PERTURBATION OF WEIGHTED SHIFT OPERATORS

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
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ABSTRACT

In this work we have studied perturbation of weighted shift operators. For our study we consider both one-variable and two-variable weighted shift operators. There already exists in the literature, different necessary and sufficient conditions for a weighted shift operator to be either hyponormal, or weakly hyponormal, or 2-hyponormal, or quadratic hyponormal, or subnormal. We observe that these necessary and sufficient conditions are all framed in terms of the 'weight sequence' of the particular weighted shift. This immediately implies that any change or perturbation in the weights would reflect upon the hyponormality or any other similar property of the weighted shift. In this work we frame conditions which can exhaustively determine the situations where a perturbed shift will still retain its original property of hyponormality/ weak hyponormality/ 2-hyponormality/ quadratic hyponormality/ subnormality.

DECLARATION

I, **Bimalendu Kalita**, hereby declare that the subject matter in this thesis entitled **Perturbation of Weighted Shift Operators** is the record of work done by me, that the contents of this thesis did not form the basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/ institute. This thesis is being submitted by me to Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

Place: Napam
Date: 24/07/2013

Bimalendu Kalita
Signature of the Candidate

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CERTIFICATE

This is to certify that the thesis entitled **Perturbation of Weighted Shift Operators** submitted to the School of Sciences Tezpur University in partial fulfillment for the award of the degree of Doctor of Philosophy in Mathematical Sciences is a record of research work carried out by **Bimalendu Kalita** under my supervision and guidance.

All help received by him from various sources have been duly acknowledged.

No part of this thesis has been submitted elsewhere for award of any other degree.

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Bimalendu Kalita
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Place: Tezpur
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Dedicated

To

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Chapter 1

Introduction

1.1 Background

Several important classes of bounded Hilbert space operators were introduced around the year 1950. We refer to three such classes of operators, namely, weighted shift operators, subnormal operators and hyponormal operators.

Weighted shifts are among the apparently simple but actually very rich examples of Hilbert space operators. They are related to subtle questions of function theory and constructive mathematics.

- If $\{e_n\}_{n=0}^{\infty}$ denotes an orthonormal basis of the space of square summable complex sequences $\ell^2(\mathbb{Z}_+)$, and $\{\alpha_n\}_{n=0}^{\infty}$ is a bounded sequence of scalars, then the unilateral weighted shift W on $\ell^2(\mathbb{Z}_+)$ is defined linearly such that $We_n = \alpha_n e_{n+1}$ for all n.

Though references to these definitions go back to the late 1950's, the first systematic study of shift operators was undertaken by R. L. Kelly in his doctoral thesis in 1966 [68]. About ten years later A. L. Shields again compiled a thorough account of subsequent developments [81]. Since then this class of operators has received much attention. Initially it was used in the investigation of isometries, but slowly it emerged as a fertile domain for providing examples in the study of general operators.

The other two classes of bounded Hilbert space operators that we have mentioned are subnormal and hyponormal operators. Motivated by the successful development of the theory of normal operators, in 1950 P.R. Halmos introduced the notion of subnormality and hyponormality for bounded Hilbert space operators.

- We recall that an operator T is subnormal if it is the restriction of a normal operator to an invariant subspace.
- T is hyponormal if $T^*T \geq TT^*$.

By simple matrix calculations it can be verified that subnormality implies hyponormality, but the converse is false. One reason is that subnormality is invariant under polynomial calculus or the calculus of analytic functions, while hyponormality is not. If we define T to be polynomially hyponormal whenever $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[\mathbb{Z}]$, then the natural question that follows is:

Question A: If T is polynomially hyponormal, then must T be subnormal?

In [75] it was shown that Question-A has an affirmative answer if and only if the corresponding problem for unilateral weighted shifts has an affirmative answer. In other words, it was proved that there exists a non subnormal polynomially hyponormal operator if and only if there exist a weighted shift operator with the same property. In [34] was given an example of an operator which is polynomially hyponormal but not subnormal. This means that there must also exist a non subnormal polynomially hyponormal weighted shift operator. However, **till date such a weighted shift operator has not yet been identified**. The reason for this could be because the gap between subnormality and hyponormality is not clearly understood. In [24] it was pointed out that we can easily construct a non subnormal polynomially hyponormal weighted shift operator, if we can give an affirmative answer to the following question regarding perturbation of weighted shift operators:

Question B: Is polynomial hyponormality of the weighted shift stable under small perturbations of the weight sequence?

Let us assume that Question-B has an affirmative answer. Under this assumption, if we consider the recursively generated weighted shift T_x with weight sequence : $1, \sqrt{x}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$, then it can be shown that T_x is subnormal if and only if $x = 2$; whereas T_x is polynomially hyponormal if and only if $2 - \delta_1 < x < 2 + \delta_2$ for some $\delta_1, \delta_2 > 0$. Thus for sufficiently small $\epsilon > 0$, the weight sequence $\alpha_\epsilon : 1, \sqrt{2+\epsilon}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$ would induce a non subnormal polynomially hyponormal weighted shift operator, as desired.

Hence it needs to be investigated whether Question-B does have an affirmative answer or not. And for this we need to develop the perturbation theory of weighted shift operators. In fact, a proper investigation of the notion of perturbation of weighted shift operators would help us to bridge the gap between subnormality and hyponormality, and to understand the position of the subnormals within the class of hyponormals.

1.2 Objectives

The basic problem we refer to is understanding the gap between the classes of subnormal and hyponormal operators. In the recent past several new classes of operators like k-hyponormal and weakly k-hyponormal operators have been introduced and studied in an attempt to bridge the gap between subnormality and hyponormality. We refer the following papers for details [12] [13] [16] [17] [22] [26] [36] [44] [51] [75].

Most of this work is carried out on the class of weighted shift operators, this being a prototype to the original question. A huge volume of literature deals with characterizations of these intermediate classes of operators by establishing necessary and sufficient conditions. These conditions are always in terms of weights of the weighted shift. This motivated us to raise the following questions: "suppose we have a k-hyponormal weighted shift W with sequence $\{\alpha_n\}$. To what extent can the weight sequence be perturbed, so that the corresponding perturbed shift still retain the property of k-hyponormality?" The ability to answer this question would contribute much towards a proper understanding of the class of k-hyponormal weighted shift operators, and also to distinguish it from the other subclasses.

Again it is known from the existing literature that the class of k-hyponormals is within the class of weakly k-hyponormals. This motivate us to ask "whether a perturbed k-hyponormal remains weakly k-hyponormal." The present work attempts to address such kinds of questions.

Hence, the objective of the present work is to contribute to the development of the theory of perturbation of weighted shift operators, with reference to the notion of hyponormality, k-hyponormality, weak k-hyponormality and subnormality. Our work aims to carry forward the ongoing research in this area and also to plug some of the holes in the existing literature.

1.3 Review of literature

We begin by taking a look at the class of weighted shift operators with reference to the classes of subnormal and hyponormal operators. We denote by W_α the weighted shift on $\ell^2(\mathbb{Z}_+)$ with a bounded weight sequence $\alpha = \{\alpha_n\}$. If, in particular, each α_n is equal to 1, then W_α is referred to as the simple unilateral shift and denoted by U_+. Since the bilateral shift on $\ell^2(\mathbb{Z})$ is a natural normal extension of U_+, hence U_+ is subnormal, and therefore also hyponormal. However, the weighted shift W_α need not always be subnormal or even hyponormal. In fact we have the following results:

- W_α is hyponormal if and only if $|\alpha_n| \leq |\alpha_{n+1}|$ for all n.
- (Berger's Theorem) W_α is subnormal if and only if there exists a Borel probability measure μ supported in $[0, \|W_\alpha\|^2]$, with $\|W_\alpha\|^2 \in \operatorname{supp}\mu$, such that $\gamma_n = \int t^n d\mu(t)$ for all $n \geq 0$, where $\gamma_0 := 1$ and $\gamma_{n+1} := \alpha_n^2\alpha_{n-1}^2 \cdots \alpha_0^2$ for $n \geq 0$.

In [51, Problem 203] Halmos asked for an example of a hyponormal operator that is not subnormal. Later on he himself comes up with one such example namely, the weighted shift operator W_α with weight sequence $\alpha = \{\alpha_n\}$, where $\alpha_0 = a, \alpha_1 = b, \alpha_n = 1$ for all $n > 1$ and $a < b < 1$. In fact Stampfli [80] was the first to address the question "which monotone shifts are subnormal?" In Theorem 4 of the same paper he provides a set of necessary and sufficient conditions for subnormality of W_α in terms of the weights α_n. These conditions make it evident that even the fi1st four weights ($\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$) may 'prevent' a shift from being subnormal.

Again in [51, Problem 209], Halmos asked for an example of a hyponormal operator whose square is not hyponormal. The example was duely provided but with much difficulty. We now recall subsequent development in the theory by which such examples can now be generated with much ease.

Let H be an infinite dimensional separable complex Hilbert space and let $B(H)$, denote the algebra of bounded linear operators on H.

- For $S, T \in B(H)$, $[S,T] := ST - TS$.
- An n-tuple $T = (T_1, \ldots, T_n)$ is hyponormal or the operators $T_1, \ldots, T_n$ are jointly hyponormal if

$$[T^*, T] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \ldots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \ldots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \ldots & [T_n^*, T_n] \end{pmatrix} \geq 0.$$

- For $k \geq 1$, $T \in B(H)$ is k-hyponormal if $(T, T^2, \ldots, T^k)$ is hyponormal i.e.,

$$\begin{pmatrix} [T^*, T] & [T^{*2}, T] & \ldots & [T^{*k}, T] \\ [T^*, T^2] & [T^{*2}, T^2] & \ldots & [T^{*k}, T^2] \\ \vdots & \vdots & \ddots & \vdots \\ [T^*, T^k] & [T^{*2}, T^k] & \ldots & [T^{*k}, T^k] \end{pmatrix} \geq 0.$$

- (Bram-Halmos)

  $T \in B(H)$ is subnormal

  $\Leftrightarrow$ T is $k-$ hyponormal for all $k \geq 1$

  $\Leftrightarrow$ $(T, T^2, \ldots, T^k)$ is hyponormal for all $k \geq 1$.

- An n-tuple $T = (T_1, \ldots, T_n)$ is weakly hyponormal if $LS(T) := \left\{ \sum_{i=1}^{n} \lambda_i T_i : \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \right\}$ consists only of hyponormal operators.

- For $k \geq 1$, $T \in B(H)$ is weakly $k-$hyponormal if $(T, T^2, \ldots, T^k)$ is weakly hyponormal.

- $T \in B(H)$ is said to be polynomially hyponormal if T is weakly k-hyponormal for all $k \geq 1$.

- W_α is k-hyponormal $\Leftrightarrow$ $(\gamma_{n+i+j})_{i,j=0}^{k} \geq 0$ for all $n \geq 0$, where $\gamma_0 := 1$ and $\gamma_{n+1} = \alpha_n^2 \gamma_n$ for $n \geq 0$, defines the moment sequence of W_α.

With this last characterization at hand, it is possible to distinguish between k-hyponormality and $(k+1)$-hyponormality for every $k \geq 1$. But while k-hyponormality of weighted shift admits a simple characterization, the same is not true for weak k-hyponormality.

In an effort to unravel how k-hyponormality and weak k-hyponormality are interrelated, different researchers have adopted different line of thoughts:

(a) A number of papers have been written describing the links for specific families of weighted shifts e.g., those with recursively generated tails and those obtained by restricting the Bergman shift to suitable invariant subspaces. Some of the relevant references are the following [5] [12] [13] [17] [19] [20] [21] [24] [25] [32] [65] [66] [71].

(b) Another approach has been to take a closer look at weighted shifts whose first few weights are unrestricted but whose tails are subnormal and recursively generated, refer to [3] [15] [16] [45] [46] [60] [63] [80].

As such we have a whole range of results leading to a better understanding of the problem in hand.

- [80] If W_α is subnormal weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ and $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \ldots$ i.e., W_α is flat

- [6] Let W_α be a unilatreral weighted shift with weight sequence $\{\alpha_n\}_{n=0}^{\infty}$ and assume that W_α is quadratically hyponormal (that is, weakly 2-hyponormal). If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then $\alpha_1 = \alpha_2 = \ldots$ i.e. W_α is subnormal.

- [12] For $x > 0$ let W_α be the weighted shift whose weight sequence is given by $\alpha_0 := x$ and $\alpha_n = \sqrt{\frac{n+1}{n+2}}$ for $n \geq 1$. Then
(i) W_α is subnormal $\Leftrightarrow 0 < x \leq \sqrt{\frac{1}{2}}$
(ii) W_α is 2-hyponormal $\Leftrightarrow 0 < x \leq \frac{3}{4}$
(iii) W_α is weakly 2-hyponormal $\Leftrightarrow 0 < x \leq \sqrt{\frac{2}{3}}$.

- [46] Let $\alpha(x) : \sqrt{x}, \sqrt{x}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots$ be a weight sequence with Bergman tail. Then $\{x \in \mathbb{R}_+ | \, W_{\alpha(x)} \text{is q.h.}\}$ is a closed interval and is equal to $[\delta_1, \delta_2]$ where $\delta_1 \approx .1673$ and $\delta_2 \approx .7439$ approximate to four places after decimal.

- [23] Let $\{\delta_n\}_{n=0}^{\infty}$ be the weight sequence given by

$$\delta_n = \begin{cases} \frac{1}{2}, & \text{if} n = 0 \\ \frac{1}{2^n}, & \text{if} n = 2, 4, 6, . \\ \frac{1}{2^{n+2}}, & \text{if} n = 1, 3, 5, \ldots \end{cases}$$

If $\alpha_n = \left(\sum_{k=0}^{n} \delta_k\right)^{\frac{1}{2}}$ for $n \geq 0$, then W_α is hyponormal but not 2-hyponormal.

- [66] Let $\alpha(x) : \sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots$ There exists $\delta \in (\frac{9}{16}, \frac{2}{3})$ such that

  (i) $W_{\alpha(x)}$ is cubically but not 2−hyponormal if $\frac{9}{16} < x \leq \delta$.

  (ii) $W_{\alpha(x)}$ is quadratically hyponormal but not cubicaly hyponormal if $\delta < x < \frac{2}{3}$.

Inspite of this huge repertoire of established results and generated examples, it should however be mentioned that the overall problem still remains largely unsolved.

The study of the multivariable analogue to these problems have also received much attention in the last few years [27] [28] [29] [30] [31] [37] [38] [39] [40].

- Consider double indexed positive bounded sequences $\alpha_k, \beta_k \in \ell^\infty(\mathbb{Z}_+^2), k \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert Space of square summable complex sequences indexed by $\mathbb{Z}_+^2$. The 2-variable weighted shift $T = (T_1, T_2)$ is defined by

$$T_1 e_k = \alpha_k e_{k+\varepsilon_1}, T_2 e_k = \beta_k e_{k+\varepsilon_2}$$

where $\varepsilon_1 = (1,0)$ and $\varepsilon_2 = (0,1)$. Here

$$T_1 T_2 = T_2 T_1 \iff \beta_{k+\varepsilon_1}\alpha_k = \alpha_{k+\varepsilon_2}\beta_k \text{ for all } k \in \mathbb{Z}_+^2.$$

Given $k \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, the moments of T of order k are

$$\gamma_k := \begin{cases} 1 & \text{if } k_1 = 0 = k_2 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 \alpha_{(0,k_2)}^2 \cdots \alpha_{(k_1-1,k_2)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1 \end{cases}$$

A multivariable weighted shift can be defined in an entirely similar way.

- [67](Berger's Theorem: characterization of subnormality for 2-variable weighted shifts) T admits a commuting normal extension if and only if there is a probability measure μ defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$, ($a_i := \|T_i\|^2$) such that $\gamma_k = \int\int_R t^k d\mu(t) := \int\int_R t_1^{k_1} t_2^{k_2} d\mu(t_1, t_2)$ ($\forall\ k \in \mathbb{Z}_+^2$).

- [25] A 2-variable weighted shift $T = (T_1, T_2)$ is $k-$hyponormal
$\Leftrightarrow \left(\gamma_u \gamma_{u+(m,n)+(p,q)} - \gamma_{u+(m,n)}\gamma_{u+(p,q)}\right)_{1 \leq m+n \leq k,\ 1 \leq p+q \leq k} \geq 0$
for all $u \in \mathbb{Z}_+^2$.

- A 2-variable weighted shift T is horizontally flat if $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ for all $k_1, k_2 \geq 1$; vertically flat if $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geq 1$; flat if it is horizontally flat and vertically flat; symmetrically flat if T is flat and $\alpha_{(1,1)} = \beta_{(1,1)}$.

1.4 Notations

We mention here a few standard notations to be followed throughout the sequel.

$\mathbb{N}$: Set of natural numbers.

$\mathbb{Z}$: Set of integers.

$\mathbb{Z}_+$: Set of non-negative integers.

$\mathbb{R}$: Set of real numbers.

$\mathbb{R}_+$: Set of non-negative real numbers.

$\mathbb{C}$: Set of complex numbers.

$\mathbb{Z}_+^2$: Set of Ordered pairs of non-negative integers.

$\ell^2(\mathbb{Z}_+)$: Hilbert space of square summable complex sequences indexed by the set $\mathbb{Z}_+$.

$\ell^2(\mathbb{Z}_+^2)$: Hilbert space of square summable complex sequences indexed by the set $\mathbb{Z}_+^2$.

$\ell^\infty(\mathbb{Z}_+)$: Space of all bounded sequences of scalars indexed by the set $\mathbb{Z}_+$

$\ell^\infty(\mathbb{Z}_+^2)$: Space of all bounded sequences of scalars indexed by the set $\mathbb{Z}_+^2$.

In addition to these we also often use the following abbreviations:

q.h.: Quadratic hyponormal or weak 2-hyponormal.

p.q.h.: Positive quadratic hyponormal.

NASC: Necessary and sufficient condition.

cl: Closure.

1.5 Chapterwise brief summary

Chapter 1: This chapter is introductory in nature. We include here the motivation and objectives of the present work, along with a brief review of literature leading to the same. A chapterwise brief summary of the work done in each chapter of the thesis is also included here.

Chapter 2: On convexity of weakly k-hyponormal region

Let $\alpha = \{\alpha_n\}_{n=0}^\infty$ be a weight sequence. Let $k \geq 1$ and $j \geq 0$. Define $\alpha[j : x] : \alpha_0, \alpha_1, \ldots, \alpha_{j-1}, x, \alpha_{j+1}, \ldots$ We say, $\alpha[j : x]$ is the perturbation of weight sequence α where the j^{th} weight of α namely, α_j is perturbed to x.

Let $\Omega_\alpha(k, j) := \{x : W_{\alpha[j\ x]}$ is k-hyponormal$\}$

and $\omega_\alpha(k, j) := \{x : W_{\alpha[j\ x]}$ is weakly k-hyponormal$\}$.

If W_α is a weighted shift then $W_{\alpha[j\ x]}$ is referred to as a rank-one perturbation of W_α where the j^{th} weight α_j is perturbed to x. If for $i < j$, α_i and α_j are perturbed to x and y respectively, then $W_{\alpha[(i\ x),(j\ y)]}$ is referred to as rank-two perturbation of W_α Similarly, we can define any finite perturbation of W_α.

In [24, Theorem 6.5] it was shown that rank-one perturbations of k-hyponormal weighted shifts which preserve k-hyponormality form a convex set. That is, if W_α is k-hyponormal then $\Omega_\alpha(k,j)$ is a convex set. The natural question that follows is "If W_α is weakly k-hyponormal, then is $\omega_\alpha(k,j)$ a convex set?"

In this chapter we answer this question in the affirmative.

For this we have used the characterization of weak k-hyponormality given in [45].

Chapter 3: On convexity of positive quadratic hyponormal region

This chapter is in continuation of Chapter 2. Here also we continue to investigate the idea of convexity.

If $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a positive weight sequence, $i \geq 0$, $k \geq 1$ and W_α is weakly k-hyponormal, then we have shown that $\omega_\alpha(k,j)$ is a nonempty convex set, where $\omega_\alpha(k,j) := \{x : W_{\alpha[j:x]} \text{ is weakly } k\text{-hyponormal}\}$.

Question: For $y \in \omega_\alpha(k, i+1)$, is there any relation between $\omega_\alpha(k,i)$ and $\omega_{\alpha[i+1,y]}(k,i)$? Here $\omega_{\alpha[i+1,y]}(k,i) := \{x : W_{\alpha[(i:x),(i+1:y)]} \text{ is weakly } k\text{-hyponormal}\}$.

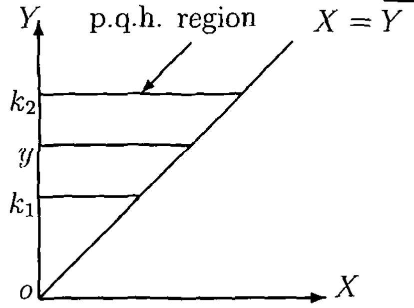
In this chapter we address this problem with reference to a positively quadratically hyponormal operator W_α with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ where $\alpha_n = \sqrt{\frac{n+1}{n+2}}$ for all n.

We have proved the following:

1. For $x \in [k_1, k_2]$, the weighted shift $W_{[(0:x),(1:x)]}$ is p.q.h., where $k_1 = 0.630435$, $k_2 = 0.737144$.

2. For $y \in [k_1, k_2]$, $\{x : W_{\alpha[(0:x),(1:y)]} \text{ is p.q.h.}\} = (0, y]$.

3. If either $y < k_1$ or $y > k_2$, then there exists $0 < x \leq y$ such that $W_{\alpha[(0:x),(1:y)]}$ is not p.q.h. If we represent the perturbations of α_0 and α_1 as x and y respectively, and represent them in the 2-dimensional plane then our result can be graphically represented as follows



Chapter 4: Finite rank perturbation of 2-hyponormal weighted shifts

In [24, Theorem 2.1] it has been shown that a non-zero finite rank perturbation of a subnormal shift is never subnormal unless the perturbation occurs at the initial weight. However, this is not necessarily true for a 2-hyponormal shift as shown in [24, Example 3.1(ii)]. In view of this, the question being addressed in this chapter is as follows:

"Given a 2-hyponormal weighted shift W_α and $j \geq 0$, does there always exist $\varepsilon > 0$ such that for $x \in (\alpha_j - \varepsilon, \alpha_j + \varepsilon)$, $W_{\alpha[j:x]}$ is again 2-hyponormal?"

In this chapter we establish a set of sufficient conditions under which there exists $\epsilon > 0$ such that for $x \in (\alpha_j - \epsilon, \alpha_j + \epsilon)$, $W_{\alpha[j:x]}$ will again be 2-hyponormal. Applying these conditions we can completely determine the situations where 2-hyponormality preserving perturbations do not exist.

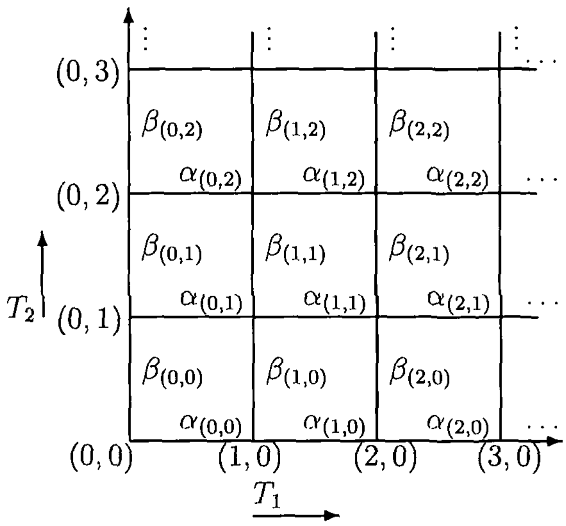
Moreover, in [24, Theorem 2.3], it was shown that a 2-hyponormal weighted shift remains quadratically hyponormal under small non-zero finite rank perturbations. The proof was based on the definition of positive quadratic hyponormality. In this chapter we give an independent proof for the same result, using a different characterization of quadratic hyponormality.

Chapter 5: Perturbation of 2-variable hyponormal weighted shift

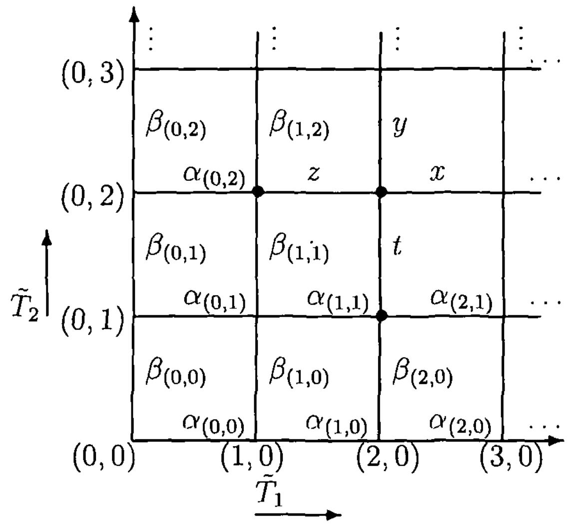
In Chapter 4 we have addressed the question of finite rank perturbation of 2-hyponormal weighted shift considering the unilateral weighted shift W_α on $\ell^2(\mathbb{Z}_+)$. In this chapter we initiate a parallel discussion for the 2-variable weighted shift on $\ell^2(\mathbb{Z}_+^2)$. For a unilateral weighted shift W_α it is well known that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all n. Hence for a strictly increasing weight sequence, any slight perturbation of the i^{th} weight still retains the hyponormality property for the perturbed shift. "Is the same true for a two variable weighted shift?" The answer is negative as is shown in the work done in this chapter. We also frame a set of positivity conditions which can completely determine hyponormality of the perturbed shift.

Chapter 6: *On weak hyponormality of 2-variable weighted shifts*

In Chapter 5 it was shown that if for a 2-variable hyponormal shift $T = (T_1, T_2)$, a weight $\alpha_{(k_1,k_2)}$ is perturbed, then the resulting perturbed shift may not remain hyponormal. For example, say we have the 2-variable hyponormal shift $T = (T_1, T_2)$ with respective weight sequences $\{\alpha_{(k_1,k_2)}\}$ and $\{\beta_{(k_1,k_2)}\}$, as shown in the following diagram



Suppose the weight $\alpha_{(2,2)}$ is perturbed slightly to x. Then to preserve commutativity, we need to perturb at least a minimum number of adjacent weights. So accordingly, $\beta_{(2,2)}$ changes to y, $\alpha_{(1,2)}$ changes to z, and $\beta_{(2,1)}$ changes to t. The weight diagram of the perturbed shift $\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$ will be as follows:



In Chapter 5 it was shown that $\tilde{T}$ may not remain hyponormal. In fact the conditions under which $\tilde{T}$ will still be hyponormal is completely given in that chapter.

In this chapter, we show that though $\tilde{T}$ may not be hyponormal, it will however still remain weakly hyponormal for sufficiently small perturbations x of $\alpha_{(k_1,k_2)}$.

Chapter 7: Back-step extension of weighted shifts

In this chapter we address the question of perturbation of subnormal weighted shifts. It was shown in [24, Theorem 2.1] that a non-zero finite rank perturbation of a subnormal shift is never subnormal, unless the perturbation occurs at the initial weight α_0. So the idea is to begin with a subnormal shift and create a back-step extension preserving subnormality The necessary and sufficient conditions (NASC) for subnormal backward extension of a 1-variable weighted shift was first given by Curto [12, Proposition 8]. Later an improved version of this result was given by Curto and Yoon [37, Proposition 1.5]. In the same paper, they have also given the NASC for subnormal backward extension of a 2-variable weighted shift [37, Proposition 2.9]. However, these results only deal with 1-step extension. In this chapter we extend these results to 2-step extension, and following a similar technique we propose NASC for n-step backward extension of 1-variable and 2-variable weighted shifts. In the last section we show how these results can also be derived applying Schur product technique.

Chapter 2

On convexity of weakly k-hyponormal region

2.1 Introduction

To express ourself clearly and systematically we begin by specifying the notations being followed. Let W_α be a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. Here α_n is referred to as the n^{th} weight. So α_0 is the 0^{th} weight, α_1 is the 1^{th} weight and so on. If for $y > 0$, and $j \geq 0$, $\alpha[j : y]$ denote the weight sequence $\alpha_0, \dots, \alpha_{j-1}, y, \alpha_{j+1}, \dots$ then $W_{\alpha[j:y]}$ is called the perturbed shift where the j^{th} weight α_j is perturbed to y. $W_{\alpha[j:y]}$ is a rank one perturbation of W_α. If the i^{th} and j^{th} weights of α are perturbed to x and y respectively, then the perturbed shift $W_{\alpha[(i:x),(j:y)]}$ is called a rank two perturbation of W_α. Similarly, we can define any finite rank perturbation of W_α.

The issue of perturbation of weights in a weighted shift operator, is intricately related to the question of convexity of the domain of perturbation. In [24, Theorem 6.5] it was shown that rank-one perturbations of k-hyponormal weighted shifts which preserve k-hyponormality form a convex set. That is, if W_α is k-hyponormal then $\Omega_\alpha(k, j) := \{x : W_{\alpha[j:x]} \text{ is } k\text{-hyponormal}\}$ is a convex set. However, it is not known whether a similar result holds for weakly k-hyponormal weighted shift W_α. In this chapter we show that, " if W_α is weakly k-hyponormal, then $\omega_\alpha(k,j) := \{x : W_{\alpha[j:x]}$ is weakly k-hyponormal$\}$ is a convex set."

In trying to ascertain this result, our first attempt is to come up with an example having this property. We try to achieve this for the case of a weak 2-hyponormal (i.e. quadratic hyponormal) operator using the characterization of quadratic hyponormality given in [65].

In examples 2.2.1 and 2.2.2 we construct two quadratically hyponormal weighted shifts W_α and W_β where the weight sequences α and β are as follows:

$$\alpha : \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{43}{80}}, \sqrt{\frac{2}{3}}, \dots \quad \text{and}\ \beta : \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{47}{80}}, \sqrt{\frac{2}{3}}, \dots$$

Let $\gamma(x)$ denote the weight sequence $\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{x}, \sqrt{\frac{2}{3}}, \dots$ In Proposition 2.2.3 we show that $W_{\gamma(x)}$ is quadratically hyponormal for all $x \in \left[\frac{43}{80}, \frac{47}{80}\right]$. For this we make use of the Theorem 2.2.2 and Mathematica graphs. The insight gained from this example enables us to prove that for a weighted shift operator W_α, if $W_{\alpha[j:x]}$ and $W_{\alpha[j:y]}$ are weakly k-hyponormal, then $W_{\alpha[j:z]}$ is weakly k- hyponormal for all z between x and y. In other words, $\omega_\alpha(k,j)$ is a convex set, and this is shown in Theorem 2.4.3.

2.2 Examples for quadratic hyponormality

We begin by recapitulation of definitions introduced in [12, 17, 65]. Let $\{e_n\}_{n=0}^\infty$ be the canonical orthonormal basis for $\ell^2(\mathbb{Z}_+)$ and let $\alpha := \{\alpha_n\}_{n=0}^\infty$ be a bounded sequence of positive numbers. Let W_α be the unilateral weighted shift defined by $W_\alpha e_n = \alpha_n e_{n+1}$ $(\forall n \geq 0)$.

