher [43], Ramanujan [81] and others (see also Andrews and Rose [7]).

In [6], Andrews, Lewis and Lovejoy studied partitions with designated summands which are constructed by taking ordinary partitions and tagging exactly one of each part size. For example, there are 10 partitions of 4 with designated summands, namely,

$4'$, $3'+1'$, $2'+2$, $2+2'$, $2'+1'+1$, $2'+1+1'$, $1'+1+1+1$, $1+1'+1+1$, $1+1+1'+1$, $1+1+1+1'$.

Let $PD(n)$ denotes the number of partitions of n with designated summands. Thus, $PD(4)=10$. They [6] also studied $PDO(n)$, the number of partitions of n with designated summands in which all parts are odd. From the above example, we note that $PDO(4)=5$.

The generating functions found by Andrews, Lewis and Lovejoy [6] for $PD(n)$ and $PDO(n)$ are

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{(q^6;q^6)_\infty^2}{(q;q)_\infty(q^2;q^2)_\infty(q^3;q^3)_\infty}$$

and

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty^2}{(q;q)_\infty(q^3;q^3)_\infty(q^{12};q^{12})_\infty}.$$

By using modular forms and q-series identities they found many interesting divisibility properties. They [6] proved that for $n \geq 0$,

$$PD(3n+2) \equiv 0 \pmod{3}. \tag{1.3.1}$$

They also obtained explicit formulas in terms of q- products for the generating functions for $PD(2n)$ and $PD(2n+1)$ by using Euler's algorithm for infinite products and Sturm's criterion. Chen, Ji, Jin, and Shen [36] gave proofs of the generating functions $PD(3n)$, $PD(3n+1)$, $PD(3n+2)$ by employing H.-C. Chan's [30] identity on Ramanujan's cubic continued fraction. In particular, they proved that

$$\sum_{n=0}^{\infty} PD(3n+2)q^n = 3\frac{(q^3;q^6)_\infty^3(q^6;q^6)_\infty^6}{(q;q^2)_\infty^5(q^2;q^2)_\infty^8},$$

which readily implies (1.3.1).

1.4 Overpartitions and overpartition pairs

An overpartition of a positive integer n is a non increasing sequence of positive integers whose sum is n in which first occurrence of a distinct number may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n and $\overline{p_o}(n)$ denote the number of overpartitions of n in which all parts are odd . For example, the overpartitions of 3 are

$(3),\ (\overline{3}),\ (2,1),\ (\overline{2},1),\ (2,\overline{1}),\ (\overline{2},\overline{1}),\ (1,1,1),\ (\overline{1},1,1).$

Thus, $\overline{p}(3)=8$ and $\overline{p_o}(3)=4$.

An overpartition pair of n is a pair of overpartitions (λ,μ) such that the sum of all of the parts is n. For convenience, it is assumed that there is only one overpartition of zero denoted by $\emptyset$. Let $\overline{pp}(n)$ denote the number of overpartition pairs of n and $\overline{pp}_o(n)$ denote the number of overpartition pairs of n into odd parts. For example, overpartition pairs of 3 are,

$((3),\emptyset),\ ((\overline{3}),\emptyset),\ ((2,1),\emptyset),\ ((\overline{2},1),\emptyset),\ ((2,\overline{1}),\emptyset),\ ((\overline{2},\overline{1}),\emptyset),\ ((1,1,1),\emptyset),$
$((\overline{1},1,1),\emptyset),\ ((2),(1)),\ ((2),(\overline{1})),\ ((\overline{2}),(1)),\ ((\overline{2}),(\overline{1})),\ ((1),(2)),\ ((\overline{1}),(2)),$
$((\overline{1}),(\overline{2})),\ ((1),(\overline{2})),\ ((1,1),(1)),\ ((\overline{1},1),(1)),\ ((1,1),(\overline{1})),\ ((\overline{1},1),(\overline{1})),$
$((1),(1,1)),\ ((1),(\overline{1},1)),\ ((\overline{1}),(1,1)),\ ((\overline{1}),(\overline{1},1)),\ (\emptyset,(3)),\ (\emptyset,(\overline{3})),$
$(\emptyset,(2,1)),\ (\emptyset,(\overline{2},1)),\ (\emptyset,(2,\overline{1})),\ (\emptyset,\overline{2},\overline{1})),\ (\emptyset,(1,1,1)),\ (\emptyset,(\overline{1},1,1)).$

Thus, $\overline{pp}(3)=32$ and $\overline{pp}_o(3)=16$.

The generating functions for $\overline{pp}(n)$ and $\overline{pp}_o(n)$ given in [37] and [59] are

$$\sum_{n=0}^{\infty}\overline{pp}(n)q^n=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^4}$$

and

$$\sum_{n=0}^{\infty}\overline{pp}_o(n)q^n=\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^4(q^4;q^4)_\infty^2}.$$

The function $\overline{p}(n)$ has been considered recently by number of mathematicians including Corteel and Lovejoy [40], Hirschhorn and Sellers [51, 52], Mahlburg [67] and Kim [54]. Overpartitions have been used in combinatorial proofs of many q- series identities and these partitions arise quite naturally in the study of hypergeometric series (see [38, 39, 40, 60, 76]). Overpartitions also arise in theoretical physics as jagged partitions in the solution of certain problems regarding seas of particles and fields (see [41]), where a jagged partition of n is an ordered sequence of nonnegative integers $(\lambda_m, \ldots, \lambda_1)$ that sum to n and satisfy the weakly decreasing conditions, $\lambda_j \geq \lambda_{j-1} - 1$ and $\lambda_j \geq \lambda_{j-2}$.

In [67], Mahlburg proved bijectively that the overpartitions correspond to the jagged partitions.

Recently, arithmetic properties of $\overline{pp}(n)$, the number of overpartition pairs of n, have been considered by Bringmann and Lovejoy [28], Chen and Lin [37] and Kim [55]. It has become clear that overpartition pairs play an important role in the theory of q- series and partitions. They provide a natural and general setting for the study of q-series identities and q-difference equations [61, 62, 64]. In [53], Hirschhorn and Sellers studied the arithmetic properties of overpartitions having only odd parts. More recently, Lin [59] investigated various arithmetic properties of overpartition pairs into odd parts.

1.5 Ramanujan's theta functions and modular equations

Define Ramanujan's general theta function $f(a, b)$ as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{1.5.1}$$

Three special cases of $f(a, b)$ are defined, for $|q| < 1$, by [20, p. 36, Entry 22]

$$\varphi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \tag{1.5.2}$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \tag{1.5.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{k=0}^{\infty} (-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty} (-1)^k q^{k(3k+1)/2} = (q;q)_\infty, \tag{1.5.4}$$

where the product representations in (1.5.2)–(1.5.4) arise from Jacobi's triple product identity

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty,$$

and the last equality in (1.5.4) is Euler's famous pentagonal number theorem.

Furthermore, the q-product representations of $\varphi(-q)$ and $\psi(-q)$ are given as

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} \quad \text{and} \quad \psi(-q) = \frac{(q;q)_\infty(q^4;q^4)_\infty}{(q^2;q^2)_\infty}. \tag{1.5.5}$$

Now, we define a modular equation as given by Ramanujan. The complete elliptic integral of the first kind associated with the modulus k, $0 < k < 1$, is given by

$$K := K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

The complementary modulus k' is defined by $k' = \sqrt{1-k^2}$. Set $K' = K(k')$. Let K, K', L, L' denote the complete elliptic integrals of first kind associated with the moduli k, k', ℓ, ℓ', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.5.6}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and ℓ that is implied by (1.5.6). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = \ell^2$. We then say that β has degree n over α. For example, we recall from [20, Entry 5(ii), p. 230] that if β has degree 3 over α, then

$$(\alpha\beta)^{1/4} + ((1-\alpha)(1-\beta))^{1/4} = 1.$$

1.6 The Rogers-Ramanujan continued fraction and Ramanujan's cubic continued fraction

The famous Rogers-Ramanujan continued fraction $R(q)$ is defined by

$$R(q) := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \cdots, \quad |q| < 1. \tag{1.6.1}$$

This continued fraction first appeared in a paper by L. J. Rogers [83] in 1894. Ramanujan later rediscovered the Rogers-Ramanujan continued fraction, and developed an extensive and deep theory for it (see [24]). In his notebooks, Ramanujan recorded many identities involving $R(q)$ which can be found in [24, 20, 79, 80]. Two important formulas for $R(q)$ are

$$\frac{1}{R(q)} - 1 - R(q) = \frac{(q^{1/5};q^{1/5})_\infty}{q^{1/5}(q^5;q^5)_\infty}$$

and

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}.$$

Ramanujan [80] derived (1.1.5), by employing the above identities.

Another continued fraction of Ramanujan, known as Ramanujan's cubic continued fraction $G(q)$, is defined by

$$G(q) := \frac{q^{1/3}}{1} \, \frac{}{+} \, \frac{q+q^2}{1} \, \frac{}{+} \, \frac{q^2+q^4}{1} \, \frac{}{+} \cdots, \quad |q| < 1. \tag{1.6.2}$$

Several results on $G(q)$ are recorded by Ramanujan in his notebook [79, p. 237, vol II] and his lost notebook [82, p. 366]. In particular, he recorded that

$$G(q) = q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3} = \frac{\chi(-q)}{\chi^3(-q^3)}, \tag{1.6.3}$$

where

$$\chi(-q) = (q;q^2)_\infty = \frac{(q;q)_\infty}{(q^2;q^2)_\infty}. \tag{1.6.4}$$

We also note here that

$$\chi(q) := (-q;q^2)_\infty = \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty(q^4;q^4)_\infty}. \tag{1.6.5}$$

Proofs of (1.6.3) can be found in papers by Selberg [85], Gordon [45] and Andrews [4].

1.7 Work done in this thesis

In this thesis, we study analogues of Ramanujan's partition identities and congruences by using his cubic continued fraction, theta function identities and modular equations. We also find several new partition identities and congruences for partitions with designated summands into odd parts and overpartition pairs into odd parts.

In the following few paragraphs, we briefly explain our work.

In Chapter 2 of this thesis, we present 3-dissections of $1/\psi(q)$, $1/\varphi(-q)$ and $1/(q;q)_\infty^3$ from identities involving Ramanujan's cubic continued fraction and derive some congruences of the coefficients of these functions.

For example, if

$$\sum_{n=0}^{\infty} p_3(n)q^n = 1/(q;q)_\infty^3,$$

then

$$\sum_{n=0}^{\infty} p_3(9n+8)q^n = 81\,\frac{(q^3;q^3)_\infty^{36}}{(q;q)_\infty^{39}w^8(q)}\{10w^2(q) + 672qw^5(q) + 8313q^2w^8(q) + 33536q^3w^{11}(q) + 66048q^4w^{14}(q) + 61440q^5w^{17}(q) + 40960q^6w^{20}(q)\},$$

where

$$w(q) := \frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3} = \frac{(q;q)_\infty(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty(q^3;q^3)_\infty^3},$$

and consequently,

$$p_3(9n+8) \equiv 0 \pmod{81}.$$

By using the 3-dissections of $1/\psi(q)$ and $1/\varphi(-q)$, we derive an analogue of Ramanujan's "Most Beautiful Identity"(1.1.5), namely,

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^4}, \tag{1.7.1}$$

where $a(n)$ is defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty(q^2;q^2)_\infty}, \tag{1.7.2}$$

which was first proved by H.-C. Chan [30].

In Section 2.4, we apply the 3-dissection for $1/(q;q)_\infty^3$ to prove some congruences proved by L. Kolitsch [57, 58] and Sellers [87], and a new congruence for the function $\overline{c\phi_m}(n)$. In particular, we prove that

$$\overline{c\phi_3}(3n) \equiv 0 \pmod{3^4}$$

and

$$\overline{c\phi_3}(3n+2) \equiv 0 \pmod{3^3}.$$

In Chapter 3, we deal with modular equations and identities involving Ramanujan's cubic continued fraction. In his notebooks [79] and his lost notebook [82], Ramanujan recorded identities giving relations between the Rogers-Ramanujan continued fraction $R(q)$ defined in (1.6.1) and the five continued fractions $R(-q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$ and $R(q^5)$. C. Gugg in his paper [47] gave a new proof of Ramanujan's modular identity relating $R(q)$ and $R(q^5)$, namely,

$$R^5(q) = R(q^5)\frac{1-2R(q^5)+4R^2(q^5)-3R^3(q^5)+R^4(q^5)}{1+3R(q^5)+4R^2(q^5)+2R^3(q^5)+R^4(q^5)}. \tag{1.7.3}$$

H.H. Chan [34] established several modular identities connecting $G(q)$ defined in (3.1.5) with $G(-q)$, $G(q^2)$ and $G(q^3)$. One of the modular relations is

$$\frac{1-G(q^3)+G^2(q^3)}{1+2G(q^3)+4G^2(q^3)} = \frac{G^3(q)}{G(q^3)}, \tag{1.7.4}$$

which is a perfect analogue of (1.7.3).

H.H. Chan [34] proved (1.7.4) by using Ramanujan's modular equations of degree 3. N.D. Baruah [11], C. Adiga, T. Kim, M.S.M. Naika and H.S. Madhusudhan [1] also found alternative proofs of (1.7.4). Baruah [12] also established two modular identities connecting $G(q)$ with $G(q^5)$ and $G(q^7)$ respectively. Further modular identities for $G(q)$ have been found by Naika, S. Chandankumar and K.S. Bairy [68].

In our work, we prove (1.7.4) by deriving product representations for $\dfrac{1}{\sqrt{G(q)}}+\sqrt{G(q)}$ and $\dfrac{1}{\sqrt{G(q)}}-2\sqrt{G(q)}$, namely,

$$\frac{1}{\sqrt{G(q)}}+\sqrt{G(q)}=\frac{\sqrt{\chi^3(-q^3)}}{q^{1/6}\varphi(-q^3)}\cdot\sqrt{\chi(-q)}\psi(q^{1/3})$$

and

$$\frac{1}{\sqrt{G(q)}}-2\sqrt{G(q)}=\frac{\sqrt{\chi^3(-q^3)}}{q^{1/6}\varphi(-q^3)}\cdot\frac{\varphi(-q^{1/3})}{\sqrt{\chi(-q)}}.$$

We also find some other interesting identities.

Chapter 4 of this thesis is devoted to some analogues of Ramanujan's partition identities and congruences arising from his theta functions and modular equations.

We define the generalized partition function $p_{[c^l d^m]}(n)$ by

$$\sum_{n=0}^{\infty} p_{[c^l d^m]}(n)q^n := \frac{1}{(q^c;q^c)_\infty^l (q^d;q^d)_\infty^m}.$$

Note that in this notation $a(n)$ defined in (1.7.2) is $p_{[1^1 2^1]}(n)$ and therefore (1.7.1) can be written as

$$\sum_{n=0}^{\infty} p_{[1^1 2^1]}(3n+2)q^n = 3\,\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 (q^2;q^2)_\infty^4}. \tag{1.7.5}$$

Combinatorially, $p_{[1^1 2^1]}(n)$ is the number of 2-colored partitions of n with one of the colors, say red, appearing only in multiples of 2. Thus, $p_{[1^1 2^1]}(5) = 12$, where the corresponding partitions, with the other color being, say blue, are given by

$$\begin{aligned}5_b &= 4_b+1_b = 4_r+1_b = 3_b+2_b = 3_b+2_r = 3_b+1_b+1_b = 2_b+2_b+1_b = 2_b+2_r+1_b\\ &= 2_r+2_r+1_b = 2_b+1_b+1_b+1_b = 2_r+1_b+1_b+1_b = 1_b+1_b+1_b+1_b+1_b,\end{aligned}$$

verifying H.-C. Chan's congruence

$$p_{[1^1 2^1]}(3n+2) \equiv 0 \pmod 3,$$

for $n = 1$.

H.H. Chan and P.C. Toh [33] also gave some beautiful analogues of (1.7.5). Some of their identities are

$$\sum_{n=0}^{\infty} p_{[1^2 3^2]}(2n+1)q^n = 2\,\frac{(q^2;q^2)_\infty^4 (q^6;q^6)_\infty^4}{(q;q)_\infty^6 (q^3;q^3)_\infty^6}.$$

$$\sum_{n=0}^{\infty} p_{[1^4 5^4]}(2n+1)q^n = 4\,\frac{(q^2;q^2)_\infty^4 (q^{10};q^{10})_\infty^4}{(q;q)_\infty^8 (q^5;q^5)_\infty^8} + 8q\,\frac{(q^2;q^2)_\infty^8 (q^{10};q^{10})_\infty^8}{(q;q)_\infty^{12} (q^5;q^5)_\infty^{12}},$$

$$\sum_{n=0}^{\infty} p_{[1^1 7^1]}(2n+1)q^n = \frac{(q^2;q^2)_\infty^2 (q^{14};q^{14})_\infty^2}{(q;q)_\infty^3 (q^7;q^7)_\infty^3},$$

$$\sum_{n=0}^{\infty} p_{[1^2 11^2]}(2n+1)q^n = 2\,\frac{(q^2;q^2)_\infty^2 (q^{22};q^{22})_\infty^2}{(q;q)_\infty^4 (q^{11};q^{11})_\infty^4} + 2q\,\frac{(q^2;q^2)_\infty^4 (q^{22};q^{22})_\infty^4}{(q;q)_\infty^6 (q^{11};q^{11})_\infty^6},$$

$$\sum_{n=0}^{\infty} p_{[1^1 23^1]}(2n+1)q^n = \frac{(q^2;q^2)_\infty (q^{46};q^{46})_\infty}{(q;q)_\infty^2 (q^{23};q^{23})_\infty^2} + q\,\frac{(q^2;q^2)_\infty^2 (q^{46};q^{46})_\infty^2}{(q;q)_\infty^3 (q^{23};q^{23})_\infty^3}.$$

In Section 4.2, we present new proofs of the above identities by employing Ramanujan's modular equations of degrees 3, 5, 7, 11, and 23, respectively. In Section 4.3, we present some new results analogous to (1.1.5) and new partition congruences deducible from them.

In Chapter 5, we deal with identities and congruences involving $PDO(n)$, the number of partitions of n with designated summands in which all parts are odd.

By using modular forms, Andrews, Lewis and Lovejoy [6] found that

$$PDO(12n+6) \equiv 0 \pmod 3$$

and

$$PDO(12n+10) \equiv 0 \pmod 3.$$

In this thesis, we prove that

$$\sum_{n=0}^{\infty} PDO(12n+6)q^n = 12\left\{\frac{(q^2;q^2)_\infty^{11}(q^3;q^3)_\infty^{13}}{(q;q)_\infty^{19}(q^6;q^6)_\infty^5} + 10q\,\frac{(q^2;q^2)_\infty^8 (q^3;q^3)_\infty^4 (q^6;q^6)_\infty^4}{(q;q)_\infty^{16}}\right\}$$

and

$$\sum_{n=0}^{\infty} PDO(12n+10)q^n = 6\left\{7\,\frac{(q^2;q^2)_\infty^{10}(q^3;q^3)_\infty^{10}}{(q;q)_\infty^{18}(q^6;q^6)_\infty^2} + 16q\,\frac{(q^2;q^2)_\infty^7 (q^3;q^3)_\infty (q^6;q^6)_\infty^7}{(q;q)_\infty^{15}}\right\},$$

which give stronger versions of the previous congruences.

We also find some more identities and congruences for $PDO(n)$.

In the concluding chapter of this thesis, we deal with arithmetic properties of overpartition pairs into odd parts. Recently, Lin [59] obtained a number of Ramanujan-type congruences modulo 3 and modulo powers of 2. In particular, he found that

$$\overline{pp}_o(4n+3) \equiv 0 \pmod{16} \tag{1.7.6}$$

and

$$\overline{pp}_o(8n+7) \equiv 0 \pmod{32}. \tag{1.7.7}$$

In derivation of the above congruences, Lin worked on taking modulo powers of 2 .

In our thesis, we obtain the above congruences and several new congruences from their respective generating functions. For example, (1.7.6) and (1.7.7) immediately follow from

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n+3)q^n = 16\frac{(q^2;q^2)_\infty^6(q^4;q^4)_\infty^4}{(q;q)_\infty^{10}}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(8n+7)q^n = 32 \times \Bigg\{5\frac{(q^4;q^4)_\infty^{19}(q^2;q^2)_\infty^6}{(q;q)_\infty^{19}(q^8;q^8)_\infty^6} + 40q\,\frac{(q^2;q^2)_\infty^{10}(q^4;q^4)_\infty^7(q^8;q^8)_\infty^2}{(q;q)_\infty^{19}} \\ + 16q^2\,\frac{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^{10}}{(q;q)_\infty^{19}(q^4;q^4)_\infty^5}\Bigg\},\end{aligned}$$

respectively.

We also find several new congruences modulo powers of 2 by employing elementary generating function techniques.

Chapter 2

Congruences Deducible from Ramanujan's Cubic Continued Fraction

2.1 Introduction

Recall from (1.1.5), Ramanujan's famous identity

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}. \tag{2.1.1}$$

The above identity was described by G. H. Hardy [81, p. xxxv] as Ramanujan's "Most Beautiful Identity". Ramanujan derived (2.1.1) by employing the identities

$$\frac{1}{R(q)} - 1 - R(q) = \frac{(q^{1/5};q^{1/5})_\infty}{q^{1/5}(q^5;q^5)_\infty}, \tag{2.1.2}$$

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}, \tag{2.1.3}$$

where $R(q)$ is the Rogers-Ramanujan continued fraction defined in (1.6.1).

Recently, H.-C. Chan [30] proved an analogue of (2.1.1), namely,

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^4}, \tag{2.1.4}$$

Note: The contents of this chapter appeared in *International Journal of Number Theory* [17] of World Scientific Publishing Company.

and consequently,

$$a(3n+2) \equiv 0 \pmod 3, \tag{2.1.5}$$

where $a(n)$ is defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}. \tag{2.1.6}$$

H.-C. Chan [30] proved (2.1.4) by using two results, analogous to (2.1.2) and (2.1.3), closely connected to Ramanujan's cubic continued fraction $G(q)$, defined in (1.6.2).

$$G(q) := \frac{q^{1/3}}{1} \, + \, \frac{q+q^2}{1} \, + \, \frac{q^2+q^4}{1} \, + \cdots, \quad |q|<1. \tag{2.1.7}$$

Z. Cao [29] has also proved the same result by applying a 3-dissection of $(q;q)_\infty (q^2;q^2)_\infty$. Recently H. Zhao and Z. Zhong [95] have proved (2.1.4) by using a 3-dissection of $1/(q;q)_\infty (q^2;q^2)_\infty$, deducible from cubic theta functions. We refer to [31], [32] and [33], and [90] for further references on $G(q)$.

In Section 2.2 of this chapter, we present 3-dissections of $1/\psi(q)$ and $1/\varphi(-q)$, where $\psi(q)$ and $\varphi(-q)$ are defined in (1.5.3) and (1.5.2), respectively. The 3-dissections are deduced from identities involving Ramanujan's cubic continued fraction. We find some congruences of the coefficients of these two functions and also find a simple proof of H.-C. Chan's congruence (2.1.4).

In Section 2.3, we give a 3-dissection for $1/(q;q)_\infty^3$ deducible from identities involving Ramanujan's cubic continued fraction $G(q)$ and derive results analogous to (2.1.1) and (2.1.4) and deduce congruences for $p_3(n)$ modulo 27 and 81 (see Theorem 2.3.1 and Theorem 2.3.3). In Section 2.4, we apply the same 3-dissection for $1/(q;q)_\infty^3$ to prove some congruences proved by L. Kolitsch [57], [58] and Sellers [87], and a new congruence for the function $\overline{c\phi_m}(n)$. which denotes the number of Frobenius partitions of n with m colors.

2.2 3-Dissections of $1/\psi(q)$ and $1/\varphi(-q)$, and Chan's congruence

Theorem 2.2.1. *The 3-dissection of* $\dfrac{1}{\psi(q)}$ *is given by*

$$\frac{1}{\psi(q)} = \frac{\psi^3(q^9)}{\psi^4(q^3)w^2(q^3)} - q\frac{\psi^3(q^9)}{\psi^4(q^3)w(q^3)} + q^2\frac{\psi^3(q^9)}{\psi^4(q^3)}, \tag{2.2.1}$$

and the 3-dissection of $\dfrac{1}{\varphi(-q)}$ *is given by*

$$\frac{1}{\varphi(-q)} = \frac{\varphi^3(-q^9)}{\varphi^4(-q^3)} + 2q\frac{\varphi^3(-q^9)w(q^3)}{\varphi^4(-q^3)} + 4q^2w^2(q^3)\frac{\varphi^3(-q^9)}{\varphi^4(-q^3)}, \tag{2.2.2}$$

where $w(q)$ *is given by* (2.2.6).

Proof. From the later two equalities in Entry 1(i) of [20, p. 345], we have

$$1 + \frac{1}{G(q^3)} = \frac{\psi(q)}{q\psi(q^9)} \tag{2.2.3}$$

and

$$1 + \frac{1}{G^3(q^3)} = \frac{\psi^4(q^3)}{q^3\psi^4(q^9)}, \tag{2.2.4}$$

where $G(q)$ is defined in (2.1.7).

Now, from the first equality of Entry 1(i) of [20, p. 345], we note that

$$G(q) = q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty} = q^{1/3}w(q), \tag{2.2.5}$$

where

$$w(q) := \frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty} = \frac{(q;q)_\infty(q^6;q^6)^3_\infty}{(q^2;q^2)_\infty(q^3;q^3)^3_\infty}. \tag{2.2.6}$$

With the help of (2.2.5), we rewrite (2.2.3) and (2.2.4) as

$$1 + \frac{1}{qw(q^3)} = \frac{\psi(q)}{q\psi(q^9)} \tag{2.2.7}$$

and

$$1+\frac{1}{q^3w^3(q^3)}=\frac{\psi^4(q^3)}{q^3\psi^4(q^9)}. \tag{2.2.8}$$

Employing (2.2.8) and (2.2.7), we find that

$$\begin{aligned}\frac{1}{\psi(q)}&=\frac{q^2\psi^3(q^9)}{\psi^4(q^3)}\cdot\frac{\psi^4(q^3)}{q^3\psi^4(q^9)}\cdot\frac{q\psi(q^9)}{\psi(q)}\\&=\frac{q^2\psi^3(q^9)}{\psi^4(q^3)}\cdot\frac{1+\dfrac{1}{q^3w^3(q^3)}}{1+\dfrac{1}{qw(q^3)}}\\&=\frac{\psi^3(q^9)}{\psi^4(q^3)w^2(q^3)}\left(1-qw(q^3)+q^2w^2(q^3)\right),\end{aligned}$$

which is the first part of Theorem 2.2.1.

Again, from Entry 1(ii) and Entry 1(iii) of [20, p. 345], we note that

$$\frac{\varphi(-q)}{\varphi(-q^9)}=1-2G(q^3)=1-2qw(q^3) \tag{2.2.9}$$

and

$$9\frac{\varphi^4(-q^3)}{\varphi^4(-q)}-1=\left(3\frac{\varphi(-q^9)}{\varphi(-q)}-1\right)^3. \tag{2.2.10}$$

Employing (2.2.9) in (2.2.10) we find that

$$\frac{\varphi^4(-q^3)}{\varphi^4(-q)}=\frac{3}{(1-2qw(q^3))^3}-\frac{3}{(1-2qw(q^3))^2}+\frac{1}{1-2qw(q^3)}. \tag{2.2.11}$$

Now, cubing both sides of (2.2.9) and then multiplying with (2.2.11), we arrive at

$$\frac{\varphi^4(-q^3)}{\varphi(-q)\varphi^3(-q^9)}=1+2qw(q^3)+4q^2w^2(q^3), \tag{2.2.12}$$

which is equivalent to (2.2.2). □

The following congruences follow readily from the 3-dissection of $\dfrac{1}{\varphi(-q)}$ given in Theorem 2.2.1.