By definition, an operator T is quadratically hyponormal (q.h.) if $T + sT^2$ is hyponormal for every $s \in \mathbb{C}$.

Lemma 2.2.1. *[25] W_α is quadratically hyponormal if and only if $W_\alpha + sW_\alpha^2$ is hyponormal for every $s \geq 0$.*

Proof. By the definition of quadratic hyponormality it is trivial that W_α is quadratically hyponormal implies $W_\alpha + sW_\alpha^2$ is hyponormal for every $s \geq 0$. Conversely, suppose $W_\alpha + sW_\alpha^2$ is hyponormal for every $s \geq 0$. We need to show that $W_\alpha + sW_\alpha^2$ is hyponormal for all $s \in \mathbb{C}$.
Let $s \in \mathbb{C}$ and $s = re^{i\theta}$ for $r > 0$. Define $u_\theta : \ell^2 \longrightarrow \ell^2$ as $u_\theta e_n = e^{-in\theta}e_n$. Then $u_\theta^* e_n = e^{in\theta}e_n$, and so, u_θ is unitary.
Also, $u_\theta W_\alpha u_\theta^* e_n = e^{-i\theta}\alpha_n e_{n+1} = e^{-i\theta}W_\alpha e_n$ that is, $u_\theta W_\alpha u_\theta^* = e^{-i\theta}W_\alpha$.

$$\begin{aligned} u_\theta(W_\alpha + sW_\alpha^2)u_\theta^* &= u_\theta W_\alpha u_\theta^* + su_\theta W_\alpha^2 u_\theta^* \\ &= u_\theta W_\alpha u_\theta^* + s\big(u_\theta W_\alpha u_\theta^*\big)^2 \\ &= e^{-i\theta}W_\alpha + re^{i\theta}e^{-2i\theta}W_\alpha^2 \\ &= e^{-i\theta}(W_\alpha + rW_\alpha^2) \end{aligned}$$

Since $W_\alpha + rW_\alpha^2$ is hyponormal, therefore $W_\alpha + sW_\alpha^2$ is hyponormal ($\forall\, s \in \mathbb{C}$). □

For a hyponormal weighted shift W_α and $s \geq 0$, let $D(s) := [(W_\alpha + sW_\alpha^2)^*, (W_\alpha + sW_\alpha^2)]$. Then we have,

$$\begin{aligned} D(s) &= [(W_\alpha + sW_\alpha^2)^*, (W_\alpha + sW_\alpha^2)] \\ &= (W_\alpha + sW_\alpha^2)^*(W_\alpha + sW_\alpha^2) - (W_\alpha + sW_\alpha^2)(W_\alpha + sW_\alpha^2)^* \\ &= [W_\alpha^*, W_\alpha] + s[W_\alpha^*, W_\alpha^2] + s[W_\alpha^{*2}, W_\alpha] + s^2[W_\alpha^{*2}, W_\alpha^2] \end{aligned}$$

It can be easily shown that

$$\begin{aligned} [W_\alpha^*, W_\alpha]e_n &= (\alpha_n^2 - \alpha_{n-1}^2)e_n \ (\forall n \geq 0) \\ [W_\alpha^*, W_\alpha^2]e_n &= \alpha_n(\alpha_n^2 - \alpha_{n-1}^2)e_{n+1} \ (\forall n \geq 0) \end{aligned}$$

$$[W_\alpha^{*^2}, W_\alpha]e_n = \begin{cases} 0 & \text{if } n = 0 \\ \alpha_{n-1}(\alpha_n^2 - \alpha_{n-2}^2)e_{n-1} & \text{if} n \geq 1 \end{cases}$$

$$[W_\alpha^{*^2}, W_\alpha^2]e_n = (\alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2)e_n \ (\forall n \geq 0)$$

Let P_n be the projection of $\ell^2(\mathbb{Z})$ onto $\bigvee_{i=0}^n\{e_i\}$ and for $(n \geq 0)$, let $D_n := D_n(s) = P_nD(s)P_n$. Then

$$D_n = \begin{pmatrix} q_0 & r_0 & 0 & \dots & 0 & 0 \\ r_0 & q_1 & r_1 & \dots & 0 & 0 \\ 0 & r_1 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_{n-1} & r_{n-1} \\ 0 & 0 & 0 & \dots & r_{n-1} & q_n \end{pmatrix},$$

where

$$q_k := u_k + s^2v_k$$

$$r_k := s\sqrt{w_k}$$

$$u_k := \alpha_k^2 - \alpha_{k-1}^2$$

$$v_k := \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2$$

$$w_k := \alpha_k^2(\alpha_{k+1}^2 - \alpha_{k-1}^2)^2$$

for $k \geq 0$ and $\alpha_{-1} = \alpha_{-2} := 0$

By the definition of quadratically hyponormal operator, we immediately see that W_α is q.h. if and only if $D_n(s) \geq 0$ for every $s \geq 0$ and every $n \geq 0$.

For $x_0, x_1, \dots, x_n$ and s in $\mathbb{R}_+$ we define the following:

$$\begin{aligned} F_n &:= F_n(x_0, x_1, \dots, x_n; s) \\ &= \sum_{i=0}^{n} q_i x_i^2 - 2\sum_{i=0}^{n-1} r_i x_i x_{i+1} \end{aligned}$$

$$= \sum_{i=0}^{n} u_i x_i^2 - 2s \sum_{i=0}^{n-1} \sqrt{w_i} x_i x_{i+1} + s^2 \sum_{i=0}^{n} v_i x_i^2$$

and recall, for further use, the following result :

Theorem 2.2.2. *[65]: Let W_α be a weighted shift with a weight sequence α. Then the followings are equivalent :*

(i) *W_α is quadratically hyponormal;*

(ii) *$F_n(x_0, x_1, \ldots, x_n, s) \geq 0$ for any $x_0, x_1, \ldots, x_n, s \in \mathbb{R}_+$ $(n \geq 2)$;*

(iii) *There exists a positive integer N such that $F_n(x_0, x_1, \ldots, x_n, s) \geq 0$ for any $x_0, x_1, \ldots, x_n, s \in \mathbb{R}_+$ $(n \geq N)$.*

Example 2.2.1. *Let α be the positive weight sequence given by $\alpha : \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{43}{80}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$ We will show that the weighted shift operator W_α is quadratically hyponormal.*

For this, let $\{\alpha_n\}_{n=0}^{\infty}$ denote the sequence α so that $\alpha_{n+1} = \sqrt{\frac{n}{n+1}}$ $\forall\, n \geq 2$.
In view of Theorem 2.2.2, it is sufficient to show that $F_n \geq 0$ $\forall n \geq 5$.
For $x_0, x_1, \ldots, x_5, s$ reals, we denote a function $G_5 = G_5(x_0, x_1, \ldots, x_5, s)$ by

$$\begin{aligned} G_5 &:= F_5 - v_5 t x_5^2 \text{ where } s^2 = t \\ &= \sum_{i=0}^{4} (u_i + t v_i) x_i^2 - 2 \sum_{i=0}^{4} \sqrt{w_i t}\, x_i x_{i+1} + u_5 x_5^2 \end{aligned}$$

Then,

$$\begin{aligned} F_6 &= \sum_{i=0}^{6} (u_i + t v_i) x_i^2 - 2 \sum_{i=0}^{5} \sqrt{w_i t}\, x_i x_{i+1} \\ &= G_5 + t v_5 x_5^2 + (u_6 + t v_6) x_6^2 - 2\sqrt{w_5 t}\, x_5 x_6 \\ &= G_5 + \left(v_5 t - \frac{w_5 t}{u_6 + t v_6} \right) x_5^2 + \left(\frac{\sqrt{w_5 t}}{\sqrt{u_6 + t v_6}} x_5 - \sqrt{u_6 + t v_6}\, x_6 \right)^2 \end{aligned}$$

Suppose $F_6(x_0,\ldots,x_6,s)\geq 0$ for any $x_0,\ldots,x_6,s\in\mathbb{R}_+$. Then since x_6 is arbitrary non-negative real, we will take $x_6=\frac{\sqrt{w_5t}}{u_6+tv_6}x_5$ so that

$$F_6\geq 0\Rightarrow G_5+\left(v_5t-\frac{w_5t}{u_6+tv_6}\right)x_5^2\geq 0$$

.

Conversely, if $G_5+(v_5t-\frac{w_5t}{u_6+tv_6})x_5^2\geq 0$ then

$$F_6=G_5+\left(v_5t-\frac{w_5t}{u_6+tv_6}\right)x_5^2+\left(\frac{\sqrt{w_5t}}{\sqrt{u_6+tv_6}}x_5-\sqrt{u_6+tv_6}x_6\right)^2\geq 0$$

.

Hence,

$$\begin{aligned}
&F_6(x_0,\ldots,x_6,s)\geq 0\ \text{ for any } x_0,\ldots,x_6,s\in\mathbb{R}_+\\
&\Leftrightarrow G_5(x_0,\ldots,x_5,s)+\left(v_5t-\frac{w_5t}{u_6+tv_6}\right)x_5^2\geq 0\ \text{ for any } x_0,\ldots,x_5,s\in\mathbb{R}_+\\
&\Leftrightarrow G_5(x_0,\ldots,x_5,s)+\frac{z_6t}{1+z_6t}v_5tx_5^2\geq 0\ \text{ for any } x_0,\ldots,x_5,s\in\mathbb{R}_+\\
&\left(\text{using } w_n=u_{n+1}v_n\ \forall n\geq 5 \text{ and } z_n=\frac{v_n}{u_n}\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
F_7=G_5&+\left(v_5t-\frac{w_5t}{(u_6+tv_6)-\left(\frac{w_6t}{u_7+tv_7}\right)}\right)x_5^2\\
&+\left(\frac{\sqrt{w_5t}}{\sqrt{(u_6+tv_6)-\left(\frac{w_6t}{u_7+tv_7}\right)}}x_5-\sqrt{(u_6+tv_6)-\left(\frac{w_6t}{u_7+tv_7}\right)}\,x_6\right)^2\\
&+\left(\frac{\sqrt{w_6t}}{\sqrt{u_7+tv_7}}\,x_6-\sqrt{u_7+tv_7}\,x_7\right)^2
\end{aligned}$$

and so

$$
\begin{aligned}
& F_7(x_0, \ldots, x_7, s) \geq 0 \ \text{ for any } x_0, \ldots, x_7, s \in \mathbb{R}_+ \\
\Leftrightarrow & G_5(x_0, \ldots, x_5, s) + \left(v_5 t - \frac{w_5 t}{(u_6 + tv_6) - \left(\frac{w_6 t}{u_7 + tv_7}\right)} \right) x_5^2 \geq 0 \\
& \quad \text{for any } x_0, \ldots, x_5, s \in \mathbb{R}_+ \\
\Leftrightarrow & G_5(x_0, \ldots, x_5, s) + \frac{z_7 z_6 t^2}{1 + z_7 t + z_7 z_6 t^2} v_5 t x_5^2 \geq 0 \ \text{ for any } x_0, \ldots, x_5, s \in \mathbb{R}_+
\end{aligned}
$$

So, by Mathematical induction, for $n \geq 6$ we have

$$
\begin{aligned}
F_n \geq 0 \Leftrightarrow G_5 + & \frac{(z_n z_{n-1} \ldots z_6 t^{n-5}) v_5 t x_5^2}{1 + z_n t + z_n z_{n-1} t^2 + \cdots + z_n z_{n-1} \ldots z_6 t^{n-5}} \geq 0 \\
\Leftrightarrow G_5 + & \frac{1}{1 + \frac{1}{z_6 t} + \frac{1}{z_6 z_7 t^2} + \cdots + \frac{1}{z_6 z_7 \ldots z_n t^{n-5}}} v_5 t x_5^2 \geq 0 \qquad (2.2.1)
\end{aligned}
$$

<u>Claim1:</u> $G_5(x_0, \ldots, x_5, s) \geq 0$ for $0 \leq s \leq \sqrt{0.299}$

The corresponding symmetric matrix to the quadratic form G_5 is

$$
A(t) = \begin{pmatrix}
u_0 + tv_0 & -\sqrt{w_0 t} & 0 & 0 & 0 & 0 \\
-\sqrt{w_0 t} & u_1 + tv_1 & -\sqrt{w_1 t} & 0 & 0 & 0 \\
0 & -\sqrt{w_1 t} & u_2 + tv_2 & -\sqrt{w_2 t} & 0 & 0 \\
0 & 0 & -\sqrt{w_2 t} & u_3 + tv_3 & -\sqrt{w_3 t} & 0 \\
0 & 0 & 0 & -\sqrt{w_3 t} & u_4 + tv_4 & -\sqrt{w_4 t} \\
0 & 0 & 0 & 0 & -\sqrt{w_4 t} & u_5
\end{pmatrix}
$$

We discuss the positivity of A(t) by Nested Determinant Test. By direct Computation, we have

$$
\begin{aligned}
d_0 &= \frac{1}{2} + \frac{1}{4} t \\
d_1 &= \frac{3t}{320} + \frac{43t^2}{640} \\
d_2 &= \frac{43t^2}{12800} + \frac{559t^3}{76800}
\end{aligned}
$$

$$d_3 = \frac{301t^2}{1024000} + \frac{731t^3}{1024000} + \frac{20683t^4}{12288000}$$
$$d_4 = \frac{301t^2}{12288000} + \frac{301t^3}{10240000} + \frac{34529t^4}{368640000} + \frac{599807t^5}{1474560000}$$
$$d_5 = \frac{301t^2}{245760000} - \frac{301t^3}{122880000} - \frac{35647t^4}{7372800000} - \frac{20683t^5}{9830400000}$$

If $0 < t \leq 0.299$, then $d_0, \ldots, d_4 > 0$ and $d_5 > 0$, which implies that $A(t) \geq 0$ for $0 < t \leq 0.299$ and $G_5(x_0, \ldots, x_5, s) \geq 0$ for $0 < s \leq \sqrt{0.299}$ and Claim 1 is established.

Hence by (2.2.1),

$$F_n(x_0, \ldots, x_n, s) \geq 0 \text{ for any } x_0, \ldots, x_n \in \mathbb{R}_+ \text{ and } 0 < s \leq \sqrt{0.299}.$$

Again, $z_n = \frac{v_n}{u_n} = \frac{4(n+1)}{n+2}$, $(n \geq 5)$ and also $\{z_n\}_{n=6}^{\infty}$ is an increasing sequence converging to 4. Thus,

$$\begin{aligned}
&1 + \frac{1}{z_6 t} + \frac{1}{z_6 z_7 t^2} + \cdots + \frac{1}{z_6 z_7 \ldots z_n t^{n-5}} \\
&\leq 1 + \frac{1}{z_6 t} + \left(\frac{1}{z_6 t}\right)^2 + \cdots + \left(\frac{1}{z_6 t}\right)^{n-5} \\
&\leq \sum_{n=0}^{\infty} \left(\frac{1}{z_6 t}\right)^n = \frac{1}{1 - \frac{1}{z_6 t}}
\end{aligned}$$

Hence if $t > 0.299$, then

$$\begin{aligned}
&G_5 + \frac{1}{1 + \frac{1}{z_6 t} + \frac{1}{z_6 z_7 t^2} + \cdots + \frac{1}{z_6 z_7 \ldots z_n t^{n-5}}} v_5 t x_5^2 \\
&\geq G_5 + \left(1 - \frac{1}{z_6 t}\right) v_5 t x_5^2 \\
&= G_5 + \left(1 - \frac{7}{24t}\right) \frac{t}{6} x_5^2 \quad \left(\text{since } z_6 = \frac{24}{7} \text{ and } v_5 = \frac{1}{6}\right) \\
&= G_5 + \left(\frac{24t - 7}{144}\right) x_5^2
\end{aligned}$$

Now we consider the corresponding symmetric matrix B(t) to the quadratic form $G_5 + \left(\frac{24t-7}{144}\right) x_5^2$ as follows:

$$B(t) = \begin{pmatrix} u_0 + tv_0 & -\sqrt{w_0 t} & 0 & 0 & 0 & 0 \\ -\sqrt{w_0 t} & u_1 + tv_1 & -\sqrt{w_1 t} & 0 & 0 & 0 \\ 0 & -\sqrt{w_1 t} & u_2 + tv_2 & -\sqrt{w_2 t} & 0 & 0 \\ 0 & 0 & -\sqrt{w_2 t} & u_3 + tv_3 & -\sqrt{w_3 t} & 0 \\ 0 & 0 & 0 & -\sqrt{w_3 t} & u_4 + tv_4 & -\sqrt{w_4 t} \\ 0 & 0 & 0 & 0 & -\sqrt{w_4 t} & u_5 + \frac{24t-7}{144} \end{pmatrix}$$

As was done in Claim 1, $d_i > 0$ for $t \geq 0.299$ and $i = 0, 1, 2, 3, 4$. Also, d_5 of B(t) is

$$\frac{301t^2}{8847360000} + \frac{301t^3}{1474560000} - \frac{1191487t^4}{265420800000} - \frac{6653089t^5}{1061683200000} + \frac{599807t^6}{8847360000} \geq 0$$

for $t \geq 0.299$

This is because d_5 is an increasing graph as is seen from the following Mathematica graph of d_5:

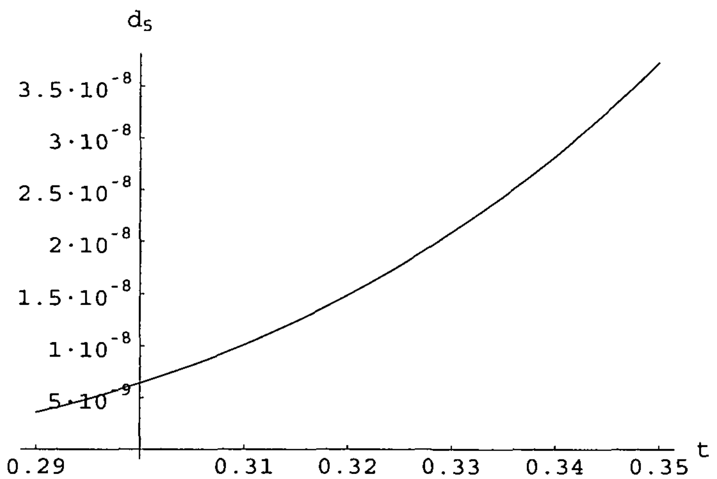


Figure 1

Therefore, $F_n \geq 0$ for $n \geq 5$ and $s \geq \sqrt{0.299}$

Thus for all $t \geq 0$, $F_n \geq 0$ $(n \geq 5)$. So by Theorem 2.2.2, W_α is quadratically hyponormal.

Example 2.2.2. *Let α be the positive weight sequence given by $\alpha : \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{47}{80}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$ Then the weighted shift operator W_α with weight sequence α is quadratically hyponormal.*

This can be shown by a method similar to that used in Example 2.2.1.

Proposition 2.2.3. *Let $\gamma(z)$ denote the positive weight sequence $\{\alpha_n\}$ given by $\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{z}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$. Then the weighted shift operator $W_{\gamma(z)}$ is quadratically hyponormal for all $z \in \left[\frac{43}{80}, \frac{47}{80}\right]$.*

Proof. As $\alpha_{n+1} = \sqrt{\frac{n}{n+1}}$ for all $n \geq 2$, therefore we have, $w_n = u_{n+1}v_n$ for all $n \geq 5$.

In view of Theorem 2.2.2, it is sufficient to show that $F_n \geq 0 \ \ \forall\, n \geq 5$.

For $x_0, x_1, \ldots, x_5, s$ reals, we denote a function $G_5 = G_5(x_0, x_1, \ldots, x_5, s)$ by

$$\begin{aligned} G_5 : &= F_5 - v_5 t x_5^2 \ \text{ where } s^2 = t \\ &= \sum_{i=0}^{4} (u_i + t v_i) x_i^2 - 2 \sum_{i=0}^{4} \sqrt{w_i t}\, x_i\, x_{i+1} + u_5 x_5^2 \end{aligned}$$

Then,

$$F_6 = G_5(x_0, \ldots, x_5, s) + \frac{z_6 t}{1 + z_6 t} v_5 t x_5^2 \ \left(\text{since } w_n = u_{n+1} v_n \ \forall\, n \geq 5 \text{ and } z_n = \frac{v_n}{u_n}\right)$$

Hence,

$$\begin{aligned} &F_6(x_0, \ldots, x_6, s) \geq 0 \ \text{ for any } x_0, \ldots, x_6, s \in \mathbb{R}_+ \\ &\Leftrightarrow G_5(x_0, \ldots, x_5, s) + \frac{z_6 t}{1 + z_6 t} v_5 t x_5^2 \geq 0 \ \text{ for any } x_0, \ldots, x_5, s \in \mathbb{R}_+ \end{aligned}$$

Similarly,

$$\begin{aligned} F_7 = &G_5 + \left(v_5 t - \frac{w_5 t}{(u_6 + t v_6) - \left(\frac{w_6 t}{u_7 + t v_7}\right)} \right) x_5^2 + \left(\frac{\sqrt{w_5 t}}{\sqrt{(u_6 + t v_6) - \left(\frac{w_6 t}{u_7 + t v_7}\right)}} x_5 \right. \\ &\left. - \sqrt{(u_6 + t v_6) - \left(\frac{w_6 t}{u_7 + t v_7}\right)}\, x_6 \right)^2 + \left(\frac{\sqrt{w_6 t}}{\sqrt{u_7 + t v_7}} x_6 - \sqrt{u_7 + t v_7}\, x_7 \right)^2 \end{aligned}$$

and so

$$\begin{aligned}
&F_7(x_0,\ldots,x_7;s)\geq 0 \text{ for any } x_0,\ldots,x_7,s\in\mathbb{R}_+\\
&\Leftrightarrow G_5(x_0,\ldots,x_5;s)+\left(v_5t-\frac{w_5t}{(u_6+tv_6)-\left(\frac{w_6t}{u_7+tv_7}\right)}\right)x_5^2\geq 0\\
&\quad \text{for any } x_0,\ldots,x_5,s\in\mathbb{R}_+\\
&\Leftrightarrow G_5(x_0,\ldots,x_5;s)+\frac{z_7z_6t^2}{1+z_7t+z_7z_6t^2}v_5tx_5^2\geq 0 \text{ for any } x_0,\ldots,x_5,s\in\mathbb{R}_+
\end{aligned}$$

So, by Mathematical induction, for $n\geq 6$ we have

$$\begin{aligned}
F_n(x_0,\ldots,x_n;s)\geq 0 \Leftrightarrow G_5+&\frac{\left(z_nz_{n-1}\cdots z_6t^{n-5}\right)v_5tx_5^2}{1+z_nt+z_nz_{n-1}t^2+\cdots+z_nz_{n-1}\cdots z_6t^{n-5}}\geq 0\\
\Leftrightarrow G_5+&\frac{1}{1+\frac{1}{z_6t}+\frac{1}{z_6z_7t^2}+\cdots+\frac{1}{z_6z_7\ldots z_nt^{n-5}}}v_5tx_5^2\geq 0
\end{aligned} \tag{2.2.2}$$

<u>Claim1:</u> $G_5(x_0,\ldots,x_5,s)\geq 0$ for $0\leq s\leq\sqrt{0.299}$.

The corresponding symmetric matrix to the quadratic form G_5 is

$$A(t)=\begin{pmatrix}
u_0+tv_0 & -\sqrt{w_0t} & 0 & 0 & 0 & 0\\
-\sqrt{w_0t} & u_1+tv_1 & -\sqrt{w_1t} & 0 & 0 & 0\\
0 & -\sqrt{w_1t} & u_2+tv_2 & -\sqrt{w_2t} & 0 & 0\\
0 & 0 & -\sqrt{w_2t} & u_3+tv_3 & -\sqrt{w_3t} & 0\\
0 & 0 & 0 & -\sqrt{w_3t} & u_4+tv_4 & -\sqrt{w_4t}\\
0 & 0 & 0 & 0 & -\sqrt{w_4t} & u_5
\end{pmatrix}$$

We discuss the positivity of A(t) by Nested Determinant Test. By direct Computation, we have

$$\begin{aligned}
d_0 &=\alpha_0^2+t\alpha_0^2\alpha_1^2\\
d_1 &=t\alpha_0^2\alpha_1^2(\alpha_2^2-\alpha_1^2)+t^2\alpha_0^2\alpha_1^4\alpha_2^2
\end{aligned}$$

$$
\begin{aligned}
d_2 =& t^2 \left\{\alpha_0^4\alpha_1^4(\alpha_2^2-\alpha_0^2)+\alpha_0^2\alpha_1^2(\alpha_2^2-\alpha_1^2)(\alpha_2^2\alpha_3^2-\alpha_1^2\alpha_0^2)\right\} \\
&+ t^3\left\{(\alpha_2^2\alpha_3^2-\alpha_1^2\alpha_0^2)\alpha_0^2\alpha_1^4\alpha_2^2\right\} \\
d_3 =& t^2\left(\frac{3z^2}{16}-\frac{5z}{96}-\frac{z^3}{6}\right)+t^3\left(\frac{5z^2}{24}-\frac{z}{16}-\frac{z^3}{6}\right)+t^4\left(\frac{11z^2}{192}-\frac{z}{64}-\frac{z^3}{24}\right) \\
=& t^2P_2(z)+t^3P_3(z)+t^4P_4(z) \\
d_4 =& t^2\left(\frac{z^2}{64}-\frac{11z}{1152}-\frac{z^3}{72}\right)+t^3\left(\frac{3z^2}{160}-\frac{z}{192}-\frac{z^3}{60}\right) \\
&+ t^4\left(\frac{251z^2}{2304}-\frac{13z}{480}-\frac{199z^3}{1440}+\frac{z^4}{18}\right)+t^5\left(\frac{43z^2}{960}-\frac{3z}{320}-\frac{91z^3}{1440}+\frac{z^4}{36}\right) \\
=& t^2Q_2(z)+t^3Q_3(z)+t^4Q_4(z)+t^5Q_5(z)
\end{aligned}
$$

All d_0, d_1, d_2 are positive by their expressions for $\alpha_0 = \alpha_1$ and d_3, d_4 are positive for all $z \in [\frac{43}{80}, \frac{47}{80}]$ and $\forall t \geq 0$, since all $P_i(z)$ $(i = 2, 3, 4)$ and $Q_i(z)$ $(i = 2, 3, 4, 5)$ are positive for all $z \in [\frac{43}{80}, \frac{47}{80}]$.

We use Mathematica graph to show the positivity for d_5 of the matrix $A(t)$.

$$
\begin{aligned}
d_5(z,t) = & -\frac{t^2z}{4608}+\frac{t^3z}{2304}-\frac{t^4z}{1920}-\frac{t^5z}{3840}+\frac{t^2z^2}{1280}-\frac{t^3z^2}{640}+\frac{41t^4z^2}{15360}+\frac{t^5z^2}{11520} \\
& -\frac{t^2z^3}{1440}+\frac{t^3z^3}{720}-\frac{3t^4z^3}{640}-\frac{t^5z^3}{384}+\frac{t^4z^4}{360}+\frac{t^5z^4}{720}
\end{aligned}
$$

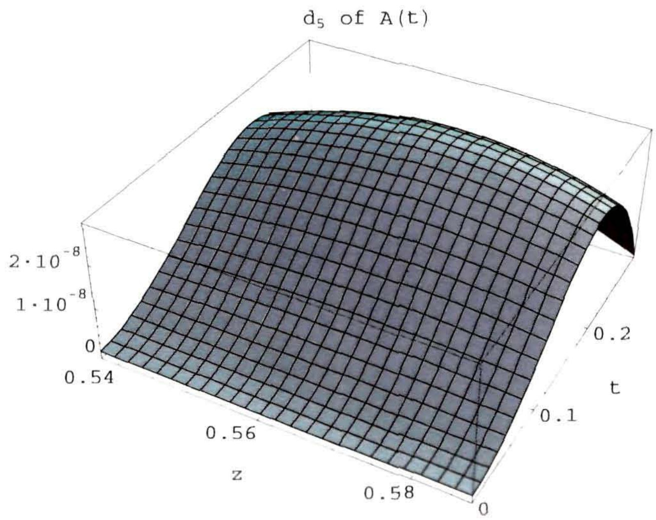


Figure 2

From the above Mathematica graph it is clear that if $0 < t \leq 0.299$ then $d_5 > 0$, which implies that $A(t) \geq 0$ for $0 < t \leq 0.299$ and $G_5(x_0, \ldots, x_5, s) \geq 0$ for $0 < s \leq \sqrt{0.299}$ and Claim 1 is established. Hence by (2.2.2) $F_n(x_0, \ldots, x_n, s) \geq 0\,(n \geq 5)$ for any $x_0, \ldots, x_n \in \mathbb{R}_+$ and $0 < s \leq \sqrt{0.299}$.

Now we will show for $t \geq 0.299$.

As, $z_n = \frac{v_n}{u_n} = \frac{4(n+1)}{n+2}$, $(n \geq 5)$, so $\{z_n\}_{n=6}^{\infty}$ is an increasing sequence and hence

$$\begin{aligned}1 + \frac{1}{z_6 t} + \frac{1}{z_6 z_7 t^2} + \cdots + \frac{1}{z_6 z_7 \ldots z_n t^{n-5}} &\leq 1 + \frac{1}{z_6 t} + \frac{1}{(z_6 t)^2} + \cdots + \frac{1}{(z_6 t)^{n-5}} \\ &\leq \frac{1}{1 - \frac{1}{z_6 t}}\end{aligned}$$

Now,

$$G_5 + \frac{1}{1 + \frac{1}{z_6 t} + \frac{1}{z_6 z_7 t^2} + \cdots + \frac{1}{z_6 z_7 . . z_n t^{n-5}}} v_5 t x_5^2 \geq G_5 + \left(\frac{24t - 7}{144}\right) x_5^2$$

$$\left(\because\ z_6 = \frac{24}{7} \text{ and } v_5 = \frac{1}{6}\right)$$

Now considering the corresponding symmetric matrix B(t) to the quadratic form $G_5 + \left(\frac{24t-7}{144}\right) x_5^2$, we have

$$B(t) = \begin{pmatrix} u_0 + t v_0 & -\sqrt{w_0 t} & 0 & 0 & 0 & 0 \\ -\sqrt{w_0 t} & u_1 + t v_1 & -\sqrt{w_1 t} & 0 & 0 & 0 \\ 0 & -\sqrt{w_1 t} & u_2 + t v_2 & -\sqrt{w_2 t} & 0 & 0 \\ 0 & 0 & -\sqrt{w_2 t} & u_3 + t v_3 & -\sqrt{w_3 t} & 0 \\ 0 & 0 & 0 & -\sqrt{w_3 t} & u_4 + t v_4 & -\sqrt{w_4 t} \\ 0 & 0 & 0 & 0 & -\sqrt{w_4 t} & u_5 + \frac{24t-7}{144} \end{pmatrix}$$

$$\begin{aligned}d_5(z, t) = &- \frac{t^2 z}{165888} - \frac{t^3 z}{27648} - \frac{t^4 z}{13824} - \frac{199 t^5 z}{46080} - \frac{t^6 z}{640} + \frac{t^2 z^2}{46080} + \frac{t^3 z^2}{7680} \\ &+ \frac{827 t^4 z^2}{1658880} + \frac{2413 t^5 z^2}{138240} + \frac{43 t^6 z^2}{5760} - \frac{t^2 z^3}{51840} - \frac{t^3 z^3}{8640} - \frac{31 t^4 z^3}{41472} - \frac{4679 t^5 z^3}{207360} \\ &- \frac{91 t^6 z^3}{8640} + \frac{t^4 z^4}{12960} + \frac{241 t^5 z^4}{25920} + \frac{t^6 z^4}{216}\end{aligned}$$

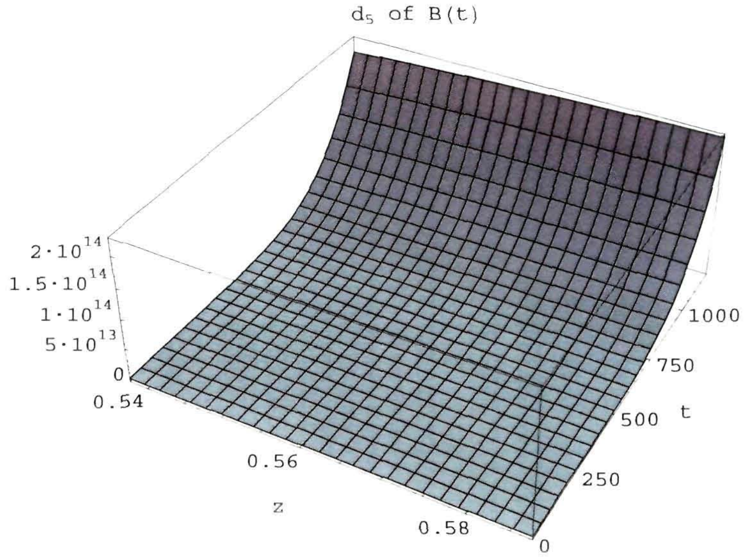


Figure 3

From the above Mathematica graph it is clear that the graph is an increasing graph and hence $d_5 > 0$ for $t \geq 0.299$ and $z \in \left[\frac{43}{80}, \frac{47}{80}\right]$.

Therefore $F_n(x_0, \ldots, x_n, s) \geq 0$ for all $t > 0$ and $z \in \left[\frac{43}{80}, \frac{47}{80}\right]$. So by Theorem 2.2.2, $W_{\gamma(z)}$ is quadratically hyponormal for $z \in \left[\frac{43}{80}, \frac{47}{80}\right]$.

□

2.3 NASC for weak k-hyponormality

Let $\mathbb{C}[z, w]$ denote the polynomials in two variables and $\mathbb{C}[z]$ denote all polynomials with one variable z. We first give a construction given by J. Agler [2] which associates operators T on a Hilbert space H with linear functionals $\lambda : \mathbb{C}[z, w] \longrightarrow \mathbb{C}$ which obey certain positivity conditions.

For $h(z, w) = \sum_{i,j} h_{ij} z^i w^j \in \mathbb{C}[z, w]$ and an operator $T \in B(H)$, define $h(T, T^*) = \sum_{i,j} h_{ij} T^{*j} T^i$. In particular, $(zw)(T, T^*) = T^*T$ and $(z^i w^j)(T, T^*) = T^{*j} T^i$. If $x \in H$, then define a linear functional $\Lambda_T : \mathbb{C}[z, w] \longrightarrow \mathbb{C}$ by the formula

$\Lambda_T(h) = \langle h(T, T^*)x, x \rangle$.

Lemma 2.3.1. *For a polynomial* $p \in \mathbb{C}[z]$, $\left(\overline{p(\bar{z})}\right)(T^*) = (p(T))^*$.

Proof. Let $p(z) = \sum_{i=0}^n \alpha_i z^i$. Then $\overline{p(\bar{z})} = \sum_{i=0}^n \bar{\alpha}_i z^i$ and so $\overline{p(\bar{z})}(T^*) = \sum_{i=0}^n \bar{\alpha}_i T^{*i}$.

$$\therefore\ p(T)^* = \left(\sum_{i=0}^n \alpha_i T^i\right)^* = \sum_{i=0}^n \bar{\alpha}_i T^{*i} = \overline{p(\bar{z})}(T^*).$$

□

Observed that for $p \in \mathbb{C}[z]$ with $p(z) = \sum_{i=0}^n a_i z^i$,

$$p(w)\,p(z) = \left(\sum_{i=0}^n a_i w^i\right)\left(\sum_{i=0}^n a_i z^i\right) \in \mathbb{C}[z, w].$$

Lemma 2.3.2. *If x is a cyclic vector for T, then $\|T\| \le 1$ if and only if*

$$\Lambda_T\left(\overline{p(\bar{w})}(1 - zw)\,p(z)\right) \ge 0,\ \ \forall p(z) \in \mathbb{C}[z].$$

Proof. Since x is a cyclic vector for T so $H = \mathrm{cl}\{p(T)x : p \in \mathbb{C}[z]\}$.

Now,

$$\begin{aligned}
\|T\| \le 1 &\Leftrightarrow \|Ty\|^2 \le \|y\|^2\ \ (\forall y \in H)\\
&\Leftrightarrow \langle (I - TT^*)y, y \rangle \ge 0\ \ (\forall y \in H)\\
&\Leftrightarrow \langle (I - TT^*)\,p(T)x, p(T)x \rangle \ge 0\\
&\Leftrightarrow \langle p(T)^*(I - TT^*)\,p(T)x, x \rangle \ge 0\\
&\Leftrightarrow \left\langle \overline{p(\bar{w})}(1 - zw)p(z)(T, T^*)x, x \right\rangle \ge 0\\
&\Leftrightarrow \Lambda_T\left(\overline{p(\bar{w})}(1 - zw)\,p(z)\right) \ge 0\ \ (\forall p(z) \in \mathbb{C}[z])
\end{aligned}$$

□

The following result was given by Agler [2]:

Lemma 2.3.3. *[2] Let T be a cyclic contraction in $B(H)$. Then T is weakly k-hyponormal if and only if*

$$\Lambda_T\left[\left(\overline{q(\bar{w})}+\overline{p(\bar{w})}\phi(z)\right)\left(q(z)+p(z)\overline{\phi(\bar{w})}\right)\right]\geq 0$$

for all polynomials $p(z), q(z)$ and $\phi(z)$ with degree $\phi(z)\leq k$.

Now let us consider a weighted shift W_α. Then e_0 is the standard cyclic vector for W_α. Thus, taking $T = W_\alpha$ we get $\Lambda_T(z^i w^j) = \left\langle (z^i w^j)(T,T^*)\, e_0, e_0\right\rangle = \left\langle T^{*j}T^i e_0, e_0\right\rangle = 0$ (for $i\neq j$).

Using this fact, a reformulation of Lemma 2.3.3 was given in [45] for the case where T is a contractive weighted shift.

Lemma 2.3.4. *[45] Suppose W_α is a contractive hyponormal weighted shift with weight sequence $\alpha := \{\alpha_i\}_{i=0}^{\infty}$. Then W_α is weakly k-hyponormal if and only if*

$$\Delta_k^\alpha(\phi,p,q) := \gamma_k|\phi_k p_0|^2+\left\langle\begin{pmatrix}\gamma_{k-1} & \gamma_k\\ \gamma_k & \gamma_{k+1}\end{pmatrix}\begin{pmatrix}\phi_{k-1}p_0\\ \phi_k p_1\end{pmatrix},\begin{pmatrix}\phi_{k-1}p_0\\ \phi_k p_1\end{pmatrix}\right\rangle$$

$$+\cdots+\left\langle\begin{pmatrix}\gamma_1 & \gamma_2 & \cdots & \gamma_k\\ \gamma_2 & \gamma_3 & \cdots & \gamma_{k+1}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1}\end{pmatrix}\begin{pmatrix}\phi_1 p_0\\ \phi_2 p_1\\ \vdots\\ \phi_k p_{k-1}\end{pmatrix},\begin{pmatrix}\phi_1 p_0\\ \phi_2 p_1\\ \vdots\\ \phi_k p_{k-1}\end{pmatrix}\right\rangle$$

$$+\sum_{j=0}^{\infty}\left\langle\begin{pmatrix}\gamma_j & \gamma_{j+1} & \cdots & \gamma_{j+k}\\ \gamma_{j+1} & \gamma_{j+2} & \cdots & \gamma_{j+k+1}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_{j+k} & \gamma_{j+k+1} & \cdots & \gamma_{j+2k}\end{pmatrix}\begin{pmatrix}q_j\\ \phi_1 p_{j+1}\\ \vdots\\ \phi_k p_{j+k}\end{pmatrix},\begin{pmatrix}q_j\\ \phi_1 p_{j+1}\\ \vdots\\ \phi_k p_{j+k}\end{pmatrix}\right\rangle$$

$$\geq 0$$

for $\phi := \{\phi_i\}_{i=1}^{k}$, $p := \{p_i\}_{i=0}^{\infty}$ and $q := \{q_i\}_{i=0}^{\infty}$ in $\mathbb{C}$.

Lemma 2.3.5.

$$\Delta_{k+1}^\alpha(\phi,p,q) = \Delta_k^\alpha(\phi,p,q)+\sum_{j=k+1}^{\infty}\gamma_j\left[|\phi_{k+1}p_{j-k-1}|^2+2Re\left\{\bar{\phi}_{k+1}p_{j-k-1}\left(\sum_{l=1}^{k}\phi_l\bar{p}_{j-l}\right)\right.\right.$$
$$\left.\left.+\,\phi_{k+1}p_j\bar{q}_{j-k-1}\right\}\right]$$

for all $k\geq 1$ and $\phi := \{\phi_i\}_{i=1}^{k}$, $p := \{p_i\}_{i=0}^{\infty}$ and $q := \{q_i\}_{i=0}^{\infty}$ in $\mathbb{C}$.

Proof.

$$\Delta^{\alpha}_{k+1}(\phi,p,q) := \gamma_{k+1}|\phi_{k+1}p_0|^2 + \left\langle \begin{pmatrix} \gamma_k & \gamma_{k+1} \\ \gamma_{k+1} & \gamma_{k+2} \end{pmatrix} \begin{pmatrix} \phi_k p_0 \\ \phi_{k+1}p_1 \end{pmatrix}, \begin{pmatrix} \phi_k p_0 \\ \phi_{k+1}p_1 \end{pmatrix} \right\rangle$$

$$+ \cdots + \left\langle \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\ \gamma_2 & \gamma_3 & \cdots & \gamma_{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k+1} & \gamma_{k+2} & & \gamma_{2k+1} \end{pmatrix} \begin{pmatrix} \phi_1 p_0 \\ \phi_2 p_1 \\ \vdots \\ \phi_{k+1}p_k \end{pmatrix}, \begin{pmatrix} \phi_1 p_0 \\ \phi_2 p_1 \\ \vdots \\ \phi_{k+1}p_k \end{pmatrix} \right\rangle$$

$$+ \sum_{j=0}^{\infty} \left\langle \begin{pmatrix} \gamma_j & \gamma_{j+1} & \cdots & \gamma_{j+k+1} \\ \gamma_{j+1} & \gamma_{j+2} & \cdots & \gamma_{j+k+2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{j+k+1} & \gamma_{j+k+2} & \cdots & \gamma_{j+2k+2} \end{pmatrix} \begin{pmatrix} q_j \\ \phi_1 p_{j+1} \\ \vdots \\ \phi_{k+1}p_{j+k+1} \end{pmatrix}, \begin{pmatrix} q_j \\ \phi_1 p_{j+1} \\ \vdots \\ \phi_{k+1}p_{j+k+1} \end{pmatrix} \right\rangle$$

$$= \Delta^{\alpha}_{k}(\phi,p,q) + \gamma_{k+1}\left[|\phi_{k+1}p_0|^2 + 2Re\left\{\bar{\phi}_{k+1}p_0\left(\sum_{l=1}^{k}\phi_l\bar{p}_{k+1-l}\right) + \phi_{k+1}p_{k+1}\bar{q}_0\right\}\right]$$

$$+ \gamma_{k+2}\left[|\phi_{k+1}p_1|^2 + 2Re\left\{\bar{\phi}_{k+1}p_1\left(\sum_{l=1}^{k}\phi_l\bar{p}_{k+2-l}\right) + \phi_{k+1}p_{k+2}\bar{q}_1\right\}\right]$$

$$+ \gamma_{k+3}\left[|\phi_{k+1}p_2|^2 + 2Re\left\{\bar{\phi}_{k+1}p_2\left(\sum_{l=1}^{k}\phi_l\bar{p}_{k+3-l}\right) + \phi_{k+1}p_{k+3}\bar{q}_2\right\}\right]$$

$$+ \cdots$$

$$= \Delta^{\alpha}_{k}(\phi,p,q) + \sum_{j=k+1}^{\infty}\gamma_j\left[|\phi_{k+1}p_{j-k-1}|^2 + 2Re\left\{\bar{\phi}_{k+1}p_{j-k-1}\left(\sum_{l=1}^{k}\phi_l\bar{p}_{j-l}\right) + \phi_{k+1}p_j\bar{q}_{j-k-1}\right\}\right]$$

□

Lemma 2.3.6. *Let $\alpha := \{\alpha_n\}_{n=0}^{\infty}$ be a positive weight sequence and $x = \varepsilon\alpha_n$ for $0 < \varepsilon < 1$, $n \geq 0$. Then for $k \geq 1$,*

$$\Delta^{\alpha[n:x]}_{k}(\phi,p,q) = \varepsilon^2\Delta^{\alpha}_{k}(\phi,p,q) + \left(1-\varepsilon^2\right)\left(|q_0|^2 + \sum_{i=1}^{n}\gamma_i z_i\right)$$

where

$$z_i = \sum_{j=1}^{k}|\phi_j p_{i-j}|^2 + |q_i|^2 + 2Re\left[p_i\left(\sum_{j=1}^{k}\phi_j\bar{q}_{i-j}\right) + \sum_{l=2}^{k}\bar{\phi}_l p_{i-l}\left(\sum_{j=1}^{l-1}\phi_j\bar{p}_{i-j}\right)\right]$$

Proof. Let γ'_j denote the moment sequence of $\alpha[n:x]$. Then

$$\gamma'_j = \begin{cases} \gamma_j, & \text{for } j \leq n \\ \varepsilon^2\gamma_j, & \text{for } j > n \end{cases}$$

Case I: If $n = 0$, that is α_0 be perturbed to $x = \varepsilon\alpha_0$, then

$$\gamma'_j = \begin{cases} \gamma_0 = 1, & \text{for } j = 0 \\ \varepsilon^2\gamma_j. & \text{for } j > 0, \end{cases}$$

$$\begin{aligned} \Delta_1^{\alpha[0\,x]}(\phi, p, q) &= \gamma'_1|\phi_1 p_0|^2 + \sum_{j=0}^{\infty} \left\langle \begin{pmatrix} \gamma'_j & \gamma'_{j+1} \\ \gamma'_{j+1} & \gamma'_{j+2} \end{pmatrix} \begin{pmatrix} q_j \\ \phi_1 p_{j+1} \end{pmatrix}, \begin{pmatrix} q_j \\ \phi_1 p_{j+1} \end{pmatrix} \right\rangle \\ &= \varepsilon^2 \Delta_1^{\alpha}(\phi, p, q) + (1 - \varepsilon^2)|q_0|^2 \end{aligned}$$

By Lemma 2.3.5, we get

$$\begin{aligned} \Delta_2^{\alpha[0\,x]}(\phi, p, q) =& \Delta_1^{\alpha[0\,x]}(\phi, p, q) + \sum_{j=2}^{\infty} \gamma'_j \left[|\phi_2 p_{j-2}|^2 + 2Re\left\{\bar{\phi}_2 p_{j-2}\phi_1\bar{p}_{j-1} + \phi_2 p_j \bar{q}_{j-2}\right\}\right] \\ =& \varepsilon^2\Delta_1^{\alpha}(\phi, p, q) + (1 - \varepsilon^2)|q_0|^2 \\ &+ \sum_{j=2}^{\infty} \varepsilon^2\gamma_j \left[|\phi_2 p_{j-2}|^2 + 2Re\left\{\bar{\phi}_2 p_{j-2}\phi_1\bar{p}_{j-1} + \phi_2 p_j \bar{q}_{j-2}\right\}\right] \\ =& \varepsilon^2\Delta_2^{\alpha}(\phi, p, q) + (1 - \varepsilon^2)|q_0|^2 \end{aligned}$$

Similarly,

$$\Delta_k^{\alpha[0\,x]}(\phi, p, q) = \varepsilon^2\Delta_k^{\alpha}(\phi, p, q) + (1 - \varepsilon^2)|q_0|^2$$

for all $k \geq 1$

Case II: If $n = 1$, that is α_1 be perturbed to $x = \varepsilon\alpha_1$, then

$$\gamma'_j = \begin{cases} \gamma_j, & \text{for } j \leq 1 \\ \varepsilon^2\gamma_j, & \text{for } j > 1, \end{cases}$$

$$\Delta_1^{\alpha[1\,x]}(\phi, p, q) = \varepsilon^2\Delta_1^{\alpha}(\phi, p, q) + (1 - \varepsilon^2)(|q_0|^2 + \gamma_1 z_1),$$

where $z_1 = |\phi_1 p_0|^2 + |q_1|^2 + 2Re(\phi_1 p_1 \bar{q}_0)$

Also, as $\gamma'_j = \varepsilon^2\gamma_j$ for $j \geq 2$, so by Lemma 2 3.5, we get

$$\Delta_k^{\alpha[1\,x]}(\phi, p, q) = \varepsilon^2\Delta_k^{\alpha}(\phi, p, q) + (1 - \varepsilon^2)(|q_0|^2 + \gamma_1 z_1),$$

for all $k \geq 1$

Case III: If $n = 2$, that is α_2 be perturbed to $x = \varepsilon\alpha_2$, then

$$\gamma'_j = \begin{cases} \gamma_j, & \text{for } j \leq 2 \\ \varepsilon^2\gamma_j, & \text{for } j > 2, \end{cases}$$

$$\Delta_1^{\alpha[2.x]}(\phi, p, q) = \varepsilon^2\Delta_1^{\alpha}(\phi, p, q) + (1 - \varepsilon^2)(|q_0|^2 + \gamma_1 z'_1 + \gamma_2 z'_2),$$

where $z'_1 = |\phi_1 p_0|^2 + |q_1|^2 + 2Re(\phi_1 p_1 \bar{q}_0)$ and $z'_2 = |\phi_1 p_1|^2 + |q_2|^2 + 2Re(\phi_1 p_2 \bar{q}_1)$
Again, by Lemma 2.3.5

$$\begin{aligned}
\Delta_2^{\alpha[2:x]}(\phi, p, q) &= \Delta_1^{\alpha[2:x]}(\phi, p, q) + \sum_{j=2}^{\infty} \gamma'_j \left[|\phi_2 p_{j-2}|^2 + 2Re\left\{\bar{\phi}_2 p_{j-2}\phi_1\bar{p}_{j-1} + \phi_2 p_j \bar{q}_{j-2}\right\}\right] \\
&= \varepsilon^2\Delta_2^{\alpha}(\phi, p, q) + (1 - \varepsilon^2)(|q_0|^2 + \gamma_1 z'_1 + \gamma_2 z'_2) + (1 - \varepsilon^2)\gamma_2\big[|\phi_2 p_0|^2 \\
&\quad + 2Re\left\{\bar{\phi}_2 p_0 \phi_1 \bar{p}_1 + \phi_2 p_2 \bar{q}_0\right\}\big] \\
&= \varepsilon^2\Delta_2^{\alpha}(\phi, p, q) + (1 - \varepsilon^2)(|q_0|^2 + \gamma_1 z''_1 + \gamma_2 z''_2),
\end{aligned}$$

where

$$z''_1 = |\phi_1 p_0|^2 + |q_1|^2 + 2Re(\phi_1 p_1 \bar{q}_0)$$

and

$$z''_2 = |\phi_1 p_1|^2 + |\phi_2 p_0|^2 + |q_2|^2 + 2Re\{\bar{\phi}_2 p_0 \phi_1 \bar{p}_1 + p_2(\phi_1 \bar{q}_1 + \phi_2 \bar{q}_0)\}.$$

As $\gamma'_j = \varepsilon^2\gamma_j$ for $j > 2$, so for $k > 2$, we get

$$\Delta_k^{\alpha[2:x]}(\phi, p, q) = \varepsilon^2\Delta_k^{\alpha}(\phi, p, q) + (1 - \varepsilon^2)(|q_0|^2 + \gamma_1 z''_1 + \gamma_2 z''_2),$$

Thus, for $k \geq 1$,

$$\Delta_k^{\alpha[2:x]}(\phi, p, q) = \varepsilon^2\Delta_k^{\alpha}(\phi, p, q) + (1 - \varepsilon^2)(|q_0|^2 + \gamma_1 z_1 + \gamma_2 z_2),$$

where

$$z_1 = |\phi_1 p_0|^2 + |q_1|^2 + 2Re(\phi_1 p_1 \bar{q}_0)$$

and

$$z_2 = \sum_{j=1}^{k} |\phi_j p_{2-j}|^2 + |q_2|^2 + 2Re\left[p_2\left(\sum_{j=1}^{k}\phi_j\bar{q}_{2-j}\right) + \bar{\phi}_2 p_0 \phi_1 \bar{p}_1\right],$$

assuming $p_m = 0 = q_m$ for $m < 0$.

Case IV: If $n = 3$, that is α_3 be perturbed to $x = \varepsilon\alpha_3$, then

$$\gamma_j' = \begin{cases} \gamma_j, & \text{for } j \le 3 \\ \varepsilon^2\gamma_j, & \text{for } j > 3, \end{cases}$$

As in Case III, here we get for $k \ge 1$,

$$\Delta_k^{\alpha[3\,x]}(\phi,p,q) = \varepsilon^2\Delta_k^{\alpha}(\phi,p,q) + (1-\varepsilon^2)\left(|q_0|^2 + \sum_{i=1}^{3}\gamma_i z_i\right),$$

where

$$z_1 = |\phi_1 p_0|^2 + |q_1|^2 + 2Re(\phi_1 p_1 \bar{q}_0)$$

$$z_2 = \sum_{j=1}^{k} |\phi_j p_{2-j}|^2 + |q_2|^2 + 2Re\left[p_2\left(\sum_{j=1}^{k}\phi_j\bar{q}_{2-j}\right) + \bar{\phi}_2 p_0 \phi_1 \bar{p}_1\right]$$

and

$$z_3 = \sum_{j=1}^{k} |\phi_j p_{3-j}|^2 + |q_3|^2 + 2Re\left[p_3\left(\sum_{j=1}^{k}\phi_j\bar{q}_{3-j}\right) + \sum_{l=2}^{k}\bar{\phi}_l p_{3-l}\left(\sum_{j=1}^{l-1}\phi_j\bar{p}_{3-j}\right)\right]$$

assuming $p_m = 0 = q_m$ for $m < 0$. That is, for $i = 1,2,3$

$$z_i = \sum_{j=1}^{k} |\phi_j p_{i-j}|^2 + |q_i|^2 + 2Re\left[p_i\left(\sum_{j=1}^{k}\phi_j\bar{q}_{i-j}\right) + \sum_{l=2}^{k}\bar{\phi}_l p_{i-l}\left(\sum_{j=1}^{l-1}\phi_j\bar{p}_{i-j}\right)\right]$$

Continuing in this way, if α_n is perturbed to $\varepsilon\alpha_n$ $(n = 0,1,2,\ldots)$, then for all $k \ge 1$,

$$\Delta_k^{\alpha[n\,x]}(\phi,p,q) = \varepsilon^2\Delta_k^{\alpha}(\phi,p,q) + (1-\varepsilon^2)\left(|q_0|^2 + \sum_{i=1}^{n}\gamma_i z_i\right),$$

where

$$z_i = \sum_{j=1}^{k} |\phi_j p_{i-j}|^2 + |q_i|^2 + 2Re\left[p_i\left(\sum_{j=1}^{k}\phi_j\bar{q}_{i-j}\right) + \sum_{l=2}^{k}\bar{\phi}_l p_{i-l}\left(\sum_{j=1}^{l-1}\phi_j\bar{p}_{i-j}\right)\right].$$

□

2.4 Perturbation and convexity

Lemma 2.4.1. *Let W_α be a weakly k-hyponormal weighted shift and $\varepsilon\alpha_n \in \omega_\alpha(k,n)$ for some $\varepsilon \in (0,1)$. Then $[\sqrt{\varepsilon}\alpha_n, \alpha_n] \subseteq \omega_\alpha(k,n)$.*

Proof. Let $x = \varepsilon\,\alpha_n$ and for $0 < t < 1$, let $z_t = \sqrt{\delta}\alpha_n$ where $\delta = t\varepsilon + (1-t)$.

Claim 1: $z_t \in \omega_\alpha(k,n)$ for all $0 < t < 1$.

As $\varepsilon < \delta < 1$, so $\delta < \sqrt{\delta} < 1$ and therefore $z_t \in (x, \alpha_n)$.

Now by Lemma 2.3.6, for $\phi := \{\phi_i\}_{i=1}^k$, $p := \{p_i\}_{i=0}^\infty$ and $q := \{q_i\}_{i=0}^\infty$ in $\mathbb{C}$,

$$\Delta_k^{\alpha[n:x]}(\phi,p,q) = \varepsilon^2\Delta_k^{\alpha}(\phi,p,q) + (1-\varepsilon^2)\left(|q_0|^2 + \sum_{i=1}^n \gamma_i z_i\right) \tag{2.4.1}$$

and

$$\Delta_k^{\alpha[n:z_t]}(\phi,p,q) = \delta\Delta_k^{\alpha}(\phi,p,q) + (1-\delta)\left(|q_0|^2 + \sum_{i=1}^n \gamma_i z_i\right) \tag{2.4.2}$$

Thus from (2.4.1) and (2.4.1), we get

$$\begin{aligned}\Delta_k^{\alpha[n:z_t]}(\phi,p,q) &= \delta\Delta_k^{\alpha}(\phi,p,q) + \left(\frac{1-\delta}{1-\varepsilon^2}\right)\left[\Delta_k^{\alpha[n:x]}(\phi,p,q) - \varepsilon^2\Delta_k^{\alpha}(\phi,p,q)\right] \\ &= \left(\delta-\varepsilon^2\right)\Delta_k^{\alpha}(\phi,p,q) + \left(\frac{1-\delta}{1-\varepsilon^2}\right)\Delta_k^{\alpha[n:x]}(\phi,p,q) \geq 0\end{aligned}$$

as $\alpha_1, x \in \omega_\alpha(k,n)$.

Therefore by Lemma 2.3.4, $z_t \in \omega_\alpha(k,n)$ and Claim 1 is established.

Now let $\xi \in [\sqrt{\varepsilon}\alpha_n, \alpha_n]$. Then

$$\begin{aligned}&\xi = \lambda\alpha_n \text{ for } \sqrt{\varepsilon} \leq \lambda \leq 1 \\ \Rightarrow\, &y = \lambda^2\alpha_n \in [\varepsilon\alpha_n, \alpha_n] \\ \Rightarrow\, &\lambda^2 = t\varepsilon + (1-t) \text{ for some } 0 < t < 1 \\ \Rightarrow\, &\xi = \lambda\alpha_n \in \omega_\alpha(k,n), \text{ by Claim 1.}\end{aligned}$$

□

Corollary 2.4.2. *If $x, y \in \omega_\alpha(k,n)$, $x < y$ and $x = \varepsilon y$ $(0 < \varepsilon < 1)$ Then* $[\sqrt{\varepsilon}y, y] \subset \omega_\alpha(k,n)$

Proof. If γ_i' denotes the moment sequence of $\alpha[n:y]$, then by Lemma 2.3.6,

$$\Delta_k^{\alpha[n:x]}(\phi,p,q) = \varepsilon^2 \Delta_k^{\alpha[n:y]}(\phi,p,q) + \left(1-\varepsilon^2\right)\left(|q_0|^2 + \sum_{i=1}^{n} \gamma_i' z_i\right)$$

and so the result follows as in Lemma 2.4.1. □

Theorem 2.4.3. *Let W_α be a contractive weakly k-hyponormal weighted shift and $\omega_\alpha(k,j) := \{x : W_{\alpha[j:x]}$ is weakly k-hyponormal$\}$ Then $\omega_\alpha(k,j)$ is a convex set.*

Proof. Let $x, y \in \omega_\alpha(k,j)$. Without loss of generality we choose $x < y$ Then $x = \varepsilon y$ for some $0 < \varepsilon < 1$ By Corollary 2.4 2,

$$[\varepsilon^{\frac{1}{2}}y, y] \subset \omega_\alpha(k,j) \tag{2.4.3}$$

Step 1:

Let $x_1 = \varepsilon^{\frac{1}{2}}y$ Then $x = \varepsilon y = \varepsilon^{\frac{1}{2}}x_1$ As $\varepsilon^{\frac{1}{2}}x_1, x_1 \in \omega_\alpha(k.j)$, so by Corollary 2.4.2, $[\varepsilon^{\frac{1}{4}}x_1, x_1] \subset \omega_\alpha(k,j)$. That is,

$$[\varepsilon^{\frac{3}{4}}y, \varepsilon^{\frac{1}{2}}y] \subset \omega_\alpha(k,j) \tag{2.4.4}$$

Therefore from (2.4.3) and (2 4.4), we get $[\varepsilon^{\frac{3}{4}}y, y] \subset \omega_\alpha(k,j)$.

Step 2:

Let $x_2 = \varepsilon^{\frac{3}{4}}y$ Then $x = \varepsilon y = \varepsilon^{\frac{1}{4}}x_2$ As $\varepsilon^{\frac{1}{4}}x_2, x_2 \in \omega_\alpha(k,j)$, so again by Corollary 2.4.2, $[\varepsilon^{\frac{1}{8}}x_2, x_2] \subset \omega_\alpha(k,j)$ That is,

$$[\varepsilon^{\frac{7}{8}}y, \varepsilon^{\frac{3}{4}}y] \subset \omega_\alpha(k,j) \tag{2.4.5}$$

Therefore from (2.4.3), (2.4.4) and (2.4.5), we get $[\varepsilon^{\frac{7}{8}}y, y] \subset \omega_\alpha(k,j)$.

Continuing this process, after n^{th} step we have $\left[\varepsilon^{\left(\frac{2^{n+1}-1}{2^{n+1}}\right)}y, y\right] \subset \omega_\alpha(k,j)$. But $\frac{2^{n+1}-1}{2^{n+1}} \uparrow 1$ as $n \to \infty$ and so $\varepsilon^{\left(\frac{2^{n+1}-1}{2^{n+1}}\right)} \downarrow \varepsilon$ as $n \to \infty$. Thus we get $(x, y] = (\varepsilon y, y] \subset \omega_\alpha(k,j)$. Therefore if $x, y \in \omega_\alpha(k,j)$, then $[x,y] \subset \omega_\alpha(k,j)$ and so $\omega_\alpha(k,j)$ is convex. $\square$

Chapter 3

On convexity of positive quadratic hyponormal region

3.1 Introduction

This chapter is in continuation of Chapter 2. Here also we continue to investigate the idea of convexity.