Corollary 2.2.2. *If $b(n)$ is defined by $\sum_{n=0}^{\infty} b(n)q^n = \dfrac{1}{\varphi(-q)}$, then*

$$\sum_{n=0}^{\infty} b(3n)q^n = \frac{\varphi^3(-q^3)}{\varphi^4(-q)} = \frac{f^6(-q^3)f^4(-q^2)}{f^3(-q^6)f^8(-q)}, \tag{2.2.13}$$

$$\sum_{n=0}^{\infty} b(3n+1)q^n = 2\, w(q)\frac{\varphi^3(-q^3)}{\varphi^4(-q)} = 2\,\frac{f^3(-q^2)f^3(-q^3)}{f^7(-q)}, \tag{2.2.14}$$

and

$$\sum_{n=0}^{\infty} b(3n+2)q^n = 4\, w^2(q)\frac{\varphi^3(-q^3)}{\varphi^4(-q)} = 4\,\frac{f^2(-q^2)f^3(-q^6)}{f^6(-q)}. \tag{2.2.15}$$

Corollary 2.2.3. *We have*

$$b(3n+1) \equiv 0 \pmod 2$$

and

$$b(3n+2) \equiv 0 \pmod 4.$$

Proof. Follow readily from (2.2.14) and (2.2.15). □

Corollary 2.2.4. *We have*

$$\sum_{n=0}^{\infty} b(6n+5)q^n = 24\,\frac{f^4(-q^2)f^3(-q^3)f^2(-q^8)}{f^9(-q)f(-q^4)} + 128q\,\frac{f^6(-q^2)f^3(-q^3)f^6(-q^8)}{f^{13}(-q)f^3(-q^4)}. \tag{2.2.16}$$

Consequently,

$$b(6n+5) \equiv 0 \pmod 8.$$

Proof. From (2.2.15), we have

$$\sum_{n=0}^{\infty} b(3n+2)q^n = 4\,\frac{f^3(-q^6)}{f^4(-q^2)(q;q^2)_\infty^6}, \tag{2.2.17}$$

where we have used the trivial fact that $f(-q) = (q;q)_\infty = (q;q^2)_\infty(q^2;q^2)_\infty = f(-q^2)(q;q^2)_\infty$. Replacing q by $-q$ in (2.2.17) and then subtracting the resulting identity from (2.2.17), we find that

$$\sum_{n=0}^{\infty} b(3n+2)q^n - \sum_{n=0}^{\infty} b(3n+2)(-q)^n = 4\,\frac{f^3(-q^6)}{f^4(-q^2)}\left(\frac{1}{(q;q^2)_\infty^6} - \frac{1}{(-q;q^2)_\infty^6}\right)$$
$$= 4\,\frac{f^3(-q^6)}{f^4(-q^2)(q^2;q^4)_\infty^6}\left((-q;q^2)_\infty^6 - (q;q^2)_\infty^6\right)$$
$$= 4\,\frac{f^3(-q^6)f^6(-q^4)}{f^{10}(-q^2)}\left((-q;q^2)_\infty^6 - (q;q^2)_\infty^6\right). \tag{2.2.18}$$

Now, from Entry 25(ii) [20, p. 40], we have

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8). \tag{2.2.19}$$

Writing (2.2.19) in q-products, with the help of (1.5.2) and (1.5.3), we find that

$$(q^2;q^2)_\infty\left\{(-q;q^2)_\infty^2 - (q;q^2)_\infty^2\right\} = 4q\frac{(q^{16};q^{16})_\infty}{(q^8;q^{16})_\infty}. \tag{2.2.20}$$

Thus,

$$(-q;q^2)_\infty^2 - (q;q^2)_\infty^2 = 4q\frac{(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty(q^8;q^{16})_\infty}$$
$$= 4q\frac{f^2(-q^{16})}{f(-q^2)f(-q^8)}.$$

Therefore,

$$(-q;q^2)_\infty^6 - (q;q^2)_\infty^6 = \left((-q;q^2)_\infty^2 - (q;q^2)_\infty^2\right)^3 + 3(q^2;q^4)_\infty^2\left((-q;q^2)_\infty^2 - (q;q^2)_\infty^2\right)$$
$$= 64q^3\,\frac{f^6(-q^{16})}{f^3(-q^2)f^3(-q^8)} + 12q\,\frac{f(-q^2)f^2(-q^{16})}{f^2(-q^4)f(-q^8)}. \tag{2.2.21}$$

Employing (2.2.21) in (2.2.18), we find that

$$\sum_{n=0}^{\infty} b(3n+2)q^n - \sum_{n=0}^{\infty} b(3n+2)(-q)^n = 48q\,\frac{f^4(-q^4)f^3(-q^6)f^2(-q^{16})}{f^9(-q^2)f(-q^8)}$$
$$+ 256q^3\,\frac{f^6(-q^4)f^3(-q^6)f^6(-q^{16})}{f^{13}(-q^2)f^3(-q^8)}. \tag{2.2.22}$$

Extracting from both sides of (2.2.22) those terms that involve only q^{2n+1}, and then dividing both sides by q and replacing q^2 by q, we arrive at (2.2.16) to complete the proof. □

Next, we prove H.-C. Chan's identity (2.1.4) by employing the 3-dissections given in Theorem 2.2.1.

Corollary 2.2.5. *Identity* (2.1.4) *holds.*

Proof. By the product representations of $\varphi(-q)$ and $\psi(q)$ in (1.5.2) and (1.5.3), we have

$$\frac{1}{\psi(q)}\cdot\frac{1}{\varphi(-q)}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\cdot\frac{1}{(q;q^2)_\infty^2(q^2;q^2)_\infty}=\frac{1}{(q;q)_\infty(q^2;q^2)_\infty}. \tag{2.2.23}$$

Employing (2.1.6), (2.2.1), and (2.2.2) in (2.2.23), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}a(n)q^n=&\left(\frac{\psi^3(q^9)}{\psi^4(q^3)w^2(q^3)}-q\frac{\psi^3(q^9)}{\psi^4(q^3)w(q^3)}+q^2\frac{\psi^3(q^9)}{\psi^4(q^3)}\right)\\&\times\left(\frac{\varphi^3(-q^9)}{\varphi^4(-q^3)}+2q\frac{\varphi^3(-q^9)w(q^3)}{\varphi^4(-q^3)}+4q^2w^2(q^3)\frac{\varphi^3(-q^9)}{\varphi^4(-q^3)}\right).\end{aligned} \tag{2.2.24}$$

Extracting from both sides of (2.2.24) those terms that involve q^{3n+2}, we obtain

$$\sum_{n=0}^{\infty}a(3n+2)q^{3n+2}=3q^2\frac{\psi^3(q^9)\varphi^3(-q^9)}{\psi^4(q^3)\varphi^4(-q^3)}. \tag{2.2.25}$$

Dividing both sides of (2.2.25) by q^2 and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty}a(3n+2)q^n=3\frac{\psi^3(q^3)\varphi^3(-q^3)}{\psi^4(q)\varphi^4(-q)}. \tag{2.2.26}$$

With the help of the product representations of $\varphi(-q)$ and $\psi(q)$ in (1.5.2) and (1.5.3), we can easily deduce (2.1.4) from (2.2.26). □

2.3 Congruences for the partition function $p_3(n)$

Setting $r=3$ in (1.1.6), we have

$$\sum_{n=0}^{\infty}p_3(n)q^n=1/(q;q)_\infty^3,$$

where $p_3(n)$ is the number of 3-colored partitions of n. Then

$$\sum_{n=0}^{\infty} p_3(n)q^n = \frac{1}{(q;q)_\infty^3} \equiv \frac{1}{(q^3;q^3)_\infty} \pmod 3. \tag{2.3.1}$$

It is clear from (2.3.1) that $p_3(3n+1) \equiv 0 \pmod 3$ and $p_3(3n+2) \equiv 0 \pmod 3$. We have the following stronger results.

Theorem 2.3.1. *If* $p_r(n)$ *is defined by* (1.1.6), *then*

$$\sum_{n=0}^{\infty} p_3(3n)q^n = \frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^{12}}\left(\frac{1}{w^2(q)} + 8qw(q) + 16q^2w^4(q)\right), \tag{2.3.2}$$

$$\sum_{n=0}^{\infty} p_3(3n+1)q^n = 3\frac{(q^3;q^3)_\infty^{12}(q^2;q^2)_\infty}{(q;q)_\infty^{13}(q^6;q^6)_\infty^3} + 12q\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^6}{(q;q)_\infty^{10}(q^2;q^2)_\infty^2}, \tag{2.3.3}$$

and

$$\sum_{n=0}^{\infty} p_3(3n+2)q^n = 9\frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^{12}}. \tag{2.3.4}$$

Proof. From Entry 1(iv) of [20, p. 345], we note that

$$\frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)} = 4G^2(q) - 3 + \frac{1}{G(q)} \tag{2.3.5}$$

and

$$\frac{f^{12}(-q)}{qf^{12}(-q^3)} = \left(4G^2(q) + \frac{1}{G(q)}\right)^3 - 27, \tag{2.3.6}$$

where $G(q)$ is Ramanujan's cubic continued fraction as defined in (2.1.7). Replacing q by q^3 in (2.3.5) and (2.3.6), and then employing (1.5.4) and (2.2.5), we find that

$$\frac{(q;q)_\infty^3}{q(q^9;q^9)_\infty^3} = 4q^2w^2(q^3) - 3 + \frac{1}{qw(q^3)} \tag{2.3.7}$$

and

$$\frac{(q^3;q^3)_\infty^{12}}{q^3(q^9;q^9)_\infty^{12}} = \left(4q^2w^2(q^3) + \frac{1}{qw(q^3)}\right)^3 - 27. \tag{2.3.8}$$

Therefore,

$$\sum_{n=0}^{\infty} p_3(n)q^n = \frac{1}{(q;q)_\infty^3} = \frac{q^2(q^9;q^9)_\infty^9}{(q^3;q^3)_\infty^{12}}\left(\frac{(q^3;q^3)_\infty^{12}}{q^3(q^9;q^9)_\infty^{12}}\cdot\frac{q(q^9;q^9)_\infty^3}{(q;q)_\infty^3}\right)$$
$$= \frac{q^2(q^9;q^9)_\infty^9}{(q^3;q^3)_\infty^{12}}\left(\frac{\left(4q^2w^2(q^3)+\dfrac{1}{qw(q^3)}\right)^3-27}{4q^2w^2(q^3)-3+\dfrac{1}{qw(q^3)}}\right)$$
$$= \frac{(q^9;q^9)_\infty^9}{(q^3;q^3)_\infty^{12}}\left\{\frac{1}{w^2(q^3)}+\frac{3q}{w(q^3)}+9q^2+8q^3w(q^3)\right.$$
$$\left.+12q^4w^2(q^3)+16q^6w^4(q^3)\right\}. \qquad (2.3.9)$$

Extracting from both sides of (2.3.9) those terms that involve q^{3n}, q^{3n+1}, and q^{3n+2}, respectively, we obtain

$$\sum_{n=0}^{\infty} p_3(3n)q^{3n} = \frac{(q^9;q^9)_\infty^9}{(q^3;q^3)_\infty^{12}}\left(\frac{1}{w^2(q^3)}+8q^3w(q^3)+16q^6w^4(q^3)\right), \qquad (2.3.10)$$

$$\sum_{n=0}^{\infty} p_3(3n+1)q^{3n+1} = \frac{(q^9;q^9)_\infty^9}{(q^3;q^3)_\infty^{12}}\left(\frac{3q}{w(q^3)}+12q^4w^2(q^3)\right), \qquad (2.3.11)$$

and

$$\sum_{n=0}^{\infty} p_3(3n+2)q^{3n+2} = 9q^2\frac{(q^9;q^9)_\infty^9}{(q^3;q^3)_\infty^{12}}. \qquad (2.3.12)$$

It is now easy to derive (2.3.2)-(2.3.4) from (2.3.10)-(2.3.12), respectively. □

The following results are immediate from Theorem 2.3.1.

Corollary 2.3.2. *We have*

$$p_3(3n+1) \equiv 0 \pmod 3$$

and

$$p_3(3n+2) \equiv 0 \pmod{3^2}.$$

Now we derive the following two interesting congruences from (2.3.4).

Theorem 2.3.3. *We have*

$$p_3(9n+5) \equiv 0 \pmod{3^3} \tag{2.3.13}$$

and

$$p_3(9n+8) \equiv 0 \pmod{3^4}. \tag{2.3.14}$$

Proof. Employing (2.3.9) in (2.3.4), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p_3(3n+2)q^n &= 9\,\frac{(q^9;q^9)_\infty^{36}}{(q^3;q^3)_\infty^{39} w^8(q^3)}\Big\{(1+3qw(q^3)+9q^2w^2(q^3)+8q^3w^3(q^3)\\ &\quad +12q^4w^4(q^3)+16q^6w^6(q^3)\Big\}^4\\ &= 9\,\frac{(q^9;q^9)_\infty^{36}}{(q^3;q^3)_\infty^{39} w^8(q^3)}\Big\{1+12qw(q^3)+90q^2w^2(q^3)+464q^3w^3(q^3)\\ &\quad +1875q^4w^4(q^3)+6048q^5w^5(q^3)+16378q^6w^6(q^3)\\ &\quad +37404q^7w^7(q^3)+74817q^8w^8(q^3)+131024q^9w^9(q^3)\\ &\quad +209616q^{10}w^{10}(q^3)+301824q^{11}w^{11}(q^3)+411040q^{12}w^{12}(q^3)\\ &\quad +501504q^{13}w^{13}(q^3)+594432q^{14}w^{14}(q^3)+610304q^{15}w^{15}(q^3)\\ &\quad +652032q^{16}w^{16}(q^3)+552960q^{17}w^{17}(q^3)+557056q^{18}w^{18}(q^3)\\ &\quad +344064q^{19}w^{19}(q^3)+368640q^{20}w^{20}(q^3)+131072q^{21}w^{21}(q^3)\\ &\quad +196608q^{22}w^{22}(q^3)+65536q^{24}w^{24}(q^3)\Big\}. \end{aligned} \tag{2.3.15}$$

Extracting from both sides of (2.3.15) those terms that involve only q^{3n+1} in (2.3.15), and then dividing both sides by q and replacing q^3 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p_3(9n+5)q^n &= 27\,\frac{(q^3;q^3)_\infty^{36}}{(q;q)_\infty^{39} w^8(q)}\Big\{4w(q)+625qw^4(q)+12468q^2w^7(q)\\ &\quad +69872q^3w^{10}(q)+127128q^4w^{13}(q)+217344q^5w^{16}(q)\\ &\quad +114688q^6w^{19}(q)+65536q^7w^{21}(q)\Big\}, \end{aligned} \tag{2.3.16}$$

from which we readily arrive at (2.3.13).

Next, extracting from both sides of (2.3.15) those terms that involve only q^{3n+2} in (2.3.15), and then dividing both sides by q^2 and replacing q^3 by q in the resulting identity, we find that

$$\sum_{n=0}^{\infty} p_3(9n+8)q^n = 81\,\frac{(q^3;q^3)_\infty^{36}}{(q;q)_\infty^{39}w^8(q)}\Big\{10w^2(q)+672qw^5(q)+8313q^2w^8(q)$$
$$+33536q^3w^{11}(q)+66048q^4w^{14}(q)+61440q^5w^{17}(q)$$
$$+40960q^6w^{20}(q)\Big\}. \qquad (2.3.17)$$

We readily deduce (2.3.14) from (2.3.17) to complete the proof. □

2.4 Some congruences for Frobenius partitions

L. Kolitsch [57, 58] introduced the partition function $\overline{c\phi_m}(n)$. Kolitsch proved that, for all $n \geq 1$ and for any $m \geq 2$, $\overline{c\phi_m}(n) \equiv 0 \pmod{m^2}$. In particular, in [57], Kolitsch found that

$$\sum_{n=0}^{\infty} \overline{c\phi_3}(n)q^n = \frac{9q(q^9;q^9)_\infty^3}{(q;q)_\infty^3(q^3;q^3)_\infty}, \qquad (2.4.1)$$

which readily implies that $\overline{c\phi_3}(n) \equiv 0 \pmod{3^2}$. In a short note, J. Sellers [86], found that, for all $n \geq 1$,

$$\overline{c\phi_5}(5n) \equiv 0 \pmod{5^3},$$
$$\overline{c\phi_7}(7n) \equiv 0 \pmod{7^3},$$

and

$$\overline{c\phi_5}(11n) \equiv 0 \pmod{11^3}.$$

Furthermore, by employing a well-known result of Jacobi in (2.4.1), Sellers [88] proved an analogous result (see(2.4.2) below) involving $\overline{c\phi_3}(3n)$ modulo 3^4. Recently, Baruah and Sarmah [19] have found an expression for the generating function for

$\overline{c\phi_4}(n)$ and also deduced the congruences

$$\overline{c\phi}_4(2n) \equiv 0 \pmod{4^3},$$
$$\overline{c\phi}_4(4n+3) \equiv 0 \pmod{4^4},$$
$$\overline{c\phi}_4(4n) \equiv 0 \pmod{4^4}.$$

In the following, we find a simple proof of Sellers's result and a new result with the help of (2.3.9) and (2.4.1).

Theorem 2.4.1. *We have*

$$\overline{c\phi_3}(3n) \equiv 0 \ (\mathrm{mod}\ 3^4) \tag{2.4.2}$$

and

$$\overline{c\phi_3}(3n+2) \equiv 0 \ (\mathrm{mod}\ 3^3). \tag{2.4.3}$$

Proof. Employing (2.3.9) in (2.4.1), we find that

$$\sum_{n=0}^{\infty} \overline{c\phi_3}(n)q^n = \frac{9q(q^9;q^9)_\infty^{18}(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty^{15}(q^{18};q^{18})_\infty^6}\{1+3qw(q^3)+9q^2w^2(q^3)+8q^3w^3(q^3)$$
$$+12q^4w^4(q^3)+16q^6w^6(q^3)\}. \tag{2.4.4}$$

Extracting the terms involving q^{3n} and q^{3n+2}, respectively, from both sides of the above, we find that

$$\sum_{n=0}^{\infty} \overline{c\phi_3}(3n)q^{3n} = 81q^3w^2(q^3)\frac{(q^9;q^9)_\infty^{18}(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty^{15}(q^{18};q^{18})_\infty^6} \tag{2.4.5}$$

and

$$\sum_{n=0}^{\infty} \overline{c\phi_3}(3n+2)q^{3n+2} = 27q^2\Big\{w(q^3)\frac{(q^9;q^9)_\infty^{18}(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty^{15}(q^{18};q^{18})_\infty^6}$$
$$+4q^3w^4(q^3)\frac{(q^9;q^9)_\infty^{18}(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty^{15}(q^{18};q^{18})_\infty^6}\Big\}. \tag{2.4.6}$$

Thus,

$$\sum_{n=0}^{\infty}\overline{c\phi_3}(3n)q^n = 81qw^2(q)\frac{(q^3;q^3)_\infty^{18}(q^2;q^2)_\infty^2}{(q;q)_\infty^{15}(q^6;q^6)_\infty^6} \tag{2.4.7}$$

and

$$\sum_{n=0}^{\infty}\overline{c\phi_3}(3n+2)q^n = 27\left\{w(q)\frac{(q^3;q^3)_\infty^{18}(q^2;q^2)_\infty^2}{(q;q)_\infty^{15}(q^6;q^6)_\infty^6} + 4qw^4(q)\frac{(q^3;q^3)_\infty^{18}(q^2;q^2)_\infty^2}{(q;q)_\infty^{15}(q^6;q^6)_\infty^6}\right\}, \tag{2.4.8}$$

from which we readily arrive at (2.4.2) and (2.4.3), respectively. With the help of (2.2.6), we can recast (2.4.7) and (2.4.8) in the form

$$\sum_{n=0}^{\infty}\overline{c\phi_3}(3n)q^n = 81q\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^{13}}$$

and

$$\sum_{n=0}^{\infty}\overline{c\phi_3}(3n+2)q^n = 27\left\{\frac{(q^3;q^3)_\infty^{15}(q^2;q^2)_\infty}{(q;q)_\infty^{14}(q^6;q^6)_\infty^3} + 4q\frac{(q^3;q^3)_\infty^{6}(q^6;q^6)_\infty^6}{(q;q)_\infty^{11}(q^2;q^2)_\infty^2}\right\},$$

respectively. □

Chapter 3

A New Proof of a Modular Relation for Ramanujan's Cubic Continued Fraction and Related Results

3.1 Introduction

In the introductory chapter we have defined Roger- Ramanujan continued fraction $R(q)$ and discussed modular identities for $R(q)$. Ramanujan's well-known modular identity relating $R(q)$ and $R(q^5)$ is

$$R^5(q) = R(q^5)\frac{1 - 2R(q^5) + 4R^2(q^5) - 3R^3(q^5) + R^4(q^5)}{1 + 3R(q^5) + 4R^2(q^5) + 2R^3(q^5) + R^4(q^5)}. \tag{3.1.1}$$

Proof of this result was given by Ramanujan [81], Rogers [84], Watson [92], Ramanathan [78].

In his notebooks, Ramanujan recorded many identities involving $R(q)$ which can be found in [24, 20, 79, 80]. One of the most important formulas for $R(q)$ is

$$\frac{1}{R(q)} - 1 - R(q) = \frac{(q^{1/5}; q^{1/5})_\infty}{q^{1/5}(q^5; q^5)_\infty}. \tag{3.1.2}$$

Furthermore, on p. 206 of his lost notebook [80], Ramanujan recorded the identities, namely,

$$\frac{1}{\sqrt{R(q)}} - \alpha\sqrt{R(q)} = \frac{1}{q^{1/10}}\sqrt{\frac{(q;q)_\infty}{(q^5;q^5)_\infty}} \prod_{n=1}^{\infty} \frac{1}{1 + \alpha q^{n/5} + q^{2n/5}} \tag{3.1.3}$$

and

$$\frac{1}{\sqrt{R(q)}} - \beta\sqrt{R(q)} = \frac{1}{q^{1/10}}\sqrt{\frac{(q;q)_\infty}{(q^5;q^5)_\infty}}\prod_{n=1}^{\infty}\frac{1}{1+\beta q^{n/5}+q^{2n/5}}, \tag{3.1.4}$$

where $\alpha = \dfrac{1-\sqrt{5}}{2}$ and $\beta = \dfrac{1+\sqrt{5}}{2}$.

Proofs of these identities are given by Ramanathan [78] and Berndt et al. [23]. See also [24, pp. 21-24].

It has been observed that these identities provides an amazing factorization of the result in (3.1.2).

Recently, C. Gugg in his paper [47], gave product identities for the expressions appearing in the numerator and denominator of the Ramanujan identity (3.1.1) by employing (3.1.3) and (3.1.4).

Now, we recall Ramanujan's cubic continued fraction

$$G(q) = \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \cdots, \quad |q| < 1. \tag{3.1.5}$$

In 1995, H. H. Chan [34] established several modular relations connecting $G(q)$ with $G(-q)$, $G(q^2)$ and $G(q^3)$. Some of those relations are

$$G(q) + G(-q) + 2G^2(-q)G^2(q) = 0, \tag{3.1.6}$$

$$G^2(q) + 2G^2(q^2)G(q) - G(q^2) = 0, \tag{3.1.7}$$

and

$$\frac{1 - G(q^3) + G^2(q^3)}{1 + 2G(q^3) + 4G^2(q^3)} = \frac{G^3(q)}{G(q^3)}. \tag{3.1.8}$$

H.H. Chan [34] proved (3.1.8) by using Ramanujan's modular equations of degree 3. N.D. Baruah [11], C. Adiga, T. Kim, M.S.M. Naika and H.S. Madhusudhan [1] also found alternative proofs of (3.1.8). Baruah [12] also established two modular identities connecting $G(q)$ with $G(q^5)$ and $G(q^7)$ respectively. Further modular identities for $G(q)$ have been found by Naika, S. Chandankumar and K.S. Bairy [68].

In our work, we prove (3.1.8) by deriving product representations namely,

$$\frac{1}{\sqrt{G(q)}} + \sqrt{G(q)} = \frac{\sqrt{\chi^3(-q^3)}}{q^{1/6}\varphi(-q^3)} \cdot \sqrt{\chi(-q)}\psi(q^{1/3}) \tag{3.1.9}$$

and

$$\frac{1}{\sqrt{G(q)}} - 2\sqrt{G(q)} = \frac{\sqrt{\chi^3(-q^3)}}{q^{1/6}\varphi(-q^3)} \cdot \frac{\varphi(-q^{1/3})}{\sqrt{\chi(-q)}}. \tag{3.1.10}$$

We also find some interesting identities related to $G(q)$. In the next section, we give some preliminary results and in the final section, we present some new identities involving Ramanujan's cubic continued fraction $G(q)$ and give a new proof of (3.1.8).

3.2 Main results and their proofs

Theorem 3.2.1. *We have*

$$\frac{1}{\sqrt{G(q)}} + \sqrt{G(q)} = \frac{\sqrt{\chi^3(-q^3)}}{q^{1/6}\varphi(-q^3)} \cdot \sqrt{\chi(-q)}\psi(q^{1/3}) \tag{3.2.1}$$

and

$$\frac{1}{\sqrt{G(q)}} - 2\sqrt{G(q)} = \frac{\sqrt{\chi^3(-q^3)}}{q^{1/6}\varphi(-q^3)} \cdot \frac{\varphi(-q^{1/3})}{\sqrt{\chi(-q)}}. \tag{3.2.2}$$

Proof of (3.2.1). We recall from [20, p. 49, corollary(ii)], [20, p. 350, Eq.(2.3)] and [20, p. 39, Entriy 24(iii)] that

$$\psi(q) = f(q^3, q^6) + q\psi(q^9), \tag{3.2.3}$$

$$\chi(-q) = \frac{\varphi(-q^3)}{f(q, q^2)}, \tag{3.2.4}$$

and

$$\chi^3(q) = \frac{\varphi(q)}{\psi(-q)}. \tag{3.2.5}$$

From (1.6.3), (3.2.4), and (3.2.5), we have

$$G(q) = q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty} = q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)} = q^{1/3}\frac{\psi(q^3)}{f(q,q^2)}. \tag{3.2.6}$$

Thus,

$$\begin{aligned}\frac{1}{\sqrt{G(q)}} + \sqrt{G(q)} &= \frac{1}{q^{1/6}}\sqrt{\frac{f(q,q^2)}{\psi(q^3)}} + q^{1/6}\sqrt{\frac{\psi(q^3)}{f(q,q^2)}} \\ &= \frac{f(q,q^2) + q^{1/3}\psi(q^3)}{q^{1/6}\sqrt{\psi(q^3)f(q,q^2)}}.\end{aligned}$$

Using (3.2.3) and (3.2.4) in the numerator and the denominator of the right side of above, we find that

$$\frac{1}{\sqrt{G(q)}} + \sqrt{G(q)} = \frac{\sqrt{\varphi(-q^3)}}{q^{1/6}\sqrt{\psi(q^3)\varphi(-q^3)}} \cdot \sqrt{\chi(-q)}\psi(q^{1/3}). \tag{3.2.7}$$

Employing (3.2.5) in the right side of (3.2.7), we arrive at (3.2.1). □

Proof of (3.2.2). From [20, p. 345, Entries 1(i), (ii)], we obtain

$$1 - 2G(q) = \frac{\varphi(-q^{1/3})}{q^{1/3}\varphi(-q^3)}. \tag{3.2.8}$$

Using (3.2.8)

$$\frac{1}{\sqrt{G(q)}} - 2\sqrt{G(q)} = \frac{\varphi(-q^{1/3})}{q^{1/3}\varphi(-q^3)\sqrt{G(q)}}. \tag{3.2.9}$$

Employing (3.2.6) in the right side of (3.2.9), we arrive at (3.2.2). □

Theorem 3.2.2. *We have*

$$1 - G(q) + G^2(q) = G(q) \cdot \frac{\chi^3(-q^3)\chi(-q)\psi^4(q)}{q^{1/3}\varphi^2(-q^3)\psi(q^3)\psi(q^{1/3})} \tag{3.2.10}$$

and

$$1 + 2G(q) + 4G^2(q) = G(q) \cdot \frac{\chi^3(-q^3)\phi^4(-q)}{q^{1/3}\chi(-q)\varphi^3(-q^3)\varphi(-q^{1/3})}. \tag{3.2.11}$$

Proof of (3.2.10). Here, we require (3.2.1). Note that for each i=1,2, we obtain an identity from (3.2.1) by replacing $q^{1/3}$ with $\omega^i q^{1/3}$, where $\omega = e^{2\pi i/3}$. Multiplying these two identities, we have

$$\prod_{i=1,2}\left(\frac{1}{\sqrt{\omega^i G(q)}}+\sqrt{\omega^i G(q)}\right)=\prod_{i=1,2}\frac{\sqrt{\chi^3(-q^3)\chi(-q)}}{\omega^{i/2}q^{1/6}\varphi(-q^3)}\cdot\psi(\omega^i q^{1/3})$$
$$=\frac{\chi^3(-q^3)\chi(-q)}{q^{1/3}\varphi^2(-q^3)}\cdot\psi(\omega q^{1/3})\psi(\omega^2 q^{1/3}). \qquad (3.2.12)$$

Since

$$(1-q)(1-\omega q)(1-\omega^2 q)=1-q^3,$$

we have

$$(q;q)_\infty(\omega q;\omega q)_\infty(\omega^2 q;\omega^2 q)_\infty=\prod_{n=1}^{\infty}(1-q^n)(1-\omega^n q^n)(1-\omega^{2n}q^n)$$
$$=\prod_{3|n}(1-q^n)^3\prod_{3\nmid n}(1-q^{3n})$$
$$=\frac{(q^3;q^3)_\infty^4}{(q^9;q^9)_\infty}. \qquad (3.2.13)$$

From (1.5.3)

$$\psi(q)=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \qquad (3.2.14)$$

Using (3.2.14) and (3.2.13), we find that

$$\psi(\omega q^{1/3})\psi(\omega^2 q^{1/3})=\frac{(\omega^2 q^{2/3};\omega^2 q^{2/3})_\infty^2(\omega q^{2/3};\omega q^{2/3})_\infty^2}{(\omega q^{1/3};\omega q^{1/3})_\infty(\omega^2 q^{1/3};\omega^2 q^{1/3})_\infty}$$
$$=\frac{(\omega^2 q^{2/3};\omega^2 q^{2/3})_\infty^2(\omega q^{2/3};\omega q^{2/3})_\infty^2(q^{2/3};q^{2/3})_\infty^2(q^{1/3};q^{1/3})_\infty}{(\omega q^{1/3};\omega q^{1/3})_\infty(\omega^2 q^{1/3};\omega^2 q^{1/3})_\infty(q^{1/3};q^{1/3})_\infty(q^{2/3};q^{2/3})_\infty^2}$$
$$=\frac{(q^2;q^2)_\infty^8(q^3;q^3)_\infty(q^{1/3};q^{1/3})_\infty}{(q^6;q^6)_\infty^2(q;q)_\infty^4(q^{2/3};q^{2/3})_\infty^2}$$
$$=\frac{\psi^4(q)}{\psi(q^3)\psi(q^{1/3})}. \qquad (3.2.15)$$

Employing (3.2.15) in (3.2.12) and expanding the product on the left side, we obtain

$$\frac{1-G(q)+G^2(q)}{G(q)}=\frac{\chi^3(-q^3)\chi(-q)\psi^4(q)}{q^{1/3}\varphi^2(-q^3)\psi(q^3)\psi(q^{1/3})}. \qquad (3.2.16)$$

Thus, we complete the proof of (3.2.10). □

Proof of (3.2.11). The proof of the second part of Theorem 3.2.2 is quite similar to the proof of first part. Here we require (3.2.2).