If $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a positive weight sequence, $\imath \geq 0$, $k \geq 1$ and W_α is weakly k-hyponormal, then we have shown that $\omega_\alpha(k,\imath)$ is a nonempty convex set, where $\omega_\alpha(k,\imath) := \{x : W_{\alpha[\imath\, x]}$ is weakly k-hyponormal$\}$. Now suppose $y \in \omega_\alpha(k,\imath{+}1) = \{x : W_{\alpha[\imath+1\, x]}$ is weakly k-hyponormal$\}$ and let $\omega_{\alpha[\imath+1,y]}(k,\imath) := \{x : W_{\alpha[(\imath\, x),(\imath+1\, y)]}$ is weakly k-hyponormal$\}$ Here $\omega_{\alpha[\imath+1,y]}(k,\imath) \neq \phi$ as $\alpha_\imath \in \omega_{\alpha[\imath+1,y]}(k,\imath)$. Moreover $\omega_{\alpha[\imath+1,y]}(k,\imath)$ is a convex set, by Theorem 2.4.3.

Question: What is the relation between $\omega_\alpha(k,\imath)$ and $\omega_{\alpha[\imath+1,y]}(k,\imath)$?

In this chapter we address this problem with reference to a positively quadratically hyponormal operator W_α with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ where $\alpha_n = \sqrt{\frac{n+1}{n+2}}$ for all n.

3.2 Positive quadratic hyponormality

To define a positive quadratically hyponormal weighted shift W_α, we recall the definition of quadratic hyponormal shift from section 2.2. W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for all $s \geq 0$ and $n \geq 0$, where

$$D_n = \begin{pmatrix} q_0 & r_0 & 0 & \dots & 0 & 0 \\ r_0 & q_1 & r_1 & \dots & 0 & 0 \\ 0 & r_1 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_{n-1} & r_{n-1} \\ 0 & 0 & 0 & \dots & r_{n-1} & q_n \end{pmatrix}$$

and

$$q_k := u_k + s^2 v_k$$

$$r_k := s\sqrt{w_k}$$

$$u_k := \alpha_k^2 - \alpha_{k-1}^2$$

$$v_k := \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2$$

$$w_k := \alpha_k^2(\alpha_{k+1}^2 - \alpha_{k-1}^2)^2$$

for $k \geq 0$ and $\alpha_{-1} = \alpha_{-2} := 0$

Let $d_n(\cdot) := det(D_n(\cdot))$. Then it follows from [17] that

$$d_0 = q_0$$

$$d_1 = q_0 q_1 - r_0^2$$

$$d_{n+2} = q_{n+2} d_{n+1} - r_{n+1}^2 d_n \ (\forall n \geq 0)$$

and that d_n is actually a polynomial in $t := s^2$ of degree $n+1$, with Maclaurine expansion $d_n(t) := \sum_{i=0}^{n+1} c(n,i)t^i$. This immediately gives that for $1 \leq i \leq n+1$,

$$c(0,0) = u_0,\ c(0,1) = v_0,\ c(1,0) = u_1u_0,$$

$$c(1,1) = u_1v_0 - u_0v_1 - w_0,\ c(1,2) = v_1v_0,$$

$$c(n,i) = u_nc(n-1,i) + v_nc(n-1,i-1) - w_{n-1}c(n-2,i-1)(\forall n \geq 2)$$

$$c(n,1) = u_nc(n-1,1) + (v_nu_{n-1} - w_{n-1}u_0 \cdots u_{n-1})\ (\forall n \geq 2)$$

Observed that $c(n,0) = u_0u_1 \ldots u_n \geq 0$ and $c(n,n+1) = v_0v_1 \ldots v_n \geq 0$ for all $(n \geq 0)$.

Definition 3.2.1. [17] A hyponormal weighted shift W_α is said to be positively quadratically hyponormal (p.q.h.) if $c(n,i) \geq 0$ for all $n, i \geq 0$ with $0 \leq i \leq n+1$ and $c(n,n+1) > 0$ for all $n \geq 0$

3.3 Statement of problem

Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be the positive weight sequence given by $\alpha_n = \sqrt{\frac{n+1}{n+2}}$ for all $n \geq 0$

In [12] it was shown that W_α is p.q.h., and $\rho_\alpha(0) := \{x : W_{\alpha[0\ x]} \text{ is p.q.h.}\} = (0, \frac{2}{3}]$ Recall that $\alpha[0:x]$ denotes the weight sequence $x, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$

Question 1: Can α_1 be perturbed to y such that $W_{\alpha[1\ y]}$ is again p.q.h.?

Question 2: If answer to Question 1 is 'yes', then what is the relation between $\rho_\alpha(0)$ and $\rho_{\alpha[1\ y]}(0)$? Note, $\rho_{\alpha[1\ y]}(0) := \{x : W_{\alpha[(0\ x),(1\ y)]} \text{ is p.q.h.}\}$.

In this chapter we answer these two questions. We show that there exists an interval (k_1, k_2) about $\alpha_1 = \sqrt{\frac{2}{3}}$ such that for $y \in (k_1, k_2)$, $W_{\alpha[1\ y]}$ is p.q.h., and $\rho_{\alpha[1\ y]}(0) = (0, y]$. Thus, if $k_1 < y < \sqrt{\frac{2}{3}}$ then $\rho_{\alpha[1\ y]}(0) \subset \rho_\alpha(0)$, and if $\sqrt{\frac{2}{3}} < y < k_2$ then $\rho_\alpha(0) \subset \rho_{\alpha[1\ y]}(0)$.

Now suppose $W_{\alpha[1:y_0]}$ is p.q.h. for $y_0 < k_1$ or $y_0 > k_2$. Then we have shown that in such cases $\rho_{\alpha[1:y_0]}(0) \neq (0, y_0]$ because there will always exist $x \in (0, y_0]$ such that $W_{\alpha[(0:x),(1:y_0)]}$ is not p.q.h.

Thus, in this chapter we determine k_1, k_2 such that if $y \in (k_1, k_2)$ then either $\rho_\alpha(0) \subset \rho_{\alpha[1:y]}(0)$ or $\rho_{\alpha[1:y]}(0) \subset \rho_\alpha(0)$.

3.4 Determination of k_1 and k_2

Consider the weighted shift $W_{\alpha(x,y)}$ with a positive weight sequence $\alpha(x, y)$: $\sqrt{x}, \sqrt{y}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots$ having a Bergman tail. In [12] it was shown that for $y = \frac{2}{3}$ and $0 < x \leq y$, the weighted shift $W_{\alpha(x,y)}$ is p.q.h. So, does there exist an interval (k_1, k_2) about $\frac{2}{3}$ such that for $y \in (k_1, k_2)$ and $0 < x \leq y$, the weighted shift $W_{\alpha(x,y)}$ is p.q.h. ?

Remark 3.4.1. We must have $x \leq y \leq \frac{3}{4}$ because $W_{\alpha(x,y)}$ cannot be p.q.h. if it is not hyponormal in the first place.

Remark 3.4.2. If (k_1, k_2) exists then it must be contained in $[\delta_1, \delta_2]$. This is in view of [46, Theorem 2.2] which states the following :

Let $\alpha(x)$: $\sqrt{x}, \sqrt{x}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots$ be a weight sequence with Bergman tail and let $QH(W_{\alpha(x)}) = \{x \in \mathbb{R}_+ : W_{\alpha(x)} \text{ is q.h.}\}$. Then $QH(W_{\alpha(x)}) = [\delta_1, \delta_2]$ where $\delta_1 \approx 0.1673$ and $\delta_2 \approx 0.7439$ with errors less than .001.

Now suppose (k_1, k_2) exists. Then for $y \in (k_1, k_2)$ and $0 < x \leq y$, $W_{\alpha(x,y)}$ is p.q.h. and hence q.h. In particular $W_{\alpha(y)}$ is q.h. and so $y \in QH(W_{\alpha(y)}) = [\delta_1, \delta_2]$.

Remark 3.4.3. If (k_1, k_2) exists then it must be contained in $(0.625, \delta_2]$. This is in view of [65, Theorem 3.7] where it was shown that for $y = \frac{5}{8} = 0.625$, $W_{\alpha(x,y)}$ is not p.q.h.

We now proceed to show that (k_1, k_2) exists and also to determine the biggest such interval. Before that we record a few definitions and results from [3] which are to be used in solving our problem.

Definition 3.4.1. [3] Let $\alpha : \alpha_0, \alpha_1, \ldots$ be a weight sequence.

(1) A weighted shift W_α has property $B(k)$ if $u_{n+1}v_n \geq w_n$, $(n \geq k)$

(2) A weighted shift W_α has property $C(k)$ if $v_{n+1}u_n \geq w_n$, $(n \geq k)$,

where u_n, v_n, w_n are defined as in section 3.2.

Corollary 3.4.1. *[3] Let W_α be a weighted shift with property $C(2)$. Then W_α is p.q.h. if and only if $c(n+1, n) \geq 0 \, \forall n \in \mathbb{N}$*

Lemma 3.4.2. *[3] If W_α has property $B(n+1)$ for some $n \geq 1$, then W_α has property $C(n)$.*

Theorem 3.4.3. *[3] If W_α be a weighted shift with property $B(k)$ for some $k \geq 2$, then W_α is p.q.h. if and only if $c(n+i-1, i) \geq 0$ for $n = 1, 2, \ldots, k$*

In view of Remark 3.4.3, we shall consider $y \in (0.625, \frac{2}{3}]$ for determining k_1, and we consider $y \in [\frac{2}{3}, 0.7439]$ for determining k_2.

CASE I : Determining k_1

Choose $y \in (0.625, \frac{2}{3}]$, $0 < x \leq y$ and denote the sequence $\alpha(x, y)$ as $\alpha_0, \alpha_1, \alpha_2, \ldots$. Then we have $\alpha_0 = \sqrt{x}$, $\alpha_1 = \sqrt{y}$ and $\alpha_n = \sqrt{\frac{n+1}{n+2}}$ for $n \geq 2$. Using the expressions of u_n, v_n and w_n as given in §3.2, we see that $u_{n+1}v_n - w_n = \frac{1}{40}(\frac{2}{3} - y) \geq 0$ for $n = 3$, and for $n \geq 4$ we have $u_{n+1} = \frac{1}{(n+2)(n+3)}, v_n = \frac{4}{(n+1)(n+3)}, w_n =$

$\frac{4}{(n+1)(n+2)(n+3)^2}$ and so $u_{n+1}v_n = w_n$. Therefore, we have $u_{n+1}v_n - w_n \geq 0$ for $n \geq 3$ and so by Definition 3.4.1, $W_{\alpha(x,y)}$ has property $B(3)$.

Since $W_{\alpha(x,y)}$ has property $B(3)$ so by Theorem 3.4.3, $W_{\alpha(x,y)}$ is p.q.h. if and only if $c(n+i-1, i) \geq 0$ for $n = 1, 2$ and $i = 1, 2, 3$.

Again, since $W_{\alpha(x,y)}$ has property $B(3)$, so by Lemma 3.4.2, $W_{\alpha(x,y)}$ has property $C(2)$ and hence by Corollary 3.4.1, $W_{\alpha(x,y)}$ is p.q.h. if and only if $c(n+1, n) \geq 0$ for all $n \in \mathbb{N}$.

Combining the above two results we get that $W_{\alpha(x,y)}$ is p.q.h. if and only if $c(2,1), c(3,2)$ and $c(4,3)$ are ≥ 0. Using the expressions of $c(n,i)$ from §3.2 and simplifying we get,

$c(2,1) = \frac{3}{4}\, x\, (\frac{4}{5} - y)(y - x)$

$c(3,2) = \frac{1}{80}\, x\, [(5y - 4y^2 - 3y^3) - x(32 - 112y + 138y^2 - 60y^3)]$

$c(4,3) = \frac{1}{2800}\, x\, [(41y - 79y^2 + 37y^3) - x(128 - 420y + 475y^2 - 184y^3)]$

Clearly, for $y \in (0.625, \frac{2}{3}]$ and $0 < x \leq y$ we get $c(2,1) \geq 0$.

Regarding $c(3,2)$, if we define $f(y) := \frac{5y-4y^2-3y^3}{32-112y+138y^2+60y^3}$ then it is seen from the Mathematica graph and also by rigorous calculation that for $y \in (0.625, \frac{2}{3}]$ and $0 < x \leq y$, $f(y) \geq y$ and so $c(3,2) \geq 0$.

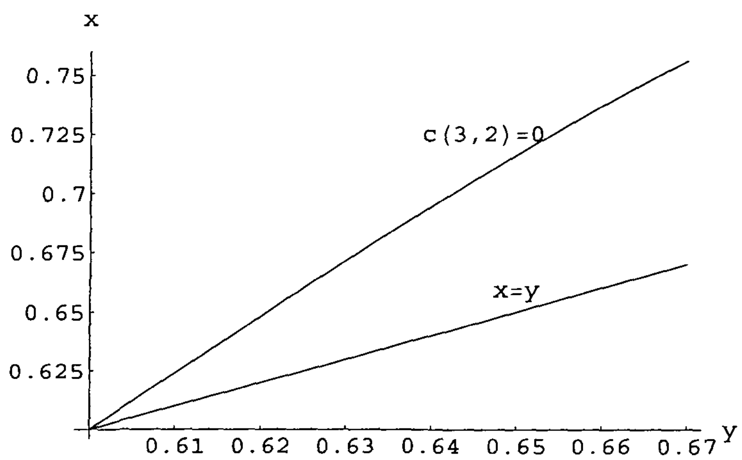


Figure 4

To check whether $c(4,3) \geq 0$, we define $f(y) := \frac{41y-79y^2+37y^3}{128-420y+475y^2-184y^3}$. Then

(i) for $y \in (0.625, \frac{29}{46})$, $f(y) < y$ and so for $f(y) < x \leq y$ we have $c(4,3) < 0$

(ii) for $y \in [\frac{29}{46}, \frac{2}{3}]$, $f(y) \geq y$ and so for $0 < x \leq y$ we have $c(4,3) \geq 0$

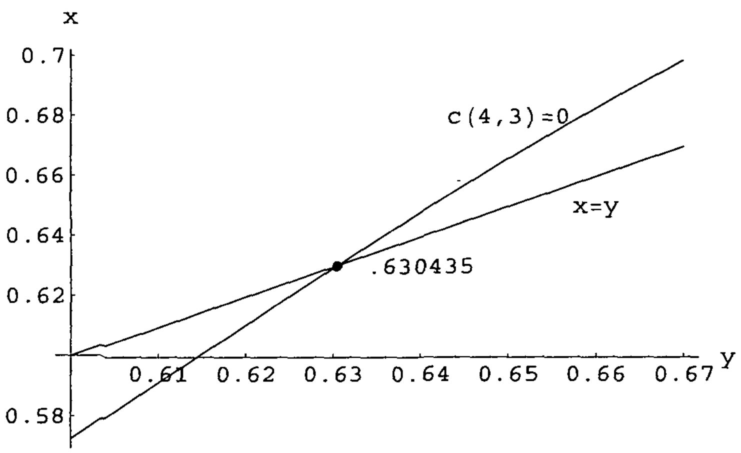


Figure 5

Hence we conclude that $W_{\alpha(x,y)}$ is p.q.h. for $0 < x \leq y$ if and only if $y \in [\frac{29}{46}, \frac{2}{3}]$. Thus, $k_1 = \frac{29}{46} \approx 0.630435$

CASE II : Determining k_2

Choosing $y \in [\frac{2}{3}, 0.7439]$ and $0 < x \leq y$ and proceeding as in Case I we see that $W_{\alpha(x,y)}$ has property $B(4)$. So by Theorem 3.4.3, $W_{\alpha(x,y)}$ is p.q.h. if and only if $c(n+i-1, i) \geq 0$ for $n = 1,2,3$ and $i = 1,2,3,4$. That is, if and only if $c(1,1), c(2,1), c(2,2), c(3,1), c(3,2), c(3,3), c(4,2), c(4,3), c(4,4), c(5,3), c(5,4)$, $c(6,4)$ are all ≥ 0.

Using the expressions of $c(n,i)$ from §3.2 and simplifying we get,

$c(1,1) = \frac{1}{4}\, xy\,(3-4x)$

$c(2,1) = \frac{3}{20}(4-5y)(y-x)$

$c(2,2) = \frac{2}{20}\, xy\,[(3-5y^2) - x(4-5y)]$

$c(3,1) = \frac{1}{60}\, x\,(3-4y)(y-x)$

$c(3,2) = \frac{1}{80}\, x\,[(-5y+4y^2+3y^3) - x(-32+112y-138y^2+60y^3)]$

$c(3,3) = \frac{1}{40}\, xy\,[(-12+27y-16y^2) - x(-16+38y-24y^2)]$

$c(4,2) = \frac{1}{16800}\, x\,[(75y-86y^2-21y^3) - x(264-842y+966y^2-420y^3)]$

$c(4,3) = \frac{1}{2800}\, x\,[(41y-79y^2+37y^3) - x(128-420y+475y^2-184y^3)]$

$c(4,4) = \frac{1}{5600}\, xy\,[(192-390y+193y^2) - x(256-608y+454y^2-105y^3)]$

$c(5,3) = \frac{1}{201600}\, x\,[(111y-194y^2+63y^3) - 2x(132-397y+417y^2-162y^3)]$

$c(5,4) = \frac{1}{33600}\, x\,[(41y-73y^2+28y^3) - x(128-420y+475y^2-174y^3-15y^4)]$

$c(6,4) = \frac{1}{50803200}\, x\,[(1776y-2942y^2+765y^3) - x(4224-12704y+13344y^2-4914y^3-405y^4)]$

Now $c(1,1), c(2,1)$ and $c(3,1)$ are obviously ≥ 0 for $0 < x \leq y \leq \frac{3}{4}$.

Thus we only need to check $c(2,2), c(3,2), c(3,3), c(4,2), c(4,3), c(4,4)$, $c(5,3)$, $c(5,4)$ and $c(6,4)$. Of these we find that other than $c(4,2), c(4,4), c(5,4)$ and $c(6,4)$, all the rest are ≥ 0 for $y \in [\frac{2}{3}, 0.7439]$ and $0 < x \leq y$. This is clear from the following figure which shows that the graphs of $c(2,2), c(3,2), c(3,3), c(4,3)$ and $c(5,3)$ are all above the $x = y$ line in the region $y \in [\frac{2}{3}, 0.7439]$.

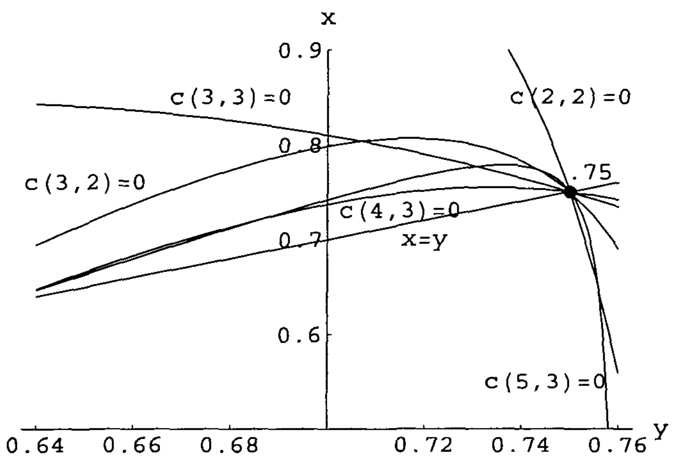


Figure 6

To check whether $c(4,2) \geq 0$, we define $f(y) := \frac{75y-86y^2-21y^3}{264-842y+966y^2-420y^3}$. Then

(i) for $y \in (0.737144, 0.7439)$, $f(y) < y$ and so for $f(y) < x \leq y$ we have $c(4,2) < 0$

(ii) for $y \in [\frac{2}{3}, 0.737144)$, $f(y) \geq y$ and so for $0 < x \leq y$ we have $c(4,2) \geq 0$

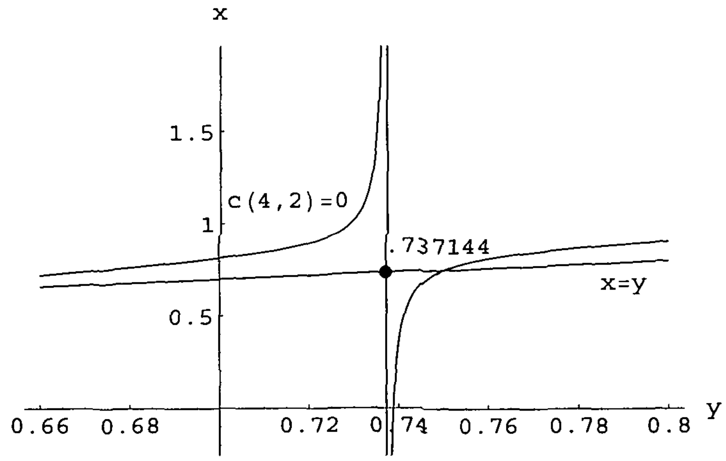


Figure 7

To check whether $c(5,4) \geq 0$, we define $f(y) := \frac{41y-73y^2+28y^3}{128-420y+475y^2-174y^3-15y^4}$. Then

(i) for $y \in (0.742207, 0.7439]$, $f(y) < y$ and so for $f(y) < x \leq y$ we have $c(5,4) < 0$

(ii) for $y \in [\frac{2}{3}, 0.742207]$, $f(y) \geq y$ and so for $0 < x \leq y$ we have $c(5,4) \geq 0$

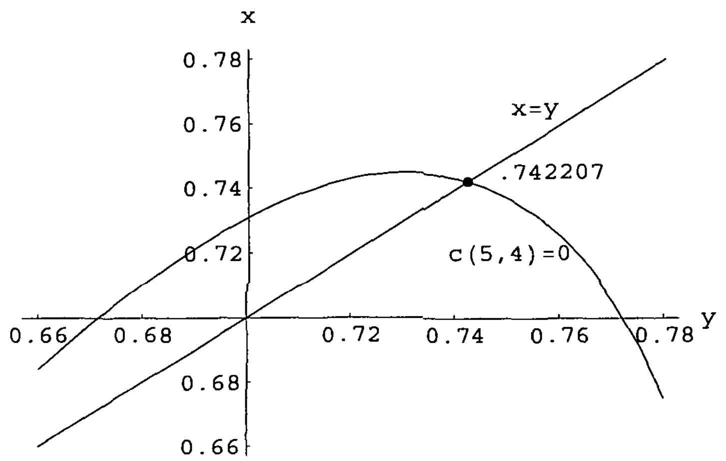


Figure 8

To check whether $c(6,4) \geq 0$, we define $f(y) := \frac{1776y-2942y^2+765y^3}{4224-12704y+13344y^2-4914y^3-405y^4}$.
Then (i) for $y \in (0.742654, 0.7439]$, $f(y) < y$ and so for $f(y) < x \leq y$ we have $c(6,4) < 0$

(ii) for $y \in [\frac{2}{3}, 0.742654]$, $f(y) \geq y$ and so for $0 < x \leq y$ we have $c(6,4) \geq 0$

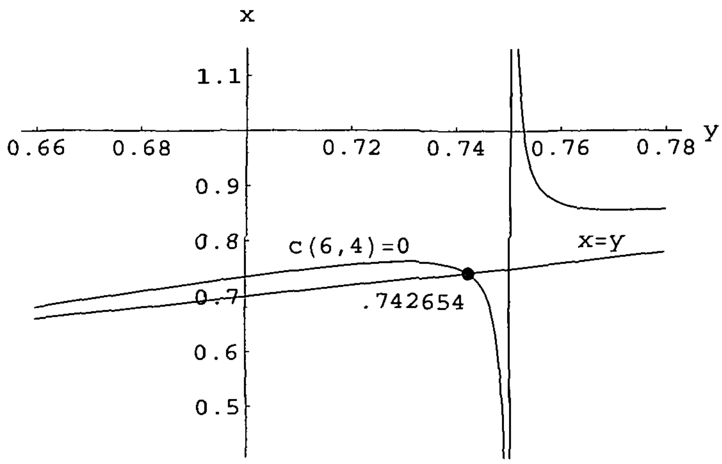


Figure 9

To check whether $c(4,4) \geq 0$, we define $f(y) := \frac{192-390y+193y^2}{256-608y+454y^2-105y^3}$. Then
(i) for $y \in (0.742847, 0.7439]$, $f(y) < y$ and so for $f(y) < x \leq y$ we have $c(4,4) < 0$

(ii) for $y \in [\frac{2}{3}, 0.742847]$, $f(y) \geq y$ and so for $0 < x \leq y$ we have $c(4,4) \geq 0$

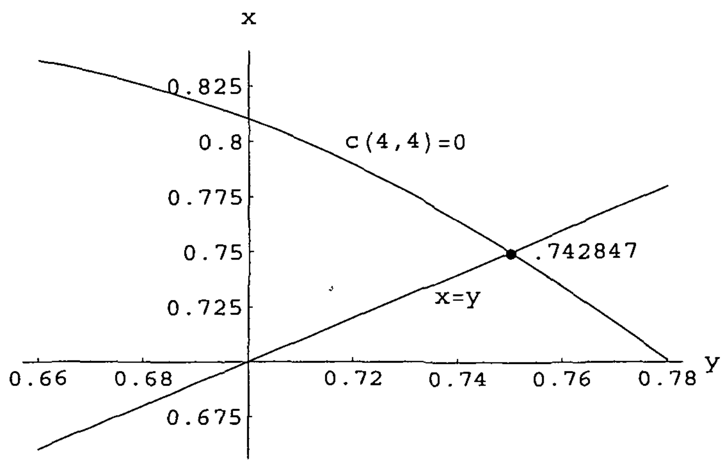


Figure 10

Hence we conclude that $W_{\alpha(x,y)}$ is p.q.h. for $0 < x \leq y$ if and only if $y \in [\frac{2}{3}, 0.737144)$. Thus, $k_2 = 0.737144$.

3.5 Conclusion

For $0 < x \leq y \leq \frac{3}{4}$, let $\alpha(x, y)$ denote the sequence with Bergman tail given by $\alpha(x, y) : \sqrt{x}\,, \sqrt{y}\,, \sqrt{\frac{3}{4}}\,, \sqrt{\frac{4}{5}}\,, \ldots$. Then,

(A) there exists an interval (k_1, k_2) about the point $\frac{2}{3}$ such that for every $y \in (k_1, k_2)$ and $0 < x \leq y$, the weighted shift operator $W_{\alpha(x,y)}$ is p.q.h.

(B) for $y \leq \frac{29}{46} = 0.630435$ there exists $0 < x \leq y$ such that $W_{\alpha(x,y)}$ is not p.q.h.

(C) for $y > 0.737144$ there exists $0 < x \leq y$ such that $W_{\alpha(x,y)}$ is not p.q.h.

(D) $W_{\alpha(x,y)}$ is p.q.h. for $0 < x \leq y$ if and only if $y \in [k_1, k_2)$, where $k_1 = \frac{29}{46} = 0.630435$ and $k_2 = 0.737144$, correct upto six places after the decimal.

Chapter 4

Finite rank perturbation of 2-hyponormal weighted shifts

4.1 Introduction

In this chapter we consider hyponormal weighted shift W_α, and then a finite rank perturbation of W_α, say $W_{\alpha[j\ x]}$ where the j^{th} weight α_j is perturbed to x. In [24, Theorem 2.1], it has been shown that a non zero finite rank perturbation of a subnormal shift is never subnormal unless the perturbation occurs at initial weight α_0. However, this is not necessarily true for a 2-hyponormal shifts as shown in [24, Example 3.1(ii)]. So the question being addressed in this chapter is as follows:

Given a 2-hyponormal weighted shift W_α and $j = 0, 1, 2, \dots$ does there always exist $\epsilon > 0$ such that for $x \in (\alpha_j - \epsilon, \alpha_j + \epsilon)$, $W_{\alpha[j\ x]}$ is again 2-hyponormal?

Here we propose a set of sufficient conditions under which there exists $\epsilon > 0$ such that for $x \in (\alpha_j - \epsilon, \alpha_j + \epsilon)$, $W_{\alpha[j\ x]}$ will again be 2-hyponormal. We also specify conditions under which there exists $\epsilon > 0$ such that for all $x \in (\alpha_j - \epsilon, \alpha_j + \epsilon)$ and $x \neq \alpha_j$, $W_{\alpha[j\ x]}$ is not 2-hyponormal; that is, conditions under which slight perturbation of the weight α_j makes the perturbed shift non 2-hyponormal. We further prove that in such cases the perturbed shift $W_{\alpha[j\ x]}$ will however be at least quadratic hyponormal.

4.2 About 2-hyponormality

A bounded linear operator T on a complex Hilbert space H is said to be 2-hyponormal if the operator matrix

$$\begin{pmatrix} [T^*,T] & [T^{*2},T] \\ [T^*,T^2] & [T^{*2},T^2] \end{pmatrix}$$

is positive on $H \oplus H$

Theorem 4.2.1. *[12] Let W_α be the weighted shift with positive weight sequence $\alpha := \{\alpha_n\}_{n=0}^{\infty}$, on the space $\ell^2(\mathbb{Z}_+)$. Then the following are equivalent:*

1. *T is 2-hyponormal.*
2. *The matrix $\left(\left\langle [T^{*j},T^i]e_{n+j}, e_{n+i} \right\rangle \right)_{i,j=1}^{2}$ is positive for all $n \geq -1$*
3. *The matrix $\left(\beta_n^2\beta_{n+i+j}^2 - \beta_{n+i}^2\beta_{n+j}^2 \right)_{i,j=1}^{2}$ is positive for all $n \geq 0$ where $\beta_0 = 1$ and $\beta_n = \alpha_0, \quad, \alpha_{n-1}$ $(n \geq 1)$*
4. *The Hankel matrix $\left(\beta_{n+i+j-2}^2 \right)_{i,j=1}^{3}$ is positive for all $n \geq 0$*

Example 4.2.1. *If $\alpha_n = \sqrt{\frac{n+1}{n+2}}$ $(\forall n \geq 0)$, then W_α with weight sequence $\alpha := \{\alpha_n\}_{n=0}^{\infty}$ is 2-hyponormal. In fact, in [24, Example 3.1] it has been shown that $W_{\alpha[1\ x]}$ is 2-hyponormal if and only if $\frac{63-\sqrt{129}}{80} \leq x \leq \frac{24}{35}$. This example highlights the fact that it is possible to perturb the 1^{th} weight of α and still keep the perturbed shift 2-hyponormal.*

4.3 NASC for 2-hyponormality of finite rank perturbation

Lemma 4.3.1. *Let W_α be a hyponormal weighted shift with positive weight sequence $\alpha := \{\alpha_n\}_{n=0}^{\infty}$. Then W_α is 2-hyponormal if and only if*

$$\Delta_n := \begin{pmatrix} u_i & \sqrt{w_i} \\ \sqrt{w_i} & v_{i+1} \end{pmatrix} \geq 0$$

for all $n \geq 0$ (Here u_i, v_i, w_i are defined as in section 2 2 and 3 2)

Proof.

$$\begin{aligned}
&W_\alpha \text{ is 2-hyponormal} \\
&\Leftrightarrow \begin{pmatrix} [W_\alpha^*, W_\alpha] & [W_\alpha^{*2}, W_\alpha] \\ [W_\alpha^*, W_\alpha^2] & [W_\alpha^{*2}, W_\alpha^2] \end{pmatrix} \geq 0 \\
&\Leftrightarrow \langle [W_\alpha^*, W_\alpha]x, x\rangle + \left\langle [W_\alpha^{*2}, W_\alpha]y, x\right\rangle + \langle [W_\alpha^*, W_\alpha^2]x, y\rangle \\
&\quad + \left\langle [W_\alpha^{*2}, W_\alpha^2]y, y\right\rangle \geq 0 \, (\forall x = (x_i) \text{ and } y = (y_i) \text{ in } \ell^2(\mathbb{Z}_+)) \\
&\Leftrightarrow \sum_{i=0}^{\infty} \left(u_i|x_i|^2 + \sqrt{w_i}(x_i\bar{y}_{i+1} + \bar{x}_i y_{i+1}) + v_i|y_i|^2\right) \geq 0 \, (\forall x, y \in \ell^2(\mathbb{Z}_+)) \\
&\Leftrightarrow v_0|y_0|^2 + \sum_{i=0}^{\infty} \left\langle \Delta_i \begin{pmatrix} x_i \\ y_{i+1} \end{pmatrix}, \begin{pmatrix} x_i \\ y_{i+1} \end{pmatrix} \right\rangle \geq 0 \, (\forall x, y \in \ell^2(\mathbb{Z}_+)) \\
&\Leftrightarrow \Delta_i \geq 0 \; (\forall i \geq 0).
\end{aligned}$$

(4.3 1)

□

We are now ready to state our results in this chapter.