Proceeding as in the proof of first part, we deduce that

$$\prod_{i=1,2}\left(\frac{1}{\sqrt{\omega^i G(q)}}-2\sqrt{\omega^i G(q)}\right)=\frac{\chi^3(-q^3)}{q^{1/3}\varphi^2(-q^3)\chi(-q)}\cdot\varphi(-\omega q^{1/3})\varphi(-\omega^2 q^{1/3}). \tag{3.2.17}$$

Again, we have from (1.5.5)

$$\varphi(-q)=\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}. \tag{3.2.18}$$

Using (3.2.18) and (3.2.13), we obtain

$$\begin{aligned}\varphi(-\omega q^{1/3})\varphi(-\omega^2 q^{1/3})&=\frac{(\omega q^{1/3};\omega q^{1/3})_\infty^2(\omega^2 q^{1/3};\omega^2 q^{1/3})_\infty^2}{(\omega q^{2/3};\omega q^{2/3})_\infty(\omega q^{2/3};\omega q^{2/3})_\infty^2}\\&=\frac{(q;q)_\infty^8(q^6;q^6)_\infty(q^{2/3};q^{2/3})_\infty}{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^2(q^{1/3};q^{1/3})_\infty^2}\\&=\frac{\varphi^4(-q)}{\varphi(-q^3)\varphi(-q^{1/3})}.\end{aligned} \tag{3.2.19}$$

Now, expanding the left side of (3.2.17) and using (3.2.19), we obtain

$$\frac{1+2G(q)+4G^2(q)}{G(q)}=\frac{\chi^3(-q^3)\varphi^4(-q)}{q^{1/3}\chi(-q)\varphi^2(-q^3)\varphi(-q^3)\varphi(-q^{1/3})}, \tag{3.2.20}$$

which is equivalent to (3.2.11). □

Corollary 3.2.3.

$$\begin{aligned}\frac{1}{\psi(q)\varphi(-q)}&=\frac{q^2\varphi^5(-q^9)\psi(q^9)}{\chi^6(-q^9)\psi^4(q^3)\varphi^4(-q^3)}\Big\{4\,G^2(q^3)-2\,G(q^3)+3+\frac{1}{G(q^3)}\\&\quad+\frac{1}{G^2(q^3)}\Big\},\end{aligned} \tag{3.2.21}$$

$$\begin{aligned}\frac{1}{\psi(q)\varphi^2(-q)}&=\frac{q^3\chi(-q^3)\varphi^8(-q^9)\psi(q^9)}{\chi^9(-q^3)\psi^4(q^3)\varphi^8(-q^3)}\Big\{16\,G^3(q^3)+12\,G(q^3)+8+\frac{9}{G(q^3)}\\&\quad+\frac{3}{G^2(q^3)}+\frac{1}{G^3(q^3)}\Big\}.\end{aligned} \tag{3.2.22}$$

Proof. Multiplying (3.2.10) and (3.2.11) and replacing q by q^3, we can easily derive (3.2.21).

Squaring (3.2.11) and multiplying with (3.2.10) and replacing q by q^3, we can easily derive (3.2.22). □

Next, as corollary we obtain the following 3-dissections, which are equivalent to some results in the previous chapter.

Corollary 3.2.4. *If* $\psi(q)$ *and* $\phi(q)$ *are defined in* (1.5.3) *and* (1.5.2), *then*

$$\frac{1}{\psi(q)} = \frac{\psi^3(q^9)}{\psi^4(q^3)w^2(q^3)} - q\frac{\psi^3(q^9)}{\psi^4(q^3)w(q^3)} + q^2\frac{\psi^3(q^9)}{\psi^4(q^3)}, \tag{3.2.23}$$

$$\frac{1}{\varphi(-q)} = \frac{\varphi^3(-q^9)}{\varphi^4(-q^3)} + 2q\frac{\varphi^3(-q^9)w(q^3)}{\varphi^4(-q^3)} + 4q^2\frac{\varphi^3(-q^9)w^2(q^3)}{\varphi^4(-q^3)}, \tag{3.2.24}$$

$$\frac{1}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{(q^9;q^9)^3_\infty(q^{18};q^{18})^3_\infty}{(q^3;q^3)^4_\infty(q^6;q^6)^4_\infty}\bigg\{\frac{1}{w^2(q^3)} + \frac{q}{w(q^3)} + 3q^2 - 2q^3w(q^3) + 4q^4w^3(q^3)\bigg\}, \tag{3.2.25}$$

$$\frac{1}{(q;q)^3_\infty} = \frac{(q^9;q^9)^9_\infty}{(q^3;q^3)^{12}_\infty}\bigg\{\frac{1}{w^2(q^3)} + \frac{3q}{w(q^3)} + 9q^2 + 8q^3w(q^3) + 12q^4w^2(q^3) + 16q^6w^4(q^3)\bigg\}, \tag{3.2.26}$$

where

$$G(q) = q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty} = q^{1/3}w(q). \tag{3.2.27}$$

Proof. Employing (3.2.27) in (3.2.10) and replacing q by q^3, we arrive at (3.2.23). Employing (3.2.27) in (3.2.11) and replacing q by q^3, we find (3.2.24). In (3.2.21), employing (3.2.27) in the right hand side and using (3.2.14), (3.2.18) in the left hand side, we obtain (3.2.25). In (3.2.22), employing (3.2.27) in the right side and using (3.2.14), (3.2.18) in the left side, we arrive at (3.2.26). □

Now, let us consider L.J. Slater's identities

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{f(-q,-q^5)}{\psi(-q)} \tag{3.2.28}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{f(-q^3,-q^3)}{\psi(-q)}, \tag{3.2.29}$$

which are analogous to the famous Roger-Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{f(-q^2,-q^3)}{f(-q)}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{f(-q,-q^4)}{f(-q)}.$$

Let us define

$$A(q) := \frac{f(-q,-q^5)}{\psi(-q)} \tag{3.2.30}$$

and

$$B(q) := \frac{f(-q^3,-q^3)}{\psi(-q)}. \tag{3.2.31}$$

Andrews [4], proved that

$$G(q) = q^{1/3}\frac{A(q)}{B(q)} = q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)}. \tag{3.2.32}$$

In the next theorem, we find q-product representations of the even and odd terms of $\frac{B(q)}{A(q)} = \frac{q^{1/3}}{G(q)}$.

Theorem 3.2.5. *If*

$$D(q) = \frac{q^{1/3}}{G(q)} = \frac{B(q)}{A(q)} = \sum_{n=0}^{\infty} c_n q^n, \tag{3.2.33}$$

then

$$\sum_{n=0}^{\infty} c_{2n}q^n = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty(q^3;q^3)_\infty(q^6;q^6)_\infty} \tag{3.2.34}$$

and

$$\sum_{n=0}^{\infty} c_{2n+1}q^n = \frac{(q;q)_\infty(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty(q^3;q^3)_\infty^3} = q^{-1/3}G(q). \tag{3.2.35}$$

Proof of Theorem 3.2.5. From (3.2.33), we have

$$\sum_{n=0}^{\infty} c_{2n}q^{2n} = \frac{1}{2}[D(q)+D(-q)].$$
$$= \frac{1}{2}\left[\frac{B(q)A(-q)+B(-q)A(q)}{A(q)A(-q)}\right], \qquad (3.2.36)$$

where we have used (3.2.33).

We recall from [20, p. 51, Example (v)] that

$$f(-q,-q^5) = \chi(-q)\psi(q^3). \qquad (3.2.37)$$

From (3.2.30) and (3.2.37),

$$A(q)A(-q) = \frac{\chi(-q)\chi(q)\psi(q^3)\psi(-q^3)}{\psi(-q)\psi(q)}. \qquad (3.2.38)$$

Again, employing (3.2.30) and (3.2.31), we find that

$$B(q)A(-q)+B(-q)A(q) = \frac{f(q,q^5)f(-q^3,-q^3)+f(-q,-q^5)f(q^3,q^3)}{\psi(-q)\psi(q)}. \qquad (3.2.39)$$

Next, recall from [20, p.45, Entry 29] that if $ab = cd$, then

$$f(a,b)f(c,d)+f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc). \qquad (3.2.40)$$

Setting $a = q$, $b = q^5$, and $c = d = -q^3$ in (3.2.40) and using in the right side of (3.2.39), we have

$$B(q)A(-q)+B(-q)A(q) = \frac{2f^2(-q^4,-q^8)}{\psi(-q)\psi(q)}$$
$$= \frac{2(q^4;q^4)_\infty^2}{\psi(-q)\psi(q)}, \qquad (3.2.41)$$

where we have also used (1.5.4). Applying (3.2.41) and (3.2.38) in (3.2.36), we arrive at

$$\sum_{n=0}^{\infty} c_{2n}q^{2n} = \frac{1}{2}\left[\frac{2(q^4;q^4)_\infty^2}{\chi(-q)\chi(q)\psi(q^3)\psi(-q^3)}\right]. \qquad (3.2.42)$$

Now, using q- product representations for $\psi(q)$ and $\chi(q)$ in (3.2.42), and then replacing q^2 by q, we arrive at (3.2.34).

The proof of the second part of Theorem 3.2.5 is quite similar to the proof of first part.

Proceeding as in the proof of first part, we have

$$\sum_{n=0}^{\infty} c_{2n+1}q^{2n+1} = \frac{1}{2}[D(q) - D(-q)] = \frac{1}{2}\left[\frac{B(q)A(-q) - B(-q)A(q)}{A(q)A(-q)}\right]. \tag{3.2.43}$$

Using (3.2.30) and (3.2.31), we find that

$$B(q)A(-q) - B(-q)A(q) = \frac{f(q,q^5)f(-q^3,-q^3) - f(-q,-q^5)f(q^3,q^3)}{\psi(-q)\psi(q)}. \tag{3.2.44}$$

Again, we recall from [20, p.45, Entry 29] that if $ab = cd$, then

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, ac^2d\right) f\left(\frac{b}{d}, acd^2\right). \tag{3.2.45}$$

Setting $a = q$, $b = q^5$, and $c = d = -q^3$ in (3.2.45), and employing in the right side of (3.2.44), we obtain

$$\begin{aligned} B(q)A(-q) - B(-q)A(q) &= \frac{2q\, f^2(-q^2,-q^{10})}{\psi(-q)\psi(q)} \\ &= \frac{2q\, \chi^2(-q^2)\psi^2(q^6)}{\psi(-q)\psi(q)}, \end{aligned} \tag{3.2.46}$$

where we have also used (3.2.37). Employing (3.2.46), (3.2.38) in (3.2.43), we deduce that

$$\sum_{n=0}^{\infty} c_{2n+1}q^{2n+1} = \frac{1}{2}\left[\frac{2q\, \chi(-q^2)\psi(q^6)}{\phi(-q^6)}\right]. \tag{3.2.47}$$

Now, using q-product representations for $\varphi(q)$, $\psi(q)$ and $\chi(q)$ on the right side of (3.2.47), diving both sides by q, and then replacing q^2 by q, we easily deduce the first equality of (3.2.35). The second equality then follows from (3.2.6). □

Remark 3.2.6. *M. Hirschhorn and Roselin [50] also proved* (3.2.34) *and* (3.2.35) *by applying a 2-dissection for* $\dfrac{(q^3;q^3)_\infty^3}{(q;q)_\infty}$.

Finally, we prove (3.1.8).

Proof of (3.1.8). Dividing (3.2.10) by (3.2.11) and replacing q by q^3, we obtain

$$\frac{1-G(q^3)+G^2(q^3)}{1+2G(q^3)+4G^2(q^3)} = \frac{\chi^2(-q^3)\psi^4(q^3)\varphi(-q^9)\varphi(-q)}{\varphi^4(-q^3)\psi(q^9)\psi(q)}$$
$$= \frac{\chi^3(-q)\chi^3(-q^9)}{\chi^{10}(-q^3)}, \tag{3.2.48}$$

where we have also used (3.2.5).

Since

$$G(q) = q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)},$$

we complete the proof of (3.1.8).

□

Chapter 4

Analogues of Ramanujan's Partition Identities and Congruences Arising from his Theta Functions and Modular Equations

4.1 Introduction

In the introductory chapter, we have defined the partition function $p_{[c^l d^m]}(n)$ as

$$\sum_{n=0}^{\infty} p_{[c^l d^m]}(n) q^n := \frac{1}{(q^c;q^c)_\infty^l (q^d;q^d)_\infty^m}. \tag{4.1.1}$$

Recently, H.-C. Chan [30] proved that

$$\sum_{n=0}^{\infty} p_{[1^1 2^1]}(3n+2) q^n = 3\,\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 (q^2;q^2)_\infty^4}, \tag{4.1.2}$$

and consequently, $p_{[1^1 2^1]}(3n+2) \equiv 0 \pmod 3$.

Again, H.-C. Chan and S. Cooper [32] considered the partition function $p_{[1^2 3^2]}(n)$ and proved the congruence $p_{[1^2 3^2]}(2n+1) \equiv 0 \pmod 2$ by showing that

$$\sum_{n=0}^{\infty} p_{[1^2 3^2]}(2n+1) q^n = 2\,\frac{(q^2;q^2)_\infty^4 (q^6;q^6)_\infty^4}{(q;q)_\infty^6 (q^3;q^3)_\infty^6}. \tag{4.1.3}$$

Note: The contents of this chapter appeared in *The Ramanujan Journal* [18] of Springer.

Here, combinatorially, $p_{[1^2 3^2]}(n)$ is the number of 4-colored partitions of n with two of the colors appearing only in multiples of 3.

Very recently, H.H. Chan and P.C. Toh [33] have given some beautiful analogues of (1.1.5) including the results in the following theorem. They have established these results using the theory of modular functions.

Theorem 4.1.1. *If the generalized partition function* $p_{[c^l d^m]}(n)$ *is defined by*(4.1.1), *then*

$$\sum_{n=0}^{\infty} p_{[1^4 5^4]}(2n+1)q^n = 4\,\frac{(q^2;q^2)_\infty^4 (q^{10};q^{10})_\infty^4}{(q;q)_\infty^8 (q^5;q^5)_\infty^8} + 8q\,\frac{(q^2;q^2)_\infty^8 (q^{10};q^{10})_\infty^8}{(q;q)_\infty^{12} (q^5;q^5)_\infty^{12}}, \tag{4.1.4}$$

$$\sum_{n=0}^{\infty} p_{[1^1 7^1]}(2n+1)q^n = \frac{(q^2;q^2)_\infty^2 (q^{14};q^{14})_\infty^2}{(q;q)_\infty^3 (q^7;q^7)_\infty^3}, \tag{4.1.5}$$

$$\sum_{n=0}^{\infty} p_{[1^2 11^2]}(2n+1)q^n = 2\,\frac{(q^2;q^2)_\infty^2 (q^{22};q^{22})_\infty^2}{(q;q)_\infty^4 (q^{11};q^{11})_\infty^4} + 2q\,\frac{(q^2;q^2)_\infty^4 (q^{22};q^{22})_\infty^4}{(q;q)_\infty^6 (q^{11};q^{11})_\infty^6}, \tag{4.1.6}$$

$$\sum_{n=0}^{\infty} p_{[1^1 23^1]}(2n+1)q^n = \frac{(q^2;q^2)_\infty (q^{46};q^{46})_\infty}{(q;q)_\infty^2 (q^{23};q^{23})_\infty^2} + q\,\frac{(q^2;q^2)_\infty^2 (q^{46};q^{46})_\infty^2}{(q;q)_\infty^3 (q^{23};q^{23})_\infty^3}. \tag{4.1.7}$$

It readily follows from (4.1.4) and (4.1.6) that

$$p_{[1^4 5^4]}(2n+1) \equiv 0 \pmod 4$$

and

$$p_{[1^2 11^2]}(2n+1) \equiv 0 \pmod 2.$$

In Section 4.2, we present new proofs of (4.1.3)–(4.1.7) by employing Ramanujan's modular equations of degrees 3, 5, 7, 11, and 23, respectively.

We also find a host of other analogous results by employing Ramanujan's theta function identities and modular equations. We also deduce some new interesting partition congruences. In Section 4.3, we present some new results analogous to (1.1.5) and new partition congruences deducible from them.

4.2 New proofs of (4.1.3)–(4.1.7)

The proofs of (4.1.3)–(4.1.7), can be given by using a certain type of modular equations independently found by Schröter, Russel, and Ramanujan. These modular equations were beautifully employed by Berndt [21] to deduce certain Farkas-Kra-type partition identities.

Proof of (4.1.3). We have

$$\sum_{n=0}^{\infty} p_{[1^2 3^2]}(n)q^n = \frac{1}{(q;q)_\infty^2 (q^3;q^3)_\infty^2} = \frac{1}{(q;q^2)_\infty^2 (q^3;q^6)_\infty^2 (q^2;q^2)_\infty^2 (q^6;q^6)_\infty^2}. \quad (4.2.1)$$

Replacing q by $-q$ in (4.2.1) and then subtracting the resulting identity from (4.2.1), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{[1^2 3^2]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^2 3^2]}(n)(-1)^n q^n \\
&= \frac{1}{(q^2;q^2)_\infty^2 (q^6;q^6)_\infty^2} \left\{ \frac{1}{(q;q^2)_\infty^2 (q^3;q^6)_\infty^2} - \frac{1}{(-q;q^2)_\infty^2 (-q^3;q^6)_\infty^2} \right\} \\
&= \frac{(q^4;q^4)_\infty^2 (q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^4 (q^6;q^6)_\infty^4} \left\{ (-q;q^2)_\infty^2 (-q^3;q^6)_\infty^2 - (q;q^2)_\infty^2 (q^3;q^6)_\infty^2 \right\}. \quad (4.2.2)
\end{aligned}$$

Now, we recall from [20, Entry 5(ii), p. 230] that if β has degree 3 over α, then

$$(\alpha\beta)^{1/4} + ((1-\alpha)(1-\beta))^{1/4} = 1,$$

which can be transformed into (see [21])

$$(-q;q^2)_\infty^2 (-q^3;q^6)_\infty^2 - (q;q^2)_\infty^2 (q^3;q^6)_\infty^2 = 4q(-q^2;q^2)_\infty^2 (-q^6;q^6)_\infty^2. \quad (4.2.3)$$

Employing (4.2.3) in (4.2.2), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{[1^2 3^2]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^2 3^2]}(n)(-1)^n q^n &= 4q \frac{(q^4;q^4)_\infty^2 (q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^4 (q^6;q^6)_\infty^4} \\
&\quad \times \left\{ (-q^2;q^2)_\infty^2 (-q^6;q^6)_\infty^2 \right\} \\
&= 4q \frac{(q^4;q^4)_\infty^4 (q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^6 (q^6;q^6)_\infty^6}. \quad (4.2.4)
\end{aligned}$$

Extracting from both sides of (4.2.4) those terms involving only q^{2n+1}, and then dividing both sides by q and replacing q^2 by q, we arrive at (4.1.3) to complete the proof. □

Proof of (4.1.4). We have

$$\sum_{n=0}^{\infty} p_{[1^4 5^4]}(n)q^n = \frac{1}{(q;q)_\infty^4 (q^5;q^5)_\infty^4} = \frac{1}{(q;q^2)_\infty^4 (q^5;q^{10})_\infty^4 (q^2;q^2)_\infty^4 (q^{10};q^{10})_\infty^4},$$

so that

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{[1^4 5^4]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^4 5^4]}(n)(-1)^n q^n \\
&= \frac{1}{(q^2;q^2)_\infty^4 (q^{10};q^{10})_\infty^4} \left\{ \frac{1}{(q;q^2)_\infty^4 (q^5;q^{10})_\infty^4} - \frac{1}{(-q;q^2)_\infty^4 (-q^5;q^{10})_\infty^4} \right\} \\
&= \frac{(q^4;q^4)_\infty^4 (q^{20};q^{20})_\infty^4}{(q^2;q^2)_\infty^8 (q^{10};q^{10})_\infty^8} \left\{ (-q;q^2)_\infty^4 (-q^5;q^{10})_\infty^4 - (q;q^2)_\infty^4 (q^5;q^{10})_\infty^4 \right\}. \qquad (4.2.5)
\end{aligned}$$

Now, we recall from [20, Entry 13(i), p. 280] the following modular equation of Ramanujan. If β has degree 5 over α, then

$$(\alpha\beta)^{1/2} + ((1-\alpha)(1-\beta))^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1$$

As shown by Berndt [21], the above modular equation can be transformed into

$$(-q;q^2)_\infty^4 (-q^5;q^{10})_\infty^4 - (q;q^2)_\infty^4 (q^5;q^{10})_\infty^4 = 8q + 16q^3 (-q^2;q^2)_\infty^4 (-q^{10};q^{10})_\infty^4. \qquad (4.2.6)$$

Employing (4.2.6) in (4.2.5), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{[1^4 5^4]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^4 5^4]}(n)(-1)^n q^n \\
&= 2q \, \frac{(q^4;q^4)_\infty^4 (q^{20};q^{20})_\infty^4}{(q^2;q^2)_\infty^8 (q^{10};q^{10})_\infty^8} \left\{ 4 + 8q^2 (-q^2;q^2)_\infty^4 (-q^{10};q^{10})_\infty^4 \right\} \\
&= 2q \, \frac{(q^4;q^4)_\infty^4 (q^{20};q^{20})_\infty^4}{(q^2;q^2)_\infty^8 (q^{10};q^{10})_\infty^8} \left\{ 4 + 8q^2 \frac{(q^4;q^4)_\infty^4 (q^{20};q^{20})_\infty^4}{(q^2;q^2)_\infty^4 (q^{10};q^{10})_\infty^4} \right\}. \qquad (4.2.7)
\end{aligned}$$

Now, extracting from both sides of (4.2.7) those terms involving only q^{2n+1}, and then dividing both sides by q and replacing q^2 by q, we arrive at (4.1.4) to complete the proof. □

Proof of (4.1.5). We have

$$\sum_{n=0}^{\infty} p_{[1^1 7^1]}(n)q^n = \frac{1}{(q;q)_\infty (q^7;q^7)_\infty} = \frac{1}{(q;q^2)_\infty (q^7;q^{14})_\infty (q^2;q^2)_\infty (q^{14};q^{14})_\infty},$$

so that

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{[1^17^1]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^17^1]}(n)(-1)^n q^n \\
&= \frac{1}{(q^2;q^2)_\infty (q^{14};q^{14})_\infty} \left\{ \frac{1}{(q;q^2)_\infty (q^7;q^{14})_\infty} - \frac{1}{(-q;q^2)_\infty(-q^7;q^{14})_\infty} \right\} \\
&= \frac{(q^4;q^4)_\infty (q^{28};q^{28})_\infty}{(q^2;q^2)_\infty^2 (q^{14};q^{14})_\infty^2} \left\{ (-q;q^2)_\infty(-q^7;q^{14})_\infty - (q;q^2)_\infty (q^7;q^{14})_\infty \right\}. \qquad (4.2.8)
\end{aligned}$$

Now, we recall from [20, Entry 19(i), p. 314] the following modular equation of Ramanujan. If β has degree 7 over α, then

$$(\alpha\beta)^{1/8} + ((1-\alpha)(1-\beta))^{1/8} = 1.$$

As shown by Berndt [21], the above modular equation can be transformed into

$$(-q;q^2)_\infty(-q^7;q^{14})_\infty - (q;q^2)_\infty(q^7;q^{14})_\infty = 2q(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty. \qquad (4.2.9)$$

With the aid of (4.2.9), we deduce from from (4.2.8) that

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{[1^17^1]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^17^1]}(n)(-1)^n q^n &= 2q\, \frac{(q^4;q^4)_\infty (q^{28};q^{28})_\infty}{(q^2;q^2)_\infty^2 (q^{14};q^{14})_\infty^2} \\
&\quad \times \left\{ (-q^2;q^2)_\infty (-q^{14};q^{14})_\infty \right\} \\
&= 2q\, \frac{(q^4;q^4)_\infty^2 (q^{28};q^{28})_\infty^2}{(q^2;q^2)_\infty^3 (q^{14};q^{14})_\infty^3}. \qquad (4.2.10)
\end{aligned}$$

Extracting the odd terms from both sides of (4.2.10) we arrive at (4.1.5) to complete the proof. □

Proof of (4.1.6). We have

$$\sum_{n=0}^{\infty} p_{[1^211^2]}(n)q^n = \frac{1}{(q;q)_\infty^2 (q^{11};q^{11})_\infty^2} = \frac{1}{(q;q^2)_\infty^2 (q^{11};q^{22})_\infty^2 (q^2;q^2)_\infty^2 (q^{22};q^{22})_\infty^2}. \qquad (4.2.11)$$

Replacing q by $-q$ in (4.2.11) and then subtracting the resulting identity from

(4.2.11), we obtain

$$\sum_{n=0}^{\infty} p_{[1^2 11^2]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^2 11^2]}(n)(-1)^n q^n$$
$$= \frac{1}{(q^2;q^2)_\infty^2 (q^{22};q^{22})_\infty^2} \left\{ \frac{1}{(q;q^2)_\infty^2 (q^{11};q^{22})_\infty^2} - \frac{1}{(-q;q^2)_\infty^2 (-q^{11};q^{22})_\infty^2} \right\}$$
$$= \frac{(q^4;q^4)_\infty^2 (q^{44};q^{44})_\infty^2}{(q^2;q^2)_\infty^4 (q^{22};q^{22})_\infty^4} \left\{ (-q;q^2)_\infty^2 (-q^{11};q^{22})_\infty^2 - (q;q^2)_\infty^2 (q^{11};q^{22})_\infty^2 \right\}. \quad (4.2.12)$$

Now, we recall from [20, Entry 7(i), p. 363] that if β has degree 11 over α, then

$$(\alpha\beta)^{1/4} + ((1-\alpha)(1-\beta))^{1/4} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1,$$

which can be transformed into (see [21])

$$(-q;q^2)_\infty^2 (-q^{11};q^{22})_\infty^2 - (q;q^2)_\infty^2 (q^{11};q^{22})_\infty^2 = 4q + 4q^3(-q^2;q^2)_\infty^2 (-q^{22};q^{22})_\infty^2. \quad (4.2.13)$$

Employing (4.2.13) in (4.2.12), we find that

$$\sum_{n=0}^{\infty} p_{[1^2 11^2]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^2 11^2]}(n)(-1)^n q^n$$
$$= 2q \frac{(q^4;q^4)_\infty^2 (q^{44};q^{44})_\infty^2}{(q^2;q^2)_\infty^4 (q^{22};q^{22})_\infty^4} \left\{ 2 + 2q^2(-q^2;q^2)_\infty^2 (-q^{22};q^{22})_\infty^2 \right\}$$
$$= 2q \frac{(q^4;q^4)_\infty^2 (q^{44};q^{44})_\infty^2}{(q^2;q^2)_\infty^4 (q^{22};q^{22})_\infty^4} \left\{ 2 + 2q^2 \frac{(q^4;q^4)_\infty^2 (q^{44};q^{44})_\infty^2}{(q^2;q^2)_\infty^2 (q^{22};q^{22})_\infty^2} \right\}.. \quad (4.2.14)$$

Extracting from both sides of (4.2.14) those terms involving only q^{2n+1}, and then dividing both sides by q and replacing q^2 by q, we arrive at (4.1.6) to finish the proof. □

Proof of (4.1.7). We have

$$\sum_{n=0}^{\infty} p_{[1^1 23^1]}(n)q^n = \frac{1}{(q;q)_\infty (q^{23};q^{23})_\infty} = \frac{1}{(q;q^2)_\infty (q^{23};q^{46})_\infty (q^2;q^2)_\infty (q^{46};q^{46})_\infty}. \quad (4.2.15)$$

Replacing q by $-q$ in (4.2.15) and then subtracting the resulting identity from

(4.2.15), we obtain

$$\sum_{n=0}^{\infty} p_{[1^1 23^1]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^1 23^1]}(n)(-1)^n q^n$$
$$= \frac{1}{(q^2;q^2)_\infty(q^{46};q^{46})_\infty}\left\{\frac{1}{(q;q^2)_\infty(q^{23};q^{46})_\infty} - \frac{1}{(-q;q^2)_\infty(-q^{23};q^{46})_\infty}\right\}$$
$$= \frac{(q^4;q^4)_\infty(q^{92};q^{92})_\infty}{(q^2;q^2)_\infty^2(q^{46};q^{46})_\infty^2}\left\{(-q;q^2)_\infty(-q^{23};q^{46})_\infty - (q;q^2)_\infty(q^{23};q^{46})_\infty\right\}. \quad (4.2.16)$$

Now, we recall from [20, Entry 15(i), p. 411] that if β has degree 23 over α, then

$$(\alpha\beta)^{1/8} + ((1-\alpha)(1-\beta))^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1,$$

which can be transformed into (see [21])

$$(-q;q^2)_\infty(-q^{23};q^{46})_\infty - (q;q^2)_\infty(q^{23};q^{46})_\infty = 2q + 2q^3(-q^2;q^2)_\infty(-q^{46};q^{46})_\infty. \quad (4.2.17)$$

Employing (4.2.17) in (4.2.16), we find that

$$\sum_{n=0}^{\infty} p_{[1^1 23^1]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^1 23^1]}(n)(-1)^n q^n$$
$$= 2q\,\frac{(q^4;q^4)_\infty(q^{92};q^{92})_\infty}{(q^2;q^2)_\infty^2(q^{46};q^{46})_\infty^2}\left\{1 + q^2(-q^2;q^2)_\infty(-q^{46};q^{46})_\infty\right\}$$
$$= 2q\,\frac{(q^4;q^4)_\infty(q^{92};q^{92})_\infty}{(q^2;q^2)_\infty^2(q^{46};q^{46})_\infty^2}\left\{1 + q^2\frac{(q^4;q^4)_\infty(q^{92};q^{92})_\infty}{(q^2;q^2)_\infty(q^{46};q^{46})_\infty}\right\}. \quad (4.2.18)$$

Extracting from both sides of (4.2.18) those terms involving only q^{2n+1}, and then dividing both sides by q and replacing q^2 by q, we arrive at (4.1.7) to complete the proof. □

Corollary 4.2.1. *We have*

$$p_{[1^2 3^2]}(4n+3) \equiv 0 \pmod 4. \quad (4.2.19)$$

Proof. From (4.1.3), we find that

$$\sum_{n=0}^{\infty} p_{[1^23^2]}(2n+1)q^n - \sum_{n=0}^{\infty} p_{[1^23^2]}(2n+1)(-q)^n$$
$$= \frac{2}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2}\left\{\frac{1}{(q;q^2)_\infty^6(q^3;q^6)_\infty^6} - \frac{1}{(-q;q^2)_\infty^6(-q^3;q^6)_\infty^6}\right\}$$
$$= 2\,\frac{(q^4;q^4)_\infty^6(q^{12};q^{12})_\infty^6}{(q^2;q^2)_\infty^8(q^6;q^6)_\infty^8}\Big\{(-q;q^2)_\infty^6(-q^3;q^6)_\infty^6$$
$$- (q;q^2)_\infty^6(q^3;q^6)_\infty^6\Big\}$$
$$= 2\,\frac{(q^4;q^4)_\infty^6(q^{12};q^{12})_\infty^6}{(q^2;q^2)_\infty^8(q^6;q^6)_\infty^8}\Big\{\Big((-q;q^2)_\infty^2(-q^3;q^6)_\infty^2$$
$$- (q;q^2)_\infty^2(q^3;q^6)_\infty^2\Big)^3 + 3(q^2;q^4)_\infty^2(q^6;q^{12})_\infty^2$$
$$\times \left((-q;q^2)_\infty^2(-q^3;q^6)_\infty^2 - (q;q^2)_\infty^2(q^3;q^6)_\infty^2\right)\Big\}. \tag{4.2.20}$$

Employing (4.2.3) in (4.2.20), we obtain

$$\sum_{n=0}^{\infty} p_{[1^23^2]}(2n+1)q^n - \sum_{n=0}^{\infty} p_{[1^23^2]}(2n+1)(-q)^n = 24q\,\frac{(q^4;q^4)_\infty^6(q^{12};q^{12})_\infty^6}{(q^2;q^2)_\infty^8(q^6;q^6)_\infty^8} + 128q^3\,\frac{(q^4;q^4)_\infty^{12}(q^{12};q^{12})_\infty^{12}}{(q^2;q^2)_\infty^{14}(q^6;q^6)_\infty^{14}}, \tag{4.2.21}$$

and therefore,

$$\sum_{n=0}^{\infty} p_{[1^23^2]}(4n+3)q^n = 12\,\frac{(q^2;q^2)_\infty^6(q^6;q^6)_\infty^6}{(q;q)_\infty^8(q^3;q^3)_\infty^8} + 64q\,\frac{(q^2;q^2)_\infty^{12}(q^6;q^6)_\infty^{12}}{(q;q)_\infty^{14}(q^3;q^3)_\infty^{14}}. \tag{4.2.22}$$

The proffered congruence in (4.2.19) now readily follows from (4.2.22). □

Remark 4.2.2. *The above congruence is a particular case of a general congruence found by Chan and Cooper [32, Eq. (1.4), Theorem 1.1].*

Corollary 4.2.3. *We have*

$$p_{[1^17^1]}(8n+7) \equiv 0 \pmod 2. \tag{4.2.23}$$

Proof. From (4.1.5), we have

$$\sum_{n=0}^{\infty} p_{[1^17^1]}(2n+1)q^n - \sum_{n=0}^{\infty} p_{[1^17^1]}(2n+1)(-1)^n q^n$$
$$= \frac{(q^4;q^4)_\infty^3(q^{28};q^{28})_\infty^3}{(q^2;q^2)_\infty^4(q^{14};q^{14})_\infty^4}\left\{(-q;q^2)_\infty^3(-q^7;q^{14})_\infty^3 - (q;q^2)_\infty^3(q^7;q^{14})_\infty^3\right\}. \quad (4.2.24)$$

Employing (4.2.9) in (4.2.24), and then comparing odd terms from both sides, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p_{[1^17^1]}(4n+3)q^n &= 3\,\frac{(q^2;q^2)_\infty^3(q^{14};q^{14})_\infty^3}{(q;q)_\infty^4(q^7;q^7)_\infty^4} + 4q\,\frac{(q^2;q^2)_\infty^6(q^{14};q^{14})_\infty^6}{(q;q)_\infty^7(q^7;q^7)_\infty^7} \\ &\equiv \frac{(q^2;q^2)_\infty^3(q^{14};q^{14})_\infty^3}{(q;q)_\infty^4(q^7;q^7)_\infty^4} \pmod 2 \\ &\equiv (q^2;q^2)_\infty(q^{14};q^{14})_\infty \pmod 2, \qquad (4.2.25)\end{aligned}$$

where we have also applied the binomial theorem. From (4.2.25), we readily deduce (4.2.23). □

4.3 Some new results and their proofs

In this section, we state and prove some results which are analogues to Ramanujan's "Most beautiful identity" (1.1.5). We further deduce some new interesting partition congruences.

Theorem 4.3.1.

$$\begin{aligned}\sum_{n=0}^{\infty} p_{[1^13^15^115^1]}(2n+1)q^n &= \frac{(q^2;q^2)_\infty(q^6;q^6)_\infty(q^{10};q^{10})_\infty(q^{30};q^{30})_\infty}{(q;q)_\infty^2(q^3;q^3)_\infty^2(q^5;q^5)_\infty^2(q^{15};q^{15})_\infty^2} \\ &\quad + 2q\,\frac{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2(q^{10};q^{10})_\infty^2(q^{30};q^{30})_\infty^2}{(q;q)_\infty^3(q^3;q^3)_\infty^3(q^5;q^5)_\infty^3(q^{15};q^{15})_\infty^3}. \qquad (4.3.1)\end{aligned}$$

Proof. The proof is similar to those in Section 4.2. We utilize the modular equation [20, Entry 11(xiv), p. 385]

$$(\alpha\beta\gamma\delta)^{1/8} + \{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8}$$
$$+ 2^{1/3}\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/24} = 1,$$

where β, γ, and δ have degrees 3, 5, and 15, respectively, over α. □

Remark 4.3.2. *We note that, $p_{[1^1 3^1 5^1 15^1]}(n)$ is the number of 4-colored partitions of n with three of the four colors appearing only in multiples of 3, 5, and 15, respectively.*

Theorem 4.3.3. *We have*

$$\sum_{n=0}^{\infty} p_{[1^1 3^1]}(2n+1)q^n = \frac{(q^2;q^2)_\infty^5 (q^{12};q^{12})_\infty^2}{(q;q)_\infty^4 (q^3;q^3)_\infty^2 (q^4;q^4)_\infty^2 (q^6;q^6)_\infty} \tag{4.3.2}$$

and

$$\sum_{n=0}^{\infty} p_{[1^1 3^1]}(2n)q^n = \frac{(q^4;q^4)_\infty^2 (q^6;q^6)_\infty^5}{(q;q)_\infty^2 (q^2;q^2)_\infty (q^3;q^3)_\infty^4 (q^{12};q^{12})_\infty^2}. \tag{4.3.3}$$

Proof. We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{[1^1 3^1]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^1 3^1]}(n)(-1)^n q^n \\
&= \frac{1}{(q^2;q^2)_\infty (q^6;q^6)_\infty} \left\{ \frac{1}{(q;q^2)_\infty (q^3;q^6)_\infty} - \frac{1}{(-q;q^2)_\infty (-q^3;q^6)_\infty} \right\} \\
&= \frac{(q^4;q^4)_\infty (q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^2 (q^6;q^6)_\infty^2} \left\{ (-q;q^2)_\infty (-q^3;q^6)_\infty - (q;q^2)_\infty (q^3;q^6)_\infty \right\}.
\end{aligned} \tag{4.3.4}$$

Similarly, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{[1^1 3^1]}(n)q^n + \sum_{n=0}^{\infty} p_{[1^1 3^1]}(n)(-1)^n q^n \\
&= \frac{(q^4;q^4)_\infty (q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^2 (q^6;q^6)_\infty^2} \left\{ (-q;q^2)_\infty (-q^3;q^6)_\infty + (q;q^2)_\infty (q^3;q^6)_\infty \right\}.
\end{aligned} \tag{4.3.5}$$

Now, from [14, p. 958], we have

$$\psi(q)\psi(q^3) + \psi(-q)\psi(-q^3) = 2\varphi(q^6)\psi(q^4) \tag{4.3.6}$$

and

$$\psi(q)\psi(q^3) - \psi(-q)\psi(-q^3) = 2q\varphi(q^2)\psi(q^{12}). \tag{4.3.7}$$

Transforming (4.3.6) and (4.3.7) into q-products, we have

$$(-q;q^2)_\infty(-q^3;q^6)_\infty + (q;q^2)_\infty(q^3;q^6)_\infty = 2(-q^4;q^4)_\infty^2(-q^6;q^{12})_\infty^2 \tag{4.3.8}$$

and

$$(-q;q^2)_\infty(-q^3;q^6)_\infty - (q;q^2)_\infty(q^3;q^6)_\infty = 2q(-q^2;q^4)_\infty^2(-q^{12};q^{12})_\infty^2, \tag{4.3.9}$$

respectively.

Employing (4.3.8) and (4.3.9) in (4.3.5) and (4.3.4) and comparing the odd terms and even terms, respectively, from both sides we complete the proofs of (4.3.2) and (4.3.3). □

Corollary 4.3.4. *We have*

$$p_{[1^1 3^1]}(4n+2) \equiv 0 \pmod 2 \text{ and } p_{[1^1 3^1]}(4n+3) \equiv 0 \pmod 2. \tag{4.3.10}$$

Proof. Applying the binomial theorem in (4.3.2) and (4.3.3), we obtain

$$\sum_{n=0}^{\infty} p_{[1^1 3^1]}(2n+1)q^n \equiv \frac{(q^2;q^2)_\infty^3(q^{12};q^{12})_\infty^2}{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^2} \pmod 2$$

and

$$\sum_{n=0}^{\infty} p_{[1^1 3^1]}(2n)q^n \equiv \frac{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^2} \pmod 2,$$

from which we readily deduce (4.3.10). □

Remark 4.3.5. *We note that, $p_{[1^1 3^1]}(n)$ is the number of 2-colored partitions of n with one of the colors appearing only in multiples of 3.*

Theorem 4.3.6. *We have*

$$\sum_{n=0}^{\infty} p_{[3^1 5^1]}(2n+1)q^n = q\,\frac{(q^2;q^2)_\infty^2(q^{30};q^{30})_\infty^2}{(q^3;q^3)_\infty^2(q^5;q^5)_\infty^2(q;q)_\infty(q^{15};q^{15})_\infty} \tag{4.3.11}$$

and

$$\sum_{n=0}^{\infty} p_{[1^1 15^1]}(2n+1)q^n = q\,\frac{(q^6;q^6)_\infty^2(q^{10};q^{10})_\infty^2}{(q;q)_\infty^2(q^{15};q^{15})_\infty^2(q^3;q^3)_\infty(q^5;q^5)_\infty}. \tag{4.3.12}$$

Proof. We have

$$\sum_{n=0}^{\infty} p_{[3^15^1]}(n)q^n - \sum_{n=0}^{\infty} p_{[3^15^1]}(n)(-1)^n q^n$$
$$= \frac{1}{(q^6;q^6)_\infty (q^{10};q^{10})_\infty}\left\{\frac{1}{(q^3;q^6)_\infty (q^5;q^{10})_\infty} - \frac{1}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty}\right\}$$
$$= \frac{(q^{12};q^{12})_\infty (q^{20};q^{20})_\infty}{(q^6;q^6)_\infty^2 (q^{10};q^{10})_\infty^2}\left\{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty - (q^3;q^6)_\infty(q^5;q^{10})_\infty\right\} \quad (4.3.13)$$

and

$$\sum_{n=0}^{\infty} p_{[1^115^1]}(n)q^n + \sum_{n=0}^{\infty} p_{[1^115^1]}(n)(-1)^n q^n$$
$$= \frac{1}{(q^2;q^2)_\infty (q^{30};q^{30})_\infty}\left\{\frac{1}{(q;q^2)_\infty (q^{15};q^{30})_\infty} + \frac{1}{(-q;q^2)_\infty(-q^{15};q^{30})_\infty}\right\}$$
$$= \frac{(q^4;q^4)_\infty (q^{60};q^{60})_\infty}{(q^2;q^2)_\infty^2 (q^{30};q^{30})_\infty^2}\left\{(-q;q^2)_\infty(-q^{15};q^{30})_\infty + (q;q^2)_\infty(q^{15};q^{30})_\infty\right\}. \quad (4.3.14)$$

Now, from [20, Entries 9(i) and (iv), p. 377], we note that

$$\psi(q^3)\psi(q^5) - \psi(-q^3)\psi(-q^5) = 2q^3\varphi(q^2)\psi(q^{30})$$

and

$$\psi(q)\psi(q^{15}) + \psi(-q)\psi(-q^{15}) = 2\varphi(q^6)\psi(q^{10}).$$

The above two identities can be transformed into

$$(-q^3;q^6)_\infty(-q^5;q^{10})_\infty - (q^3;q^6)_\infty(q^5;q^{10})_\infty$$
$$= 2q^3\frac{(q^4;q^4)_\infty^2 (q^{60};q^{60})_\infty^2}{(q^2;q^2)_\infty (q^{12};q^{12})_\infty (q^{20};q^{20})_\infty (q^{30};q^{30})_\infty} \quad (4.3.15)$$

and

$$(-q;q^2)_\infty(-q^{15};q^{30})_\infty + (q;q^2)_\infty(q^{15};q^{30})_\infty$$
$$= 2\,\frac{(q^{12};q^{12})_\infty^2 (q^{20};q^{20})_\infty^2}{(q^4;q^4)_\infty (q^6;q^6)_\infty (q^{10};q^{10})_\infty (q^{60};q^{60})_\infty}, \quad (4.3.16)$$

respectively.

Employing (4.3.15) and (4.3.16) in (4.3.13) and (4.3.14) and comparing the odd terms and even terms, respectively, from both sides of the resulting identities, we readily arrive at (4.3.11) and (4.3.12). □

Theorem 4.3.7. *We have*

$$\sum_{n=0}^{\infty} p_{[1^2 27^2]}(2n+1)q^n = 2\,\frac{(q^2;q^2)_\infty^2(q^{54};q^{54})_\infty^2(q^3;q^3)_\infty(q^9;q^9)_\infty}{(q;q)_\infty^5(q^{27};q^{27})_\infty^5} + 2q^3\frac{(q^2;q^2)_\infty^4(q^{54};q^{54})_\infty^4}{(q;q)_\infty^6(q^{27};q^{27})_\infty^6}, \tag{4.3.17}$$

$$\sum_{n=0}^{\infty} p_{[1^2 35^2]}(2n+1)q^n = 2\,\frac{(q^2;q^2)_\infty^2(q^{70};q^{70})_\infty^2(q^5;q^5)_\infty(q^7;q^7)_\infty}{(q;q)_\infty^5(q^{35};q^{35})_\infty^5} + 2q^4\frac{(q^2;q^2)_\infty^4(q^{70};q^{70})_\infty^4}{(q;q)_\infty^6(q^{35};q^{35})_\infty^6}, \tag{4.3.18}$$

$$\sum_{n=0}^{\infty} p_{[5^2 7^2]}(2n+1)q^n = 2q\,\frac{(q^{10};q^{10})_\infty^4(q^{14};q^{14})_\infty^4}{(q^5;q^5)_\infty^6(q^7;q^7)_\infty^6} - 2q\,\frac{(q^{10};q^{10})_\infty^2(q^{14};q^{14})_\infty^2(q;q)_\infty(q^{35};q^{35})_\infty}{(q^5;q^5)_\infty^5(q^7;q^7)_\infty^5}. \tag{4.3.19}$$

Proof. Since the proof is similar to those in the previous theorems, we omit the details. We only give the theta function identities in Entries 4(iv), 17(i), and 17(ii) of [20, Chapter 20, p. 359 and p. 417], which were used to deduce (4.3.17) - (4.3.19) are

$$\varphi(q)\varphi(q^{27}) - \varphi(-q)\varphi(-q^{27}) = 4q(q^6;q^6)_\infty(q^{18};q^{18})_\infty + 4q^7\psi(q^2)\psi(q^{54}), \tag{4.3.20}$$

$$\varphi(q)\varphi(q^{35}) - \varphi(-q)\varphi(-q^{35}) = 4q(q^{10};q^{10})_\infty(q^{14};q^{14})_\infty + 4q^9\psi(q^2)\psi(q^{70}), \tag{4.3.21}$$

and

$$\varphi(q^5)\varphi(q^7) - \varphi(-q^5)\varphi(-q^7) = 4q^3\psi(q^{10})\psi(q^{14}) - 4q^3(q^2;q^2)_\infty(q^{70};q^{70})_\infty, \tag{4.3.22}$$

respectively.

□

Corollary 4.3.8. *We have*

$$p_{[1^2 2^2 7^2]}(2n+1) \equiv 0 \pmod{2},$$

$$p_{[1^2 3 5^2]}(2n+1) \equiv 0 \pmod{2},$$

and

$$p_{[5^2 7^2]}(2n+1) \equiv 0 \pmod{2}.$$

Proof. Readily follows from (4.3.17)–(4.3.19). □

Theorem 4.3.9. *We have*

$$\sum_{n=0}^{\infty} p_{[1^1 3^5]}(2n+1)q^n = \frac{(q^6;q^6)_\infty^7}{(q^2;q^2)_\infty (q^3;q^3)_\infty^{12}} + 8q\,\frac{(q^4;q^4)_\infty^2 (q^6;q^6)_\infty^7 (q^{12};q^{12})_\infty^2}{(q;q)_\infty^2 (q^2;q^2)_\infty (q^3;q^3)_\infty^{14}} \tag{4.3.23}$$

and

$$\sum_{n=0}^{\infty} p_{[1^1 5^3]}(2n+1)q^{2n} = \frac{(q;q)_\infty^2 (q^4;q^4)_\infty (q^{20};q^{20})_\infty^3}{(q^2;q^2)_\infty^4 (q^5;q^5)_\infty^2 (q^{10};q^{10})_\infty^4} + 2q\,\frac{(q^4;q^4)_\infty^2 (q^{20};q^{20})_\infty^6}{(q^2;q^2)_\infty^3 (q^{10};q^{10})_\infty^9}. \tag{4.3.24}$$

Proof. We have

$$\sum_{n=0}^{\infty} p_{[1^1 3^5]}(n)q^n = \frac{1}{(q;q)_\infty (q^3;q^3)_\infty^5} = \frac{1}{(q;q^2)_\infty (q^2;q^2)_\infty (q^3;q^6)_\infty^5 (q^6;q^6)_\infty^5}. \tag{4.3.25}$$

Therefore,

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{[1^1 3^5]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^1 3^5]}(n)(-1)^n q^n \\
&\qquad = \frac{1}{(q^2;q^2)_\infty (q^6;q^6)_\infty^5}\left\{\frac{1}{(q;q^2)_\infty (q^3;q^6)_\infty^5} - \frac{1}{(-q;q^2)_\infty (-q^3;q^6)_\infty^5}\right\} \\
&\qquad = \frac{(q^4;q^4)_\infty (q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^2 (q^6;q^6)_\infty^{10}}\left\{(-q;q^2)_\infty (-q^3;q^6)_\infty^5 - (q;q^2)_\infty (q^3;q^6)_\infty^5\right\}.
\end{aligned} \tag{4.3.26}$$

Now, we recall the following modular equation of degree 3 in [20, Entry 5(viii), p. 231].

If β has degree 3 over α, then

$$(\alpha\beta^5)^{1/8} + \{(1-\alpha)(1-\beta)^5\}^{1/8} = 1 - \left(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\right)^{1/8}. \tag{4.3.27}$$

The above modular equation can be transformed into (see [13, pp. 1035–1036])

$$\begin{aligned}(-q;q^2)_\infty(-q^3;q^6)_\infty^5 - (q;q^2)_\infty(q^3;q^6)_\infty^5 &= 2q\,\frac{(q;q)_\infty^4(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^4}\\ &\quad + 8q^2\frac{(q^4;q^4)_\infty(q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty(q^6;q^6)_\infty^5}.\end{aligned} \tag{4.3.28}$$

Employing (4.3.28) in (4.3.26), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p_{[1^13^5]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^13^5]}(n)(-1)^n q^n &= 2q\,\frac{(q;q)_\infty^4(q^4;q^4)_\infty(q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^6(q^6;q^6)_\infty^6(q^3;q^3)_\infty^4}\\ &\quad + 8q^2\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^{10}}{(q^2;q^2)_\infty^3(q^6;q^6)_\infty^{15}}.\end{aligned} \tag{4.3.29}$$

By comparing odd terms from both sides of (4.3.29), we deduce that

$$\sum_{n=0}^{\infty} p_{[1^13^5]}(2n+1)q^{2n} = \frac{(q;q)_\infty^4(q^4;q^4)_\infty(q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^6(q^6;q^6)_\infty^6(q^3;q^3)_\infty^4} + 4q\,\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^{10}}{(q^2;q^2)_\infty^3(q^6;q^6)_\infty^{15}}. \tag{4.3.30}$$

Replacing q by $-q$ in (4.3.30), and then adding the resulting identity with (4.3.30), we find that

$$\begin{aligned}2\sum_{n=0}^{\infty} p_{[1^13^5]}(2n+1)q^{2n} &= \frac{(q^4;q^4)_\infty(q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^{10}}\left\{\frac{(q;q^2)_\infty^4}{(q^3;q^6)_\infty^4} + \frac{(-q;q^2)_\infty^4}{(-q^3;q^6)_\infty^4}\right\}\\ &= \frac{(q^4;q^4)_\infty(q^{12};q^{12})_\infty^9}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^{14}}\Big\{(q;q^2)_\infty^4(-q^3;q^6)_\infty^4\\ &\quad + (-q;q^2)_\infty^4(q^3;q^6)_\infty^4\Big\}.\end{aligned} \tag{4.3.31}$$

Now, from [14, p. 958], we have

$$\varphi(q)\varphi(-q^3) - \varphi(-q)\varphi(q^3) = 4q\psi(-q^2)\psi(-q^6), \tag{4.3.32}$$

which can be transformed, with the aid of (1.5.2) and (1.5.3), into

$$(-q;q^2)_\infty^2(q^3;q^6)_\infty^2-(q;q^2)_\infty^2(-q^3;q^6)_\infty^2=4q\,\frac{(q^8;q^8)_\infty(q^{24};q^{24})_\infty}{(q^4;q^4)_\infty(q^{12};q^{12})_\infty}. \tag{4.3.33}$$

Therefore,

$$\begin{aligned}&(-q;q^2)_\infty^2(q^3;q^6)_\infty^4+(q;q^2)_\infty^2(-q^3;q^6)_\infty^4\\&=\left\{(-q;q^2)_\infty^2(q^3;q^6)_\infty^2-(q;q^2)_\infty^2(-q^3;q^6)_\infty^2\right\}^2+2(q^2;q^4)_\infty^2(q^6;q^{12})_\infty^2\\&=16q^2\,\frac{(q^8;q^8)_\infty^2(q^{24};q^{24})_\infty^2}{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}+2\,\frac{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2}{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}.\end{aligned} \tag{4.3.34}$$

Employing (4.3.34) in (4.3.31), we easily deduce (4.3.23).

In a similar fashion we can prove (4.3.24) by using the modular equation [20, Entry 13(vii), p. 281]

$$(\alpha\beta^3)^{1/8}+\{(1-\alpha)(1-\beta)^3\}^{1/8}=1-2^{1/3}\left(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right)^{1/24},$$

where, in this case, β has degree 5 over α. We omit the details. □

Remark 4.3.10. *Combinatorially, $p_{[1^13^5]}(n)$ is the number of 6-colored partitions of n with five of the six colors appearing only in multiples of 3. Similarly, $p_{[1^15^3]}(n)$ is the number of 4-colored partitions of n with three of the four colors appearing only in multiples of 5.*

Theorem 4.3.11. *We have*

$$\sum_{n=0}^{\infty}p_{[1^23^{-2}]}(2n+1)q^n=2\,\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^{12};q^{12})_\infty(q^3;q^3)_\infty^2}{(q;q)_\infty^4(q^6;q^6)_\infty} \tag{4.3.35}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}p_{[1^23^{-2}]}(4n+3)q^n=4\Big\{&\frac{(q^2;q^2)_\infty^7(q^6;q^6)_\infty^3(q^8;q^8)_\infty^2}{(q;q)_\infty^9(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty}\\&+\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty(q^4;q^4)_\infty^6(q^{12};q^{12})_\infty}{(q;q)_\infty^8(q^8;q^8)_\infty^2}\Big\}.\end{aligned} \tag{4.3.36}$$

Proof. We begin with

$$\sum_{n=0}^{\infty} p_{[1^2 3^{-2}]}(n) q^n = \frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^2} = \frac{(q^3;q^6)_\infty^2 (q^6;q^6)_\infty^2}{(q;q^2)_\infty^2 (q^2;q^2)_\infty^2}, \tag{4.3.37}$$

so that

$$\sum_{n=0}^{\infty} p_{[1^2 3^{-2}]}(n) q^n - \sum_{n=0}^{\infty} p_{[1^2 3^{-2}]}(n)(-1)^n q^n = \frac{(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2}\left(\frac{(q^3;q^6)_\infty^2}{(q;q^2)_\infty^2} - \frac{(-q^3;q^6)_\infty^2}{(-q;q^2)_\infty^2}\right). \tag{4.3.38}$$

Now, from (4.3.33), we find that

$$\frac{(q^3;q^6)_\infty^2}{(q;q^2)_\infty^2} - \frac{(-q^3;q^6)_\infty^2}{(-q;q^2)_\infty^2} = 4q\,\frac{(q^4;q^4)_\infty (q^8;q^8)_\infty (q^{24};q^{24})_\infty}{(q^{12};q^{12})_\infty (q^2;q^2)_\infty^2}. \tag{4.3.39}$$

Employing (4.3.39) in (4.3.38) and comparing the odd terms from both sides, we arrive at (4.3.35).