Theorem 4.3.2. *Let W_α be a 2-hyponormal weighted shift with weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$. Let the 0^{th} weight α_0 be slightly perturbed to say x, and let $W_{\alpha[0\,x]}$ denote the perturbed shift with weight sequence $\{\alpha_n'\}$ given by $\alpha_0' = x$, $\alpha_n' = \alpha_n$ for $n > 0$. Then there exists $\varepsilon > 0$ such that $W_{\alpha[0\,x]}$ is 2-hyponormal for all*

$x \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$. That is, for a slight perturbation of the 0^{th} weight α_0, the perturbed shift still remains 2-hyponormal.

Proof. Here,

$$u'_n = \begin{cases} x^2, & \text{if } n = 0 \\ \alpha_1^2 - x^2, & \text{if } n = 1 \\ u_n, & \text{if } n \geq 2 \end{cases}$$

$$v'_n = \begin{cases} x^2\alpha_1^2, & \text{if } n = 0 \\ \alpha_1^2\alpha_2^2, & \text{if } n = 1 \\ \alpha_2^2\alpha_3^2 - x^2\alpha_1^2, & \text{if } n = 2 \\ v_n, & \text{if } n \geq 3 \end{cases}$$

$$w'_n = \begin{cases} x^2\alpha_1^4, & \text{if } n = 0 \\ \alpha_1^2(\alpha_2^2 - x^2)^2, & \text{if } n = 1 \\ w_n, & \text{if } n \geq 2 \end{cases}$$

Thus, if $\Delta'_n := \begin{pmatrix} u'_n & \sqrt{w'_n} \\ \sqrt{w'_n} & v'_{n+1} \end{pmatrix}$, then $W_{\alpha[0:x]}$ is 2-hyponormal if $\Delta'_n \geq 0$ for all $n \geq 0$.

Now $\Delta'_n = \Delta_n \geq 0$ for $n \geq 2$. Thus we only need to check the positivity of Δ'_0 and Δ'_1.

$$det\Delta'_0 = \alpha_1^2x^2(\alpha_2^2 - x^2) \geq 0 \text{ for all } 0 < x \leq \alpha_1. \text{ So } \Delta'_0 \geq 0.$$

Let $f(x) := det\Delta'_1$. Then

$$f(x) = x^2[\alpha_1^2(\alpha_2^2 - \alpha_1^2) - \alpha_2^2(\alpha_3^2 - \alpha_1^2)] + \alpha_1^2\alpha_2^2\alpha_3^2.$$

If $det\Delta_1 > 0$ then $f(\alpha_0) = det\Delta_1 > 0$ and so by continuity of f, there exists $\varepsilon > 0$ such that $f(x) > 0$ for all $x \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$.

But suppose $det\Delta_1 = 0$. Then $f(\alpha_0) = 0$. Also

$$f'(x) = 2x[\alpha_1^2(\alpha_2^2 - \alpha_1^2) - \alpha_2^2(\alpha_3^2 - \alpha_1^2)] < 0 \text{ for all } x.$$

Thus, the continuous function f is decreasing and as $f(\alpha_0) = 0$, we conclude that there exists $\varepsilon > 0$ such that $f(x) > 0$ for $x \in (\alpha_0 - \varepsilon, \alpha_0)$ and $f(x) < 0$ for

$x \in (\alpha_0, \alpha_0 + \varepsilon)$. So if $det\Delta_1 = 0$ then there exists $\varepsilon > 0$ such that $\Delta'_1 \geq 0$ for $x \in (\alpha_0 - \varepsilon, \alpha_0)$ but $\Delta'_1 \ngeq 0$ for $x \in (\alpha_0, \alpha_0 + \varepsilon)$

We can thus, give the following conclusion

1. If $det\Delta_1 > 0$ then there exists $\varepsilon > 0$ such that $W_{\alpha[0\ x]}$ is 2-hyponormal for all $x \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$.

2. If $det\Delta_1 = 0$ then there exists $\varepsilon > 0$ such that $W_{\alpha[0\ x]}$ is 2-hyponormal for all $x \in (\alpha_0 - \varepsilon, \alpha_0)$ but $W_{\alpha[0\ x]}$ is not 2-hyponormal for $x \in (\alpha_0, \alpha_0 + \varepsilon)$

□

Following a similar line of argument, we now give an exhaustive set of conditions under which perturbation of the i^{th} weight of a 2-hyponormal weighted shift again keeps the perturbed shift as 2-hyponormal.

Theorem 4.3.3. *Let $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ be a strictly increasing positive weight sequence and W_α be a 2-hyponormal weighted shift. Choose any i from $0, 1, 2,$ Then (referring to notations already introduced) we have:*

(a) *If either $det\Delta_{i+1} = 0$ or $det\Delta_{i-1} = 0$ on the one hand, and $det\Delta_i = 0$ or $det\Delta_{i-2} = 0$ on the other, then there exists $\epsilon > 0$ such that for all x in the deleted neighborhood $(\alpha_i - \epsilon, \alpha_i + \epsilon)$ of α_i, $W_{\alpha[i\ x]}$ is not 2-hyponormal.*

(b) *If $det\Delta_{i+1} > 0, det\Delta_{i-1} > 0$ but either $det\Delta_i = 0$ or $det\Delta_{i-2} = 0$, then there always exist $\epsilon > 0$ such that for $x \in (\alpha_i - \epsilon, \alpha_i), W_{\alpha[i\ x]}$ is not 2-hyponormal, and for $x \in (\alpha_i, \alpha_i + \epsilon), W_{\alpha[i\ x]}$ is 2-hyponormal.*

(c) *If either $det\Delta_{i+1} = 0$ or $det\Delta_{i-1} = 0$, but $det\Delta_i > 0, det\Delta_{i-2} > 0$, then there always exist $\epsilon > 0$ such that for $x \in (\alpha_i - \epsilon, \alpha_i), W_{\alpha[i\,x]}$ is 2-hyponormal, and for $x \in (\alpha_i, \alpha_i + \epsilon), W_{\alpha[i\,x]}$ is not 2-hyponormal.*

(d) *If $det\Delta_j > 0$ for $j = i-2, i-1, i, i+1$, then there exists $\epsilon > 0$ such that for $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon), W_{\alpha[i\,x]}$ is again 2-hyponormal.*

Note: For $i = 0$, we need consider only Δ_i and Δ_{i+1}; and for $i = 1$, we need consider only Δ_{i-1}, Δ_i and Δ_{i+1}.

Proof. Choose i arbitrarily and fix it. Here $\alpha[i : x]$ denotes the perturbed weight sequence α where the i^{th} weight α_i is replaced by x, for $\alpha_{i-1} < x < \alpha_{i+1}$. Note $\alpha_{-n} := 0$ for $n \in \mathbb{N}$. Then by Lemma 4.3.1, $W_{\alpha[i\,x]}$ will be 2-hyponormal if and only if $\Delta'_n \geq 0$ for all $n \in \mathbb{N}$, where

$$\Delta'_n := \begin{pmatrix} u'_n & \sqrt{w'_n} \\ \sqrt{w'_n} & v'_{n+1} \end{pmatrix},$$

$$u'_n = \begin{cases} u_n, & \text{for } n < i, \\ x^2 - \alpha_{i-1}^2, & \text{for } n = i; \\ \alpha_{i+1}^2 - x^2, & \text{for } n = i+1, \\ u_n, & \text{for } n \geq i+2, \end{cases}$$

$$v'_n = \begin{cases} v_n, & \text{for } n < i-1; \\ \alpha_{i-1}^2 x^2 - \alpha_{i-2}^2\alpha_{i-3}^2, & \text{for } n = i-1; \\ x^2\alpha_{i+1}^2 - \alpha_{i-1}^2\alpha_{i-2}^2, & \text{for } n = i; \\ \alpha_{i+1}^2\alpha_{i+2}^2 - x^2\alpha_{i-1}^2, & \text{for } n = i+1, \\ \alpha_{i+2}^2\alpha_{i+3}^2 - \alpha_{i+1}^2 x^2, & \text{for } n = i+2, \\ v_n, & \text{for } n \geq i+3, \end{cases}$$

$$w'_n = \begin{cases} w_n, & \text{for } n < i-1; \\ \alpha_{i-1}^2(x^2 - \alpha_{i-2}^2)^2, & \text{for } n = i-1, \\ x^2(\alpha_{i+1}^2 - \alpha_{i-1}^2)^2, & \text{for } n = i; \\ \alpha_{i+1}^2(\alpha_{i+2}^2 - x^2)^2, & \text{for } n = i+1; \\ w_n, & \text{for } n \geq i+2. \end{cases}$$

Clearly $\Delta'_n = \Delta_n$ for all n except for $n = i-2, i-1, i, i+1$. Hence we only need to check the positivity of Δ'_n for these four particular values of n.

(i) To determine x for which $\Delta'_{i+1} \geq 0$.

Let

$$\begin{aligned} h(x) &:= det\Delta'_{i+1} \\ &= x^2\left[\alpha_{i+1}^2(\alpha_{i+2}^2 - \alpha_{i+1}^2) - \alpha_{i+2}^2(\alpha_{i+3}^2 - \alpha_{i+1}^2)\right] + \alpha_{i+1}^2\alpha_{i+2}^2(\alpha_{i+3}^2 - \alpha_{i+2}^2) \end{aligned}$$

As $h(\alpha_i) = det\Delta_{i+1} \geq 0$ so we have the following two cases:

Case I: If $det\Delta_{i+1} > 0$, then h being continuous in x, there exists $\epsilon > 0$ such that $h(x) > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$.

Case II: If $det\Delta_{i+1} = 0$ then we consider $h'(x)$ given by

$$h'(x) = 2x[\alpha_{i+1}^2(\alpha_{i+2}^2 - \alpha_{i+1}^2) - \alpha_{i+2}^2(\alpha_{i+3}^2 - \alpha_{i+1}^2)]$$

As $\alpha_{i+1} < \alpha_{i+2} < \alpha_{i+3}$ so $\alpha_{i+1}^2(\alpha_{i+2}^2 - \alpha_{i+1}^2) < \alpha_{i+2}^2(\alpha_{i+3}^2 - \alpha_{i+1}^2)$. Hence $h'(x) < 0$ for all $x > 0$. In particular, h is a decreasing function at $x = \alpha_i$ and since $h(\alpha_i) = 0$ so there exists $\epsilon > 0$ such that $h(x) > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i)$, and $h(x) < 0$ for all $x \in (\alpha_i, \alpha_i + \epsilon)$.

So we can summarize that if $det\Delta_{i+1} > 0$ then there exists $\epsilon > 0$ such that $\Delta'_{i+1} > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$. Otherwise $\Delta'_{i+1} > 0$ for $x \in (\alpha_i - \epsilon, \alpha_i)$, and $\Delta'_{i+1} < 0$ for $x \in (\alpha_i, \alpha_i + \epsilon)$.

(ii) To determine x for which $\Delta'_i \geq 0$.

Let

$$\begin{aligned} g(x) &:= det\Delta'_i \\ &= -\alpha_{i-1}^2 x^4 + x^2\left[\alpha_{i+1}^2\alpha_{i+2}^2 + \alpha_{i-1}^4 - (\alpha_{i+1}^2 - \alpha_{i-1}^2)^2\right] - \alpha_{i-1}^2\alpha_{i+1}^2\alpha_{i+2}^2 \end{aligned}$$

Now $g(\alpha_i) = det\Delta_i \geq 0$. If $det\Delta_i > 0$ then by continuity of g there exists $\epsilon > 0$ such that $g(x) > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$. On the other hand if $det\Delta_i = 0$ then since

$$g'(x) = 2x[2\alpha_{i-1}^2(\alpha_{i+1}^2 - x^2) + \alpha_{i+1}^2(\alpha_{i+2}^2 - \alpha_{i+1}^2)] > 0 \text{ for all } x > 0,$$

so g is an increasing function at $x = \alpha_i$ Also since $g(\alpha_i) = 0$ so there exists $\epsilon > 0$ such that $g(x) < 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i)$, and $g(x) > 0$ for all $x \in (\alpha_i, \alpha_i + \epsilon)$.

So we can summarize that if $det\Delta_i > 0$ then there exists $\epsilon > 0$ such that $\Delta'_i > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$. Otherwise $\Delta'_i < 0$ for $x \in (\alpha_i - \epsilon, \alpha_i)$, and $\Delta'_i > 0$ for $x \in (\alpha_i, \alpha_i + \epsilon)$.

(iii) To determine x for which $\Delta'_{i-1} \geq 0$

Let

$$\begin{aligned} f(x) &:= det\Delta'_{i-1} \\ &= -\alpha_{i-1}^2 x^4 + x^2[\alpha_{i+1}^2(\alpha_{i-1}^2 - \alpha_{i-2}^2) + 2\alpha_{i-1}^2\alpha_{i-2}^2] + \alpha_{i-1}^2\alpha_{i-2}^4 \end{aligned}$$

As $f(\alpha_i) = det\Delta_{i-1} \geq 0$, so two cases may arise:

Case I: If $det\Delta_{i-1} > 0$, then f being continuous in x, there exists $\epsilon > 0$ such that $f(x) > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$.

Case II: If $det\Delta_{i-1} = 0$ then we consider $f'(x)$ given by

$$f'(x) = 2x[-2\alpha_{i-1}^2 x^2 + \alpha_{i+1}^2(\alpha_{i-1}^2 - \alpha_{i-2}^2) + 2\alpha_{i-1}^2\alpha_{i-2}^2]$$

As,

$$\begin{aligned}
f'(\alpha_i) &= 2\alpha_i[\alpha_{i+1}^2 u_{i-1} - 2\alpha_{i-1}^2(\alpha_i^2 - \alpha_{i-2}^2)] \\
&= \frac{2\alpha_i}{v_i}[\alpha_{i+1}^2 w_{i-1} - 2v_i\alpha_{i-1}^2(\alpha_i^2 - \alpha_{i-2}^2)] \ (\because det\Delta_{i-1} = 0 \Rightarrow u_{i-1}v_i = w_{i-1}) \\
&= -\frac{2\alpha_i\alpha_{i-1}^2}{v_i}(\alpha_i^2 - \alpha_{i-2}^2)[(\alpha_i^2\alpha_{i+1}^2 - \alpha_{i-1}^2\alpha_{i-2}^2) + \alpha_{i-2}^2(\alpha_{i+1}^2 - \alpha_{i-1}^2)] \\
&< 0,
\end{aligned}$$

so the function f is decreasing at $x = \alpha_i$, and hence, there exists $\epsilon > 0$ such that $f(x) > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i)$, and $f(x) < 0$ for all $x \in (\alpha_i, \alpha_i + \epsilon)$. Thus, if $det\Delta_{i-1} > 0$ then there exists $\epsilon > 0$ such that $\Delta'_{i-1} > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$. Otherwise $\Delta'_{i-1} > 0$ for $x \in (\alpha_i - \epsilon, \alpha_i)$, and $\Delta'_{i-1} < 0$ for $x \in (\alpha_i, \alpha_i + \epsilon)$.

(iv) To determine x for which $\Delta'_{i-2} \geq 0$.

Let

$$\begin{aligned}
e(x) &:= det\Delta'_{i-2} \\
&= \alpha_{i-1}^2(\alpha_{i-2}^2 - \alpha_{i-3}^2)x^2 - [\alpha_{i-2}^2\alpha_{i-3}^2(\alpha_{i-2}^2 - \alpha_{i-3}^2) + \alpha_{i-2}^2(\alpha_{i-1}^2 - \alpha_{i-3}^2)^2].
\end{aligned}$$

Now $e(\alpha_i) = det\Delta_{i-2} \geq 0$. If $det\Delta_{i-2} > 0$, then by continuity of e there exists $\epsilon > 0$ such that $e(x) > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$. On the other hand if $det\Delta_{i-2} = 0$ then since

$$e'(x) = 2x\alpha_{i-1}^2(\alpha_{i-2}^2 - \alpha_{i-3}^2) > 0 \ \text{ for all } x > 0,$$

so e is an increasing function at $x = \alpha_i$ Also since $e(\alpha_i) = 0$ so there exists $\epsilon > 0$ such that $e(x) < 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i)$, and $e(x) > 0$ for all $x \in (\alpha_i, \alpha_i + \epsilon)$. So we can summarize that if $det\Delta_{i-2} > 0$ then there exists $\epsilon > 0$ such that

${\Delta'}_{i-2} > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$. Otherwise ${\Delta'}_{i-2} < 0$ for $x \in (\alpha_i - \epsilon, \alpha_i)$, and ${\Delta'}_{i-2} > 0$ for $x \in (\alpha_i, \alpha_i + \epsilon)$.

Considering all the above possibilities, the conclusion of the theorem now follows obviously.

□

4.4 Small perturbation of 2-hyponormal is quadratically hyponormal

We know from [1, 12, 13] that if W_α is 2-hyponormal then it is necessarily quadratically hyponormal. The converse however is not true as is seen from the following example:

Example 4.4.1. *[12] If* $\alpha_0 = \sqrt{\frac{2}{3}}$ *and* $\alpha_n = \sqrt{\frac{n+1}{n+2}}$ $(\forall n \geq 1)$ *then* W_α *is quadratic hyponormal but* W_α *is not 2-hyponormal.*

We have also seen from Theorem 4.3.3 above that under certain conditions there may exist $\epsilon > 0$ such that $W_{\alpha[i:x]}$ is not 2-hyponormal $\forall x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$, $x \neq \alpha_i$. However in this section we show that for each $i = 0, 1, \ldots$ there will always exist $\epsilon > 0$ such that $W_{\alpha[i:x]}$ is quadratically hyponormal $\forall x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$.

Remark 4.4.1. In [24, Theorem 2.3], it was shown that a 2-hyponormal weighted shift remains quadratically hyponormal under small non-zero finite rank perturbations. The proof was based on the definition of positive quadratic hyponormality. In this chapter we give an independent proof for the same result, using a different characterization of quadratic hyponormality.

We begin with a few results and notations introduced in [65]. We recall from Chapter 2, that W_α is q.h. if and only if $D_n(s) \geq 0$ for every $s \geq 0$ and for every $n \geq 0$.

For $s \in \mathbb{R}, x_0, \ldots, x_n \in \mathbb{C}$ and $X_n = (x_0, \ldots, x_n)^T$, define

$$F_n(x_0, \ldots, x_n, s) := \langle D_n(s) X_n, X_n \rangle .$$

Then we have the following lemma:

Lemma 4.4.1. *[65] For $s \in \mathbb{R}, x_0, \ldots, x_n \in \mathbb{C}$, it holds that*

$$F_n(x_0, \ldots, x_n, s) = \sum_{i=0}^{n} u_i |x_i|^2 + \sum_{i=0}^{n-1} s\sqrt{w_i}(x_i \bar{x}_{i+1} + \bar{x}_i x_{i+1}) + \sum_{i=0}^{n} s^2 v_i |x_i|^2$$

Proof.

$$\begin{aligned} F_n(x_0, \ldots, x_n, s) &= \left\langle \begin{pmatrix} q_0 x_0 + r_0 x_1 \\ r_0 x_0 + q_1 x_1 + r_1 x_2 \\ \vdots \\ r_{n-2} x_{n-2} + q_{n-1} x_{n-1} + r_{n-1} x_n \\ r_{n-1} x_{n-1} + q_n x_n \end{pmatrix}, \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \right\rangle \\ &= \sum_{i=0}^{n} q_i |x_i|^2 + \sum_{i=0}^{n-1} r_i (x_i \bar{x}_{i+1} + \bar{x}_i x_{i+1}) \\ &= \sum_{i=0}^{n} u_i |x_i|^2 + \sum_{i=0}^{n-1} s\sqrt{w_i}(x_i \bar{x}_{i+1} + \bar{x}_i x_{i+1}) + \sum_{i=0}^{n} s^2 v_i |x_i|^2 \end{aligned}$$

□

In fact,

$$F_n(x_0, \ldots, x_n, s) = \left\langle \begin{pmatrix} \sum_{i=0}^{n} u_i |x_i|^2 & \sum_{i=0}^{n-1} \sqrt{w_i} x_i \bar{x}_{i+1} \\ \sum_{i=0}^{n-1} \sqrt{w_i} \bar{x}_i x_{i+1} & \sum_{i=0}^{n} v_i |x_i|^2 \end{pmatrix} \begin{pmatrix} 1 \\ s \end{pmatrix}, \begin{pmatrix} 1 \\ s \end{pmatrix} \right\rangle$$

For $x = (x_i) \in \ell^2$, we define $F(x, s) := \langle D(s)x, x \rangle$, so that

$$\begin{aligned} F(x, s) &= \left\langle \begin{pmatrix} \sum_{i=0}^{\infty} u_i |x_i|^2 & \sum_{i=0}^{\infty} \sqrt{w_i} x_i \bar{x}_{i+1} \\ \sum_{i=0}^{\infty} \sqrt{w_i} \bar{x}_i x_{i+1} & \sum_{i=0}^{\infty} v_i |x_i|^2 \end{pmatrix} \begin{pmatrix} 1 \\ s \end{pmatrix}, \begin{pmatrix} 1 \\ s \end{pmatrix} \right\rangle \\ &= s^2 v_0 |x_0|^2 + \sum_{i=0}^{\infty} \left[u_i |x_i|^2 + s\sqrt{w_i}(x_i \bar{x}_{i+1} + \bar{x}_i x_{i+1}) + s^2 v_{i+1} |x_{i+1}|^2 \right] \\ &= s^2 v_0 |x_0|^2 + \sum_{i=0}^{\infty} \left\langle \Delta_i \begin{pmatrix} x_i \\ s x_{i+1} \end{pmatrix}, \begin{pmatrix} x_i \\ s x_{i+1} \end{pmatrix} \right\rangle , \end{aligned}$$

where $\Delta_i = \begin{pmatrix} u_i & \sqrt{w_i} \\ \sqrt{w_i} & v_{i+1} \end{pmatrix}$ for all $i \geq 0$.
The following result is immediately obvious.

Proposition 4.4.2. *[65] Let W_α be a weighted shift with positive weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$. The following are equivalent*

(a) *W_α is quadratically hyponormal.*

(b) *$F(x,s) \geq 0$ for any $s \geq 0$ and $x \in \ell^2$.*

(c) *$F_n(x_0, \ldots, x_n, s) \geq 0$ for any $s \geq 0, x_0, \ldots, x_n \in \mathbb{C}$ and $n \in \mathbb{N}$.*

For $n \geq 1$, we define

$$\tilde{F}_n(x_0, \ldots, x_n) := s^2 v_0 |x_0|^2 + \sum_{i=0}^{n-1} \left\langle \Delta_i \begin{pmatrix} x_i \\ s x_{i+1} \end{pmatrix}, \begin{pmatrix} x_i \\ s x_{i+1} \end{pmatrix} \right\rangle$$

Then by Lemma 4.4.1, we have

$$\begin{aligned} F_n(x_0, \ldots, x_n, s) &= \sum_{i=0}^{n} u_i |x_i|^2 + \sum_{i=0}^{n-1} s\sqrt{w_i}(x_i \bar{x}_{i+1} + \bar{x}_i x_{i+1}) + \sum_{i=0}^{n} s^2 v_i |x_i|^2 \\ &= u_n |x_n|^2 + s^2 v_0 |x_0|^2 + \sum_{i=0}^{n-1} \Big[u_i |x_i|^2 + s\sqrt{w_i}(x_i \bar{x}_{i+1} + \bar{x}_i x_{i+1}) \\ &\quad + s^2 v_{i+1} |x_{i+1}|^2 \Big] \\ &= u_n |x_n|^2 + s^2 v_0 |x_0|^2 + \sum_{i=0}^{n-1} \left\langle \Delta_i \begin{pmatrix} x_i \\ s x_{i+1} \end{pmatrix}, \begin{pmatrix} x_i \\ s x_{i+1} \end{pmatrix} \right\rangle \\ &= u_n |x_n|^2 + \tilde{F}_n(x_0, \ldots, x_n, s) \end{aligned}$$

Since $\{\|\Delta_i\|\}_{i=0}^{\infty}$ is bounded, so $\tilde{F}_n(x_0, \ldots, x_n, s)$ converges pointwise to $F(x,s)$ for each $s \in \mathbb{R}$ and $x = (x_i) \in \ell^2$, and so

$$\lim_{n \to \infty} F_n(x_0, \ldots, x_n, s) = \lim_{n \to \infty} \tilde{F}_n(x_0, \ldots, x_n, s) = F(x,s).$$

In view of this, we can say that W_α is quadratically hyponormal if $\tilde{F}_n(x_0, \ldots, x_n, s) \geq 0$ for all $s \geq 0, x_0, \ldots, x_n \in \mathbb{C}$ and $n \in \mathbb{N}$

Theorem 4.4.3. *(Rank 1 perturbation)*

Let $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ be a strictly increasing positive weight sequence and W_α be a 2-hyponormal weighted shift For any arbitrarily fixed $i = 0, 1, 2, \ldots$ and $\alpha_{i-1} < x < \alpha_{i+1}$, let $\alpha[i\ x]$ denote the weight sequence α with the i^{th} weight α_i replaced by x Then there exists $\epsilon > 0$ such that $W_{\alpha[i\ x]}$ is quadratically hyponormal for $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$

Proof Let $\Delta'_n, u'_n, v'_n, w'_n$ be defined (as in the earlier section) with respect to $W_{\alpha[i\ x]}$ For $n \in \mathbb{N}$ $x_0, \ldots, x_n \in \mathbb{C}$ and $s \geq 0$, define

$$\tilde{F}_n(x_0, \ldots, x_n, s) = s^2 v'_0 |x_0|^2 + \sum_{j=0}^{n-1} \left\langle \Delta'_j \begin{pmatrix} x_j \\ s x_{j+1} \end{pmatrix}, \begin{pmatrix} x_j \\ s x_{j+1} \end{pmatrix} \right\rangle$$

Then $W_{\alpha[i\ x]}$ is quadratically hyponormal if $\tilde{F}_n(x_0, \ldots, x_n, s) \geq 0$ for all $s \geq 0, n \in \mathbb{N}$ and $x_0, \ldots, x_n \in \mathbb{C}$ As $\Delta'_j = \Delta_j \geq 0$ for all $j \geq i + 2$, we only need to check the positivity of $\tilde{F}_n(x_0, \ldots, x_n, s)$ for $1 \leq n \leq i + 2$

For $1 \leq n \leq i + 2$ define

$$A'_n = \begin{pmatrix} s^2 v'_0 + u'_0 & \sqrt{w'_0} & 0 & \cdots & 0 & 0 \\ \sqrt{w'_0} & v'_1 + \frac{u'_1}{s^2} & \frac{\sqrt{w'_1}}{s^2} & \cdots & 0 & 0 \\ 0 & \frac{\sqrt{w'_1}}{s^2} & \frac{v'_2}{s^2} + \frac{u'_2}{s^4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{v'_{n-1}}{s^{2(n-2)}} + \frac{u'_{n-1}}{s^{2(n-1)}} & \frac{\sqrt{w'_{n-1}}}{s^{2(n-1)}} \\ 0 & 0 & 0 & \cdots & \frac{\sqrt{w'_{n-1}}}{s^{2(n-1)}} & \frac{v'_n}{s^{2(n-1)}} \end{pmatrix}$$

If $X_n = (x_0, s x_1, \ldots, s^n x_n)^T$, then $\tilde{F}_n(x_0, \ldots, x_n, s) = \langle A'_n X_n, X_n \rangle$

Claim: $A'_n > 0$ for all $1 \leq n \leq i + 2$

For $2 \le n \le i+2$ define

$$B'_n := \begin{pmatrix} s^2 v'_0 + u'_0 & \sqrt{w'_0} & 0 & \dots & 0 \\ \sqrt{w'_0} & v'_1 + \frac{u'_1}{s^2} & \frac{\sqrt{w'_1}}{s^2} & \dots & 0 \\ 0 & \frac{\sqrt{w'_1}}{s^2} & \frac{v'_2}{s^2} + \frac{u'_2}{s^4} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{v'_{n-1}}{s^{2(n-2)}} + \frac{u'_{n-1}}{s^{2(n-1)}} \end{pmatrix}$$

Define

$$\varphi_n(x) := \begin{cases} s^2 v'_0 + u'_0, & \text{for } n = 1; \\ det B'_n, & \text{for } 2 \le n \le i+2, \end{cases}$$

and

$$\psi_n(x) := det A'_n \ \text{ for } 1 \le n \le i+2.$$

For $1 \le n \le i+1$, we have

$$\varphi_{n+1}(x) = \psi_n(x) + \frac{u'_n}{s^{2n}} \varphi_n(x)$$

and

$$\begin{aligned} \psi_{n+1}(x) &= \frac{v'_{n+1}}{s^{2n}} \varphi_{n+1}(x) - \frac{w'_n}{s^{4n}} \varphi_n(x) \\ &= \frac{v'_{n+1}}{s^{2n}} \psi_n(x) + \frac{\varphi_n(x)}{s^{4n}} det \Delta'_n \end{aligned}$$

Clearly $\varphi_1(x) > 0$ for all $\alpha_{i-1} < x < \alpha_{i+1}$. So first we show that $\psi_1(x) > 0$ for all $\alpha_{i-1} < x < \alpha_{i+1}$.

We have $\psi_1(x) = s^2 v'_0 v'_1 + (u'_0 v'_1 - w'_0)$, and

$$u'_0 = \begin{cases} x^2, & \text{if } i = 0; \\ \alpha_0^2, & \text{if } i \ge 1. \end{cases}$$

$$v'_1 = \begin{cases} \alpha_1^2 \alpha_2^2, & \text{if } i = 0; \\ x^2 \alpha_2^2, & \text{if } i = 1; \\ \alpha_1^2 x^2, & \text{if } i = 2; \\ \alpha_1^2 \alpha_2^2, & \text{if } i \ge 3. \end{cases}$$

$$w'_0 = \begin{cases} x^2 \alpha_1^4, & \text{if } i = 0; \\ \alpha_0^2 x^4, & \text{if } i = 1; \\ \alpha_0^2 \alpha_1^4, & \text{if } i \ge 2. \end{cases}$$

Therefore, $det\Delta'_0 = u'_0 v'_1 - w'_0 = \begin{cases} x^2\alpha_1^2(\alpha_2^2 - \alpha_1^2), & \text{if } i = 0; \\ \alpha_0^2 x^2(\alpha_2^2 - x^2), & \text{if } i = 1; \\ \alpha_0^2\alpha_1^2(x^2 - \alpha_1^2), & \text{if } i = 2; \\ \alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2), & \text{if } i \geq 3. \end{cases}$

So $\psi_1(x) > 0$ for all $\alpha_{i-1} < x < \alpha_{i+1}$.