To prove (4.3.36), we note from [94, Corollary 3.3, p. 84] that

$$\varphi(q)\varphi(-q^3) = \varphi(-q^4)\varphi(-q^{12}) + 2q\psi(-q^2)\psi(-q^6),$$

which can be transformed, with the aid of (1.5.2) and (1.5.3), into

$$\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^2} = \frac{(q^4;q^4)_\infty^4 (q^6;q^6)_\infty (q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^5 (q^8;q^8)_\infty (q^{24};q^{24})_\infty} + 2q\,\frac{(q^4;q^4)_\infty (q^8;q^8)_\infty (q^6;q^6)_\infty^2 (q^{24};q^{24})_\infty}{(q^2;q^2)_\infty^4 (q^{12};q^{12})_\infty}. \tag{4.3.40}$$

Now, from Entries 25(i) and 25(ii) of [20, p. 40], we obtain

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{4.3.41}$$

Employing (1.5.2) and (1.5.3) in (4.3.41), we find that

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5 (q^{16};q^{16})_\infty^2} + 2q\,\frac{(q^4;q^4)_\infty^2 (q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5 (q^8;q^8)_\infty}, \tag{4.3.42}$$

which is a 2-dissection of $1/(q;q)_\infty^2$. From (4.3.40) and (4.3.42), we arrive at

$$\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^4} = \frac{(q^4;q^4)_\infty^4(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}(q^{24};q^{24})_\infty(q^{16};q^{16})_\infty^2}$$
$$+ 2q\,\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^{10}(q^8;q^8)_\infty^2(q^{24};q^{24})_\infty}$$
$$+ 2q\,\frac{(q^4;q^4)_\infty(q^8;q^8)_\infty^6(q^6;q^6)_\infty^2(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty^9(q^{12};q^{12})_\infty(q^{16};q^{16})_\infty^2}$$
$$+ 4q^2\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2(q^{16};q^{16})_\infty^2(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty^9(q^{12};q^{12})_\infty}. \tag{4.3.43}$$

Employing (4.3.43) in (4.3.35), we find that

$$\sum_{n=0}^{\infty} p_{[1^2 3^{-2}]}(2n+1)q^n = 2\,\frac{(q^4;q^4)_\infty^5(q^{12};q^{12})_\infty^3(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^9(q^{24};q^{24})_\infty(q^{16};q^{16})_\infty^2}$$
$$+ 4q\,\frac{(q^4;q^4)_\infty^7(q^{12};q^{12})_\infty^3(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^9(q^8;q^8)_\infty^2(q^{24};q^{24})_\infty}$$
$$+ 4q\,\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^6(q^6;q^6)_\infty(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty^8(q^{16};q^{16})_\infty^2}$$
$$+ 8q^2\frac{(q^4;q^4)_\infty^4(q^6;q^6)_\infty(q^{16};q^{16})_\infty^2(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty^8}. \tag{4.3.44}$$

Replacing q by $-q$ in (4.3.44) and then subtracting the resulting identity from (4.3.44), we obtain

$$\sum_{n=0}^{\infty} p_{[1^2 3^{-2}]}(2n+1)q^n - p_{[1^2 3^{-2}]}(2n+1)(-q)^n$$
$$= 8q\,\frac{(q^4;q^4)_\infty^7(q^{12};q^{12})_\infty^3(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^9(q^8;q^8)_\infty^2(q^{24};q^{24})_\infty} + 8q\,\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^6(q^6;q^6)_\infty(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty^8(q^{16};q^{16})_\infty^2}. \tag{4.3.45}$$

Comparing the odd terms from both sides of (4.3.45), dividing by q, and then replacing q^2 by q, we arrive at (4.3.36) to complete the proof. □

Corollary 4.3.12. *We have*

$$p_{[1^2 3^{-2}]}(2n+1) \equiv 0 \pmod 2$$

and

$$p_{[1^2 3^{-2}]}(4n+3) \equiv 0 \pmod 4.$$

Proof. Follows readily from (4.3.35) and (4.3.36). □

Remark 4.3.13. *We note that, $p_{[1^23^{-2}]}(n)$ is the number of 2-colored partitions of n with no part having multiples of 3 appears.*

Theorem 4.3.14. *We have*

$$\sum_{n=0}^{\infty} p_{[1^25^{-2}]}(2n+1)q^n = 2\,\frac{(q^2;q^2)_\infty^3(q^5;q^5)_\infty(q^{10};q^{10})_\infty}{(q;q)_\infty^5} \tag{4.3.46}$$

and

$$\sum_{n=0}^{\infty} p_{[1^29^{-2}]}(2n+1)q^n = \frac{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2(q^9;q^9)_\infty}{(q;q)_\infty^5}. \tag{4.3.47}$$

Proof. As in the proof of Theorem 4.3.11, we notice that

$$\sum_{n=0}^{\infty} p_{[1^25^{-2}]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^25^{-2}]}(n)(-1)^nq^n = \frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2}\left(\frac{(q^5;q^{10})_\infty^2}{(q;q^2)_\infty^2} - \frac{(-q^5;q^{10})_\infty^2}{(-q;q^2)_\infty^2}\right). \tag{4.3.48}$$

From [20, p. 278] and [10, Eq. (2.3)], we note that

$$\varphi(q)\varphi(-q^5) - \varphi(-q)\varphi(q^5) = 4q\,(q^4;q^4)_\infty(q^{20};q^{20})_\infty, \tag{4.3.49}$$

which can be transformed into

$$\frac{(q^5;q^{10})_\infty^2}{(q;q^2)_\infty^2} - \frac{(-q^5;q^{10})_\infty^2}{(-q;q^2)_\infty^2} = 4q\,\frac{(q^4;q^4)_\infty^3(q^{20};q^{20})_\infty}{(q^2;q^2)_\infty^3(q^{10};q^{10})_\infty}. \tag{4.3.50}$$

Employing (4.3.50) in (4.3.48), and then comparing the odd terms from both sides, we readily deduce (4.3.46).

The proof of (4.3.47) can be accomplished in a similar fashion by applying the identity [15, Eq. (4.43), p. 121]

$$\varphi(q)\varphi(-q^9) - \varphi(-q)\varphi(q^9) = 2q(q^{12};q^{12})_\infty^2. \tag{4.3.51}$$

□

The following result is immediate from (4.3.46).

Corollary 4.3.15. *We have*

$$p_{[1^25^{-2}]}(2n+1) \equiv 0 \pmod{2}.$$

Remark 4.3.16. *As in Remark 4.3.13, $p_{[1^25^{-2}]}(n)$ and $p_{[1^29^{-2}]}(n)$ are the number of 2-colored partitions of n with no part having multiples of 5 and 9, respectively, appears.*

Theorem 4.3.17. *We have*

$$\sum_{n=0}^{\infty} p_{[1^33^{-1}]}(2n+1)q^n = 3\,\frac{(q^3;q^3)_\infty (q^6;q^6)_\infty^2 (q^2;q^2)_\infty^2}{(q;q)_\infty^7}, \tag{4.3.52}$$

$$\sum_{n=0}^{\infty} p_{[1^33^{-1}]}(2n)q^n = \frac{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty^3}{(q;q)_\infty^9 (q^6;q^6)_\infty^2}, \tag{4.3.53}$$

$$\sum_{n=0}^{\infty} p_{[1^33^{-1}]}(4n+2)q^n = 9\,\frac{(q^3;q^3)_\infty^2 (q^2;q^2)_\infty^5 (q^6;q^6)_\infty}{(q;q)_\infty^{10}} + 108q\,\frac{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty (q^6;q^6)_\infty^6}{(q;q)_\infty^{15}}, \tag{4.3.54}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} p_{[1^33^{-1}]}(8n+4)q^n = 24\,&\frac{(q^3;q^3)_\infty (q^2;q^2)_\infty^{11} (q^6;q^6)_\infty^2}{(q;q)_\infty^{16}} \bigg\{ 23328q^3 \frac{(q^2;q^2)_\infty^3 (q^6;q^6)_\infty^{12}}{(q;q)_\infty^{15}} \\ &+ 4536q^2 \frac{(q^2;q^2)_\infty^2 (q^3;q^3)_\infty (q^6;q^6)_\infty^7}{(q;q)_\infty^{10}} \\ &+ 225q\,\frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^2 (q^6;q^6)_\infty^2}{(q;q)_\infty^5} \\ &+ 2\,\frac{(q^3;q^3)_\infty^3}{(q^6;q^6)_\infty^3} \bigg\}.\end{aligned} \tag{4.3.55}$$

Proof. We note that

$$\sum_{n=0}^{\infty} p_{[1^33^{-1}]}(n)q^n = \frac{(q^3;q^3)_\infty}{(q;q)_\infty^3} = \frac{(q^3;q^6)_\infty (q^6;q^6)_\infty}{(q;q^2)_\infty^3 (q^2;q^2)_\infty^3}. \tag{4.3.56}$$

Replacing q by $-q$ in (4.3.56) and then subtracting the resulting identity with (4.3.56), we find that

$$\sum_{n=0}^{\infty} p_{[1^3 3^{-1}]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^3 3^{-1}]}(n)(-1)^n q^n = \frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty^3}\left(\frac{(q^3;q^6)_\infty}{(q;q^2)_\infty^3} - \frac{(-q^3;q^6)_\infty}{(-q;q^2)_\infty^3}\right). \tag{4.3.57}$$

Now, from [13, Eqs. (6.7) and (6.8), p. 1034], we obtain

$$\frac{\chi^3(q)}{\chi(q^3)} - \frac{\chi^3(-q)}{\chi(-q^3)} = 3q\left(\frac{\psi(-q^9)}{\psi(-q)} + \frac{\psi(q^9)}{\psi(q)}\right). \tag{4.3.58}$$

Again, from [16, Equation (8.15), p. 294], we note that

$$\frac{\psi(-q^9)}{\psi(-q)} + \frac{\psi(q^9)}{\psi(q)} = 2\,\frac{(q^{12};q^{12})_\infty^3}{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^6;q^6)_\infty}. \tag{4.3.59}$$

Employing (4.3.59) in (4.3.58), we obtain

$$\frac{(q^3;q^6)_\infty}{(q;q^2)_\infty^3} - \frac{(-q^3;q^6)_\infty}{(-q;q^2)_\infty^3} = 6q\,\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^4}. \tag{4.3.60}$$

Using (4.3.60) in (4.3.57), and then comparing the odd terms from both sides, we find that

$$\sum_{n=0}^{\infty} p_{[1^3 3^{-1}]}(2n+1)q^{2n+1} = 3q\,\frac{(q^6;q^6)_\infty(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^7}, \tag{4.3.61}$$

which is clearly equivalent to (4.3.52).

Again, replacing q by $-q$ in (4.3.56) and then adding the resulting identity with (4.3.56), we obtain

$$\sum_{n=0}^{\infty} p_{[1^3 3^{-1}]}(n)q^n + \sum_{n=0}^{\infty} p_{[1^3 3^{-1}]}(n)(-1)^n q^n = \frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty^3}\left(\frac{(q^3;q^6)_\infty}{(q;q^2)_\infty^3} + \frac{(-q^3;q^6)_\infty}{(-q;q^2)_\infty^3}\right). \tag{4.3.62}$$

Now, from Entry 5(i) of [20, p. 230], we note that

$$\left(\frac{\alpha^3}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} = 1, \tag{4.3.63}$$

where β has degree 3 over α. This modular equation can be transformed into (see [21, pp. 1033–1034])

$$\frac{\chi^3(q)}{\chi(q^3)}+\frac{\chi^3(-q)}{\chi(-q^3)}=2\frac{\chi(-q^6)}{\chi^3(-q^2)}, \tag{4.3.64}$$

which can also be written, with the aid of (1.6.4)–(1.6.5), as

$$\frac{(q^3;q^6)_\infty}{(q;q^2)^3_\infty}+\frac{(-q^3;q^6)_\infty}{(-q;q^2)^3_\infty}=2\,\frac{(q^4;q^4)^6_\infty(q^6;q^6)^2_\infty}{(q^2;q^2)^6_\infty(q^{12};q^{12})^2_\infty}. \tag{4.3.65}$$

Employing (4.3.65) in (4.3.62), and then comparing the even terms from both sides, we deduce (4.3.53).

Now, to prove (4.3.54), we note from (4.3.53) that

$$\sum_{n=0}^\infty p_{[1^33^{-1}]}(2n)q^n-\sum_{n=0}^\infty p_{[1^33^{-1}]}(2n)q^n=\frac{(q^6;q^6)_\infty}{(q^2;q^2)^3_\infty}\left\{\frac{(q^3;q^6)^3_\infty}{(q;q^2)^9_\infty}-\frac{(-q^3;q^6)^3_\infty}{(-q;q^2)^9_\infty}\right\}. \tag{4.3.66}$$

Now, with the aid of (4.3.60), we find that

$$\begin{aligned}&\frac{(q^3;q^6)^3_\infty}{(q;q^2)^9_\infty}-\frac{(-q^3;q^6)^3_\infty}{(-q;q^2)^9_\infty}\\&=\left\{\frac{(q^3;q^6)_\infty}{(q;q^2)^3_\infty}-\frac{(-q^3;q^6)_\infty}{(-q;q^2)^3_\infty}\right\}^3+3\,\frac{(q^6;q^{12})_\infty}{(q^2;q^4)^3_\infty}\left\{\frac{(q^3;q^6)_\infty}{(q;q^2)^3_\infty}-\frac{(-q^3;q^6)_\infty}{(-q;q^2)^3_\infty}\right\}\\&=216q^3\frac{(q^4;q^4)^6_\infty(q^{12};q^{12})^6_\infty}{(q^2;q^2)^{12}_\infty}+18q\,\frac{(q^4;q^4)^5_\infty(q^6;q^6)_\infty(q^{12};q^{12})_\infty}{(q^2;q^2)^7_\infty}.\end{aligned} \tag{4.3.67}$$

Employing (4.3.67) in (4.3.66), and then comparing the odd terms from both sides, we deduce (4.3.54).

Next, to prove (4.3.55), we again note from (4.3.53) that

$$\sum_{n=0}^\infty p_{[1^33^{-1}]}(2n)q^n+\sum_{n=0}^\infty p_{[1^33^{-1}]}(2n)q^n=\frac{(q^6;q^6)_\infty}{(q^2;q^2)^3_\infty}\left\{\frac{(q^3;q^6)^3_\infty}{(q;q^2)^9_\infty}+\frac{(-q^3;q^6)^3_\infty}{(-q;q^2)^9_\infty}\right\}. \tag{4.3.68}$$

Employing (4.3.65) in (4.3.68), we obtain

$$\begin{aligned}\sum_{n=0}^\infty p_{[1^33^{-1}]}(2n)q^n+\sum_{n=0}^\infty p_{[1^33^{-1}]}(2n)q^n&=8\,\frac{(q^4;q^4)^{18}_\infty(q^6;q^6)^7_\infty}{(q^2;q^2)^{21}_\infty(q^{12};q^{12})^6_\infty}\\&\quad-6\,\frac{(q^4;q^4)^9_\infty(q^6;q^6)^4_\infty}{(q^2;q^2)^{12}_\infty(q^{12};q^{12})^3_\infty}.\end{aligned} \tag{4.3.69}$$

Equating the even parts in (4.3.69), we deduce that

$$\sum_{n=0}^{\infty} p_{[1^3 3^{-1}]}(4n)q^n = 4\,\frac{(q^2;q^2)_\infty^{18}(q^3;q^3)_\infty^7}{(q;q)_\infty^{21}(q^6;q^6)_\infty^6} - 3\,\frac{(q^2;q^2)_\infty^{9}(q^3;q^3)_\infty^4}{(q;q)_\infty^{12}(q^6;q^6)_\infty^3}. \tag{4.3.70}$$

Replacing q by $-q$ in (4.3.70), and then subtracting the resulting identity from (4.3.70), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{[1^3 3^{-1}]}(4n)q^n - \sum_{n=0}^{\infty} p_{[1^3 3^{-1}]}(4n)(-q)^n \\
&= 4\,\frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty^3}\left\{\frac{(q^3;q^6)_\infty^7}{(q;q^2)_\infty^{21}} - \frac{(-q^3;q^6)_\infty^7}{(-q;q^2)_\infty^{21}}\right\} - 3\,\frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty^3}\left\{\frac{(q^3;q^6)_\infty^4}{(q;q^2)_\infty^{12}} - \frac{(-q^3;q^6)_\infty^4}{(-q;q^2)_\infty^{12}}\right\} \\
&= 4\,\frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty^3}\left(A^7 - B^7\right) - 3\,\frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty^3}\left(A^4 - B^4\right),
\end{aligned} \tag{4.3.71}$$

where $A = \dfrac{(q^3;q^6)_\infty}{(q;q^2)_\infty^3}$ and $B = \dfrac{(-q^3;q^6)_\infty}{(-q;q^2)_\infty^3}$. Noting the algebraic identities

$$A^4 - B^4 = (A+B)(A-B)((A-B)^2 + 2AB)$$

and

$$A^7 - B^7 = (A-B)^7 + 7AB(A^5 - B^5) - 21A^2B^2(A^3 - B^3) + 35A^3B^3(A-B),$$

and then employing (4.3.60) and (4.3.65) in (4.3.71), we arrive at (4.3.55). □

Corollary 4.3.18. *We have*

$$p_{[1^3 3^{-1}]}(2n+1) \equiv 0 \pmod 3,$$

$$p_{[1^3 3^{-1}]}(4n+2) \equiv 0 \pmod 9,$$

and

$$p_{[1^3 3^{-1}]}(8n+4) \equiv 0 \pmod{24}.$$

Proof. Follows readily from (4.3.52), (4.3.54) and (4.3.55). □

Remark 4.3.19. *We note that,* $p_{[1^3 3^{-1}]}(n)$ *is the number of* 3*-colored partitions of* n *with one of the colors appearing only in parts that are not multiples of* 3.

Theorem 4.3.20.

$$\sum_{n=0}^{\infty} p_{[1^4 3^{-4}]}(2n+1)q^n = 4\,\frac{(q^2;q^2)_\infty^5 (q^3;q^3)_\infty^3 (q^6;q^6)_\infty}{(q;q)_\infty^9} \tag{4.3.72}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} p_{[1^4 3^{-4}]}(4n+3)q^n &= 36\,\frac{(q^2;q^2)_\infty^5 (q^3;q^3)_\infty^5 (q^6;q^6)_\infty^2}{(q;q)_\infty^{11}} \\ &\quad + 432q\,\frac{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty^4 (q^6;q^6)_\infty^6}{(q;q)_\infty^{16}}.\end{aligned} \tag{4.3.73}$$

Proof. As before, we note that

$$\sum_{n=0}^{\infty} p_{[1^4 3^{-4}]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^4 3^{-4}]}(n)(-1)^n q^n = \frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4}\left\{\frac{(q^3;q^6)_\infty^4}{(q;q^2)_\infty^4} - \frac{(-q^3;q^6)_\infty^4}{(-q;q^2)_\infty^4}\right\}. \tag{4.3.74}$$

Now, working on a modular equation of degree 3, Baruah and Berndt [13, Eq. (6.19), p. 1036] proved that

$$\frac{(-q;q^2)_\infty^4}{(-q^3;q^6)_\infty^4} - \frac{(q;q^2)_\infty^4}{(q^3;q^6)_\infty^4} = 8q\,\frac{(q^4;q^4)_\infty (q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty (q^6;q^6)_\infty^5},$$

which can also be written in the form

$$\frac{(q^3;q^6)_\infty^4}{(q;q^2)_\infty^4} - \frac{(-q^3;q^6)_\infty^4}{(-q;q^2)_\infty^4} = 8q\,\frac{(q^4;q^4)_\infty^5 (q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^5 (q^6;q^6)_\infty}. \tag{4.3.75}$$

Using (4.3.75) in (4.3.74), we find that

$$\sum_{n=0}^{\infty} p_{[1^4 3^{-4}]}(n)q^n - \sum_{n=0}^{\infty} p_{[1^4 3^{-4}]}(n)(-1)^n q^n = 8q\,\frac{(q^4;q^4)_\infty^5 (q^{12};q^{12})_\infty (q^6;q^6)_\infty^3}{(q^2;q^2)_\infty^9}. \tag{4.3.76}$$

Equating the odd parts in (4.3.76), we readily deduce (4.3.72).

Next, to prove (4.3.73), we notice from (4.3.72) that

$$\sum_{n=0}^{\infty} p_{[1^4 3^{-4}]}(2n+1)q^n - \sum_{n=0}^{\infty} p_{[1^4 3^{-4}]}(2n+1)(-q)^n = \frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4}\left\{\frac{(q^3;q^6)_\infty^3}{(q;q^2)_\infty^9} - \frac{(-q^3;q^6)_\infty^3}{(-q;q^2)_\infty^9}\right\} \quad (4.3.77)$$

Employing (4.3.67) in (4.3.77), we obtain

$$\sum_{n=0}^{\infty} p_{[1^4 3^{-4}]}(2n+1)q^n - \sum_{n=0}^{\infty} p_{[1^4 3^{-4}]}(2n+1)(-q)^n$$
$$= 72q\left\{\frac{(q^4;q^4)_\infty^5 (q^6;q^6)_\infty^5 (q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^{11}} + 12q^2 \frac{(q^4;q^4)_\infty^6 (q^{12};q^{12})_\infty^6 (q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^{16}}\right\}. \quad (4.3.78)$$

Equating the odd terms in (4.3.78), we easily arrive at (4.3.73) to complete the proof. □

Corollary 4.3.21. *We have*

$$p_{[1^4 3^{-4}]}(2n+1) \equiv 0 \pmod 4$$

and

$$p_{[1^4 3^{-4}]}(4n+3) \equiv 0 \pmod{36}.$$

Proof. Follows readily from (4.3.72) and (4.3.73). □

Remark 4.3.22. *Combinatorially, $p_{[1^4 3^{-4}]}(n)$ is the number of 4-colored partitions of n, where parts having multiples of 3 do not occur.*

Chapter 5

Partitions with Designated Summands into Odd Parts

5.1 Introduction

In the introductory chapter, we have discussed partitions with designated parts and partitions into odd parts with designated parts.

By using modular forms and q-series identities, Andrews, Lewis and Lovejoy [6] showed that the partition function $PD(n)$ has many interesting divisibility properties. In particular, they obtained the following Ramanujan-type congruence.

Theorem 5.1.1. [6, Corollary 7] *For $n \geq 0$, we have*

$$PD(3n+2) \equiv 0 \pmod{3}. \tag{5.1.1}$$

They also obtained explicit formulas for the generating functions for $PD(2n)$ and $PD(2n+1)$ by using Euler's algorithm for infinite products and Sturm's criterion. Chen, Ji, Jin, and Shen [36] gave proofs of the generating functions of $PD(3n)$, $PD(3n+1)$, $PD(3n+2)$ by employing H.-C. Chan's [30] identity on Ramanujan's cubic continued fraction. By using modular forms, Andrews, Lewis and Lovejoy [6, Corollary 19] also found some Ramanujan-type identities for $PDO(n)$, namely,

$$PDO(12n+6) \equiv 0 \pmod{3}, \tag{5.1.2}$$

$$PDO(12n+10) \equiv 0 \pmod{3}, \tag{5.1.3}$$

$$PDO(24n) \equiv 0 \pmod 3, \tag{5.1.4}$$

$$PDO(24n+16) \equiv 0 \pmod 3, \tag{5.1.5}$$

$$PDO(24n+18) \equiv 0 \pmod{24}. \tag{5.1.6}$$

The generating function found by Andrews, Lewis and Lovejoy for $PDO(n)$ is given as

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{(q^4;q^4)_\infty (q^6;q^6)_\infty^2}{(q;q)_\infty (q^3;q^3)_\infty (q^{12};q^{12})_\infty}. \tag{5.1.7}$$

By using q-series and modular forms; they found (5.1.7) as well as the following identities.

Theorem 5.1.2. [6, Theorem 21 and Theorem 22] *We have*

$$\sum_{n=0}^{\infty} PDO(2n)q^n = \frac{(q^4;q^4)_\infty^2 (q^6;q^6)_\infty^4}{(q;q)_\infty^2 (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^2}, \tag{5.1.8}$$

$$\sum_{n=0}^{\infty} PDO(2n+1)q^n = \frac{(q^2;q^2)_\infty^6 (q^{12};q^{12})_\infty^2}{(q;q)_\infty^4 (q^4;q^4)_\infty^2 (q^6;q^6)_\infty^2}, \tag{5.1.9}$$

$$\sum_{n=0}^{\infty} PDO(3n)q^n = \frac{(q^2;q^2)_\infty^2 (q^6;q^6)_\infty^4}{(q;q)_\infty^4 (q^{12};q^{12})_\infty^2}, \tag{5.1.10}$$

$$\sum_{n=0}^{\infty} PDO(3n+1)q^n = \frac{(q^3;q^3)_\infty^3 (q^2;q^2)_\infty^4 (q^{12};q^{12})_\infty}{(q;q)_\infty^5 (q^4;q^4)_\infty (q^6;q^6)_\infty^2}, \tag{5.1.11}$$

and

$$\sum_{n=0}^{\infty} PDO(3n+2)q^n = 2\frac{(q^2;q^2)_\infty^3 (q^6;q^6)_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty^4 (q^4;q^4)_\infty}. \tag{5.1.12}$$

The aim of this chapter is to find proofs of (5.1.8)–(5.1.12) and the following new identities by using certain dissections of theta functions.

Theorem 5.1.3. *We have*

$$\sum_{n=0}^{\infty} PDO(4n+2)q^n = 2\,\frac{(q^2;q^2)_\infty^6 (q^6;q^6)_\infty^2}{(q;q)_\infty^6 (q^3;q^3)_\infty^2}, \tag{5.1.13}$$

$$\sum_{n=0}^{\infty} PDO(4n+3)q^n = 4\,\frac{(q^4;q^4)_\infty^4 (q^6;q^6)_\infty^2}{(q:q)_\infty^4 (q^3;q^3)_\infty^2}, \tag{5.1.14}$$

$$\sum_{n=0}^{\infty} PDO(6n+2)q^n = 2\,\frac{(q^2;q^2)_\infty^{13}(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q;q)_\infty^{11}(q^4;q^4)_\infty^4}, \tag{5.1.15}$$

$$\sum_{n=0}^{\infty} PDO(6n+3)q^n = 4\,\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^4(q^4;q^4)_\infty^4}{(q;q)_\infty^8(q^6;q^6)_\infty^2}, \tag{5.1.16}$$

$$\sum_{n=0}^{\infty} PDO(6n+5)q^n = 8\,\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty(q^4;q^4)_\infty^4(q^6;q^6)_\infty}{(q;q)_\infty^7}, \tag{5.1.17}$$

$$\sum_{n=0}^{\infty} PDO(9n+3)q^n = 4\left\{\frac{(q^2;q^2)_\infty^{11}(q^3;q^3)_\infty^9}{(q;q)_\infty^{15}(q^4;q^4)_\infty^2(q^6;q^6)_\infty^3} + 4q\frac{(q^2;q^2)_\infty^8(q^6;q^6)_\infty^6}{(q;q)_\infty^{12}(q^4;q^4)_\infty^2}\right\}, \tag{5.1.18}$$

$$\sum_{n=0}^{\infty} PDO(9n+6)q^n = 12\,\frac{(q^2;q^2)_\infty^{10}(q^3;q^3)_\infty^6}{(q;q)_\infty^{14}(q^4;q^4)_\infty^2}, \tag{5.1.19}$$

$$\sum_{n=0}^{\infty} PDO(12n+6)q^n = 12\left\{\frac{(q^2;q^2)_\infty^{11}(q^3;q^3)_\infty^{13}}{(q;q)_\infty^{19}(q^6;q^6)_\infty^5} + 10q\,\frac{(q^2;q^2)_\infty^8(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}{(q;q)_\infty^{16}}\right\}, \tag{5.1.20}$$

$$\begin{aligned}\sum_{n=0}^{\infty} PDO(12n+9)q^n = 16\Bigg\{&\frac{(q^2;q^2)_\infty^{14}(q^4;q^4)_\infty^2(q^6;q^6)_\infty^4}{(q;q)_\infty^{18}(q^{12};q^{12})_\infty^2}\\ &+\frac{(q^2;q^2)_\infty^9(q^3;q^3)_\infty(q^4;q^4)_\infty^{10}(q^6;q^6)_\infty}{(q;q)_\infty^{17}(q^8;q^8)_\infty^4}\\ &+4q\,\frac{(q^2;q^2)_\infty^{13}(q^3;q^3)_\infty(q^6;q^6)_\infty(q^8;q^8)_\infty^4}{(q;q)_\infty^{17}(q^4;q^4)_\infty^2}\\ &+4q\,\frac{(q^2;q^2)_\infty^8(q^4;q^4)_\infty^6(q^3;q^3)_\infty^2(q^{12};q^{12})_\infty^2}{(q;q)_\infty^{16}(q^6;q^6)_\infty^2}\Bigg\},\end{aligned} \tag{5.1.21}$$

$$\begin{aligned}\sum_{n=0}^{\infty} PDO(12n+10)q^n = 6\Bigg\{&7\,\frac{(q^2;q^2)_\infty^{10}(q^3;q^3)_\infty^{10}}{(q;q)_\infty^{18}(q^6;q^6)_\infty^2}\\ &+16q\,\frac{(q^2;q^2)_\infty^7(q^3;q^3)_\infty(q^6;q^6)_\infty^7}{(q;q)_\infty^{15}}\Bigg\}.\end{aligned} \tag{5.1.22}$$

From the above identities, we easily deduce the following congruences.

Corollary 5.1.4. *We have*

$$PDO(4n+2) \equiv 0 \pmod 2,$$

$$PDO(4n+3) \equiv 0 \pmod 4,$$

$$PDO(6n+2) \equiv 0 \pmod 2,$$

$$PDO(6n+3) \equiv 0 \pmod{4},$$

$$PDO(6n+5) \equiv 0 \pmod{8},$$

$$PDO(9n+3) \equiv 0 \pmod{4},$$

$$PDO(9n+6) \equiv 0 \pmod{12},$$

$$PDO(12n+6) \equiv 0 \pmod{12}, \tag{5.1.23}$$

$$PDO(12n+9) \equiv 0 \pmod{16},$$

$$PDO(12n+10) \equiv 0 \pmod{6}. \tag{5.1.24}$$

Note that, congruences (5.1.23) and (5.1.24) are improved versions of (5.1.2) and (5.1.3).

In the following two theorems, we give some congruences which we derive by using elementary generating function dissection technique.