Now $\varphi_1(x), \psi_1(x) > 0$ gives $\varphi_2(x) > 0$ for all $\alpha_{i-1} < x < \alpha_{i+1}$.

Also

$$\psi_2(\alpha_i) = \frac{v_2}{s^2}\psi_1(\alpha_i) + \frac{\varphi_1(\alpha_i)}{s^4} det\Delta_1 > 0,$$

since $\varphi_1(\alpha_i) > 0, \psi_1(\alpha_i) > 0$ and $det\Delta_1 \geq 0$. So by continuity of ψ_2, there exists $\epsilon > 0$ such that $\psi_2(x) > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$.

Repeating the same argument we can conclude that there exists $\epsilon > 0$ such that $\varphi_n(x), \psi_n(x) > 0$ for all $1 \leq n \leq i + 2$ and $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$. In other words, $A'_n > 0$ for all $1 \leq n \leq i + 2$ and $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$. This completes the proof. $\square$

Theorem 4.4.4. *(Rank 2 perturbation)*

Let $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ be a strictly increasing positive weight sequence and W_α be a 2-hyponormal weighted shift. Then for any integers $0 \leq i < j$ there exists $\epsilon > 0$ such that $W_{\alpha[(i:x),(j:y)]}$ is quadratically hyponormal for $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$ and $y \in (\alpha_j - \epsilon, \alpha_j + \epsilon)$. Here $\alpha[(i : x), (j : y)]$ is the perturbation of the weight sequence α where the weights α_i and α_j are replaced by x and y respectively.

Proof. **Step 1:** Consider $\alpha[i : x]$ for $\alpha_{i-1} < x < \alpha_{i+1}$. Then by Theorem 4.4.3 there exists $\epsilon > 0$ such that $W_{\alpha[(i:x)]}$ is quadratically hyponormal for $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$.

Step 2: Take $\alpha_{j-1} < y < \alpha_{j+1}$ and $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$ and consider $\alpha[(i : x), (j : y)]$. Define

$$\Delta''_n := \begin{pmatrix} u''_n & \sqrt{w''_n} \\ \sqrt{w''_n} & v''_{n+1} \end{pmatrix},$$

$$u''_n := \begin{cases} u'_n, & \text{for } n < j; \\ y^2 - \xi_1^2, & \text{for } n = j; \\ \alpha_{j+1}^2 - y^2, & \text{for } n = j+1; \\ u_n, & \text{for } n \geq j+2, \end{cases}$$

$$v''_n := \begin{cases} v'_n, & \text{for } n < j-1; \\ \xi_1^2 y^2 - \xi_2^2 \xi_3^2, & \text{for } n = j-1; \\ y^2 \alpha_{j+1}^2 - \xi_1^2 \xi_2^2, & \text{for } n = j; \\ \alpha_{j+1}^2 \alpha_{j+2}^2 - y^2 \xi_1^2, & \text{for } n = j+1; \\ \alpha_{j+2}^2 \alpha_{j+3}^2 - \alpha_{j+1}^2 y^2, & \text{for } n = j+2; \\ v_n, & \text{for } n \geq j+3, \end{cases}$$

$$w''_n := \begin{cases} w'_n, & \text{for } n < j-1; \\ \xi_1^2 (y^2 - \xi_2^2)^2, & \text{for } n = j-1; \\ y^2 (\alpha_{j+1}^2 - \xi_1^2)^2, & \text{for } n = j; \\ \alpha_{j+1}^2 (\alpha_{j+2}^2 - y^2)^2, & \text{for } n = j+1; \\ w_n, & \text{for } n \geq j+2, \end{cases}$$

and

$$\xi_k = \begin{cases} x, & \text{if } i = j-k; \\ \alpha_{j-k}, & \text{otherwise.} \end{cases} \quad \text{for } k = 1, 2, 3.$$

Clearly, $\Delta''_n = \Delta_n \geq 0$ for all $n \geq j+2$. So $W_{\alpha[(i \cdot x),(j \cdot y)]}$ is quadratically hyponormal if $A''_n > 0$ for all $1 \leq n \leq j+2$ and $s \geq 0$, where

$$A''_n := \begin{pmatrix} s^2 v''_0 + u''_0 & \sqrt{w''_0} & 0 & \cdots & 0 & 0 \\ \sqrt{w''_0} & v''_1 + \frac{u''_1}{s^2} & \frac{\sqrt{w''_1}}{s^2} & \cdots & 0 & 0 \\ 0 & \frac{\sqrt{w''_1}}{s^2} & \frac{v''_2}{s^2} + \frac{u''_2}{s^4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{v''_{n-1}}{s^{2(n-2)}} + \frac{u''_{n-1}}{s^{2(n-1)}} & \frac{\sqrt{w''_{n-1}}}{s^{2(n-1)}} \\ 0 & 0 & 0 & \cdots & \frac{\sqrt{w''_{n-1}}}{s^{2(n-1)}} & \frac{v''_n}{s^{2(n-1)}} \end{pmatrix}$$

For $2 \leq n \leq j+2$, define

$$B''_n := \begin{pmatrix} s^2 v''_0 + u''_0 & \sqrt{w''_0} & 0 & \cdots & 0 \\ \sqrt{w''_0} & v''_1 + \frac{u''_1}{s^2} & \frac{\sqrt{w''_1}}{s^2} & \cdots & 0 \\ 0 & \frac{\sqrt{w''_1}}{s^2} & \frac{v''_2}{s^2} + \frac{u''_2}{s^4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{v''_{n-1}}{s^{2(n-2)}} + \frac{u''_{n-1}}{s^{2(n-1)}} \end{pmatrix}$$

Define

$$\varphi_n^x(y) := \begin{cases} s^2 v''_0 + u''_0, & \text{for } n = 1; \\ det B''_n, & \text{for } 2 \le n \le j+2, \end{cases}$$

and

$$\psi_n^x(y) := det A''_n \ \text{ for} 1 \le n \le j+2.$$

Then for $1 \le n \le j+1$, we have

$$\varphi_{n+1}^x(y) = \psi_n^x(y) + \frac{u''_n}{s^{2n}} \varphi_n^x(y)$$

and

$$\psi_{n+1}^x(y) = \frac{v''_n}{s^{2n}} \psi_n^x(y) + \frac{\varphi_n^x(y)}{s^{4n}} det\Delta''_n$$

But $\varphi_{n+1}^x(\alpha_j) = \varphi_n(x) > 0$ and $\psi_{n+1}^x(\alpha_j) = \psi_n(x) > 0$ for $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$. Hence for all $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$ and $y \in (\alpha_j - \epsilon, \alpha_j + \epsilon)$ we have $\varphi_{n+1}^x(y), \psi_{n+1}^x(y) > 0$, which implies that $A''_n > 0$ as desired.

(Note that for $n > i+1$ we define $\varphi_{n+1}(x) = \psi_n(x) + \frac{u_n}{s^{2n}} \varphi_n(x)$ and $\psi_{n+1}(x) = \frac{v_n}{s^{2n}} \psi_n(x) + \frac{\varphi_n(x)}{s^{4n}} det\Delta_n$) □

Then the following theorem is obvious

Theorem 4.4.5. *Let $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ be a strictly increasing positive weight sequence and W_α be a 2-hyponormal weighted shift. Then for any $n \in \mathbb{N}$ and integers i_j with $0 \le i_1 < \cdots < i_n$ there exists $\epsilon > 0$ such that $W_{\alpha[(i_1 : t_1), \ldots, (i_n : t_n)]}$ is quadratically hyponormal for $t_j \in (\alpha_j - \epsilon, \alpha_j + \epsilon)$. Here $\alpha[(i_1 : t_1), \ldots, (i_n : t_n)]$ is the perturbation of the weight sequence α where the weight α_{i_j} is replaced by t_j for $j = 1, \ldots, n$.*

4.5 Examples of rank one perturbations

Example 4.5.1. *Consider the 2-hyponormal weighted shift W_α with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ given by $\alpha_n = \sqrt{\frac{3n+1}{3n+2}}$ $(n \ge 0)$. Then $det\Delta_n > 0$ $\forall n$. So* *by Theorem 2.1.(d) perturbation to each and every weight is possible for this weight sequence. That is, for* $i = 0, 1, 2,$ *there exists* $\epsilon > 0$ *such that* $W_{\alpha[i\ x]}$ *is again 2-hyponormal if* $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$.

Example 4.5.2. *Consider the 2-hyponormal weighted shift* W_α *with weight sequence* $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ *given by* $\alpha_0 = \sqrt{\frac{35}{52}}$, $\alpha_n = \sqrt{\frac{3n+1}{3n+2}}$ $(n \geq 1)$. *Here* $det\Delta_1 = 0$ *and* $det\Delta_n > 0$ *for* $n \neq 1$. *Thus there exists* $\epsilon > 0$ *such that*

(i) *for* $x \in (\alpha_0 - \epsilon, \alpha_0], W_{\alpha[0\ x]}$ *is again 2-hyponormal but for* $x \in (\alpha_0, \alpha_0 + \epsilon), W_{\alpha[0\ x]}$ *is not 2-hyponormal.*

(ii) *for* $x \in (\alpha_1 - \epsilon, \alpha_1], W_{\alpha[1\ x]}$ *is not 2-hyponormal but for* $x \in (\alpha_1, \alpha_1 + \epsilon), W_{\alpha[0\ x]}$ *is again 2-hyponormal.*

(iii) *for* $n \geq 2, W_{\alpha[n\ x]}$ *is 2-hyponormal for all* $x \in (\alpha_n - \epsilon, \alpha_n + \epsilon)$.

Example 4.5.3. *Consider the 2-hyponormal weighted shift* W_α *with weight sequence* $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ *given by* $\alpha_0 = \sqrt{\frac{1}{2}}, \alpha_1 = \sqrt{\frac{280}{341}}$, $\alpha_n = \sqrt{\frac{3n+1}{3n+2}}$ $(n \geq 2)$. *Here* $det\Delta_2 = 0$ *and* $det\Delta_n > 0$ *for* $n \neq 2$ *Thus there exists* $\epsilon > 0$ *such that*

(i) *for* $i = 1, 3, W_{\alpha[i\ x]}$ *is 2-hyponormal if* $x \in (\alpha_i - \epsilon, \alpha_i)$, *but is not 2-hyponormal if* $x \in (\alpha_i, \alpha_i + \epsilon)$.

(ii) *for* $i = 2, 4, W_{\alpha[i\ x]}$ *is not 2-hyponormal if* $x \in (\alpha_i - \epsilon, \alpha_i)$, *but is 2-hyponormal if* $x \in (\alpha_i, \alpha_i + \epsilon)$

(iii) *for* $i = 0$ *or* $i > 4, W_{\alpha[i\ x]}$ *is 2-hyponormal if* $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$

Example 4.5.4. *Consider the 2-hyponormal weighted shift* W_α *with weight sequence* $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ *given by* $\alpha_0 = \sqrt{\frac{1}{2}}, \alpha_1 = \sqrt{\frac{580}{689}}, \alpha_2 = \sqrt{\frac{715}{812}}$, $\alpha_n = \sqrt{\frac{3n+1}{3n+2}}$ $(n \geq 3)$. *Here* $det\Delta_2 = 0$, $det\Delta_3 = 0$ *and* $det\Delta_n > 0$ *for* $n \neq 2, 3$ *Thus there exists* $\epsilon > 0$ *such that*

(i) *for* $i = 2, 3, 4, W_{\alpha[i\ x]}$ *is not 2-hyponormal if* x *is in the deleted* $\epsilon-$*neighborhood of* α_i

(ii) *for* $i = 1, W_{\alpha[i\ x]}$ *is 2-hyponormal if* $x \in (\alpha_i - \epsilon\ \alpha_i)$, *but is not 2-hyponormal if* $x \in (\alpha_i, \alpha_i + \epsilon)$

(iii) *for* $i = 5, W_{\alpha[i\ x]}$ *is not 2-hyponormal if* $x \in (\alpha_i - \epsilon, \alpha_i)$, *but is 2-hyponormal if* $x \in (\alpha_i, \alpha_i + \epsilon)$

(iv) *for* $i = 0$ *or* $i > 5, W_{\alpha[i\ x]}$ *is 2-hyponormal if* $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$

Example 4.5.5. *Consider the 2-hyponormal weighted shift* W_α *with recursive weight sequence* α $\alpha_0, \alpha_1, \quad , \alpha_{k-2}, (\alpha_{k-1}, \alpha_k, \alpha_{k+1})^\wedge$ *Since* $u_n v_{n+1} - w_n = 0$ $(\forall\ n \geq k)$, *therefore* $det\Delta_n = 0$ $(\forall\ n \geq k)$ *and so by Theorem 2 1.(a) perturbation to the weights* α_n *is not possible for* $n \geq k$

Chapter 5

Perturbation of 2-variable hyponormal shift

5.1 Introduction

In Chapter 4 we have addressed the question of finite rank perturbation of 2-hyponormal weighted shift. Till now we have only considered the unilateral weighted shift W_α on $\ell^2(\mathbb{Z}_+)$. In this chapter we initiate a parallel discussion for the 2-variable weighted shift on $\ell^2(\mathbb{Z}_+^2)$. For a unilateral weighted shift W_α it is well known that W_α is hyponormal if and only if $|\alpha_n| \leq |\alpha_{n+1}|$ for all n. Hence for a strictly increasing weight sequence, any slight perturbation of the i^{th} weight still retains the hyponormality property for the perturbed shift. "Is the same true for a two variable weighted shift?" The answer is negative as is shown in the work done in this chapter. We also frame a set of positivity conditions which can completely determine hyponormality of the perturbed shift.

5.2 Statement of problem

Consider double indexed positive bounded sequences $\{\alpha_k\}, \{\beta_k\} \in \ell^\infty(\mathbb{Z}_+^2), k \equiv (k_1, k_2) \in \mathbb{Z}_+^2$ and let $\{e_k\}_{k\in\mathbb{Z}_+^2}$ be the orthonormal basis for $\ell^2(\mathbb{Z}_+^2)$. The 2-variable weighted shift $T = (T_1, T_2)$ is defined by

$$T_1e_k = \alpha_k e_{k+\varepsilon_1}, T_2e_k = \beta_k e_{k+\varepsilon_2}$$

where $\varepsilon_1 = (1,0)$ and $\varepsilon_2 = (0,1)$. Here we assume that T_1, T_2 commute. Thus

$$T_1T_2 = T_2T_1 \Longleftrightarrow \beta_{k+\varepsilon_1}\alpha_k = \alpha_{k+\varepsilon_2}\beta_k \text{ for all } k \in \mathbb{Z}_+^2.$$

Given $k \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, the moments of T of order k are

$$\gamma_k := \begin{cases} 1 & \text{if } k_1 = 0 = k_2 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 \alpha_{(0,k_2)}^2 \cdots \alpha_{(k_1-1,k_2)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1 \end{cases}$$

A multivariable weighted shift can be defined in an entirely similar way.

A 2-variable weighted shift T is horizontally flat if $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}\ \forall k_1, k_2 \geq 1$; vertically flat if $\beta_{(k_1,k_2)} = \beta_{(1,1)}\ \forall k_1, k_2 \geq 1$; flat if it is horizontally flat and vertically flat; symmetrically flat if T is flat and $\alpha_{(1,1)} = \beta_{(1,1)}$.

By [13, Definition 1.3, Definition 1.4] and [13, Theorem 6.1] we have the following results:

Theorem 5.2.1. *T is hyponormal if and only if*

$$\Delta_k := \begin{pmatrix} \alpha_{k+\varepsilon_1}^2 - \alpha_k^2 & \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \alpha_k\beta_k \\ \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \alpha_k\beta_k & \beta_{k+\varepsilon_2}^2 - \beta_k^2 \end{pmatrix} \geq 0 \ \ (\forall\, k \in \mathbb{Z}_+^2)$$

Theorem 5.2.2. *T is weakly hyponormal if and only if*

$$\left\langle \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix} \begin{pmatrix} x \\ \lambda x \end{pmatrix}, \begin{pmatrix} x \\ \lambda x \end{pmatrix} \right\rangle \geq 0 \ \ (\forall x \in \ell^2(\mathbb{Z}_+^2)\, and\, \lambda \in \mathbb{C}).$$

If we have a pair of unilateral weighted shifts W_α and W_β, then by defining $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ for all $k_1, k_2 \in \mathbb{Z}_+$, we can get a 2-variable weighted shift $T = (T_1, T_2)$. Under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ and

$\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$, we have $T_1 \cong I \otimes W_\alpha$ and $T_2 \cong W_\beta \otimes I$. In general, T is said to be of tensor form if $T \cong (I \otimes W_\alpha, W_\beta \otimes I)$.

Let $\mathcal{M}_1 := \bigvee\{e_{(k_1,k_2)} : k_2 \geq 1\}$ and $\mathcal{N}_1 := \bigvee\{e_{(k_1,k_2)} : k_1 \geq 1\}$. By [28, Definition 1.2] the core of a 2-variable weighted shift T is $c(T) := T|_{\mathcal{M}_1 \cap \mathcal{N}_1}$.

Coming back to the discussion of perturbation of $T = (T_1, T_2)$, first of all since commutativity has to be preserved, hence if one of the weights is perturbed, some other weights in adjacent blocks will also need to be perturbed. Ideally we try to keep the number of perturbations minimum, and see if we can still preserve hyponormality or atleast weak hyponormality.

We begin our investigation by considering T to be of tensor type. That is, consider strictly increasing positive weight sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ and let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ ($\forall\, k_2 \in \mathbb{Z}_+$) and $\beta_{(k_1,k_2)} := \beta_{k_2}$ ($\forall\, k_1 \in \mathbb{Z}_+$). Let $T = (T_1, T_2)$ be hyponormal with $T_1T_2 = T_2T_1$. Let $\alpha_{(0,0)}$ be slightly perturbed to a new weight x. To preserve commutativity of T_1 and T_2, we replace $\beta_{(0,0)}$ by $y = \frac{x\beta_0}{\alpha_0}$. If $\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$ denotes the perturbed shift, then we investigate if $\tilde{T}$ is still hyponormal. We note that $\tilde{T}$ is not of tensor type. The weight diagrams of T and $\tilde{T}$ are shown in Figure 11 and Figure 12.

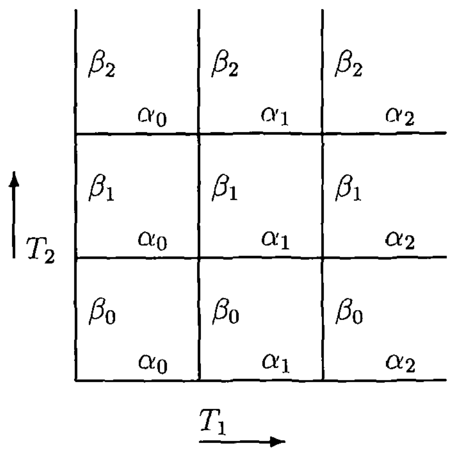


Figure 11

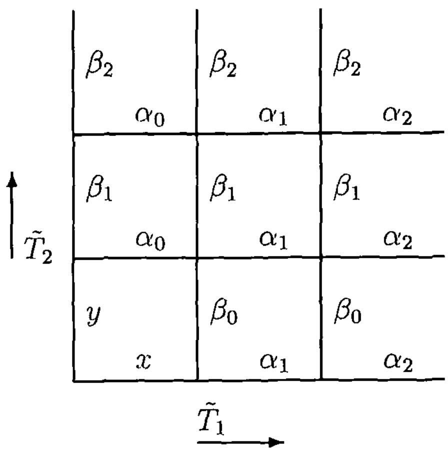


Figure 12

In view of Theorem 5.2.1, we need to check the positivity of $\tilde{\Delta}_k$ for $k = (0,0)$, where

$$\tilde{\Delta}_k := \begin{pmatrix} \tilde{\alpha}^2_{k+\varepsilon_1} - \tilde{\alpha}^2_k & \tilde{\alpha}_{k+\varepsilon_2}\tilde{\beta}_{k+\varepsilon_1} - \tilde{\alpha}_k\tilde{\beta}_k \\ \tilde{\alpha}_{k+\varepsilon_2}\tilde{\beta}_{k+\varepsilon_1} - \tilde{\alpha}_k\tilde{\beta}_k & \tilde{\beta}^2_{k+\varepsilon_2} - \tilde{\beta}^2_k \end{pmatrix}$$

and

$$\tilde{\alpha}_k = \begin{cases} x, & \text{if } k = (0,0) \\ \alpha_k, & \text{if } k \neq (0,0) \end{cases}$$

$$\tilde{\beta}_k = \begin{cases} y, & \text{if} k = (0,0) \\ \beta_k, & \text{if } k \neq (0,0) \end{cases}$$

Now $\tilde{\Delta}_{(0,0)} = \begin{pmatrix} \alpha_1^2 - x^2 & \alpha_0\beta_0 - xy \\ \alpha_0\beta_0 - xy & \beta_1^2 - y^2 \end{pmatrix} = \begin{pmatrix} \alpha_1^2 - x^2 & \alpha_0\beta_0 - \frac{x^2\beta_0}{\alpha_0} \\ \alpha_0\beta_0 - \frac{x^2\beta_0}{\alpha_0} & \beta_1^2 - \frac{x^2\beta_0^2}{\alpha_0^2} \end{pmatrix}$

If $f(x) := det\tilde{\Delta}_{(0,0)}$, then f is a continuous function of x. Also $f(\alpha_0) = det\Delta_{(0,0)} = (\alpha_1^2 - \alpha_0^2)(\beta_1^2 - \beta_0^2) > 0$. Hence there exists a neighbourhood N of α_0 such that $f(x) > 0$ $\forall x \in N$. Thus, there exists $\delta > 0$ such that for all $x \in (\alpha_0 - \delta, \alpha_0 + \delta)$, $\tilde{\Delta}_{(0,0)} \geq 0$.

Hence, $\tilde{T}$ remains hyponormal for a slight perturbation of $\alpha_{(0,0)}$.

Next we consider an arbitrary hyponormal 2-variable shift $T = (T_1, T_2)$ with positive weight sequence $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}_+^2}$ and $\beta = \{\beta_k\}_{k\in\mathbb{Z}_+^2}$. As T_1 and T_2 are both hyponormal, so we have $\alpha_k \leq \alpha_{k+\varepsilon_1}$ and $\beta_k \leq \beta_{k+\varepsilon_2}$ $\forall\, k \in \mathbb{Z}_+^2$. We assume strict inequality so as to make perturbation possible on either side. In section 5.3 we consider perturbation of the weight $\alpha_{(0,k_2)}$ for any $k_2 \in \mathbb{Z}_+$. In section 5.4 we consider perturbation of the weight $\alpha_{(k_1,0)}$ for any $k_1 \in \mathbb{Z}_+$. We try to minimise the number of necessary perturbations of adjacent weights. We also try not to disturb the weights at the core of T.

5.3 Perturbation of the weight $\alpha_{(0,k_2)}$

Here we consider $k_2 \geq 2$. The cases of $k_2 = 0$ and $k_2 = 1$ are addressed in Remark 5.3.1. We begin with a 2-variable hyponormal shift $T = (T_1, T_2)$ with weight sequences $\alpha = \{\alpha_k\}_{k \in \mathbb{Z}_+^2}$ and $\beta = \{\beta_k\}_{k \in \mathbb{Z}_+^2}$. As T_1 and T_2 are both hyponormal, so we have $\alpha_k \leq \alpha_{k+\varepsilon_1}$ and $\beta_k \leq \beta_{k+\varepsilon_2}$ for all $k \in \mathbb{Z}_+^2$. We assume strict inequality so to make perturbation possible on either side.

Let $\alpha_{(0,k_2)}$ be slightly perturbed to a new weight x. To preserve commutativity, we change $\beta_{(0,k_2)}$ to $y = \frac{x\beta_{(1,k_2)}}{\alpha_{(0,k_2+1)}}$ and $\beta_{(0,k_2-1)}$ to $t = \frac{\alpha_{(0,k_2-1)}\beta_{(1,k_2-1)}}{x}$.

We will investigate under what condition it is possible for the perturbed shift $\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$ to still remain hyponormal. The corresponding weight diagram is shown in Figure 13.

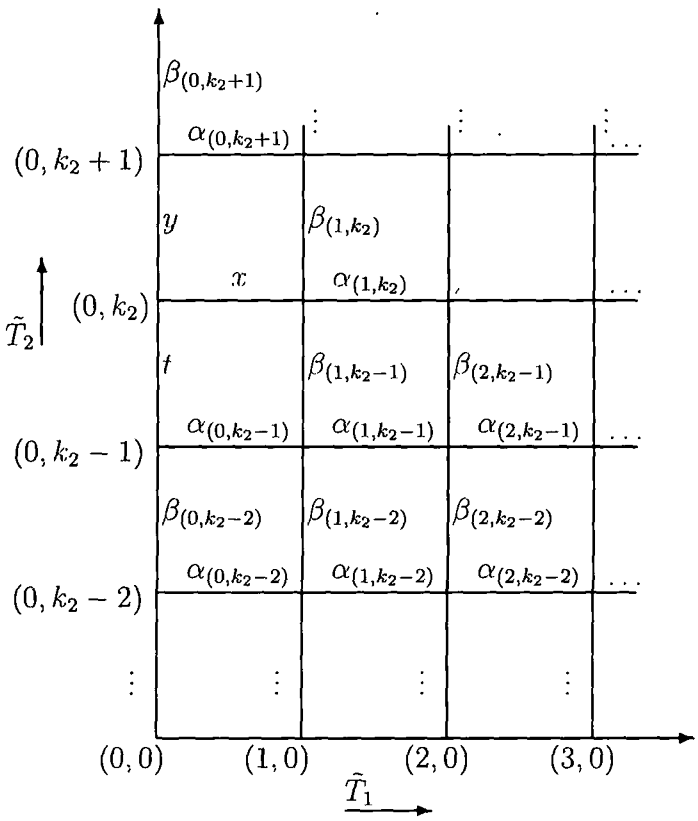


Figure 13

By Theorem 5.2.1 for hyponormality of $\tilde{T}$ we must have positivity of $\tilde{\Delta}_{(0,k_2-2)}$, $\tilde{\Delta}_{(0,k_2-1)}$ and $\tilde{\Delta}_{(0,k_2)}$.

Claim: $t < y$

We have

$$\begin{aligned} t < y &\Leftrightarrow \frac{\alpha_{(0,k_2-1)}\beta_{(1,k_2-1)}}{x} < \frac{x\beta_{(1,k_2)}}{\alpha_{(0,k_2+1)}} \\ &\Leftrightarrow \alpha_{(0,k_2-1)}\alpha_{(0,k_2+1)}\frac{\beta_{(1,k_2-1)}}{\beta_{(1,k_2)}} < x^2 \\ &\Leftrightarrow \frac{\beta_{(0,k_2-1)}\alpha_{(0,k_2)}}{\beta_{(1,k_2-1)}}\frac{\alpha_{(0,k_2)}\beta_{(1,k_2)}}{\beta_{(0,k_2)}}\frac{\beta_{(1,k_2-1)}}{\beta_{(1,k_2)}} < x^2 \\ &\Leftrightarrow \frac{\beta_{(0,k_2-1)}}{\beta_{(0,k_2)}}\alpha^2_{(0,k_2)} < x^2 \end{aligned}$$

But as $\frac{\beta_{(0,k_2-1)}}{\beta_{(0,k_2)}} < 1$, so $\frac{\beta_{(0,k_2-1)}}{\beta_{(0,k_2)}}\alpha^2_{(0,k_2)} < \alpha^2_{(0,k_2)}$.
Hence we can choose a suitable $\delta > 0$ such that $\frac{\beta_{(0,k_2-1)}}{\beta_{(0,k_2)}}\alpha^2_{(0,k_2)} < x^2$ i.e, $t < y$, for $x \in (\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)} + \delta)$. Similarly we have, $\beta_{(0,k_2-2)} < t < y < \beta_{(0,k_2+1)}$.

Positivity of $\tilde{\Delta}_{(0,k_2-2)}$:

$$\tilde{\Delta}_{(0,k_2-2)} = \begin{pmatrix} \alpha^2_{(1,k_2-2)} - \alpha^2_{(0,k_2-2)} & \alpha_{(0,k_2-1)}\beta_{(1,k_2-2)} - \alpha_{(0,k_2-2)}\beta_{(0,k_2-2)} \\ \alpha_{(0,k_2-1)}\beta_{(1,k_2-2)} - \alpha_{(0,k_2-2)}\beta_{(0,k_2-2)} & t^2 - \beta^2_{(0,k_2-2)} \end{pmatrix}$$

$$\begin{aligned} f_1(x) &:= det\tilde{\Delta}_{(0,k_2-2)} \\ &= \left(\alpha^2_{(1,k_2-2)} - \alpha^2_{(0,k_2-2)}\right)\left(\frac{\alpha^2_{(0,k_2-1)}\beta^2_{(1,k_2-1)}}{x^2} - \beta^2_{(0,k_2-2)}\right) \\ &\quad - \left(\alpha_{(0,k_2-1)}\beta_{(1,k_2-2)} - \alpha_{(0,k_2-2)}\beta_{(0,k_2-2)}\right)^2 \end{aligned}$$

If $det\Delta_{(0,k_2-2)} > 0$, then by continuity of f_1 and the fact that $f_1(\alpha_{(0,k_2)}) = det\Delta_{(0,k_2-2)} > 0$ we have $f_1(x) > 0 \; \forall x \in (\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)} + \delta)$, and suitable $\delta > 0$. Hence $\tilde{\Delta}_{(0,k_2-2)} \geq 0$ for all such x.
But if $det\Delta_{(0,k_2-2)} = 0$ then as $f_1'(x) = -\frac{2}{x}\alpha^2_{(0,k_2-1)}\beta^2_{(1,k_2-1)}\left(\alpha^2_{(1,k_2-2)} - \alpha^2_{(0,k_2-2)}\right)$, so $f'(\alpha_{(0,k_2)}) < 0$. As such the function f_1 is decreasing at $\alpha_{(0,k_2)}$ and hence there

exists $\delta > 0$ such that $f_1(x) > 0$ for $x \in (\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)})$, and $f_1(x) < 0$ for $x \in (\alpha_{(0,k_2)}, \alpha_{(0,k_2)} + \delta)$.

Thus, if $det\Delta_{(0,k_2-2)} = 0$ then $\tilde{\Delta}_{(0,k_2-2)} \geq 0$ for $x \in (\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)})$, but $\tilde{\Delta}_{(0,k_2-2)} \not\geq 0$ for $x \in (\alpha_{(0,k_2)}, \alpha_{(0,k_2)} + \delta)$.