Theorem 5.1.5. *We have*

$$PDO(8n+6) \equiv 0 \pmod{4}, \tag{5.1.25}$$

$$PDO(8n+7) \equiv 0 \pmod{8}, \tag{5.1.26}$$

$$PDO(18n+15) \equiv 0 \pmod{24}, \tag{5.1.27}$$

$$PDO(27n+9) \equiv 0 \pmod{16}, \tag{5.1.28}$$

$$PDO(27n+18) \equiv 0 \pmod{16}. \tag{5.1.29}$$

Theorem 5.1.6. *For any nonnegative integer n, we have*

$$PDO(24n+9) \equiv 0 \pmod{2^3}, \tag{5.1.30}$$

$$PDO(24n+15) \equiv 0 \pmod{2^3}, \tag{5.1.31}$$

$$PDO(24n+21) \equiv 0 \pmod{2^3}, \tag{5.1.32}$$

$$PDO(72n+51) \equiv 0 \pmod{2^4}, \tag{5.1.33}$$

and

$$PDO(72n+3) \equiv \begin{cases} 4 \pmod{2^4}, & \textit{if } n = P_k, \\ 0 \pmod{2^4}, & \textit{otherwise}, \end{cases} \tag{5.1.34}$$

where P_k is either of the kth generalized pentagonal numbers $k(3k \pm 1)/2$.

In the next section, we give some preliminary results and dissections of some theta functions. In the last section, we prove Theorems 5.1.2–5.1.6.

5.2 Preliminary results and dissections of theta functions.

Lemma 5.2.1. *If $\varphi(q)$, $\psi(q)$, and $\chi(-q)$ are defined in* (1.5.2), (1.5.3), *and* (1.6.4), *then*

$$\psi(q) = f(q^3, q^6) + q\psi(q^9), \tag{5.2.1}$$

$$f(q, q^2) = \frac{\varphi(-q^3)}{\chi(-q)}, \tag{5.2.2}$$

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \tag{5.2.3}$$

Proof. See [20, p. 49, Corollary(ii)] and [20, p. 350, Eq. (2.3)] for the proofs of (5.2.1) and (5.2.2), respectively. Adding identities (v) and (vi) of [20, p. 40, Entry 25], we can easily derive (5.2.3). □

In the remaining lemmas of this section, we state and prove certain 2- and 3-dissections.

Lemma 5.2.2. *We have*

$$\frac{1}{(q;q)_\infty(q^3;q^3)_\infty} = \frac{(q^8;q^8)^2_\infty(q^{12};q^{12})^5_\infty}{(q^2;q^2)^2_\infty(q^4;q^4)(q^6;q^6)^4_\infty(q^{24};q^{24})^2_\infty} + q\,\frac{(q^4;q^4)^5_\infty(q^{24};q^{24})^2_\infty}{(q^2;q^2)^4_\infty(q^6;q^6)^2_\infty(q^8;q^8)^2_\infty(q^{12};q^{12})} \tag{5.2.4}$$

and

$$\frac{1}{(q;q)_\infty^2(q^3;q^3)_\infty^2} = \frac{(q^8;q^8)_\infty^5(q^{24};q^{24})_\infty^5}{(q^2;q^2)_\infty^5(q^6;q^6)_\infty^5(q^{16};q^{16})_\infty^2(q^{48};q^{48})_\infty^2} + 2q\,\frac{(q^4;q^4)_\infty^4(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^6(q^6;q^6)_\infty^6}$$
$$+ 4q^4\,\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2(q^{16};q^{16})_\infty^4(q^{48};q^{48})_\infty^2}{(q^2;q^2)_\infty^5(q^6;q^6)_\infty^5(q^8;q^8)_\infty(q^{24};q^{24})_\infty}. \tag{5.2.5}$$

Proof. From [35, Corollary 8], we find that

$$\psi(q)\psi(q^3) = \psi(q^4)\varphi(q^6) + q\psi(q^{12})\varphi(q^2). \tag{5.2.6}$$

Again, from [35, Corollary 4]

$$\varphi(q)\varphi(q^3) = \varphi(q^4)\varphi(q^{12}) + 2q\ \psi(q^2)\psi(q^6) + 4q^4\ \psi(q^8)\psi(q^{24}). \tag{5.2.7}$$

Employing the q-product representations of $\varphi(q)$ and $\psi(q)$, namely,

$$\varphi(q) := \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2} \tag{5.2.8}$$

and

$$\psi(q) := \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \tag{5.2.9}$$

in (5.2.6) and (5.2.7), we easily derive (5.2.4) and (5.2.5), respectively.

□

Lemma 5.2.3. *We have*

$$\frac{1}{(q;q)_\infty^4} = \frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\,\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}. \tag{5.2.10}$$

Proof. From (5.2.3)

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \tag{5.2.11}$$

Employing (5.2.8) and (5.2.9) in (5.2.11), we readily arrive at (5.2.10). □

Proofs of the results in the next lemma can be found in Chapter 4. It can also be found in Hirschhorn, Garvan, Borwein [48].

Lemma 5.2.4. *We have*

$$\frac{(q^3;q^3)_\infty}{(q;q)_\infty^3} = \frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty^9(q^{12};q^{12})_\infty^2} + 3q\,\frac{(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2(q^4;q^4)_\infty^2}{(q^2;q^2)^7}, \tag{5.2.12}$$

$$\begin{aligned}\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^4} &= \frac{(q^4;q^4)_\infty^4(q^6;q^6)_\infty(q^8;q^8)_\infty^4(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^{10}(q^{16};q^{16})_\infty^2(q^{24};q^{24})_\infty}\\ &+ 2q\,\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^{10}(q^8;q^8)_\infty^2(q^{24};q^{24})_\infty}\\ &+ 2q\,\frac{(q^4;q^4)_\infty(q^6;q^6)_\infty^2(q^8;q^8)_\infty^6(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty^9(q^{12};q^{12})_\infty(q^{16};q^{16})^2}\\ &+ 4q^2\,\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2(q^{16};q^{16})_\infty^2(q^{24};q^{24})_\infty}{(q^2;q^2)_\infty^9(q^{12};q^{12})_\infty},\end{aligned} \tag{5.2.13}$$

Proof. From (4.3.52) and (4.3.53) of Chapter 4, we readily arrive at (5.2.12). From (4.3.43), we obtain (5.2.13). □

The proof of the result in next lemma can be found in Chapter 2.

Lemma 5.2.5. *We have*

$$\frac{1}{\varphi(-q)} = \frac{\varphi^3(-q^9)}{\varphi^4(-q^3)} + 2q\frac{\varphi^3(-q^9)w(q^3)}{\varphi^4(-q^3)} + 4q^2w^2(q^3)\frac{\varphi^3(-q^9)}{\varphi^4(-q^3)}, \tag{5.2.14}$$

where

$$w(q) = \frac{(q;q)_\infty(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty(q^3;q^3)_\infty^3}. \tag{5.2.15}$$

Lemma 5.2.6. *We have*

$$\begin{aligned}\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^6} &= \frac{(q^6;q^6)_\infty^{10}(q^9;q^9)_\infty^{16}}{(q^3;q^3)_\infty^{18}(q^{18};q^{18})_\infty^8} + 6q\,\frac{(q^6;q^6)_\infty^9(q^9;q^9)_\infty^{13}}{(q^3;q^3)_\infty^{17}(q^{18};q^{18})_\infty^5}\\ &+ 21q^2\,\frac{(q^6;q^6)_\infty^8(q^9;q^9)_\infty^{10}}{(q^3;q^3)_\infty^{16}(q^{18};q^{18})_\infty^2} + 44q^3\,\frac{(q^6;q^6)_\infty^7(q^9;q^9)_\infty^7(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty^{15}}\\ &+ 60q^4\,\frac{(q^6;q^6)_\infty^6(q^9;q^9)_\infty^4(q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^{14}}\\ &+ 48q^5\,\frac{(q^6;q^6)_\infty^5(q^9;q^9)_\infty(q^{18};q^{18})_\infty^7}{(q^3;q^3)_\infty^{13}}\\ &+ 16q^6\,\frac{(q^6;q^6)_\infty^4(q^{18};q^{18})_\infty^{10}}{(q^9;q^9)_\infty^2(q^3;q^3)_\infty^{12}}.\end{aligned} \tag{5.2.16}$$

Proof. Squaring both sides of (5.2.1) and then employing (5.2.2), we have

$$\psi^2(q) = \frac{\varphi^2(-q^9)}{\chi^2(-q^3)} + q^2\psi^2(q^9) + 2q\frac{\varphi(-q^9)\psi(q^9)}{\chi(-q^3)}. \tag{5.2.17}$$

Again, squaring both sides of (5.2.14), we find that

$$\frac{1}{\varphi^2(-q)} = \frac{\varphi^6(-q^9)}{\varphi^8(-q^3)}\{1 + 4qw(q^3) + 12q^2w^2(q^3) + 16q^3w^3(q^3) + 16q^4w^4(q^3)\}. \tag{5.2.18}$$

From (1.5.5) and (1.6.4), we have

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)}. \tag{5.2.19}$$

$$\chi(-q) = (q;q^2)_\infty = \frac{(q;q)_\infty}{(q^2;q^2)_\infty}. \tag{5.2.20}$$

Multiplying (5.2.17) and (5.2.18) and then employing (5.2.9), (5.2.20), (5.2.15) *and* (5.2.19), *we easily arrive at* (5.2.16) *to complete the proof.* □

Lemma 5.2.7. *We have*

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty} = \frac{(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^3(q^{36};q^{36})_\infty^2} + q\,\frac{(q^6;q^6)_\infty^2(q^9;q^9)_\infty^3(q^{36};q^{36})_\infty}{(q^3;q^3)_\infty^4(q^{18};q^{18})_\infty^2} + 2q^2\,\frac{(q^6;q^6)_\infty(q^{18};q^{18})_\infty(q^{36};q^{36})_\infty}{(q^3;q^3)_\infty^3}. \tag{5.2.21}$$

Proof. We recall from [26] that the cubic theta function $c(q)$:

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} = 3q^{1/3}\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}. \tag{5.2.22}$$

From [22], we have

$$\frac{c(q)}{c(q^4)} = 1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)}. \tag{5.2.23}$$

Employing (5.2.22) in (5.2.23), we find that

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty} = q\frac{(q^{12};q^{12})_\infty^3}{(q^3;q^3)^3}\left\{1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)}\right\}. \tag{5.2.24}$$

Next, replacing q by q^2 in (5.2.17),

$$\psi^2(q^2)=\frac{\varphi^2(-q^{18})}{\chi^2(-q^6)}+q^4\psi^2(q^{18})+2q^2\frac{\varphi(-q^{18})\psi(q^{18})}{\chi(-q^6)}. \tag{5.2.25}$$

Using (5.2.25) in (5.2.24), we obtain

$$\begin{aligned}\frac{(q^4;q^4)_\infty}{(q;q)_\infty}&=q\frac{(q^{12};q^{12})^3_\infty}{(q^3;q^3)^3_\infty}\Big\{1+\frac{1}{q\psi^2(q^6)}\Big(\frac{\varphi^2(-q^{18})}{\chi^2(-q^6)}+q^4\psi^2(q^{18})\\&\quad+2q^2\frac{\varphi(-q^{18})\psi(q^{18})}{\chi(-q^6)}\Big)\Big\}\\&=q\frac{(q^{12};q^{12})^3_\infty}{(q^3;q^3)^3_\infty}\Big\{1+q^3\frac{\psi^2(q^{18})}{\psi^2(q^6)}\Big\}+q\frac{(q^{12};q^{12})^3_\infty}{(q^3;q^3)^3_\infty}\Big\{\frac{\varphi^2(-q^{18})}{q\psi^2(q^6)\chi^2(-q^6)}\\&\quad+2q\,\frac{\varphi(-q^{18})\psi(q^{18})}{\psi^2(q^6)\chi(-q^6)}\Big\}.\end{aligned} \tag{5.2.26}$$

Employing (5.2.9), (5.2.20) and (5.2.19) in (5.2.26), we find that

$$\begin{aligned}\frac{(q^4;q^4)_\infty}{(q;q)_\infty}&=q\frac{(q^{12};q^{12})^3_\infty}{(q^3;q^3)^3_\infty}\Big\{1+q^3\frac{\psi^2(q^{18})}{\psi^2(q^6)}\Big\}+\frac{(q^{12};q^{12})_\infty(q^{18};q^{18})^4_\infty}{(q^3;q^3)^3_\infty(q^{36};q^{36})^2_\infty}\\&\quad+2q^2\,\frac{(q^{18};q^{18})_\infty(q^{36};q^{36})_\infty(q^6;q^6)_\infty}{(q^3;q^3)^3_\infty}.\end{aligned} \tag{5.2.27}$$

Now, multiplying both sides of (5.2.24) by $\psi^2(q^6)/\psi^2(q^2)$, replacing q by q^3, and then employing (5.2.9), we deduce that

$$1+q^3\,\frac{\psi^2(q^{18})}{\psi^2(q^6)}=\frac{(q^6;q^6)^2_\infty(q^9;q^9)^3_\infty(q^{36};q^{36})_\infty}{(q^3;q^3)_\infty(q^{18};q^{18})^2_\infty(q^{12};q^{12})^3_\infty}. \tag{5.2.28}$$

Employing (5.2.28) in (5.2.27), we arrive at (5.2.21) to finish the proof. □

5.3 Proofs of Theorems 5.1.2–5.1.6

Proof of (5.1.8). Using (5.2.4) in (5.1.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty}PDO(n)q^n&=\frac{(q^8;q^8)^2_\infty(q^{12};q^{12})^4_\infty}{(q^2;q^2)^2_\infty(q^4;q^4)_\infty(q^6;q^6)^2_\infty(q^{24};q^{24})^2_\infty}\\&\quad+q\,\frac{(q^4;q^4)^6_\infty(q^{24};q^{24})^2_\infty}{(q^2;q^2)^4_\infty(q^8;q^8)^2_\infty(q^{12};q^{12})^2_\infty}.\end{aligned} \tag{5.3.1}$$

Extracting from both sides of (5.3.1), those terms involving only q^{2n}, and then replacing q^2 by q, we arrive at (5.1.8). □

Proof of (5.1.9). Extracting from both sides of (5.3.1), those terms involving only q^{2n+1}, and then dividing both sides by q and replacing q^2 by q, we arrive at (5.1.9). □

Proofs of (5.1.10)–(5.1.12). Using (5.2.21) in (5.1.7), we find that

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{(q^6;q^6)_\infty^2(q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^4(q^{36};q^{36})_\infty^2} + q\,\frac{(q^6;q^6)_\infty^4(q^9;q^9)_\infty^3(q^{36};q^{36})_\infty}{(q^3;q^3)_\infty^5(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty^2} + 2q^2\,\frac{(q^6;q^6)_\infty^3(q^{18};q^{18})_\infty(q^{36};q^{36})_\infty}{(q^3;q^3)_\infty^4(q^{12};q^{12})_\infty}. \quad (5.3.2)$$

Extracting from both sides of (5.3.2), the terms involving q^{3n}, q^{3n+1}, and q^{3n+2}, respectively, we arrive at (5.1.10)–(5.1.12), respectively. □

Proof of (5.1.13). Employing (5.2.5) in (5.1.8), we have

$$\sum_{n=0}^{\infty} PDO(2n)q^n = \frac{(q^4;q^4)_\infty^2(q^8;q^8)^5(q^{24};q^{24})_\infty^5}{(q^2;q^2)_\infty^5(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2(q^{16};q^{16})_\infty^2(q^{48};q^{48})_\infty^2} + 2q\,\frac{(q^4;q^4)_\infty^6(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^6(q^6;q^6)_\infty^2} + 4q^4\,\frac{(q^4;q^4)_\infty^4(q^{16};q^{16})_\infty^4(q^{48};q^{48})_\infty^2}{(q^2;q^2)_\infty^5(q^6;q^6)_\infty(q^8;q^8)_\infty(q^{24};q^{24})_\infty}. \quad (5.3.3)$$

We can arrive at (5.1.13) by extracting the terms involving only q^{2n+1} from both sides of (5.3.3), and then dividing both sides by q and replacing q^2 by q. □

Proof of (5.1.14). Employing (5.2.10) in (5.1.9), we have

$$\sum_{n=0}^{\infty} PDO(2n+1)q^n = \frac{(q^2;q^2)_\infty^6(q^{12};q^{12})_\infty^2}{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^2}\left\{\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\,\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}\right\}.$$

Extracting the terms involving only q^{2n+1} from the above, we easily deduce (5.1.14). □

Proofs of (5.1.15) *and* (5.1.17). Employing (5.2.10) in (5.1.12), we arrive at

$$\sum_{n=0}^{\infty} PDO(3n+2)q^n = 2\,\frac{(q^2;q^2)_\infty^3(q^6;q^6)_\infty(q^{12};q^{12})_\infty}{(q^4;q^4)_\infty}\left\{\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4}\right.$$

$$\left.+\,4q\,\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}\right\}. \tag{5.3.4}$$

Extracting the even and odd powers of q from both sides of (5.3.4), we readily deduce (5.1.15) and (5.1.17), respectively. □

Proof of (5.1.16). Using (5.2.10) in (5.1.10), we arrive at

$$\sum_{n=0}^{\infty} PDO(3n)q^n = \frac{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^4}{(q^{12};q^{12})_\infty^2}\left\{\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4}+4q\,\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}\right\}. \tag{5.3.5}$$

Now (5.1.16) can be deduced by extracting the odd powers of q from both sides of (5.3.5). □

Proofs of (5.1.18) *and* (5.1.19). Squaring both sides of (5.2.14) and then employing the resultant identity in (5.1.10), we obtain

$$\sum_{n=0}^{\infty} PDO(3n)q^n = \frac{\varphi^2(-q^6)\varphi^6(-q^9)}{\varphi^8(-q^3)}\left\{1+4q\ w(q^3)+12q^2\ w^2(q^3)+16q^3w^3(q^3)\right.$$

$$\left.+\,16q^4\ w^4(q^3)\right\}. \tag{5.3.6}$$

Extracting from both sides of (5.3.6), those terms involving only q^{3n+1} and q^{3n+2}, respectively, we find that

$$\sum_{n=0}^{\infty} PDO(9n+3)q^n = 4\,\frac{w(q)\varphi^2(-q^2)\varphi^6(-q^3)}{\varphi^8(-q)}\{1+4q\ w^3(q)\}$$

and

$$\sum_{n=0}^{\infty} PDO(9n+6)q^n = 12\,\frac{w^2(q)\varphi^2(-q^2)\varphi^6(-q^3)}{\varphi^8(-q)},$$

which by (5.2.15) and (5.2.19) reduce to (5.1.18) and (5.1.19), respectively. □

Proofs of (5.1.20) *and* (5.1.22). Employing (5.2.16) in (5.1.13), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} PDO(4n+2)q^n \\
&= 2\left\{\frac{(q^6;q^6)_\infty^{12}(q^9;q^9)_\infty^{16}}{(q^3;q^3)_\infty^{11}(q^{18};q^{18})_\infty^{4}} + 6q\,\frac{(q^6;q^6)_\infty^{11}(q^9;q^9)_\infty^{13}}{(q^3;q^3)_\infty^{19}(q^{18};q^{18})_\infty^{5}} + 21q^2\,\frac{(q^6;q^6)_\infty^{10}(q^9;q^9)_\infty^{10}}{(q^3;q^3)_\infty^{18}(q^{18};q^{18})_\infty^{2}}\right. \\
&\quad + 44q^3\,\frac{(q^6;q^6)_\infty^{9}(q^9;q^9)_\infty^{7}(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty^{17}} + 60q^4\,\frac{(q^6;q^6)_\infty^{8}(q^9;q^9)_\infty^{4}(q^{18};q^{18})_\infty^{4}}{(q^3;q^3)_\infty^{16}} \\
&\quad \left. + 48q^5\,\frac{(q^6;q^6)_\infty^{7}(q^9;q^9)_\infty(q^{18};q^{18})_\infty^{7}}{(q^3;q^3)_\infty^{15}}\right\}. \qquad (5.3.7)
\end{aligned}$$

Extracting from both sides of (5.3.7), those terms involving only q^{3n+1} and q^{3n+2}, respectively, we deduce (5.1.20) and (5.1.22).

Now we present a second proof of (5.1.20).

Extracting the terms involving q^{2n} from both sides of (5.3.5), we find that

$$\sum_{n=0}^{\infty} PDO(6n)q^n = \frac{(q^2;q^2)_\infty^{14}(q^3;q^3)_\infty^{4}}{(q;q)_\infty^{12}(q^4;q^4)_\infty^{4}(q^6;q^6)_\infty^{2}}. \qquad (5.3.8)$$

Next, from (5.2.12), we obtain

$$\begin{aligned}
&\frac{(q^3;q^3)_\infty^{4}}{(q;q)_\infty^{12}} \\
&= \frac{(q^4;q^4)_\infty^{24}(q^6;q^6)_\infty^{12}}{(q^2;q^2)_\infty^{36}(q^{12};q^{12})_\infty^{8}} + 12q\,\frac{(q^4;q^4)_\infty^{20}(q^6;q^6)_\infty^{10}}{(q^2;q^2)_\infty^{34}(q^{12};q^{12})_\infty^{4}} + 54q^2\,\frac{(q^4;q^4)_\infty^{16}(q^6;q^6)_\infty^{8}}{(q^2;q^2)_\infty^{32}} \\
&\quad + 108q^3\,\frac{(q^4;q^4)_\infty^{12}(q^6;q^6)_\infty^{6}(q^{12};q^{12})_\infty^{4}}{(q^2;q^2)_\infty^{30}} + 81q^4\,\frac{(q^4;q^4)_\infty^{8}(q^6;q^6)_\infty^{4}(q^{12};q^{12})_\infty^{8}}{(q^2;q^2)_\infty^{28}}.
\end{aligned}$$

(5.3.9)

Employing (5.3.9) in (5.3.8), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} PDO(6n)q^n &= \frac{(q^2;q^2)_\infty^{14}}{(q^4;q^4)_\infty^{4}(q^6;q^6)_\infty^{2}}\left\{\frac{(q^4;q^4)_\infty^{24}(q^6;q^6)_\infty^{12}}{(q^2;q^2)_\infty^{36}(q^{12};q^{12})_\infty^{8}}\right. \\
&\quad + 12q\,\frac{(q^4;q^4)_\infty^{20}(q^6;q^6)_\infty^{10}}{(q^2;q^2)_\infty^{34}(q^{12};q^{12})_\infty^{4}} + 54q^2\,\frac{(q^4;q^4)_\infty^{16}(q^6;q^6)_\infty^{8}}{(q^2;q^2)_\infty^{32}} \\
&\quad + 108q^3\,\frac{(q^4;q^4)_\infty^{12}(q^6;q^6)_\infty^{6}(q^{12};q^{12})_\infty^{4}}{(q^2;q^2)_\infty^{30}} \\
&\quad \left. + 81q^4\,\frac{(q^4;q^4)_\infty^{8}(q^6;q^6)_\infty^{4}(q^{12};q^{12})_\infty^{8}}{(q^2;q^2)_\infty^{28}}\right\}. \qquad (5.3.10)
\end{aligned}$$

Extracting the terms involving only q^{2n+1} from both sides of (5.3.10), we readily arrive at (5.1.20) to finish the proof. □

Proof of (5.1.21). Squaring both sides of (5.2.13), we have

$$\begin{aligned}
&\frac{(q^3;q^3)_\infty^4}{(q;q)_\infty^8}\\
&= \frac{(q^4;q^4)_\infty^8(q^6;q^6)_\infty^2(q^8;q^8)_\infty^8(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^{20}(q^{16};q^{16})_\infty^4(q^{24};q^{24})_\infty^2} + 4q\,\frac{(q^4;q^4)_\infty^{10}(q^6;q^6)_\infty^2(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^{20}(q^{24};q^{24})_\infty^2}\\
&\quad + 4q\,\frac{(q^4;q^4)_\infty^5(q^6;q^6)_\infty^3(q^8;q^8)_\infty^{10}(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^{19}(q^{16};q^{16})_\infty^4}\\
&\quad + 16q^2\,\frac{(q^4;q^4)_\infty^7(q^6;q^6)_\infty^3(q^8;q^8)_\infty^4(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^{19}}\\
&\quad + 4q^2\,\frac{(q^4;q^4)_\infty^{12}(q^6;q^6)_\infty^2(q^{12};q^{12})_\infty^4(q^{16};q^{16})_\infty^4}{(q^2;q^2)_\infty^{20}(q^8;q^8)_\infty^4(q^{24};q^{24})_\infty^2}\\
&\quad + 4q^2\,\frac{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^4(q^8;q^8)_\infty^{12}(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^{18}(q^{12};q^{12})_\infty^2(q^{16};q^{16})_\infty^4}\\
&\quad + 16q^3\,\frac{(q^4;q^4)_\infty^9(q^6;q^6)_\infty^3(q^{12};q^{12})_\infty(q^{16};q^{16})_\infty^4}{(q^2;q^2)_\infty^{19}(q^8;q^8)_\infty^2}\\
&\quad + 16q^3\,\frac{(q^4;q^4)_\infty^4(q^8;q^8)_\infty^6(q^6;q^6)_\infty^4(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^{18}(q^{12};q^{12})_\infty^2}\\
&\quad + 16q^4\,\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^4(q^{16};q^{16})_\infty^4(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^{18}(q^{12};q^{12})_\infty^2}.
\end{aligned} \tag{5.3.11}$$

Now, using (5.3.11) in (5.1.16), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} PDO(6n+3)q^n\\
&= 4\left\{\frac{(q^4;q^4)_\infty^{12}(q^8;q^8)_\infty^8(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^{18}(q^{16};q^{16})_\infty^4(q^{24};q^{24})_\infty^2} + 4q\,\frac{(q^4;q^4)_\infty^{14}(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^{18}(q^{24};q^{24})_\infty^2}\right.\\
&\quad + 4q\,\frac{(q^4;q^4)_\infty^9(q^6;q^6)_\infty(q^8;q^8)_\infty^{10}(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^{17}(q^{16};q^{16})_\infty^4}\\
&\quad + 16q^2\,\frac{(q^4;q^4)_\infty^{11}(q^6;q^6)_\infty(q^8;q^8)_\infty^4(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^{17}}\\
&\quad + 4q^2\,\frac{(q^4;q^4)_\infty^{16}(q^{12};q^{12})_\infty^4(q^{16};q^{16})_\infty^4}{(q^2;q^2)_\infty^{18}(q^8;q^8)_\infty^4(q^{24};q^{24})_\infty^2}
\end{aligned}$$

$$+ 4q^2 \frac{(q^4;q^4)_\infty^6 (q^6;q^6)_\infty^2 (q^8;q^8)_\infty^{12} (q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^{16} (q^{12};q^{12})_\infty^2 (q^{16};q^{16})_\infty^4}$$
$$+ 16q^3 \frac{(q^4;q^4)_\infty^{13} (q^6;q^6)_\infty (q^{12};q^{12})_\infty (q^{16};q^{16})_\infty^4}{(q^2;q^2)_\infty^{17} (q^8;q^8)_\infty^2}$$
$$+ 16q^3 \frac{(q^4;q^4)_\infty^8 (q^8;q^8)_\infty^6 (q^6;q^6)_\infty^2 (q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^{16} (q^{12};q^{12})_\infty^2}$$
$$+ 16q^4 \frac{(q^4;q^4)_\infty^{10} (q^6;q^6)_\infty^2 (q^{16};q^{16})_\infty^4 (q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^{16} (q^{12};q^{12})_\infty^2} \Bigg\}. \tag{5.3.12}$$

Extracting the terms involving q^{2n+1} from both sides of the above, we arrive at (5.1.21) to complete the proof. □

Now we prove (5.1.25)–(5.1.27).

Proofs of (5.1.25)–(5.1.27). By binomial theorem, it is easy to deduce that

$$(q;q)_\infty^2 \equiv (q^2;q^2)_\infty \pmod 2. \tag{5.3.13}$$

Employing (5.3.13) in (5.1.13), (5.1.14), and (5.1.19), we find that

$$\sum_{n=0}^{\infty} \frac{PDO(4n+2)}{2} q^n \equiv (q^2;q^2)_\infty^3 (q^6;q^6)_\infty \pmod 2,$$
$$\sum_{n=0}^{\infty} \frac{PDO(4n+3)}{4} q^n \equiv (q^2;q^2)_\infty^6 (q^6;q^6)_\infty \pmod 2,$$

and

$$\sum_{n=0}^{\infty} \frac{PDO(9n+6)}{12} q^n \equiv \frac{(q^6;q^6)_\infty^3}{(q^2;q^2)} \pmod 2,$$

respectively. Now (5.1.25)–(5.1.27) are apparent from the above.