Positivity of $\tilde{\Delta}_{(0,k_2-1)}$:

$$\begin{aligned}
\tilde{\Delta}_{(0,k_2-1)} &= \begin{pmatrix} \alpha^2_{(1,k_2-1)} - \alpha^2_{(0,k_2-1)} & x\beta_{(1,k_2-1)} - \alpha_{(0,k_2-1)}y \\ x\beta_{(1,k_2-1)} - \alpha_{(0,k_2-1)}y & y^2 - t^2 \end{pmatrix} \\
&= \begin{pmatrix} \alpha^2_{(1,k_2-1)} - \alpha^2_{(0,k_2-1)} & x\beta_{(1,k_2-1)} - \frac{\alpha^2_{(0,k_2-1)}\beta_{(1,k_2-1)}}{x} \\ x\beta_{(1,k_2-1)} - \frac{\alpha^2_{(0,k_2-1)}\beta_{(1,k_2-1)}}{x} & \frac{\beta^2_{(1,k_2)}x^2}{\alpha^2_{(0,k_2+1)}} - \frac{\alpha^2_{(0,k_2-1)}\beta^2_{(1,k_2-1)}}{x^2} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
f_2(x) &:= det\tilde{\Delta}_{(0,k_2-1)} \\
&= \frac{1}{\alpha^2_{(0,k_2+1)}}\Bigg[x^2\left\{\beta^2_{(1,k_2)}(\alpha^2_{(1,k_2-1)} - \alpha^2_{(0,k_2-1)}) - \alpha^2_{(0,k_2+1)}\beta^2_{(1,k_2-1)}\right\} \\
&\quad + 2\alpha^2_{(0,k_2-1)}\alpha^2_{(0,k_2+1)}\beta^2_{(1,k_2-1)} - \frac{\alpha^2_{(0,k_2-1)}\alpha^2_{(1,k_2-1)}\alpha^2_{(0,k_2+1)}\beta^2_{(1,k_2-1)}}{x^2}\Bigg]
\end{aligned}$$

If $det\Delta_{(0,k_2-1)} > 0$ then $f_2(\alpha_{(0,k_2)}) > 0$ and hence by continuity of f_2, $f_2(x) > 0$ in a neibhourhood of $\alpha_{(0,k_2)}$. Thus there exists $\delta > 0$ such that $\tilde{\Delta}_{(0,k_2-1)} \geq 0$ for all $x \in (\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)} + \delta)$.

If $det\Delta_{(0,k_2-1)} = 0$ then $f_2(\alpha_{(0,k_2)}) = 0$, so

$$\alpha^2_{(0,k_2)}\lambda = \frac{\alpha^2_{(0,k_2-1)}\alpha^2_{(1,k_2-1)}\alpha^2_{(0,k_2+1)}\beta^2_{(1,k_2-1)}}{\alpha^2_{(0,k_2)}} - 2\alpha^2_{(0,k_2-1)}\alpha^2_{(0,k_2+1)}\beta^2_{(1,k_2-1)} \quad (5.3.1)$$

where $\lambda = \beta^2_{(1,k_2)}(\alpha^2_{(1,k_2-1)} - \alpha^2_{(0,k_2-1)}) - \alpha^2_{(0,k_2+1)}\beta^2_{(1,k_2-1)}$.

Also,

$$f_2'(x) = \frac{2}{\alpha^2_{(0,k_2+1)}x}\left[x^2\lambda + \frac{\alpha^2_{(0,k_2-1)}\alpha^2_{(1,k_2-1)}\alpha^2_{(0,k_2+1)}\beta^2_{(1,k_2-1)}}{x^2}\right].$$

Therefore

$$f_2'(\alpha_{(0,k_2)}) = \frac{4\,\alpha^2_{(0,k_2-1)}\beta^2_{(1,k_2-1)}}{\alpha^3_{(0,k_2)}}\left(\alpha^2_{(1,k_2-1)} - \alpha^2_{(0,k_2)}\right) \text{ (using (5.3.1))}$$
$$\begin{cases} = 0, & \text{if } \alpha_{(1,k_2-1)} = \alpha_{(0,k_2)} \\ > 0, & \text{if } \alpha_{(1,k_2-1)} > \alpha_{(0,k_2)} \\ < 0, & \text{if } \alpha_{(1,k_2-1)} < \alpha_{(0,k_2)}. \end{cases}$$

Thus, if $det\Delta_{(0,k_2-1)} = 0$ then

1. if $\alpha_{(1,k_2-1)} = \alpha_{(0,k_2)}$ then $\tilde{\Delta}_{(0,k_2-1)} \geq 0 \;\; \forall\, x \in \left(\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)} + \delta\right)$;
2. if $\alpha_{(1,k_2-1)} > \alpha_{(0,k_2)}$ then $\tilde{\Delta}_{(0,k_2-1)} \geq 0$ for $x \in \left(\alpha_{(0,k_2)}, \alpha_{(0,k_2)} + \delta\right)$ but $\tilde{\Delta}_{(0,k_2-1)} \ngeq 0$ for $x \in \left(\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)}\right)$;
3. if $\alpha_{(1,k_2-1)} < \alpha_{(0,k_2)}$ then $\tilde{\Delta}_{(0,k_2-1)} \geq 0$ for $x \in \left(\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)}\right)$ but $\tilde{\Delta}_{(0,k_2-1)} \ngeq 0$ for $x \in \left(\alpha_{(0,k_2)}, \alpha_{(0,k_2)} + \delta\right)$.

<u>Positivity of $\tilde{\Delta}_{(0,k_2)}$:</u>

$$\tilde{\Delta}_{(0,k_2)} = \begin{pmatrix} \alpha^2_{(1,k_2)} - x^2 & \alpha_{(0,k_2+1)}\beta_{(1,k_2)} - xy \\ \alpha_{(0,k_2+1)}\beta_{(1,k_2)} - xy & \beta^2_{(0,k_2+1)} - y^2 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha^2_{(1,k_2)} - x^2 & \alpha_{(0,k_2+1)}\beta_{(1,k_2)} - \frac{x^2\beta_{(1,k_2)}}{\alpha_{(0,k_2+1)}} \\ \alpha_{(0,k_2+1)}\beta_{(1,k_2)} - \frac{x^2\beta_{(1,k_2)}}{\alpha_{(0,k_2+1)}} & \beta^2_{(0,k_2+1)} - \frac{x^2\beta^2_{(1,k_2)}}{\alpha^2_{(0,k_2+1)}} \end{pmatrix}.$$

So

$$\begin{aligned} f_3(x) &:= det\tilde{\Delta}_{(0,k_2)} \\ &= \frac{1}{\alpha^2_{(0,k_2+1)}}\left[x^2\mu + \alpha^2_{(0,k_2+1)}\{\alpha^2_{(1,k_2)}\beta^2_{(0,k_2+1)} - \alpha^2_{(0,k_2+1)}\beta^2_{(1,k_2)}\}\right], \qquad (5.3.2) \end{aligned}$$

where

$$\mu = 2\alpha^2_{(0,k_2+1)}\beta^2_{(1,k_2)} - \alpha^2_{(0,k_2+1)}\beta^2_{(0,k_2+1)} - \alpha^2_{(1,k_2)}\beta^2_{(1,k_2)}$$

As earlier, if $det\Delta_{(0,k_2)} > 0$ then there exists $\delta > 0$ such that $\tilde{\Delta}_{(0,k_2)} \geq 0$ for all $x \in \left(\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)} + \delta\right)$.

But if $det\Delta_{(0,k_2)} = 0$ then $f_3\left(\alpha_{(0,k_2)}\right) = det\Delta_{(0,k_2)} = 0$.

Therefore,

$$\alpha^2_{(0,k_2)}\,\mu = \alpha^2_{(0,k_2+1)}\left(\alpha^2_{(0,k_2+1)}\beta^2_{(1,k_2)} - \alpha^2_{(1,k_2)}\beta^2_{(0,k_2+1)}\right). \tag{5.3.3}$$

Also,

$$f_3'(x) = \frac{2x\mu}{\alpha^2_{(0,k_2+1)}}$$

and so

$$\begin{aligned} f_3'\left(\alpha_{(0,k_2)}\right) &= \frac{2}{\alpha_{(0,k_2)}}\left(\alpha^2_{(0,k_2+1)}\beta^2_{(1,k_2)} - \alpha^2_{(1,k_2)}\beta^2_{(0,k_2+1)}\right) \quad \text{(using (5.3.3))} \\ &\begin{cases} = 0, & \text{if } \alpha_{(0,k_2+1)}\beta_{(1,k_2)} = \alpha_{(1,k_2)}\beta_{(0,k_2+1)} \\ > 0, & \text{if } \alpha_{(0,k_2+1)}\beta_{(1,k_2)} > \alpha_{(1,k_2)}\beta_{(0,k_2+1)} \\ < 0, & \text{if } \alpha_{(0,k_2+1)}\beta_{(1,k_2)} < \alpha_{(1,k_2)}\beta_{(0,k_2+1)}. \end{cases} \end{aligned}$$

Thus if $det\Delta_{(0,k_2)} = 0$ then

1. if $\alpha_{(0,k_2+1)}\beta_{(1,k_2)} = \alpha_{(1,k_2)}\beta_{(0,k_2+1)}$ then there exists $\delta > 0$ such that $\tilde{\Delta}_{(0,k_2)} \geq 0 \ \forall\, x \in \left(\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)} + \delta\right)$;

2. if $\alpha_{(0,k_2+1)}\beta_{(1,k_2)} > \alpha_{(1,k_2)}\beta_{(0,k_2+1)}$ then $\tilde{\Delta}_{(0,k_2)} \geq 0$ for $x \in \left(\alpha_{(0,k_2)}, \alpha_{(0,k_2)} + \delta\right)$ but $\tilde{\Delta}_{(0,k_2)} \ngeq 0$ for $x \in \left(\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)}\right)$;

3. if $\alpha_{(0,k_2+1)}\beta_{(1,k_2)} < \alpha_{(1,k_2)}\beta_{(0,k_2+1)}$ then $\tilde{\Delta}_{(0,k_2)} \geq 0$ for $x \in \left(\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)}\right)$ but $\tilde{\Delta}_{(0,k_2)} \ngeq 0$ for $x \in \left(\alpha_{(0,k_2)}, \alpha_{(0,k_2)} + \delta\right)$.

From the above analysis we can exhaustively determine whether perturbation of $\alpha_{(0,k_2)}$ will again give us a hyponormal shift $\tilde{T}$ or not.

For example if we have the following situation, say:

1. $det\Delta_{(0,k_2-2)} > 0,\ det\Delta_{(0,k_2)} > 0;$

2. $det\Delta_{(0,k_2-1)} = 0$, and $\alpha_{(1,k_2-1)} < \alpha_{(0,k_2)}$.

Then $\tilde{T}$ will still be hyponormal for a slight left perturbation of $\alpha_{(0,k_2)}$, but will not be hyponormal for any right perturbation of $\alpha_{(0,k_2)}$.

Similarly, we have another situation say:

1. $det\Delta_{(0,k_2-2)} = 0;$

2. $det\Delta_{(0,k_2)} = 0$ and $\alpha_{(0,k_2+1)}\beta_{(1,k_2)} > \alpha_{(1,k_2)}\beta_{(0,k_2+1)}$.

Then there exists $\delta > 0$ such that $\tilde{\Delta}_{(0,k_2-2)} \ngeq 0$ for any $x \in \left(\alpha_{(0,k_2)}, \alpha_{(0,k_2)} + \delta\right)$ and $\tilde{\Delta}_{(0,k_2)} \ngeq 0$ for any $x \in \left(\alpha_{(0,k_2)} - \delta, \alpha_{(0,k_2)}\right)$. So $\tilde{T}$ is not hyponormal for any perturbation of $\alpha_{(0,k_2)}$.

Remark 5.3.1. If $k_2 = 1$ then we need to consider only $\tilde{\Delta}_{(0,k_2-1)}$ and $\tilde{\Delta}_{(0,k_2)}$ for positivity. Similarly, if $k_2 = 0$, then we need only consider positivity of $\tilde{\Delta}_{(0,k_2)}$ to check whether the perturbed shift $\tilde{T}$ is still hyponormal or not.

Remark 5.3.2. Since the perturbation do not affect the core of T, so these results can be applied to 2-variable shifts whose core is of tensor type.

5.4 Perturbation of the weight $\alpha_{(k_1,0)}$

For $k_1 \geq 0$, let $\alpha_{(k_1,0)}$ be slightly perturbed to a new weight x. For commutativity we change $\beta_{(k_1,0)}$ to $y = \frac{x\beta_{(k_1,0)}}{\alpha_{(k_1,0)}}$ and $\alpha_{(k_1-1,0)}$ to $z = \frac{\alpha_{(k_1-1,0)}\alpha_{(k_1,0)}}{x}$.

As

$$y < \beta_{(k_1,1)} \Leftrightarrow x < \frac{\beta_{(k_1,1)}}{\beta_{(k_1,0)}}\alpha_{(k_1,0)}$$

so, by keeping x suitably near $\alpha_{(k_1,0)}$, we can preserve the conditions $y < \beta_{(k_1,1)}$ and $\alpha_{(k_1-2,0)} < z < x < \alpha_{(k_1+1,0)}$.

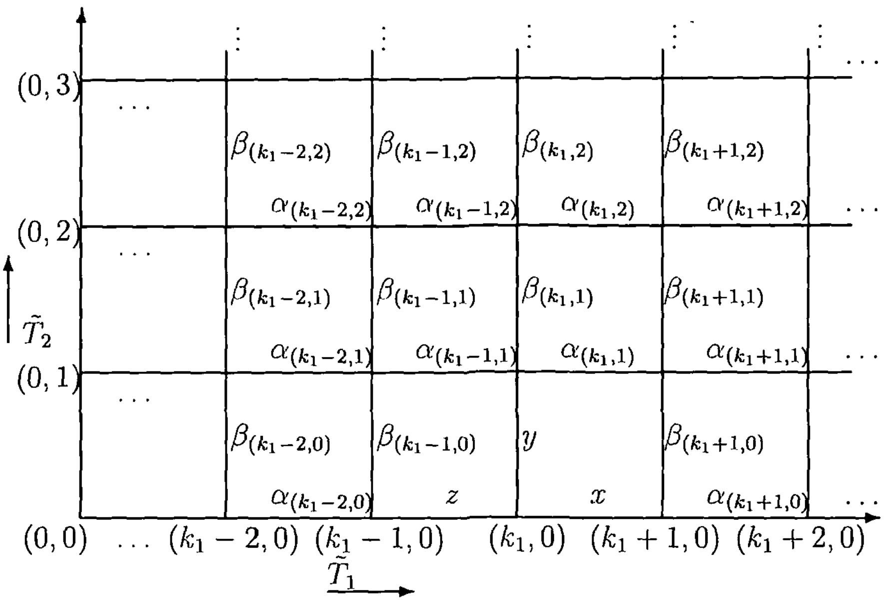


Figure 14

For hyponormality of $\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$, we need to check the positivity of $\tilde{\Delta}_{(k_1-2,0)}$, $\tilde{\Delta}_{(k_1-1,0)}$, $\tilde{\Delta}_{(k_1,0)}$.

Positivity of $\tilde{\Delta}_{(k_1-2,0)}$:

$$\tilde{\Delta}_{(k_1-2,0)} = \begin{pmatrix} z^2 - \alpha^2_{(k_1-2,0)} & \alpha_{(k_1-2,1)}\beta_{(k_1-1,0)} - \alpha_{(k_1-2,0)}\beta_{(k_1-2,0)} \\ \alpha_{(k_1-2,1)}\beta_{(k_1-1,0)} - \alpha_{(k_1-2,0)}\beta_{(k_1-2,0)} & \beta^2_{(k_1-2,1)} - \beta^2_{(k_1-2,0)} \end{pmatrix}$$

We consider

$$\begin{aligned} f_1(x) &:= det\tilde{\Delta}_{(k_1-2,0)} \\ &= \left(\frac{\alpha^2_{(k_1-1,0)}\alpha^2_{(k_1,0)}}{x^2} - \alpha^2_{(k_1-2,0)} \right) \left(\beta^2_{(k_1-2,1)} - \beta^2_{(k_1-2,0)}\right) \\ &\quad - \left(\alpha_{(k_1-2,1)}\beta_{(k_1-1,0)} - \alpha_{(k_1-2,0)}\beta_{(k_1-2,0)}\right)^2 \end{aligned}$$

So,

$$f_1'(x) = -\frac{2\alpha^2_{(k_1-1,0)}\alpha^2_{(k_1,0)}}{x^3} \left(\beta^2_{(k_1-2,1)} - \beta^2_{(k_1-2,0)}\right)$$

$$\therefore f_1'(\alpha_{(k_1,0)}) = -\frac{2\alpha^2_{(k_1-1,0)}}{\alpha_{(k_1,0)}}\left(\beta^2_{(k_1-2,1)} - \beta^2_{(k_1-2,0)}\right) < 0.$$

Case 1: If $det\Delta_{(k_1-2,0)} > 0$ then by continuity of f_1, there exists $\delta > 0$ such that $f_1(x) > 0\ \forall x \in \left(\alpha_{(k_1,0)} - \delta, \alpha_{(k_1,0)} + \delta\right)$. Thus for all such x, $\tilde{\Delta}_{(k_1-2,0)} \geq 0$

Case 2: If $det\Delta_{(k_1-2,0)} = 0$ then since $f_1'\left(\alpha_{(k_1,0)}\right) < 0$ and hence f_1 is decreasing at $\alpha_{(k_1,0)}$, so $\tilde{\Delta}_{(k_1-2,0)} > 0$ for $x \in \left(\alpha_{(k_1,0)} - \delta, \alpha_{(k_1,0)}\right)$ and $\tilde{\Delta}_{(k_1-2,0)} \not\geq 0$ for $x \in \left(\alpha_{(k_1,0)}, \alpha_{(k_1,0)} + \delta\right)$.

Positivity of $\tilde{\Delta}_{(k_1-1,0)}$:

$$\tilde{\Delta}_{(k_1-1,0)} := \begin{pmatrix} x^2 - z^2 & \alpha_{(k_1-1,1)}y - z\beta_{(k_1-1,0)} \\ \alpha_{(k_1-1,1)}y - z\beta_{(k_1-1,0)} & \beta^2_{(k_1-1,1)} - \beta^2_{(k_1-1,0)} \end{pmatrix}$$

We consider

$$\begin{aligned}
f_2(x) &:= det\tilde{\Delta}_{(k_1-1,0)} \\
&= \left(x^2 - z^2\right)\left(\beta^2_{(k_1-1,1)} - \beta^2_{(k_1-1,0)}\right) - \left(\alpha_{(k_1-1,1)}y - z\beta_{(k_1-1,0)}\right)^2 \\
&= \left(x^2 - \frac{\alpha^2_{(k_1-1,0)}\alpha^2_{(k_1,0)}}{x^2}\right)\left(\beta^2_{(k_1-1,1)} - \beta^2_{(k_1-1,0)}\right) \\
&\quad - \left(\alpha_{(k_1-1,1)}\frac{x\beta_{(k_1,0)}}{\alpha_{(k_1,0)}} - \frac{\alpha_{(k_1-1,0)}\alpha_{(k_1,0)}}{x}\beta_{(k_1-1,0)}\right)^2 \\
&= \frac{x^2}{\alpha^2_{(k_1,0)}}\left[\alpha^2_{(k_1,0)}\left(\beta^2_{(k_1-1,1)} - \beta^2_{(k_1-1,0)}\right) - \alpha^2_{(k_1-1,1)}\beta^2_{(k_1,0)}\right] \\
&\quad + 2\alpha_{(k_1-1,0)}\beta_{(k_1,0)}\alpha_{(k_1-1,1)}\beta_{(k_1-1,0)} - \frac{1}{x^2}\left[\alpha^2_{(k_1-1,0)}\alpha^2_{(k_1,0)}\beta^2_{(k_1-1,1)}\right] \\
&= \frac{x^2\mu}{\alpha^2_{(k_1,0)}} + 2\alpha_{(k_1-1,0)}\beta_{(k_1,0)}\alpha_{(k_1-1,1)}\beta_{(k_1-1,0)} - \frac{1}{x^2}\left[\alpha^2_{(k_1-1,0)}\alpha^2_{(k_1,0)}\beta^2_{(k_1-1,1)}\right],
\end{aligned}$$

where $\mu = \alpha^2_{(k_1,0)}\left(\beta^2_{(k_1-1,1)} - \beta^2_{(k_1-1,0)}\right) - \alpha^2_{(k_1-1,1)}\beta^2_{(k_1,0)}$

If $det\Delta_{(k_1-1,0)} = 0$, then $f_2(\alpha_{(k_1,0)}) = 0$. So, $\mu = \alpha^2_{(k_1-1,0)}\left(\beta^2_{(k_1-1,1)} - 2\beta^2_{(k_1,0)}\right)$

Now

$$
\begin{aligned}
f_2'(x) &= \frac{2x\mu}{\alpha^2_{(k_1,0)}} + \frac{2}{x^3}\left[\alpha^2_{(k_1-1,0)}\alpha^2_{(k_1,0)}\beta^2_{(k_1-1,0)}\right] \\
\therefore\ f_2'(\alpha_{(k_1,0)}) &= \frac{2}{\alpha_{(k_1,0)}}\left[\mu + \alpha^2_{(k_1-1,0)}\beta^2_{(k_1-1,0)}\right] \\
&= \frac{4}{\alpha_{(k_1,0)}}\left[\alpha^2_{(k_1-1,0)}\left(\beta^2_{(k_1-1,1)} - \beta^2_{(k_1,0)}\right)\right] \\
&\begin{cases} > 0, & \text{if } \beta_{(k_1-1,1)} > \beta_{(k_1,0)} \\ < 0, & \text{if } \beta_{(k_1-1,1)} < \beta_{(k_1,0)} \\ = 0, & \text{if } \beta_{(k_1-1,1)} = \beta_{(k_1,0)}. \end{cases}
\end{aligned} \tag{5.4.1}
$$

Thus, if $det\Delta_{(k_1-1,0)} = 0$, then

1. if $\beta_{(k_1-1,1)} = \beta_{(k_1,0)}$ then there exists $\delta > 0$ such that $\tilde{\Delta}_{(k_1-1,0)} \geq 0$ for all $x \in \left(\alpha_{(k_1,0)} - \delta, \alpha_{(k_1,0)} + \delta\right)$.

2. if $\beta_{(k_1-1,1)} > \beta_{(k_1,0)}$ then $\tilde{\Delta}_{(k_1-1,0)} \geq 0$ for $x \in \left(\alpha_{(k_1,0)}, \alpha_{(k_1,0)} + \delta\right)$, and $\tilde{\Delta}_{(k_1-1,0)} \ngeq 0$ for $x \in \left(\alpha_{(k_1,0)} - \delta, \alpha_{(k_1,0)}\right)$.

3. if $\beta_{(k_1-1,1)} < \beta_{(k_1,0)}$ then $\tilde{\Delta}_{(k_1-1,0)} \geq 0$ for $x \in \left(\alpha_{(k_1,0)} - \delta, \alpha_{(k_1,0)}\right)$, and $\tilde{\Delta}_{(k_1-1,0)} \ngeq 0$ for $x \in \left(\alpha_{(k_1,0)}, \alpha_{(k_1,0)} + \delta\right)$.

Positivity of $\tilde{\Delta}_{(k_1,0)}$:

$$
\tilde{\Delta}_{(k_1,0)} = \begin{pmatrix} \alpha^2_{(k_1+1,0)} - x^2 & \alpha_{(k_1,1)}\beta_{(k_1+1,0)} - xy \\ \alpha_{(k_1,1)}\beta_{(k_1+1,0)} - xy & \beta^2_{(k_1,1)} - y^2 \end{pmatrix}
$$

$$
\begin{aligned}
f_3(x) &:= \left(\alpha^2_{(k_1+1,0)} - x^2\right)\left(\beta^2_{(k_1,1)} - \frac{x^2\beta^2_{(k_1,0)}}{\alpha^2_{(k_1,0)}}\right) - \left(\alpha_{(k_1,1)}\beta_{(k_1+1,0)} - \frac{x^2\beta_{(k_1,0)}}{\alpha_{(k_1,0)}}\right)^2 \\
&= \left(\alpha^2_{(k_1+1,0)}\beta^2_{(k_1,1)} - \alpha^2_{(k_1,1)}\beta^2_{(k_1+1,0)}\right) \\
&\quad + \frac{x^2}{\alpha^2_{(k_1,0)}}\left(2\alpha_{(k_1,0)}\alpha_{(k_1,1)}\beta_{(k_1+1,0)}\beta_{(k_1,0)} - \beta^2_{(k_1,0)}\alpha^2_{(k_1+1,0)} - \beta^2_{(k_1,1)}\alpha^2_{(k_1,0)}\right)
\end{aligned}
$$

$$= \left(\alpha^2_{(k_1+1,0)}\beta^2_{(k_1,1)} - \alpha^2_{(k_1,1)}\beta^2_{(k_1+1,0)}\right) + \frac{x^2\gamma}{\alpha^2_{(k_1,0)}},$$

where $\gamma = 2\alpha_{(k_1,0)}\alpha_{(k_1,1)}\beta_{(k_1+1,0)}\beta_{(k_1,0)} - \beta^2_{(k_1,0)}\alpha^2_{(k_1+1,0)} - \beta^2_{(k_1,1)}\alpha^2_{(k_1,0)}$.

If $det\Delta_{(k_1,0)} = 0$ then $f_3(\alpha_{(k_1,0)}) = 0$. Therefore

$$\gamma = \alpha^2_{(k_1,1)}\beta^2_{(k_1+1,0)} - \alpha^2_{(k_1+1,0)}\beta^2_{(k_1,1)}.$$

Again,

$$f_3'(x) = \frac{2x\gamma}{\alpha^2_{(k_1,0)}} = \frac{2x}{\alpha^2_{(k_1,0)}}\left(\alpha^2_{(k_1,1)}\beta^2_{(k_1+1,0)} - \alpha^2_{(k_1+1,0)}\beta^2_{(k_1,1)}\right)$$
$$\begin{cases} > 0, & \text{if } \alpha_{(k_1,1)}\beta_{(k_1+1,0)} > \alpha_{(k_1+1,0)}\beta_{(k_1,1)} \\ < 0, & \text{if } \alpha_{(k_1,1)}\beta_{(k_1+1,0)} < \alpha_{(k_1+1,0)}\beta_{(k_1,1)} \\ = 0, & \text{if } \alpha_{(k_1,1)}\beta_{(k_1+1,0)} = \alpha_{(k_1+1,0)}\beta_{(k_1,1)}. \end{cases}$$

From the continuity of f_3 we can make the conclusions:

1. $det\Delta_{(k_1,0)} > 0$ or $\alpha_{(k_1,1)}\beta_{(k_1+1,0)} = \alpha_{(k_1+1,0)}\beta_{(k_1,1)}$ then there exists $\delta > 0$ such that $\tilde{\Delta}_{(k_1,0)} \geq 0$ for all $x \in \left(\alpha_{(k_1,0)} - \delta, \alpha_{(k_1,0)} + \delta\right)$.

2. If $det\Delta_{(k_1,0)} = 0$ then there exists $\delta > 0$ such that

   (i) if $\alpha_{(k_1,1)}\beta_{(k_1+1,0)} > \alpha_{(k_1+1,0)}\beta_{(k_1,1)}$ then $\tilde{\Delta}_{(k_1,0)} \geq 0$ for all $x \in \left(\alpha_{(k_1,0)}, \alpha_{(k_1,0)} + \delta\right)$ and $\tilde{\Delta}_{(k_1,0)} \ngeq 0$ for $x \in \left(\alpha_{(k_1,0)} - \delta, \alpha_{(k_1,0)}\right)$;

   (ii) if $\alpha_{(k_1,1)}\beta_{(k_1+1,0)} < \alpha_{(k_1+1,0)}\beta_{(k_1,1)}$ then $\tilde{\Delta}_{(k_1,0)} \geq 0$ for all $x \in \left(\alpha_{(k_1,0)} - \delta, \alpha_{(k_1,0)}\right)$ and $\tilde{\Delta}_{(k_1,0)} \ngeq 0$ for $x \in \left(\alpha_{(k_1,0)}, \alpha_{(k_1,0)} + \delta\right)$.

From the above analysis we can exhaustively determine whether perturbation of $\alpha_{(k_1,0)}$ will again result in a hyponormal shift $\tilde{T}$ or not.

Remark 5.4.1. For $k_1 = 0$, we only need to consider $\tilde{\Delta}_{(k_1,0)}$ for positivity and for $k_1 = 1$, we need to consider only positivity of $\tilde{\Delta}_{(k_1,0)}$ and $\tilde{\Delta}_{(k_1-1,0)}$.

Remark 5.4.2. Since the perturbation do not affect the core of T, so these results can be applied to 2-variable shifts whose core is of tensor type.

5.5 Perturbation of the weight $\alpha_{(k_1,k_2)}$

In general if we want to perturb a weight $\alpha_{(k_1,k_2)}$ for $k_1 > 0,\ k_2 > 0$, then for commutativity we need to change at least three other adjacent weights. The further perturbation of weights in adjacent blocks are as follows:

1. β_k changes to $y = \frac{\beta_k x}{\alpha_k}$
2. $\alpha_{k-\varepsilon_1}$ changes to $z = \frac{\alpha_{k-\varepsilon_1}\alpha_k}{x}$
3. $\beta_{k-\varepsilon_2}$ changes to $t = \frac{\beta_{k-\varepsilon_2}\alpha_k}{x}$

with the understanding that if $k = (0, k_2)$ then we neglect (2), and if $k = (k_1, 0)$ we neglect (3).

$\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$ is the perturbed shift with weight sequences $\{\tilde{\alpha}_\tau\}_{\tau\in\mathbf{Z}_+^2}$ and $\{\tilde{\beta}_\tau\}_{\tau\in\mathbf{Z}_+^2}$ given as follows:

$$\tilde{\alpha}_\tau = \begin{cases} x, & \text{if } \tau = k \\ z, & \text{if } \tau = k - \varepsilon_1 \\ \alpha_k, & \text{if } \tau \neq k, \tau \neq k - \varepsilon_1 \end{cases} \quad \text{and } \tilde{\beta}_\tau = \begin{cases} y, & \text{if } \tau = k \\ t, & \text{if } \tau = k - \varepsilon_2 \\ \beta_k, & \text{if } \tau \neq k, \tau \neq k - \varepsilon_2 \end{cases}$$

The perturbed weight diagram is given in Figure 15.

As $\alpha_k < \left(\frac{\beta_{k-\epsilon_2}}{\beta_{k-2\epsilon_2}}\right)\alpha_k$, so by keeping $x < \left(\frac{\beta_{k-\epsilon_2}}{\beta_{k-2\epsilon_2}}\right)\alpha_k$ we will preserve the condition $\beta_{k-2\epsilon_2} < t$. Similarly, by keeping x suitably near α_k, we can preserve the conditions $\beta_{k-2\epsilon_2} < t < y < \beta_{k+\epsilon_2}$ and $\alpha_{k-2\epsilon_1} < z < x < \alpha_{k+\epsilon_1}$.