From (5.3.6)

$$\sum_{n=0}^{\infty} PDO(3n)q^n = \frac{\varphi^2(-q^6)\varphi^6(-q^9)}{\varphi^8(-q^3)} \Big\{ 1 + 4q\, w(q^3) + 12q^2\, w^2(q^3) + 16q^3 w^3(q^3) + 16q^4\, w^4(q^3) \Big\}$$
$$\equiv \frac{\varphi^2(-q^6)\varphi^6(-q^9)}{\varphi^8(-q^3)} \left\{ 1 + 4q\, w(q^3) - 4q^2\, w^2(q^3) \right\} \pmod{16}. \tag{5.3.14}$$

Therefore,

$$\sum_{n=0}^{\infty} PDO(9n)q^n \equiv \frac{\varphi^2(-q^2)\varphi^6(-q^3)}{\varphi^8(-q)} \pmod{16}.$$
$$\equiv \frac{(q^2;q^2)^{12}(q^3;q^3)^{12}}{(q;q)^{16}(q^4;q^4)^2(q^6;q^6)} \pmod{16}. \quad \text{by (5.2.19)}$$
$$\equiv (q^3;q^3)^{10} \pmod{16}, \tag{5.3.15}$$

which implies that

$$\sum_{n=0}^{\infty} PDO(27n+9)q^n \equiv 0 \pmod{16}$$

and

$$\sum_{n=0}^{\infty} PDO(27n+18)q^n \equiv 0 \pmod{16}.$$

Now, (5.1.28) and (5.1.29) are apparent from the above. □

Proof of Theorem 5.1.6. Taking modulo 16 in (5.3.12), we find that

$$\sum_{n=0}^{\infty} PDO(6n+3)q^n \equiv 4\ \frac{(q^4;q^4)_\infty^{12}(q^8;q^8)_\infty^8(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^{18}(q^{16};q^{16})_\infty^4(q^{24};q^{24})_\infty^2} \pmod{16}.$$
$$\equiv 4\ (q^4;q^4)_\infty^3 \pmod{16}, \qquad \text{(by binomial theorem)} \tag{5.3.16}$$

which yields that

$$\sum_{n=0}^{\infty} PDO(24n+9)q^n \equiv 0 \pmod{16}, \tag{5.3.17}$$
$$\sum_{n=0}^{\infty} PDO(24n+15)q^n \equiv 0 \pmod{16}, \tag{5.3.18}$$
$$\sum_{n=0}^{\infty} PDO(24n+21)q^n \equiv 0 \pmod{16}, \tag{5.3.19}$$

and

$$\sum_{n=0}^{\infty} PDO(24n+3)q^n \equiv 4\ (q;q)_\infty^3 \pmod{16}. \tag{5.3.20}$$

Now, (5.1.30)–(5.1.32) follow from (5.3.17)–(5.3.19).

Again, from (5.3.20), we have

$$\sum_{n=0}^{\infty} PDO(24n+3)q^n \equiv 4\ (q;q)_\infty^3 \equiv 4\ \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \equiv 4\ \psi(q) \pmod{16}. \tag{5.3.21}$$

Using (5.2.2) in (5.2.1), we have

$$\psi(q) = \frac{\phi(-q^9)}{\chi(-q^3)} + q\psi(q^9). \tag{5.3.22}$$

Employing (5.2.19), (5.2.20) in (5.3.22),

$$\psi(q) = \frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}{(q^3;q^3)_\infty(q^{18};q^{18})_\infty} + q\psi(q^9). \tag{5.3.23}$$

Thus, from (5.3.21), we obtain

$$\sum_{n=0}^{\infty} PDO(24n+3)q^n \equiv 4\ \frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}{(q^3;q^3)_\infty(q^{18};q^{18})_\infty} + 4q\psi(q^9) \pmod{16}, \tag{5.3.24}$$

which implies

$$\sum_{n=0}^{\infty} PDO(72n+51)q^n \equiv 0 \pmod{2^4} \tag{5.3.25}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} PDO(72n+3)q^n &\equiv 4\ \frac{(q^2;q^2)_\infty(q^3;q^3)_\infty^2}{(q;q)_\infty(q^6;q^6)_\infty} \pmod{16}\\ &\equiv 4\ (q;q)_\infty \pmod{16}\\ &\equiv 4+4\ \sum_{n=0}^{\infty}(-1)^k(q^{k(3k-1)/2}+q^{k(3k+1)/2}) \pmod{16}. \end{aligned}\tag{5.3.26}$$

Thus, (5.1.33) and (5.1.34) follow from (5.3.25) and (5.3.26). This completes the proof. □

Chapter 6

Some Identities of Overpartition Pairs into Odd Parts

6.1 Introduction

In the introductory chapter, we defined overpartitions and overpartition pairs. Recently, arithmetic properties of overpartition pairs $\overline{pp}(n)$ have been considered by Bringmann and Lovejoy [28], Chen and Lin [37] and Kim [55].

An overpartition pair into odd parts is a pair of overpartitions (λ, μ) such that the parts of both overpartitions λ and μ are restricted to be odd integers. Note that either λ or μ may be an overpartition of zero, which, for convenience, assumed to be $\emptyset$. Let $\overline{pp}_o(n)$ denote the number of overpartition pairs of n into odd parts. Then the generating function for $\overline{pp}_o(n)$ is

$$\sum_{n=0}^{\infty} \overline{pp}_o(n) q^n = \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^4 (q^4;q^4)_\infty^2}. \tag{6.1.1}$$

Recently, Lin [59] investigated various arithmetic properties of $\overline{pp}_o(n)$. He obtained a number of Ramanujan-type congruences for modulo 3 and for modulo powers of 2. In this chapter, we derive the congruences for the function $\overline{pp}_o(n)$ from the respective generating functions.

We list our main results in the following theorems.

Theorem 6.1.1. [59, Theorem 2.1] *We have*

$$\sum_{n=0}^{\infty} \overline{pp}_0(2n)q^n = \frac{(q^2;q^2)_\infty^{12}}{(q;q)_\infty^8 (q^4;q^4)_\infty^4}, \tag{6.1.2}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(2n+1)q^n = 4\frac{(q^4;q^4)_\infty^4}{(q;q)_\infty^4}. \tag{6.1.3}$$

Theorem 6.1.2. *We have*

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n)q^n = 4\frac{(q^2;q^2)_\infty^{24}}{(q;q)_\infty^{16}(q^4;q^4)_\infty^8} + 16q\,\frac{(q^4;q^4)_\infty^8}{(q;q)_\infty^8}, \tag{6.1.4}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n+1)q^n = 4\frac{(q^2;q^2)_\infty^{18}}{(q;q)_\infty^{14}(q^4;q^4)_\infty^4}, \tag{6.1.5}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n+2)q^n = 8\frac{(q^2;q^2)_\infty^{12}}{(q;q)_\infty^{12}}, \tag{6.1.6}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n+3)q^n = 16\frac{(q^2;q^2)_\infty^6 (q^4;q^4)_\infty^4}{(q;q)_\infty^{10}}. \tag{6.1.7}$$

Lin [59] has also proved (6.1.6).

Theorem 6.1.3. *We have*

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+4)q^n = 80 \times \left\{ \frac{(q^2;q^2)_\infty^{36}}{(q;q)_\infty^{28}(q^4;q^4)_\infty^8} + 16q\,\frac{(q^2;q^2)_\infty^{12}(q^4;q^4)_\infty^8}{(q;q)_\infty^{20}} \right\}, \tag{6.1.8}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+6)q^n = 32 \times \left\{ 3\frac{(q^2;q^2)_\infty^{30}}{(q;q)_\infty^{26}(q^4;q^4)_\infty^4} + 16q\,\frac{(q^2;q^2)_\infty^6(q^4;q^4)_\infty^{12}}{(q;q)_\infty^{18}} \right\}, \tag{6.1.9}$$

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(8n+7)q^n = 32 \times \Bigg\{ & 5\frac{(q^4;q^4)_\infty^{19}(q^2;q^2)_\infty^6}{(q;q)_\infty^{19}(q^8;q^8)_\infty^6} + 40q\,\frac{(q^2;q^2)_\infty^{10}(q^4;q^4)_\infty^7(q^8;q^8)_\infty^2}{(q;q)_\infty^{19}} \\ & + 16q^2\,\frac{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^{10}}{(q;q)_\infty^{19}(q^4;q^4)_\infty^5} \Bigg\}. \end{aligned}\tag{6.1.10}$$

Theorem 6.1.4. *We have*

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(12n+6)q^n = 24\,\Bigg\{ & 4\frac{(q^2;q^2)_\infty^{19}(q^3;q^3)_\infty^{29}}{(q;q)_\infty^{35}(q^6;q^6)_\infty^{13}} + 363q\,\frac{(q^2;q^2)_\infty^{16}(q^3;q^3)_\infty^{20}}{(q;q)_\infty^{32}(q^6;q^6)_\infty^4} \\ & + 2496q^2\,\frac{(q^2;q^2)_\infty^{13}(q^3;q^3)_\infty^{11}(q^6;q^6)_\infty^5}{(q;q)_\infty^{29}} \\ & + 1408q^3\,\frac{(q^2;q^2)_\infty^{10}(q^3;q^3)_\infty^2(q^6;q^6)_\infty^{14}}{(q;q)_\infty^{26}} \Bigg\}, \end{aligned}\tag{6.1.11}$$

$$\sum_{n=0}^{\infty}\overline{pp}_o(12n+10)q^n = 48\left\{13\frac{(q^2;q^2)_\infty^{18}(q^3;q^3)_\infty^{26}}{(q;q)_\infty^{34}(q^6;q^6)_\infty^{10}} + 444q\,\frac{(q^2;q^2)_\infty^{15}(q^3;q^3)_\infty^{17}}{(q;q)_\infty^{21}(q^6;q^6)_\infty}\right.$$
$$+ 1416q^2\,\frac{(q^2;q^2)_\infty^{12}(q^3;q^3)_\infty^{8}(q^6;q^6)_\infty^{8}}{(q;q)_\infty^{28}}$$
$$\left.+ 256q^3\,\frac{(q^2;q^2)_\infty^{9}(q^6;q^6)_\infty^{17}}{(q;q)_\infty^{25}(q^3;q^3)_\infty}\right\}. \tag{6.1.12}$$

From (6.1.3)–(6.1.12), we easily arrive at the following congruences.

Corollary 6.1.5. *We have*

$$\overline{pp}_o(2n+1) \equiv 0 \pmod 4,$$
$$\overline{pp}_o(4n) \equiv 0 \pmod 4,$$
$$\overline{pp}_o(4n+1) \equiv 0 \pmod 4,$$
$$\overline{pp}_o(4n+2) \equiv 0 \pmod 8,$$
$$\overline{pp}_o(4n+3) \equiv 0 \pmod{16}, \tag{6.1.13}$$
$$\overline{pp}_o(8n+4) \equiv 0 \pmod{80},$$
$$\overline{pp}_o(8n+6) \equiv 0 \pmod{32},$$
$$\overline{pp}_o(8n+7) \equiv 0 \pmod{32}, \tag{6.1.14}$$
$$\overline{pp}_o(12n+6) \equiv 0 \pmod{24}, \tag{6.1.15}$$
$$\overline{pp}_o(12n+10) \equiv 0 \pmod{48}. \tag{6.1.16}$$

Lin [59] also proved (6.1.13), (6.1.14), and (6.1.15) by taking modulo powers of 2. The identity in (6.1.16) is an improved version of Lin's identity in [59, Eq. (2.11), Corollary 2.1]

$$\overline{pp}_o(12n+10) \equiv 0 \pmod{24}.$$

In the following theorems, we prove some new congruences for $\overline{pp}_o(n)$ by employing elementary generating function technique.

Theorem 6.1.6. *We have*

$$\overline{pp}_o(8n+5) \equiv 0 \pmod{2^3}, \tag{6.1.17}$$

$$\overline{pp}_o(16n+8) \equiv 0 \pmod{2^3}, \tag{6.1.18}$$

$$\overline{pp}_o(16n+10) \equiv 0 \pmod{2^4}, \tag{6.1.19}$$

$$\overline{pp}_o(16n+12) \equiv 0 \pmod{2^3}, \tag{6.1.20}$$

$$\overline{pp}_o(16n+14) \equiv 0 \pmod{2^4}, \tag{6.1.21}$$

$$\overline{pp}_o(32n+20) \equiv 0 \pmod{160}, \tag{6.1.22}$$

$$\overline{pp}_o(32n+28) \equiv 0 \pmod{160}, \tag{6.1.23}$$

$$\overline{pp}_o(48n+10) \equiv 0 \pmod{2^5}, \tag{6.1.24}$$

$$\overline{pp}_o(48n+18) \equiv 0 \pmod{2^3}, \tag{6.1.25}$$

$$\overline{pp}_o(48n+26) \equiv 0 \pmod{2^5}, \tag{6.1.26}$$

$$\overline{pp}_o(48n+34) \equiv 0 \pmod{2^5}, \tag{6.1.27}$$

$$\overline{pp}_o(48n+42) \equiv 0 \pmod{2^5}. \tag{6.1.28}$$

Theorem 6.1.7. *We have*

$$\overline{pp}_o(3n+1) \equiv 0 \pmod{2^2}, \tag{6.1.29}$$

$$\overline{pp}_o(3n+2) \equiv 0 \pmod{2^2}, \tag{6.1.30}$$

$$\overline{pp}_o(6n+3) \equiv 0 \pmod{2^4}, \tag{6.1.31}$$

$$\overline{pp}_o(9n+3) \equiv 0 \pmod{2^4}, \tag{6.1.32}$$

$$\overline{pp}_o(9n+6) \equiv 0 \pmod{2^4}. \tag{6.1.33}$$

Theorem 6.1.8. *For all nonnegative integers n, we have*

$$\overline{pp}_o(24n+17) \equiv 0 \pmod{2} \tag{6.1.34}$$

and

$$\overline{pp}_o(24n+1) \equiv \begin{cases} 4 \pmod{2}, & \text{if } n = P_k, \\ 0 \pmod{2}, & \text{otherwise}, \end{cases} \tag{6.1.35}$$

where P_k is either of the kth generalized pentagonal numbers $k(3k \pm 1)/2$.

Theorem 6.1.9. *For all nonnegative integers n, we have*

$$\overline{pp}_o(48n+34) \equiv 0 \pmod{2^5} \tag{6.1.36}$$

and

$$\overline{pp}_o(48n+2) \equiv \begin{cases} 8 \pmod{2^5}, & \text{if } n = P_k, \\ 0 \pmod{2^5}, & \text{otherwise}, \end{cases} \tag{6.1.37}$$

where P_k is either of the kth generalized pentagonal numbers $k(3k \pm 1)/2$.

Theorem 6.1.10. *For all nonnegative integers n, we have*

$$\overline{pp}_o(96n+68) \equiv 0 \pmod{2} \tag{6.1.38}$$

and

$$\overline{pp}_o(96n+4) \equiv \begin{cases} 80 \pmod{2}, & \text{if } n = P_k, \\ 0 \pmod{2}, & \text{otherwise}, \end{cases} \tag{6.1.39}$$

where P_k is either of the kth generalized pentagonal numbers $k(3k \pm 1)/2$.

In the next section, we state some lemmas which will be used in the final section to prove the above identities and congruences.

6.2 Preliminary results and dissections of theta functions

In this section, we state some lemmas containing certain dissections.

Lemma 6.2.1. *The following 2-dissections hold:*

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5 (q^{16};q^{16})_\infty^2} + 2q\,\frac{(q^4;q^4)_\infty^2 (q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5 (q^8;q^8)_\infty}, \tag{6.2.1}$$

$$\frac{1}{(q;q)_\infty^4} = \frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14} (q^8;q^8)_\infty^4} + 4q\,\frac{(q^4;q^4)_\infty^2 (q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}, \tag{6.2.2}$$

$$\frac{1}{(q;q)_\infty^8} = \frac{(q^4;q^4)_\infty^{28}}{(q^2;q^2)_\infty^{28} (q^8;q^8)_\infty^8} + 8q\frac{(q^4;q^4)_\infty^{16}}{(q^2;q^2)_\infty^{24}} + 16q^2\,\frac{(q^4;q^4)_\infty^4 (q^8;q^8)_\infty^8}{(q^2;q^2)_\infty^{20}}, \tag{6.2.3}$$

$$\frac{1}{(q;q)_\infty^{16}} = \frac{(q^4;q^4)_\infty^{56}}{(q^2;q^2)_\infty^{56}(q^8;q^8)_\infty^{16}} + 16q\frac{(q^4;q^4)_\infty^{44}}{(q^2;q^2)_\infty^{52}(q^8;q^8)_\infty^{8}} + 96q^2\frac{(q^4;q^4)_\infty^{32}}{(q^2;q^2)_\infty^{48}} + 256q^3\frac{(q^4;q^4)_\infty^{20}(q^8;q^8)_\infty^{8}}{(q^2;q^2)_\infty^{44}} + 256q^4\frac{(q^4;q^4)_\infty^{8}(q^8;q^8)_\infty^{16}}{(q^2;q^2)_\infty^{40}}, \tag{6.2.4}$$

$$\frac{1}{(q;q)_\infty^{12}} = \frac{(q^4;q^4)_\infty^{42}}{(q^2;q^2)_\infty^{42}(q^8;q^8)_\infty^{12}} + 12q\frac{(q^4;q^4)_\infty^{30}}{(q^2;q^2)_\infty^{38}(q^8;q^8)_\infty^{4}} + 48q^2\frac{(q^4;q^4)_\infty^{18}(q^8;q^8)_\infty^{4}}{(q^2;q^2)_\infty^{34}} + 64q^3\frac{(q^4;q^4)_\infty^{6}(q^8;q^8)_\infty^{12}}{(q^2;q^2)_\infty^{30}}. \tag{6.2.5}$$

Proof. Adding Entry 25(i) and Entry 25(ii) of [20, p. 40], we have

$$\varphi(q) = \varphi(q^4) + 2q\psi^2(q^8). \tag{6.2.6}$$

From (5.2.3), we obtain

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \tag{6.2.7}$$

Squaring (6.2.7), we arrive at

$$\varphi^4(q) = \varphi^4(q^2) + 8q\varphi^2(q^2)\psi^2(q^4) + 16q^2\psi^4(q^4), \tag{6.2.8}$$

where $\varphi(q)$ and $\psi(q)$ are defined in (1.5.2) and (1.5.3). Employing (1.5.2) and (1.5.3) in (6.2.6), (6.2.7), (6.2.8), we readily arrive at (6.2.1), (6.2.2) and (6.2.3). From (6.2.3), we can easily deduce (6.2.4) and (6.2.5). □

Proofs of the identities in the next lemma can be found in Chapter 5.

Lemma 6.2.2. *The following 3-dissections hold:*

$$\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^6} = \frac{(q^6;q^6)_\infty^{10}(q^9;q^9)_\infty^{16}}{(q^3;q^3)_\infty^{18}(q^{18};q^{18})_\infty^{8}} + 6q\frac{(q^6;q^6)_\infty^{9}(q^9;q^9)_\infty^{13}}{(q^3;q^3)_\infty^{17}(q^{18};q^{18})_\infty^{5}} + 21q^2\frac{(q^6;q^6)_\infty^{8}(q^9;q^9)_\infty^{10}}{(q^3;q^3)_\infty^{16}(q^{18};q^{18})_\infty^{2}} + 44q^3\frac{(q^6;q^6)_\infty^{7}(q^9;q^9)_\infty^{7}(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty^{15}} + 60q^4\frac{(q^6;q^6)_\infty^{6}(q^9;q^9)_\infty^{4}(q^{18};q^{18})_\infty^{4}}{(q^3;q^3)_\infty^{14}} + 48q^5\frac{(q^6;q^6)_\infty^{5}(q^9;q^9)_\infty(q^{18};q^{18})_\infty^{7}}{(q^3;q^3)_\infty^{13}} + 16q^6\frac{(q^6;q^6)_\infty^{4}(q^{18};q^{18})_\infty^{10}}{(q^9;q^9)_\infty^{2}(q^3;q^3)_\infty^{12}} \tag{6.2.9}$$

and

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty} = \frac{(q^{12};q^{12})_\infty(q^{18};q^{18})^4_\infty}{(q^3;q^3)^3_\infty(q^{36};q^{36})^2_\infty} + q\ \frac{(q^6;q^6)^2_\infty(q^9;q^9)^3_\infty(q^{36};q^{36})_\infty}{(q^3;q^3)^4_\infty(q^{18};q^{18})^2_\infty}$$
$$+ 2q^2\ \frac{(q^6;q^6)_\infty(q^{18};q^{18})_\infty(q^{36};q^{36})_\infty}{(q^3;q^3)^3_\infty}. \tag{6.2.10}$$

Lemma 6.2.3. *The following 3-dissection holds:*

$$(q;q)^3_\infty = \frac{(q^6;q^6)_\infty(q^9;q^9)^6_\infty}{(q^3;q^3)_\infty(q^{18};q^{18})^3_\infty} - 3q(q^9;q^9)^3_\infty + 4q^3\frac{(q^3;q^3)^2_\infty(q^{18};q^{18})^6_\infty}{(q^6;q^6)^2_\infty(q^9;q^9)^3_\infty}. \tag{6.2.11}$$

Proof. From [20, p. 49, Corollary (i)] and [20, p. 51, Ex. (v)], we note that

$$\varphi(-q) = \varphi(-q^9) - 2q\ f(-q^3,-q^{15}) \tag{6.2.12}$$

and

$$f(-q,-q^5) = \chi(-q)\psi(q^3). \tag{6.2.13}$$

Squaring (6.2.12) and then employing (6.2.13), we obtain

$$\phi^2(-q) = \varphi^2(-q^9) - 4q\ \varphi(-q^9)\chi(-q^3)\psi(q^9) + 4q^2\chi^2(-q^3)\psi^2(q^9). \tag{6.2.14}$$

Again, we recall from (5.2.1) and (5.2.2) of Chapter 5, that

$$\psi(q) = f(q^3,q^6) + q\ \psi(q^9) \tag{6.2.15}$$

and

$$f(q,q^2) = \frac{\varphi(-q^3)}{\chi(-q)}. \tag{6.2.16}$$

Employing (6.2.16) in (6.2.15), we find that

$$\psi(q) = \frac{\varphi(-q^9)}{\chi(-q^3)} + q\ \psi(q^9). \tag{6.2.17}$$

Multiplying (6.2.14) and (6.2.17), we arrive at

$$\phi^2(-q).\psi(q) = \frac{\phi^3(-q^9)}{\chi(-q^3)} - 3q\ \phi^2(-q^9)\psi(q^9) + 4q^3\chi^2(-q^3)\psi^3(q^9). \tag{6.2.18}$$

Now, applying q-product representations for $\varphi(q)$, $\psi(q)$, and $\chi(q)$ in (6.2.18), we can easily derive (6.2.11). □

6.3 Proofs of Theorems 6.1.1–6.1.10

Proof of Theorem 6.1.1. Employing (6.2.2) in (6.1.1), we obtain

$$\sum_{n=0}^{\infty}\overline{pp}_o(n)q^n=\frac{(q^2;q^2)_\infty^6}{(q^4;q^4)_\infty^2}\left\{\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4}+4q\,\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}\right\}. \tag{6.3.1}$$

Extracting from both sides of (6.3.1), those terms involving only q^{2n}, and then replacing q^2 by q, we arrive at (6.1.2). Again, extracting from both sides of (6.3.1), those terms involving only q^{2n+1}, and then replacing q^2 by q, we arrive at (6.1.3). □

Proof of Theorem 6.1.2. Using (6.2.3) in (6.1.2),

$$\sum_{n=0}^{\infty}\overline{pp}_o(2n)q^n=\frac{(q^2;q^2)_\infty^{12}}{(q^4;q^4)_\infty^4}\Big\{\frac{(q^4;q^4)_\infty^{28}}{(q^2;q^2)_\infty^{28}(q^8;q^8)_\infty^8}+8q\frac{(q^4;q^4)_\infty^{16}}{(q^2;q^2)_\infty^{24}}$$
$$+16q^2\,\frac{(q^4;q^4)_\infty^4(q^8;q^8)_\infty^8}{(q^2;q^2)_\infty^{20}}\Big\}. \tag{6.3.2}$$

Extracting from both sides of (6.3.2), those terms involving only q^{2n}, and then replacing q^2 by q, we arrive at (6.1.4). Again, extracting from both sides of (6.3.2), those terms involving only q^{2n+1}, and then replacing q^2 by q, we easily deduce (6.1.6).

Again, employing (6.2.2) in (6.1.3),

$$\sum_{n=0}^{\infty}\overline{pp}_o(2n+1)q^n=4(q^4;q^4)_\infty^4\left\{\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4}+4q\,\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}\right\}. \tag{6.3.3}$$

Extracting from both sides of (6.3.3), those terms involving only q^{2n}, and then replacing q^2 by q, we arrive at (6.1.5). Again, extracting from both sides of (6.3.3), those terms involving only q^{2n+1}, and then replacing q^2 by q, we arrive at (6.1.7).

□

Proof of Theorem 6.1.3. Using (6.2.3) and (6.2.4) in (6.1.4), we obtain

$$\sum_{n=0}^{\infty}\overline{pp}_o(4n)q^n=4\frac{(q^2;q^2)_\infty^{24}}{(q^4;q^4)_\infty^8}\Big\{\frac{(q^4;q^4)_\infty^{56}}{(q^2;q^2)_\infty^{56}(q^8;q^8)_\infty^{16}}+16q\frac{(q^4;q^4)_\infty^{44}}{(q^2;q^2)_\infty^{52}(q^8;q^8)_\infty^8}$$

$$+ 96q^2 \frac{(q^4;q^4)_\infty^{32}}{(q^2;q^2)_\infty^{48}} + 256q^3 \frac{(q^4;q^4)_\infty^{20}(q^8;q^8)_\infty^{8}}{(q^2;q^2)_\infty^{44}}$$
$$+ 256q^4 \frac{(q^4;q^4)_\infty^{8}(q^8;q^8)_\infty^{16}}{(q^2;q^2)_\infty^{40}}\Big\} + 16q\,(q^4;q^4)_\infty^{8}\Big\{\frac{(q^4;q^4)_\infty^{28}}{(q^2;q^2)_\infty^{28}(q^8;q^8)_\infty^{8}}$$
$$+ 8q\frac{(q^4;q^4)_\infty^{16}}{(q^2;q^2)_\infty^{24}} + 16q^2 \frac{(q^4;q^4)_\infty^{4}(q^8;q^8)_\infty^{8}}{(q^2;q^2)_\infty^{20}}\Big\}. \tag{6.3.4}$$

Now, extracting from both sides of (6.3.4), those terms involving only q^{2n+1}, and then replacing q^2 by q, we easily derive (6.1.8).

Again, employing (6.2.5) in (6.1.6), we find that

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n+2)q^n = 8(q^2;q^2)_\infty^{12}\Big\{\frac{(q^4;q^4)_\infty^{42}}{(q^2;q^2)_\infty^{42}(q^8;q^8)_\infty^{12}} + 12q\frac{(q^4;q^4)_\infty^{30}}{(q^2;q^2)_\infty^{38}(q^8;q^8)_\infty^{4}}$$
$$+ 48q^2 \frac{(q^4;q^4)_\infty^{18}(q^8;q^8)_\infty^{4}}{(q^2;q^2)_\infty^{34}} + 64q^3\frac{(q^4;q^4)_\infty^{6}(q^8;q^8)_\infty^{12}}{(q^2;q^2)_\infty^{30}}\Big\}. \tag{6.3.5}$$

Now, extracting from both sides of (6.3.5), those terms involving only q^{2n+1}, and then replacing q^2 by q, we arrive at (6.1.9). □

Proof of Theorem 6.1.4. Employing (6.2.9) in (6.1.6), we obtain

$$\sum_{n=0}^{\infty} \overline{pp}_o(4n+2)q^n$$
$$= 8\Big\{\frac{(q^6;q^6)_\infty^{20}(q^9;q^9)_\infty^{32}}{(q^3;q^3)_\infty^{36}(q^{18};q^{18})_\infty^{16}} + 12q\,\frac{(q^6;q^6)_\infty^{19}(q^9;q^9)_\infty^{29}}{(q^3;q^3)_\infty^{35}(q^{18};q^{18})_\infty^{13}}$$
$$+ 78q^2\,\frac{(q^6;q^6)_\infty^{18}(q^9;q^9)_\infty^{26}}{(q^3;q^3)_\infty^{34}(q^{18};q^{18})_\infty^{10}} + 340q^3\,\frac{(q^6;q^6)_\infty^{17}(q^9;q^9)_\infty^{23}}{(q^3;q^3)_\infty^{33}(q^{18};q^{18})_\infty^{7}}$$
$$+ 1089q^4\,\frac{(q^6;q^6)_\infty^{16}(q^9;q^9)_\infty^{20}}{(q^3;q^3)_\infty^{32}(q^{18};q^{18})_\infty^{4}} + 2664q^5\,\frac{(q^6;q^6)_\infty^{15}(q^9;q^9)_\infty^{17}}{(q^3;q^3)_\infty^{21}(q^{18};q^{18})_\infty}$$
$$+ 5064q^6\,\frac{(q^6;q^6)_\infty^{14}(q^9;q^9)_\infty^{14}(q^{18};q^{18})_\infty^{2}}{(q^3;q^3)_\infty^{30}}$$
$$+ 7488q^7\,\frac{(q^6;q^6)_\infty^{13}(q^9;q^9)_\infty^{11}(q^{18};q^{18})_\infty^{5}}{(q^3;q^3)_\infty^{29}}$$
$$+ 8496q^8\,\frac{(q^6;q^6)_\infty^{12}(q^9;q^9)_\infty^{8}(q^{18};q^{18})_\infty^{8}}{(q^3;q^3)_\infty^{28}}$$
$$+ 7168q^9\,\frac{(q^6;q^6)_\infty^{11}(q^9;q^9)_\infty^{5}(q^{18};q^{18})_\infty^{11}}{(q^3;q^3)_\infty^{27}}$$

$$+ 4224q^{10} \frac{(q^6;q^6)_\infty^{10}(q^9;q^3)_\infty^2(q^{18};q^{18})_\infty^{14}}{(q^3;q^3)_\infty^{26}}$$

$$+ 1536q^{11} \frac{(q^6;q^6)_\infty^{9}(q^{18};q^{18})_\infty^{17}}{(q^3;q^3)_\infty^{25}(q^9;q^9)_\infty} + 256q^{12} \frac{(q^6;q^6)_\infty^{8}(q^{18};q^{18})_\infty^{20}}{(q^3;q^3)_\infty^{24}(q^9;q^9)_\infty^4}\Big\}. \qquad (6.3.6)$$

Extracting from both sides of (6.3.6), those terms involving only q^{3n+1}, and then replacing q^3 by q, we arrive at (6.1.11). Again, extracting from both sides of (6.3.6), those terms involving only q^{3n+2}, and then replacing q^3 by q, we easily deduce (6.1.12). □

Proof of Theorem 6.1.6. From (6.1.3), we find that

$$\sum_{n=0}^{\infty} \frac{\overline{pp}_o(2n+1)}{4} q^n = \frac{(q^4;q^4)_\infty^4}{(q;q)_\infty^4}.$$

$$\equiv (q^4;q^4)_\infty^3 \pmod 4, \qquad (6.3.7)$$

which implies that

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+1) q^n \equiv 4\ (q;q)_\infty^3 \pmod 2 \qquad (6.3.8)$$

and

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+3) q^n \equiv 0 \pmod 8, \qquad (6.3.9)$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+5) q^n \equiv 0 \pmod 8, \qquad (6.3.10)$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+7) q^n \equiv 0 \pmod 8. \qquad (6.3.11)$$

It follows from (6.3.10) and (6.3.11) that (6.1.17) and (6.1.14) hold.

Again, from (6.1.6), we obtain

$$\sum_{n=0}^{\infty} \frac{\overline{pp}_o(4n+2)}{8} q^n = \frac{(q^2;q^2)_\infty^{12}}{(q;q)_\infty^{12}}.$$

$$\equiv \frac{(q^8;q^8)_\infty^3}{(q^4;q^4)_\infty^3} \pmod 4, \qquad (6.3.12)$$

which yields

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+6)q^n \equiv 0 \pmod{16}, \tag{6.3.13}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(16n+6)q^n \equiv 0 \pmod{16}, \tag{6.3.14}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(16n+10)q^n \equiv 0 \pmod{16}, \tag{6.3.15}$$

and

$$\sum_{n=0}^{\infty} \overline{pp}_o(16n+14)q^n \equiv 0 \pmod{16}. \tag{6.3.16}$$

It follows from (6.3.13), (6.3.15) and (6.3.16) that (6.1.13), (6.1.19) and (6.1.21) hold.

From (6.1.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\overline{pp}_o(4n)}{4} q^n &= \frac{(q^2;q^2)_\infty^{24}}{(q;q)_\infty^{16}(q^4;q^4)_\infty^8} + 4q\,\frac{(q^4;q^4)_\infty^8}{(q;q)_\infty^8} \\ &\equiv \frac{(q^8;q^8)_\infty^6}{(q^4;q^4)_\infty^{12}} + 4q\,(q^4;q^4)_\infty^6 \pmod{2},\end{aligned} \tag{6.3.17}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{pp}_o(16n+8)q^n \equiv 0 \pmod{8} \tag{6.3.18}$$

and

$$\sum_{n=0}^{\infty} \overline{pp}_o(16n+12)q^n \equiv 0 \pmod{8}. \tag{6.3.19}$$

It follows from (6.3.18) and (6.3.19) that (6.1.18) and (6.1.20) hold.

From (6.1.8),

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\overline{pp}_o(8n+4)}{80} q^n &= \left\{ \frac{(q^2;q^2)_\infty^{36}}{(q;q)_\infty^{28}(q^4;q^4)_\infty^8} + 16q\,\frac{(q^2;q^2)_\infty^{12}(q^4;q^4)_\infty^8}{(q;q)_\infty^{20}} \right\} \\ &\equiv \frac{(q^8;q^8)_\infty^9}{(q^4;q^4)_\infty^{15}} + 16q\,(q^4;q^4)_\infty^3(q^8;q^8)_\infty^3 \pmod{4},\end{aligned} \tag{6.3.20}$$

which yields

$$\sum_{n=0}^{\infty} \overline{pp}_o(32n+20)q^n \equiv 0 \pmod{160} \tag{6.3.21}$$

and

$$\sum_{n=0}^{\infty} \overline{pp}_o(32n+28)q^n \equiv 0 \pmod{160}. \tag{6.3.22}$$

Now, (6.1.22) and (6.1.23) easily follow from (6.3.21) and (6.3.22).

Taking modulo 32 in (6.3.5), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(8n+2)q^n &\equiv 8\frac{(q^2;q^2)_\infty^{42}}{(q;q)_\infty^{30}(q^4;q^4)_\infty^{12}} \pmod{32}\\ &\equiv 8\ (q^2;q^2)_\infty^3 \pmod{32}.\end{aligned} \tag{6.3.23}$$

Employing (6.2.11) in (6.3.23), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(8n+2)q^n &\equiv 8\frac{(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty^6}{(q^6;q^6)_\infty(q^{36};q^{36})_\infty^3} - 24q^2(q^{18};q^{18})_\infty^3\\ &+ 32q^6\frac{(q^6;q^6)_\infty^2(q^{36};q^{36})_\infty^6}{(q^{12};q^{12})_\infty^2(q^{18};q^{18})_\infty^3} \pmod{32}.\end{aligned} \tag{6.3.24}$$

It follows from (6.3.24) that

$$\sum_{n=0}^{\infty} \overline{pp}_o(48n+2)q^n \equiv 0 \pmod{8}, \tag{6.3.25}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(48n+10)q^n \equiv 0 \pmod{32}, \tag{6.3.26}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(48n+18)q^n \equiv 0 \pmod{8}, \tag{6.3.27}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(48n+26)q^n \equiv 0 \pmod{32}, \tag{6.3.28}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(48n+34)q^n \equiv 0 \pmod{32}, \tag{6.3.29}$$

and

$$\sum_{n=0}^{\infty} \overline{pp}_o(48n+42)q^n \equiv 0 \pmod{32}. \tag{6.3.30}$$

Thus, (6.1.24)–(6.1.28) are apparent.

□

Proof of Theorem 6.1.7. Employing (5.2.19) in (6.1.1), we find that

$$\sum_{n=0}^{\infty} \overline{pp}_o(n)q^n = \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^4(q^4;q^4)_\infty^2} = \frac{\phi^2(-q^2)}{\phi^2(-q)}. \tag{6.3.31}$$

Again, using (6.2.13) in (6.2.12), we obtain

$$\varphi(-q) = \varphi(-q^9) - 2q\ \chi(-q^3)\psi(q^9). \tag{6.3.32}$$

Squaring (6.3.32) and replacing q by q^2, we find that

$$\varphi^2(-q^2) = \varphi^2(-q^{18}) - 4q^2\ \chi(-q^6)\psi(q^{18})\varphi(-q^{18}) + 4q^4\ \chi^2(-q^6)\psi^2(q^{18}). \tag{6.3.33}$$

Now, employing (6.3.33) and (5.2.18) in (6.3.31), we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(n)q^n = \frac{\varphi^6(-q^9)}{\varphi^8(-q^3)}\Big\{&\varphi^2(-q^{18}) + 4q\ w(q^3)\varphi^2(-q^{18}) - 4q^2\ \chi(-q^6)\psi(q^{18})\varphi(-q^{18})\\
&+ 12q^2\ w^2(q^3)\varphi^2(-q^{18}) - 16q^3\ w(q^3)\chi(-q^6)\psi(q^{18})\varphi(-q^{18})\\
&+ 16q^3\ w^3(q^3)\varphi^2(-q^{18}) + 4q^4\ \chi^2(-q^6)\psi^2(q^{18}) + 16q^4\ w^4(q^3)\varphi^2(-q^{18})\\
&- 48q^4\ w^2(q^3)\chi(-q^6)\psi(q^{18})\varphi(-q^{18}) + 16q^5\ w(q^3)\chi^2(-q^6)\psi^2(q^{18})\\
&- 64q^5\ w^3(q^3)\chi(-q^6)\psi(q^{18})\varphi(-q^{18}) + 48q^6\ w^2(q^3)\chi^2(-q^6)\psi^2(q^{18})\\
&- 64q^6\ w^4(q^3)\chi(-q^6)\psi(q^{18})\varphi(-q^{18}) + 64q^7\ w^3(q^3)\chi^2(-q^6)\psi^2(q^{18})\\
&+ 64q^8\ w^4(q^3)\chi^2(-q^6)\psi^2(q^{18})\Big\},\end{aligned} \tag{6.3.34}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{pp}_o(3n+1)q^n \equiv 0 \pmod{4} \tag{6.3.35}$$

and

$$\sum_{n=0}^{\infty} \overline{pp}_o(3n+2)q^n \equiv 0 \pmod 4. \tag{6.3.36}$$

Now, (6.1.29) and (6.1.30) readily follow from (6.3.35) and (6.3.36).

Also, from (6.3.34), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(3n)q^n &\equiv \frac{\varphi^6(-q^3)\varphi^2(-q^6)}{\varphi^8(-q)} \pmod{16} \\ &\equiv \frac{(q^2;q^2)_\infty^8 (q^3;q^3)_\infty^{12}}{(q;q)^{16}(q^6;q^6)_\infty^2 (q^{12};q^{12})_\infty^2} \pmod{16}, \quad \text{(by (5.2.19))}. \end{aligned} \tag{6.3.37}$$

By binomial theorem

$$\frac{(q^2;q^2)_\infty^8 (q^3;q^3)_\infty^{12}}{(q;q)^{16}(q^6;q^6)_\infty^2 (q^{12};q^{12})_\infty^2} \equiv \frac{(q^6;q^6)_\infty^4}{(q^{12};q^{12})_\infty^2} \pmod 2. \tag{6.3.38}$$

Using (6.3.38) in (6.3.37), we arrive at

$$\sum_{n=0}^{\infty} \overline{pp}_o(3n)q^n \equiv \frac{(q^6;q^6)_\infty^4}{(q^{12};q^{12})_\infty^2} \pmod{16}. \tag{6.3.39}$$

From (6.3.39), we obtain

$$\sum_{n=0}^{\infty} \overline{pp}_o(6n+3)q^n \equiv 0 \pmod{16}, \tag{6.3.40}$$

$$\sum_{n=0}^{\infty} \overline{pp}_o(9n+3)q^n \equiv 0 \pmod{16}, \tag{6.3.41}$$

and

$$\sum_{n=0}^{\infty} \overline{pp}_o(9n+6)q^n \equiv 0 \pmod{16}. \tag{6.3.42}$$

Thus, (6.1.31)–(6.1.33) follow from (6.3.40)–(6.3.42). □

Proof of Theorem 6.1.8. From (6.3.8),

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+1)q^n \equiv 4\ (q;q)_\infty^3 \equiv 4\ \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \equiv 4\ \psi(q) \pmod 2. \tag{6.3.43}$$

Again, employing (6.2.16), (5.2.19), (5.2.20) in (6.2.15), we arrive at

$$\psi(q) = \frac{(q^6;q^6)_\infty (q^9;q^9)_\infty^2}{(q^3;q^3)_\infty (q^{18};q^{18})_\infty} + q\psi(q^9). \tag{6.3.44}$$

Using (6.3.44) in (6.3.43),

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+1)q^n \equiv 4\, \frac{(q^6;q^6)_\infty (q^9;q^9)_\infty^2}{(q^3;q^3)_\infty (q^{18};q^{18})_\infty} + 4q\psi(q^9) \pmod 2, \tag{6.3.45}$$

which yields

$$\sum_{n=0}^{\infty} \overline{pp}_o(24n+17)q^n \equiv 0 \pmod 2 \tag{6.3.46}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(24n+1)q^n &\equiv 4\, \frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^2}{(q;q)_\infty (q^6;q^6)_\infty} \pmod 2\\ &\equiv 4\ (q;q)_\infty \pmod 2\\ &\equiv 4 + 4\sum_{n=0}^{\infty} (-1)^k (q^{k(3k-1)/2} + q^{k(3k+1)/2}) \pmod 2.\end{aligned} \tag{6.3.47}$$

Thus, (6.1.34) and (6.1.35) are readily follow from (6.3.46) and (6.3.47). This completes the proof. □

Proof of Theorem 6.1.9. From (6.3.23), we have

$$\sum_{n=0}^{\infty} \overline{pp}_o(8n+2)q^n \equiv 8\ (q^2;q^2)_\infty^3 \pmod{32}, \tag{6.3.48}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{pp}_o(16n+2)q^n \equiv 8\ (q;q)_\infty^3 \equiv 8\, \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \equiv 8\ \psi(q) \pmod{32}. \tag{6.3.49}$$

Now, employing (6.3.44) in (6.3.49), we find that

$$\sum_{n=0}^{\infty} \overline{pp}_o(16n+2)q^n \equiv 8\, \frac{(q^6;q^6)_\infty (q^9;q^9)_\infty^2}{(q^3;q^3)_\infty (q^{18};q^{18})_\infty} + 8q\psi(q^9) \pmod{32}, \tag{6.3.50}$$

which yields

$$\sum_{n=0}^{\infty} \overline{pp}_o(48n+34)q^n \equiv 0 \pmod{32} \tag{6.3.51}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{pp}_o(48n+2)q^n &\equiv 8\,\frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^2}{(q;q)_\infty (q^6;q^6)_\infty} \pmod{32}\\ &\equiv 8\,(q;q)_\infty \pmod{32}\\ &\equiv 8+8\sum_{n=0}^{\infty}(-1)^k(q^{k(3k-1)/2}+q^{k(3k+1)/2}) \pmod{32}. \qquad (6.3.52)\end{aligned}$$

Now, (6.1.36) and (6.1.37) follow from (6.3.51) and (6.3.52) which completes the proof. □

Proof of Theorem 6.1.10. From (6.3.20),

$$\sum_{n=0}^{\infty} \overline{pp}_o(32n+4)q^n \equiv 80\,(q;q)_\infty^3 \equiv 80\,\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \equiv 80\,\psi(q) \pmod{2}. \qquad (6.3.53)$$

Now, employing (6.3.44) in (6.3.53), we easily deduce (6.1.38) and (6.1.39). □

Bibliography

[1] Adiga, C., Kim, T., Naika, M.S.M. & Madhusudhan, H.S. On Ramanujan's cubic continued fraction and explicit evaluations of theta functions, *Indian J. Pure Appl. Math.* **35**, 1047–1062, 2004.

[2] Andrews, G.E., Berndt, B.C., Jacobsen, L. & Lamphere, R.L. The continued fractions found in the unorganised portions of Ramanujan's notebook, *Mem. Amer. Math. Soc.* No. **477**, Providence, RI. 1992.

[3] Andrews, G.E. & Berndt, B.C. *Ramanujan's Lost Notebook, Part I*, Spriger-Verlag, New York, 2005.

[4] Andrews, G.E. An Introduction to Ramanujan's "Lost" Notebook, *Amer. Math. Monthly* **86**, 89–108, 1979.

[5] Andrews, G.E. Generalized Frobenius Partitions, *Mem. Amer. Math. Soc.* No. **301**, Providence, RI. 1984.

[6] Andrews, G.E., Lewis, R.P. & Lovejoy, J. Partitions with designated summands, *Acta Arith.* **105**, 51–66, 2002.

[7] Andrews, G.E. & Rose, S.C.F. MacMahon's sum of divisor functions, Chebyshev polynomials, and quasi-modular forms, *J. Reine Angew. Math.* **676**, 97–103, 2013.

[8] Atkin, A.O.L. Ramanujan congruences for $p_{-k}(n)$, *Canad. J. Math.* **20**, 67–78, 1968.

[9] Atkin, A.O.L. Proof of a conjecture of Ramanujan, *Glasg. Math. J.* **8**, 14–32, 1967.

[10] Baruah, N.D. A few theta function identities and some of Ramanujan's modular equations, *Ramanujan J.* **4**, 239–250, 2000.

[11] Baruah, N.D. Contributions to Ramanujan's Schläfli-type modular equations, class invariants, theta functions, and continued fractions, *Ph.D. Thesis*, Tezpur University, 2001.

[12] Baruah, N.D. Modular equations for Ramanujan's cubic continued fraction, *J. Math. Anal. Appl.* **268**, 244–255, 2002.

[13] Baruah, N.D. & Berndt, B.C. Partition identities and Ramanujan's modular equations, *J. Combin. Theory Ser. A* **114**, 1024–1045, 2007.

[14] Baruah, N.D. & Berndt, B.C. Partition identities arising from theta function identities, *Acta Math. Sin. (Engl. Ser.)* **24**, 955–970, 2008.

[15] Baruah, N.D. & Bora, J. Some new proofs of Ramanujan's modular equations of degree 9, *Indian J. Math.* **47**, 99–122, 2005.

[16] Baruah, N.D., Bora, J. & Saikia, N. Some new proofs of modular relations for the Göllnitz-Gordon functions, *Ramanujan J.* **15**, 281–301, 2008.

[17] Baruah, N.D. & Ojah, K.K. Some congruences deducible from Ramanujan's cubic continued fraction, *Int. J. Number Theory* **7**, 1331–1343, 2011.

[18] Baruah, N.D. & Ojah, K.K. Analogues of Ramanujan's partition identities and congruences arising from his theta functions and modular equations, *The Ramanujan J.* **28**, 385–407, 2012.

[19] Baruah, N.D. & Sarmah, B.K. Congruences for generalized Frobenius partitions with 4 colors, *Discrete Math.* **311**, 1892–1902, 2011.

[20] Berndt, B.C. *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[21] Berndt, B.C. Partition-theoritic interpretations of certain modular equations of Schröter, Russel, and Ramanujan, *Ann. Comb.* **11**, 115–125, 2007.

[22] Berndt, B.C. *Ramanujan's Notebooks, Part V*, Springer-Verlag, New York, 1998.

[23] Berndt, B.C., H.H. Chan, S.S. Huang, S.Y. Kang, J. Sohn, S.H. Son, The Rogers-Ramanujan continued fraction. *J. Comput, Appl. Math.* **105**, 9–24, 1999.

[24] Berndt, B.C. *Number theory in the Spirit of Ramanujan*, American Mathematical Society, Providence, RI, 2006.

[25] Berndt, B.C., Gugg, C. & Kim, S. Ramanujan's elementary method in partition congruences, *In Partition, q-series and Modular Forms, K. Alladi and F. Garvan, eds., Development in Math.* **23**, 13–22, 2011.

[26] Borwein, J.M., Borwein, P.B. & Garvan, F.G. Some cubic modular identities of Ramaujan, *Trans. Amer. Math. Soc.* **343**, 35–47, 1994.

[27] Boylan, M. Exceptional congruences for powers of the partition function, *Acta Arith.* **111**, 187–203, 2004.

[28] Bringman, K. & Lovejoy, J. Rank and congruences for overpartition pairs, *Int. J. Number Theory* **4**, 303–322, 2008.

[29] Cao, Z. On Somos' dissection identities, *J. Math. Anal. Appl.* **365**, 659–667, 2010.

[30] Chan, H.-C. Ramanujan's cubic continued fraction and a generalization of his "Most Beautiful Identity", *Int. J. Number Theory* **6**, 673–680, 2010.

[31] Chan, H.-C. Ramanujan's cubic continued fraction and Ramanujan type congruences for a certain partition function, *Int. J. Number Theory* **6**, 819–834, 2010.

[32] Chan, H.-C. & Cooper, S. Congruences modulo powers of 2 for a certain partition function, *Ramanujan J.* **22**, 101–117, 2010.

[33] Chan, H.H. & Toh, P.C. New analogues of Ramanujan's partition identities, *J. Number Theory* **129**, 1898–1913, 2010.

[34] Chan, H.H. On Ramanujan's cubic continued fraction, *Acta Arith.* **73**, 343–355, 1995.

[35] Chen, S-L. & Huang, S-S. Identities for certain products of theta functions, *The Ramnujan J.* **8**, 5–12, 2004.

[36] Chen, W.Y.C., Ji, K.Q., Jin, H.-T. & Shen, E.Y.Y, On the number of partitions with designated summands, *J. Number Theory*, **133**, 2929–2938, 2013.

[37] Chen, W.Y.C. & Lin, B.L.S. Arithmetic properties of overpartition pairs, *Acta Arith.* **151**, 263–277, 2012.

[38] Corteel, S. Particle seas and basic hypergeometric series, *Adv. Appl. Math.* **31**, 199–214, 2003.

[39] Corteel, S. & Lovejoy, J. Frobenius partitions and the combinatorics of Ramanujan's ${}_1\psi_1$ summation, *J. Combin. Theory A* **97**, 177–183, 2002.

[40] Corteel, S. & Lovejoy, J. Overpartitions, *Trans. Amer. Math. Soc.* **356**, 1623–1635, 2004.

[41] Fortin, J.-F., Jacob, P. & Mathieu, P. Jagged Partitions, *arXiv:math. CO/0310079,* 2003.

[42] Garvan, F.G. Some congruence properties of the partition function, *M.Sc. Thesis*, University of New South Wales, 1982.

[43] Glaisher, J.W.L. Expression for the first five powers of the series in which the co.eff. are the sums of the divisors of the exponent, *Messenger of Math.,* **171**, 1885.

[44] Gordon, B. Ramanujan congruences for $p_{-k}(\text{mod } 11^r)$, *Glasgow Math. J.* **24**, 107–123, 1983.

[45] Gordon, B. Some identities in combinanoinal analysis, *Q. J. Math (Oxford)* **12**, 285–290, 1961.

[46] Gugg, C. Two modular equations for squares of the Rogers-Ramanujan functions with application, *Ramanujan J.* **18**, 183–207, 2009.

[47] Gugg, C. A new proof of Ramanujan's modular equation relating $R(q)$ and $R(q^5)$, *Ramanujan J.*, **20**, 163–177, 2009.

[48] Hirschhorn, M.D., Garvan, F., & Borwein J. M. Cubic analogues of the Jacobian theta function $\theta(z, q)$, *Canad. J. Math.* **45**, 673–694, 1993.

[49] Hirschhorn, M.D. & Hunt, D.C. A simple proof of the Ramanujan conjecture for powers of 5, *J. Reine Angew. Math.* **326**, 1–17, 1981.

[50] Hirschhorn, M.D. & Roselin, On the 2-, 3-, 4- and 6- dissections of Ramanujan's cubic continued fraction and its reciprocal, *Ramanujan Rediscovered,* N.D. Baruah, B.C. Berndt, S. Cooper, T. Huber, M. Schlosser (eds.), RMS Lecture Notes Series, **14**, 125–138, 2010.

[51] Hirschhorn, M.D. & Sellers, J.A. Arithmetic relations for overpartitions, *J. Combin. Math. Combin.Comp.* **53**, 65–73, 2005.

[52] Hirschhorn, M.D. & Sellers, J.A. An infinite family of overpartition congruences modulo 12, *Integers* **5**, Article A20, 2005.

[53] Hirschhorn, M.D. & Sellers, J.A. Arithmetic properties of overpartitions into odd parts, *Annals of Combinatorics* **10**, 353–367, 2006.

[54] Kim, B. A short note on the overpartition function, *Discrete Math.* **309**, 2528–2532, 2009.

[55] Kim, B. Overpartition pairs modulo powers of 2, *Discrete Math.* **311**, 835–840, 2011.

[56] Kiming, I. & Olsson, J. Congruences like Ramanujan's for powers of the partition function, *Arch. Math.* **59**, 348–360, 1992.

[57] Kolitsch, L. A congruence for generalized Frobenius partitions with 3 colors modulo powers of 3, in *Analytic Number Theory, Proceedings of a Conference in Honor of Paul Bateman*, B. C. Berndt, H. G. Diamond, H. Halberstam, and A. Hilderbrand (eds.) Birkhauser, Basel, 343–348, 1990.

[58] Kolitsch, L. M-order generalized frobenius partitions with M colors, *J. Number Theory* **39**, 279–284, 1991.

[59] Lin, B.S. Arithmetic properties of overpartition pairs into odd parts, *Electron. J. Combin.* **19**, # P17, 2012.

[60] Lovejoy, J. Gordon's theorem for overpartitions, *J. Combin. Theory A,* **103**, 393–401, 2003.

[61] Lovejoy, J. Overpartition pairs, *Ann. Inst. Fourier,* **56**, 781–794, 2006.

[62] Lovejoy, J. Partition and overpartition with attached parts , *Arch. Math. (Basel),* **88**, 216–222, 2007.

[63] Lovejoy, J. Rank and conjugation for the second Frobenius representation of an overpartition, *Ann. Comb.,* **12**, 101–113, 2008.

[64] Lovejoy, J. & Mallet, O. Overpartition pairs and two classes of basic hypergeometric series, *Adv. Math.,* **217**, 386–416, 2008.

[65] Lovejoy, J. & Osburn, R. Rank differences for overpartitions, *Q. J. Math.,* **59**, 257–273, 2008.

[66] MacMahon, P.A. Divisors of numbers and their continuations in the theory of partitions, *Proc. London Math. Soc.*, Ser. 2 **19**, 75–113, 1919.

[67] Mahlburg, K. The overpartition function modulo small powers of 2, *Discrete Math.* **286**, 263–267, 2004.

[68] Naika, M.S.M, Chandankumar, S. & Bairy, K.S. New identities for Ramanujan's cubic continued fraction, *Functiones et Approximati Commentarii Mathematici*, **46**, 29–44, 2012.

[69] Newman, M. Remarks on some modular identities, *Trans. Amer. Math. Soc.* **73**, 313–320, 1952.

[70] Newman, M. The coefficients of certain infinite products, *Proc. Amer. Math. Soc.* **4**, 435–439, 1953.

[71] Newman, M. An identity for the coefficients of certain modular forms, *J. London Math. Soc.* **30**, 488–493, 1955.

[72] Newman, M. On the existence of identities for the coefficients of certain modular forms, *J. London Math. Soc.* **31**, 350–359, 1956.

[73] Newman, M. Some theorems about $p_r(n)$, *Canad. J. Math.* **9**, 68–70, 1957.

[74] Newman, M. Congruences for the coefficients of modular forms and some new congruences for the partition function, *Canad. J. Math.* **9**, 549–552, 1957.

[75] Newman, M. Further identities and congruences for the coefficients of modular forms, *Canad. J. Math.* **10**, 577–586, 1958.

[76] Pak, I. Partition bijections, a survey, *Ramanujan J.*, **12**, 5–75, 2006.

[77] Ramanathan, K.G. Identities and congruences of the Ramanujan type, *Canad. J. Math.* **2**, 168–178, 1950.

[78] Ramanathan, K.G. On Ramanujan's continued fraction, *Acta Arith.* **43**, 209–226, 1984.

[79] Ramanujan, S. *Notebooks (2 volumes)*, Tata Institute of Fundamental Reasearch, Bombay, 1957.

[80] Ramanujan, S. Some properties of $p(n)$, the number of partitions of n, *Proc. Cambridge Phillos. Soc.* **19**, 207–210, 1919.

[81] Ramanujan, S. *Collected Papers*, Cambridege University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by American Mathematical Soicety, Providence, 2000.

[82] Ramanujan, S. *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[83] Rogers, L.J. Second memoir on the expansion of certain infinte products, *Proc. Lond. Math. Soc.* **25**, 318–343, 1894.

[84] Rogers, L.J. On a type of modular relation, *Proc. Lond. Math. Soc.* **19**, 387–397, 1921.

[85] Selberg, A. *Collected Papers, Vol. 1*, Springer-Verlag, Berlin, 1989.

[86] Sellers, J. Congruences involving generalized Frobenius partitions, *Int. J. Math. Math. Sci.* **16**, 413–415, 1993.

[87] Sellers, J. Congruences involving F-partition functions, *Int. J. Math. Math. Sci.* **17**, 187–188, 1994.

[88] Sellers, J. New congruences for generalized Frobenius partition with 2 or 3 colors, *Discrete Math.* **131**, 367–374, 1994.

[89] Serre, J.-P. Sur la lacunarité des puisances de η, *Glasgow Math. J.* **27**, 203–221, 1985.

[90] Sinick, J. Ramanujan congruences for a class of eta quotients, *Int. J. Number Theory* **6**, 835–847, 2010.

[91] Slater, L.J. Further identities of the Rogers-Ramanujan type, *Proc. London Math. Soc.* **54**, 147–167, 1952.

[92] Watson, G.N. Theorems stated by Ramanujan (VII): Theorems on Continued fraction, *J. Lond. Math. Soc.* **4**, 39–48, 1929.

[93] Watson, G.N. Ramanujan's vermutung über zerfällungsan zahlen, *J. Reine Angew. Math.* **179**, 97–128, 1938.

[94] Yan, Q. Several identities for certain products of Theta functions, *Ramanujan J.* **19**, 79–94, 2009.

[95] Zhao, H. & Zhong, Z. Ramanujan type congruences for a partition function, *Electron. J. Combin.* **18**, # P58, 2011.