Now for hyponormality of the perturbed shift $\tilde{T}$ it is sufficient to identify the conditions of positivity for the following matrices: $\tilde{\Delta}_{k-2\varepsilon_2}, \tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2}, \tilde{\Delta}_{k-2\varepsilon_1}$, $\tilde{\Delta}_{k-\varepsilon_1}, \tilde{\Delta}_{k-\varepsilon_2}$ and $\tilde{\Delta}_k$, where

$$\tilde{\Delta}_\tau := \begin{pmatrix} \tilde{\alpha}^2_{\tau+\varepsilon_1} - \tilde{\alpha}^2_\tau & \tilde{\alpha}_{\tau+\varepsilon_2}\tilde{\beta}_{\tau+\varepsilon_1} - \tilde{\alpha}_\tau\tilde{\beta}_\tau \\ \tilde{\alpha}_{\tau+\varepsilon_2}\tilde{\beta}_{\tau+\varepsilon_1} - \tilde{\alpha}_\tau\tilde{\beta}_\tau & \tilde{\beta}^2_{\tau+\varepsilon_2} - \tilde{\beta}^2_\tau \end{pmatrix} \geq 0 \quad (\forall \tau \in \mathbb{Z}^2_+).$$

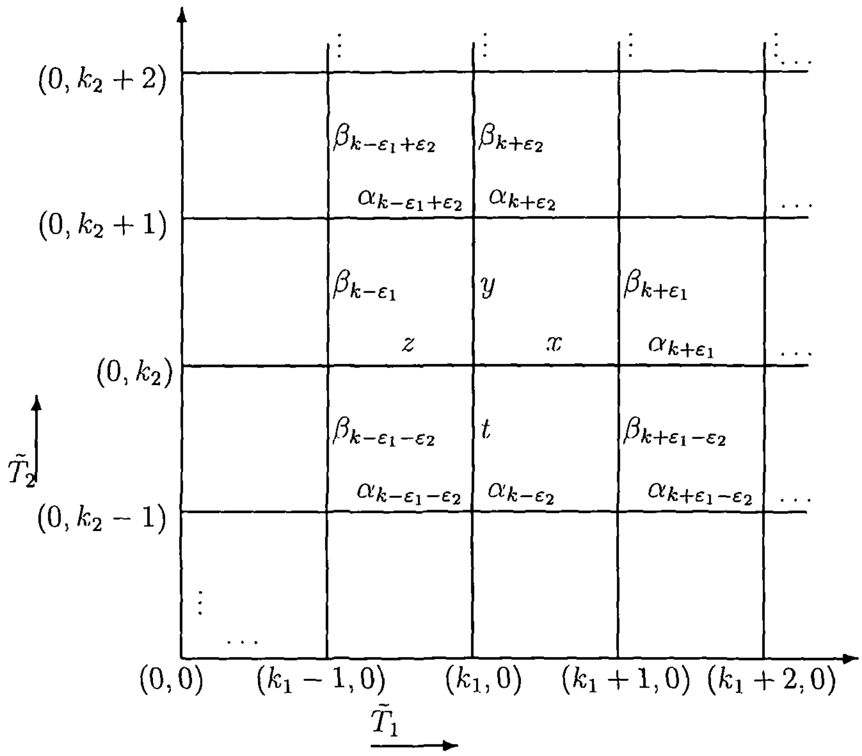


Figure 15

To check positivity of $\tilde{\Delta}_{k-2\varepsilon_2}$ we consider

$$
\begin{aligned}
f_1(x) :=& det\tilde{\Delta}_{k-2\varepsilon_2} \\
=& \left(t^2 - \beta^2_{k-2\varepsilon_2}\right)\left(\alpha^2_{k+\varepsilon_1-2\varepsilon_2} - \alpha^2_{k-2\varepsilon_2}\right) - \left(\alpha_{k-\varepsilon_2}\beta_{k+\varepsilon_1-2\varepsilon_2} - \alpha_{k-2\varepsilon_2}\beta_{k-2\varepsilon_2}\right)^2 \\
=& \left(\frac{\beta^2_{k-\varepsilon_2}\alpha^2_k}{x^2} - \beta^2_{k-2\varepsilon_2}\right)\left(\alpha^2_{k+\varepsilon_1-2\varepsilon_2} - \alpha^2_{k-2\varepsilon_2}\right) \\
& - \left(\alpha_{k-\varepsilon_2}\beta_{k+\varepsilon_1-2\varepsilon_2} - \alpha_{k-2\varepsilon_2}\beta_{k-2\varepsilon_2}\right)^2 \\
\therefore f_1'(x) =& -\frac{2\beta^2_{k-\varepsilon_2}\alpha^2_k}{x^3}\left(\alpha^2_{k+\varepsilon_1-2\varepsilon_2} - \alpha^2_{k-2\varepsilon_2}\right) < 0
\end{aligned}
$$

Now, $f_1(\alpha_k) = det\Delta_{k-2\varepsilon_2} \geq 0$. So by continuity of f_1 we can make the following conclusion <u>C1</u>:

1. If $det\Delta_{k-2\varepsilon_2} > 0$ then there exists $\delta_k > 0$ such that for all $x \in (\alpha_k - \delta_k; \alpha_k + \delta_k)$, $\tilde{\Delta}_{k-2\varepsilon_2} \geq 0$
2. If $det\Delta_{k-2\varepsilon_2} = 0$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_{k-2\varepsilon_2} \geq 0$ for all

$x \in (\alpha_k - \delta_k, \alpha_k)$, and $\tilde{\Delta}_{k-2\varepsilon_2} \not\geq 0$ for $x \in (\alpha_k, \alpha_k + \delta_k)$.

Similarly to check the positivity of $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2}$, we consider

$$\begin{aligned}
f_2(x) :=& det\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \\
=& (\beta^2_{k-\varepsilon_1} - \beta^2_{k-\varepsilon_1-\varepsilon_2})(\alpha^2_{k-\varepsilon_1} - \alpha^2_{k-\varepsilon_1-\varepsilon_2}) - (zt - \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2})^2 \\
=& (\beta^2_{k-\varepsilon_1} - \beta^2_{k-\varepsilon_1-\varepsilon_2})(\alpha^2_{k-\varepsilon_1} - \alpha^2_{k-\varepsilon_1-\varepsilon_2}) \\
&- \left(\frac{\alpha_{k-\varepsilon_1}\alpha_k^2\beta_{k-\varepsilon_2}}{x^2} - \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}\right)^2 \\
f_2'(x) =& \frac{4\alpha_{k-\varepsilon_1}\alpha_k^2\beta_{k-\varepsilon_2}}{x^3}\left(\frac{\alpha_{k-\varepsilon_1}\alpha_k^2\beta_{k-\varepsilon_2}}{x^2} - \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}\right) \\
\therefore f_2'(\alpha_k) =& \frac{4\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2}}{\alpha_k}(\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} - \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}) \\
&\begin{cases} > 0, & \text{if } \alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} > \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2} \\ < 0, & \text{if } \alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} < \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2} \\ = 0, & \text{if } \alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} = \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}. \end{cases}
\end{aligned}$$

From the continuity of f_2 we can make the following conclusion <u>C2</u>:

1. If $det\Delta_{k-\varepsilon_1-\varepsilon_2} > 0$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k + \delta_k)$.

2. If $det\Delta_{k-\varepsilon_1-\varepsilon_2} = 0$ then there exists $\delta_k > 0$ such that

   (i) if $\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} > \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}$, then $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \geq 0$ for all $x \in (\alpha_k, \alpha_k + \delta_k)$, and $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \not\geq 0$ for $x \in (\alpha_k - \delta_k, \alpha_k)$;

   (ii) if $\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} < \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}$, then $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k)$, and $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \not\geq 0$ for $x \in (\alpha_k, \alpha_k + \delta_k)$;

   (iii) if $\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_2} = \alpha_{k-\varepsilon_1-\varepsilon_2}\beta_{k-\varepsilon_1-\varepsilon_2}$, then $\tilde{\Delta}_{k-\varepsilon_1-\varepsilon_2} \geq 0$ for all $x \in (\alpha_k + \delta_k, \alpha_k - \delta_k)$.

For positivity of $\tilde{\Delta}_{k-2\varepsilon_1}$, we consider

$$\begin{aligned}
f_3(x) &:= det\tilde{\Delta}_{k-2\varepsilon_1} \\
&= \left(z^2 - \alpha_{k-2\varepsilon_1}^2\right)\left(\beta_{k-2\varepsilon_1+\varepsilon_2}^2 - \beta_{k-2\varepsilon_1}^2\right) - \left(\alpha_{k-2\varepsilon_1+\varepsilon_2}\beta_{k-\varepsilon_1} - \alpha_{k-2\varepsilon_1}\beta_{k-2\varepsilon_1}\right)^2 \\
&= \left(\frac{\alpha_{k-\varepsilon_1}^2\alpha_k^2}{x^2} - \alpha_{k-2\varepsilon_1}^2\right)\left(\beta_{k-2\varepsilon_1+\varepsilon_2}^2 - \beta_{k-2\varepsilon_1}^2\right) - \left(\alpha_{k-2\varepsilon_1+\varepsilon_2}\beta_{k-\varepsilon_1} - \alpha_{k-2\varepsilon_1}\beta_{k-2\varepsilon_1}\right)^2
\end{aligned}$$

So,

$$\begin{aligned}
f_3'(x) &= \frac{-2\alpha_{k-\varepsilon_1}^2\alpha_k^2}{x^3}\left(\beta_{k-2\varepsilon_1+\varepsilon_2}^2 - \beta_{k-2\varepsilon_1}^2\right) \\
\therefore f_3'(\alpha_k) &= \frac{-2\alpha_{k-\varepsilon_1}^2}{\alpha_k}\left(\beta_{k-2\varepsilon_1+\varepsilon_2}^2 - \beta_{k-2\varepsilon_1}^2\right) < 0.
\end{aligned}$$

Now from the continuity of f_3 we can make the conclusion <u>C3</u>:

1. If $det\Delta_{k-2\varepsilon_1} > 0$ then there exists $\delta_k > 0$ such that for all $x \in \left(\alpha_k - \delta_k, \alpha_k + \delta_k\right)$, $\tilde{\Delta}_{k-2\varepsilon_1} \geq 0$.

2. If $det\Delta_{k-2\varepsilon_1} = 0$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_{k-2\varepsilon_2} \geq 0$ for all $x \in \left(\alpha_k - \delta_k, \alpha_k\right)$, and $\tilde{\Delta}_{k-2\varepsilon_1} \not\geq 0$ for $x \in \left(\alpha_k, \alpha_k + \delta_k\right)$.

For positivity of $\tilde{\Delta}_{k-\varepsilon_2}$, we consider

$$\begin{aligned}
f_4(x) :=& det\tilde{\Delta}_{k-\varepsilon_2} \\
=& \left(\alpha_{k+\varepsilon_1-\varepsilon_2}^2 - \alpha_{k-\varepsilon_2}^2\right)\left(y^2 - t^2\right) - \left(x\beta_{k+\varepsilon_1-\varepsilon_2} - t\alpha_{k-\varepsilon_2}\right)^2 \\
=& \left(\alpha_{k+\varepsilon_1-\varepsilon_2}^2 - \alpha_{k-\varepsilon_2}^2\right)\left(\frac{\beta_k^2 x^2}{\alpha_k^2} - \frac{\beta_{k-\varepsilon_2}^2\alpha_k^2}{x^2}\right) - \left(x\beta_{k+\varepsilon_1-\varepsilon_2} - \frac{\beta_{k-\varepsilon_2}\alpha_k\alpha_{k-\varepsilon_2}}{x}\right)^2 \\
=& \frac{1}{\alpha_k^2}\Bigg[x^2 \left\{\beta_k^2(\alpha_{k+\varepsilon_1-\varepsilon_2}^2 - \alpha_{k-\varepsilon_2}^2) - \alpha_k^2\beta_{k+\varepsilon_1-\varepsilon_2}^2\right\} \\
& + 2\beta_{k-\varepsilon_2}\alpha_k^3\alpha_{k-\varepsilon_2}\beta_{k+\varepsilon_1-\varepsilon_2} - \alpha_{k+\varepsilon_1-\varepsilon_2}^2\frac{\beta_{k-\varepsilon_2}^2\alpha_k^4}{x^2}\Bigg]
\end{aligned}$$

If $det\Delta_{k-\varepsilon_2} = 0$ then $f_4(\alpha_k) = det\Delta_{k-\varepsilon_2} = 0$. Therefore,

$$\lambda := \beta_k^2\left(\alpha_{k+\varepsilon_1-\varepsilon_2}^2 - \alpha_{k-\varepsilon_2}^2\right) - \alpha_k^2\beta_{k+\varepsilon_1-\varepsilon_2}^2 = \beta_{k-\varepsilon_2}^2\left(\alpha_{k+\varepsilon_1-\varepsilon_2}^2 - 2\alpha_k^2\right).$$

Again,

$$\begin{aligned}
f_4'(x) &= \frac{1}{\alpha_k^2}\left[2x\lambda + \frac{2\alpha_k^4\alpha_{k+\varepsilon_1-\varepsilon_2}^2\beta_{k-\varepsilon_2}^2}{x^3}\right] \\
&= \frac{2}{\alpha_k^2 x}\left[x^2\lambda + \frac{\alpha_k^4\alpha_{k+\varepsilon_1-\varepsilon_2}^2\beta_{k-\varepsilon_2}^2}{x^2}\right] \\
f_4'(\alpha_k) &= \frac{2}{\alpha_k}\left[\lambda + \alpha_{k+\varepsilon_1-\varepsilon_2}^2\beta_{k-\varepsilon_2}^2\right] \\
&= \frac{4\beta_{k-\varepsilon_2}^2}{\alpha_k}\left(\alpha_{k+\varepsilon_1-\varepsilon_2}^2 - \alpha_k^2\right) \\
&\begin{cases} > 0, & \text{if } \alpha_{k+\varepsilon_1-\varepsilon_2} > \alpha_k \\ < 0, & \text{if } \alpha_{k+\varepsilon_1-\varepsilon_2} < \alpha_k \\ = 0, & \text{if } \alpha_{k+\varepsilon_1-\varepsilon_2} = \alpha_k. \end{cases}
\end{aligned}$$

Now since f_4 is a continuous function, therefore conclusion C4:

1. If $det\Delta_{k-\varepsilon_2} > 0$ or $\alpha_{k+\varepsilon_1-\varepsilon_2} = \alpha_k$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_{k-\varepsilon_2} \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k + \delta_k)$.

2. If $det\Delta_{k-\varepsilon_2} = 0$ then there exists $\delta_k > 0$ such that

   (i) if $\alpha_{k+\varepsilon_1-\varepsilon_2} > \alpha_k$ then $\tilde{\Delta}_{k-\varepsilon_2} \geq 0$ for all $x \in \left(\alpha_k, \alpha_k + \delta_k\right)$ and $\tilde{\Delta}_{k-\varepsilon_2} \ngeq 0$ for $x \in \left(\alpha_k - \delta_k, \alpha_k\right)$;

   (ii) if $\alpha_{k+\varepsilon_1-\varepsilon_2} < \alpha_k$ then $\tilde{\Delta}_{k-\varepsilon_2} \geq 0$ for all $x \in \left(\alpha_k - \delta_k, \alpha_k\right)$ and $\tilde{\Delta}_{k-\varepsilon_2} \ngeq 0$ for $x \in \left(\alpha_k, \alpha_k + \delta_k\right)$.

For positivity of $\tilde{\Delta}_{k-\varepsilon_1}$, we consider

$$\begin{aligned}
f_5(x) :=& det\tilde{\Delta}_{k-\varepsilon_1} \\
=& \left(x^2 - z^2\right)\left(\beta_{k-\varepsilon_1+\varepsilon_2}^2 - \beta_{k-\varepsilon_1}^2\right) - \left(\alpha_{k-\varepsilon_1+\varepsilon_2}y - \beta_{k-\varepsilon_1}z\right)^2 \\
=& \left(x^2 - \frac{\alpha_{k-\varepsilon_1}^2\alpha_k^2}{x^2}\right)\left(\beta_{k-\varepsilon_1+\varepsilon_2}^2 - \beta_{k-\varepsilon_1}^2\right) - \left(\frac{\alpha_{k-\varepsilon_1+\varepsilon_2}\beta_k x}{\alpha_k} - \frac{\alpha_k\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_1}}{x}\right)^2 \\
=& \frac{x^2}{\alpha_k^2}\left[\alpha_k^2(\beta_{k-\varepsilon_1+\varepsilon_2}^2 - \beta_{k-\varepsilon_1}^2) - \alpha_{k-\varepsilon_1+\varepsilon_2}^2\beta_k^2\right] + 2\alpha_{k-\varepsilon_1+\varepsilon_2}\beta_k\alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_1} \\
& - \frac{1}{x^2}\left(\alpha_{k-\varepsilon_1}^2\alpha_k^2\beta_{k-\varepsilon_1+\varepsilon_2}^2\right).
\end{aligned}$$

If $det\Delta_{k-\varepsilon_1} = 0$ then $f_5(\alpha_k) = det\Delta_{k-\varepsilon_1} = 0$. Therefore

$$\mu := \alpha_k^2(\beta_{k-\varepsilon_1+\varepsilon_2}^2 - \beta_{k-\varepsilon_1}^2) - \alpha_{k-\varepsilon_1+\varepsilon_2}^2\beta_k^2 = \alpha_{k-\varepsilon_1}^2(\beta_{k-\varepsilon_1+\varepsilon_2}^2 - 2\beta_k^2).$$

Now

$$\begin{aligned} f_5'(x) &= \frac{2x\mu}{\alpha_k^2} + \frac{2}{x^3}\alpha_{k-\varepsilon_1}^2\alpha_k^2\beta_{k-\varepsilon_1+\varepsilon_2}^2 \\ \therefore f_5'(\alpha_k) &= \frac{2}{\alpha_k}(\mu + \alpha_{k-\varepsilon_1}^2\beta_{k-\varepsilon_1+\varepsilon_2}^2) \\ &= \frac{4\alpha_{k-\varepsilon_1}^2}{\alpha_k}(\beta_{k-\varepsilon_1+\varepsilon_2}^2 - \beta_k^2) \\ &\begin{cases} > 0, & \text{if } \beta_{k-\varepsilon_1+\varepsilon_2} > \beta_k \\ < 0, & \text{if } \beta_{k-\varepsilon_1+\varepsilon_2} < \beta_k \\ = 0, & \text{if } \beta_{k-\varepsilon_1+\varepsilon_2} = \beta_k. \end{cases} \end{aligned}$$

Again from the continuity of f_5, we can make the conclusion <u>C5</u>:

1. $det\Delta_{k-\varepsilon_1} > 0$ or $\beta_{k-\varepsilon_1+\varepsilon_2} = \beta_k$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_{k-\varepsilon_1} \geq 0$ for all $x \in \big(\alpha_k - \delta_k, \alpha_k + \delta_k\big)$.

2. If $det\Delta_{k-\varepsilon_1} = 0$ then there exists $\delta_k > 0$ such that

   (i) if $\beta_{k-\varepsilon_1+\varepsilon_2} > \beta_k$ then $\tilde{\Delta}_{k-\varepsilon_1} \geq 0$ for all $x \in \big(\alpha_k, \alpha_k + \delta_k\big)$ and $\tilde{\Delta}_{k-\varepsilon_1} \not\geq 0$ for $x \in \big(\alpha_k - \delta_k, \alpha_k\big)$;

   (ii) if $\beta_{k-\varepsilon_1+\varepsilon_2} < \beta_k$ then $\tilde{\Delta}_{k-\varepsilon_1} \geq 0$ for all $x \in \big(\alpha_k - \delta_k, \alpha_k\big)$ and $\tilde{\Delta}_{k-\varepsilon_1} \not\geq 0$ for $x \in \big(\alpha_k, \alpha_k + \delta_k\big)$.

Finally, to check the positivity of $\tilde{\Delta}_k$ we consider

$$\begin{aligned} f_6(x) &:= det\tilde{\Delta}_k \\ &= \big(\alpha_{k+\varepsilon_1}^2 - x^2\big)\big(\beta_{k+\varepsilon_2}^2 - y^2\big) - \big(\alpha_{k-\varepsilon_2}\beta_{k+\varepsilon_1} - xy\big)^2 \\ &= \big(\alpha_{k+\varepsilon_1}^2 - x^2\big)\left(\beta_{k+\varepsilon_2}^2 - \frac{\beta_k^2 x^2}{\alpha_k^2}\right) - \left(\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \frac{\beta_k x^2}{\alpha_k}\right)^2 \\ &= x^2\left(- \beta_{k+\varepsilon_2}^2 - \frac{\beta_k^2\alpha_{k+\varepsilon_1}^2}{\alpha_k^2} + \frac{2\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1}\beta_k}{\alpha_k}\right) + \big(\alpha_{k+\varepsilon_1}^2\beta_{k+\varepsilon_2}^2 - \alpha_{k+\varepsilon_2}^2\beta_{k+\varepsilon_1}^2\big) \end{aligned}$$

$$= \frac{x^2}{\alpha_k^2}\left(2\alpha_k\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1}\beta_k - \alpha_k^2\beta_{k+\varepsilon_2}^2 - \beta_k^2\alpha_{k+\varepsilon_1}^2\right) + \left(\alpha_{k+\varepsilon_1}^2\beta_{k+\varepsilon_2}^2 - \alpha_{k+\varepsilon_2}^2\beta_{k+\varepsilon_1}^2\right).$$

If $det\Delta_k = 0$ then $f_6(\alpha_k) = det\Delta_k = 0$. Therefore

$$\gamma := \left(2\alpha_k\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1}\beta_k - \alpha_k^2\beta_{k+\varepsilon_2}^2 - \beta_k^2\alpha_{k+\varepsilon_1}^2\right) = (\alpha_{k+\varepsilon_2}^2\beta_{k+\varepsilon_1}^2 - \alpha_{k+\varepsilon_1}^2\beta_{k+\varepsilon_2}^2).$$

Again,

$$f_6'(x) = \frac{2x\gamma}{\alpha_k^2} = \frac{2x}{\alpha_k^2}(\alpha_{k+\varepsilon_2}^2\beta_{k+\varepsilon_1}^2 - \alpha_{k+\varepsilon_1}^2\beta_{k+\varepsilon_2}^2)$$
$$\begin{cases} > 0, & \text{if } \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} > \alpha_{k+\varepsilon_1}\beta_{k+\varepsilon_2} \\ < 0, & \text{if } \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} < \alpha_{k+\varepsilon_1}\beta_{k+\varepsilon_2} \\ = 0, & \text{if } \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} = \alpha_{k+\varepsilon_1}\beta_{k+\varepsilon_2}. \end{cases}$$

From the continuity of f_6 we can make the conclusion C6:

1. $det\Delta_k > 0$ or $\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} = \alpha_{k+\varepsilon_1}\beta_{k+\varepsilon_2}$ then there exists $\delta_k > 0$ such that $\tilde{\Delta}_k \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k + \delta_k)$.

2. If $det\Delta_k = 0$ then there exists $\delta_k > 0$ such that

   (i) if $\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} > \alpha_{k+\varepsilon_1}\beta_{k+\varepsilon_2}$ then $\tilde{\Delta}_k \geq 0$ for all $x \in (\alpha_k, \alpha_k + \delta_k)$ and $\tilde{\Delta}_k \ngeq 0$ for $x \in (\alpha_k - \delta_k, \alpha_k)$;

   (ii) if $\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} < \alpha_{k+\varepsilon_1}\beta_{k+\varepsilon_2}$ then $\tilde{\Delta}_k \geq 0$ for all $x \in (\alpha_k - \delta_k, \alpha_k)$ and $\tilde{\Delta}_k \ngeq 0$ for $x \in (\alpha_k, \alpha_k + \delta_k)$.

From the above analysis we can exhaustively determine whether perturbation of α_k will again result in a hyponormal shift $\tilde{T}$ or not.

For illustration let us consider the following examples:

Example 5.5.1. *Let $T = (T_1, T_2)$ be hyponormal with $\Delta_{(0,3)} > 0$, $\Delta_{(0,5)} > 0$, $\Delta_{(0,4)} = 0$ and $\alpha_{(1,4)} < \alpha_{(0,5)}$. We want to perturb $\alpha_{(0,5)}$.*

Applying C1 (1), C6 (1) and C4 (2)(ii) we conclude that $\tilde{T}$ *will still be hyponormal for a slight left perturbation of* $\alpha_{(0,5)}$, *but will not be hyponormal for any right perturbation of* $\alpha_{(0,5)}$

Example 5.5.2. *We want to perturb* $\alpha_{(7,11)}$ *Hence we need to consider* $\Delta_{(7,9)}$, $\Delta_{(6,10)}, \Delta_{(5,11)}, \Delta_{(7,10)}, \Delta_{(6,11)}, \Delta_{(7,11)}$ *Suppose* $\Delta_{(6,10)}, \Delta_{(5,11)}, \Delta_{(7,10)}, \Delta_{(7,11)} > 0$ *and* $\Delta_{(7,9)} = \Delta_{(6,11)} = 0$ *So by C1 (2) and C5, we make the following conclusions:*

1. *If* $\beta_{(6,12)} \leq \beta_{(7,11)}$ *then* $\tilde{T}$ *will be hyponomal for a slight left perturbation of* $\alpha_{(7,11)}$, *but will not be hyponormal for any right perturbation of* $\alpha_{(7,11)}$.

2. *If* $\beta_{(6,12)} > \beta_{(7,11)}$ *then for any slight perturbation of* $\alpha_{(7,11)}$, $\tilde{T}$ *will fail to be hyponormal.*

Chapter 6

On weak hyponormality of 2-variable weighted shifts

6.1 Introduction

In Chapter 5 it was shown that if for a 2-variable hyponormal shift $T = (T_1, T_2)$, a weight $\alpha_{(k_1,k_2)}$ is perturbed, then the resulting perturbed shift $\tilde{T}$ may not remain hyponormal. In fact the conditions under which $\tilde{T}$ will still be hyponormal is completely given in that chapter. In this chapter, we show that though $\tilde{T}$ may not be hyponormal, it will however still remain weakly hyponormal for sufficiently small perturbations x of $\alpha_{(k_1,k_2)}$.

Let $\alpha := \{\alpha_k\}_{k\in\mathbb{Z}_+^2}$ and $\beta := \{\beta_k\}_{k\in\mathbb{Z}_+^2}$ be 2-variable weight sequences and $T = (T_1, T_2)$ be a 2-variable weighted shift on $\ell^2(\mathbb{Z}_+^2)$ defined by $T_1 e_k = \alpha_k e_{k+\varepsilon_1}$ and $T_2 e_k = \beta_k e_{k+\varepsilon_2}$.

As mentioned earlier $T = (T_1, T_2)$ is weakly hyponormal if $\lambda_1 T_1 + \lambda_2 T_2$ is hyponormal $\forall\ \lambda_i \in \mathbb{C}$. Equivalently,

$$
\begin{aligned}
&T \text{ is weakly hyponormal}\\
&\Leftrightarrow T_1 + \bar{\lambda} T_2 \text{ is hyponormal } \ \forall\ \lambda \in \mathbb{C}\\
&\Leftrightarrow [(T_1 + \bar{\lambda} T_2)^*, (T_1 + \bar{\lambda} T_2)] \geq 0 \ \ \forall\ \lambda \in \mathbb{C}
\end{aligned}
$$

$$\Leftrightarrow \left\langle [(T_1+\bar{\lambda}T_2)^*, (T_1+\bar{\lambda}T_2)]x, x\right\rangle \geq 0 \ \ \forall\, \lambda \in \mathbb{C} \text{ and } x \in \ \ell^2(\mathbb{Z}_+^2)$$

$$\Leftrightarrow \left\langle [T_1^*, T_1]x, x\right\rangle + \lambda \left\langle [T_2^*, T_1]x, x\right\rangle$$

$$+\,\bar{\lambda}\left\langle [T_1^*, T_2]x, x\right\rangle + \lambda\bar{\lambda}\left\langle [T_2^*, T_2]x, x\right\rangle \geq 0 \ \ \forall\, \lambda \in \mathbb{C} \text{ and } x \in \ \ell^2(\mathbb{Z}_+^2)$$

$$\Leftrightarrow \left\langle \begin{pmatrix} \langle [T_1^*, T_1]x, x\rangle & \langle [T_2^*, T_1]x, x\rangle \\ \langle [T_1^*, T_2]x, x\rangle & \langle [T_2^*, T_2]x, x\rangle \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}, \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \right\rangle \geq 0$$

for all $\lambda \in \mathbb{C}$ and $x \in \ \ell^2(\mathbb{Z}_+^2)$. (6.1.1)

Theorem 6.1.1. *$T = (T_1, T_2)$ is weakly hyponormal if and only if*

$$\sum_{j=0}^{\infty} |c_{(0,j)}|^2 \alpha_{(0,j)}^2 + |\lambda|^2 \sum_{i=0}^{\infty} |c_{(i,0)}|^2 \beta_{(i,0)}^2 + \sum_{k \in \mathbb{Z}_+^2} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle \geq 0$$

for all $x = \sum_{k \in \mathbf{Z}_+^2} c_k\, e_k \in \ \ell^2(\mathbb{Z}_+^2)$ and

$$\Delta_k := \begin{pmatrix} \alpha_{k+\varepsilon_1}^2 - \alpha_k^2 & \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \alpha_k\beta_k \\ \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \alpha_k\beta_k & \beta_{k+\varepsilon_2}^2 - \beta_k^2 \end{pmatrix}$$

for all $k \in \mathbb{Z}_+^2$.

Proof. We have $T_1 e_k = \alpha_k e_{k+\varepsilon_1}$ for $k = (k_1, k_2) \in \mathbb{Z}_+^2$, and

$$T_1^* e_k = \begin{cases} 0 & \text{if } k_1 = 0 \\ \alpha_{k-\varepsilon_1} e_{k-\varepsilon_1} & \text{if } k_1 > 0. \end{cases}$$

Similarly, $T_2 e_k = \alpha_k e_{k+\varepsilon_2}$ for $k = (k_1, k_2) \in \mathbb{Z}_+^2$, and

$$T_2^* e_k = \begin{cases} 0 & \text{if } k_2 = 0 \\ \alpha_{k-\varepsilon_2} e_{k-\varepsilon_2} & \text{if } k_2 > 0. \end{cases}$$

Therefore,

$$\begin{aligned} [T_1^*, T_1]e_k &= T_1^*(T_1 e_k) - T_1(T_1^* e_k) \\ &= \begin{cases} T_1^* \alpha_k\, e_{k+\varepsilon_1} & \text{if } k_1 = 0 \\ T_1^* \alpha_k\, e_{k+\varepsilon_1} - T_1 \alpha_{k-\varepsilon_1} e_{k-\varepsilon_1} & \text{if } k_1 > 0 \end{cases} \\ &= \begin{cases} \alpha_k^2\, e_k & \text{if } k_1 = 0 \\ (\alpha_k^2 - \alpha_{k-\varepsilon_1}^2) e_k & \text{if } k_1 > 0 \end{cases} \\ &= (\alpha_k^2 - \alpha_{k-\varepsilon_1}^2) e_k \end{aligned}$$

assuming $\alpha_{(t_1,t_2)} = 0$ for all $t_1 < 0$, $t_2 \in \mathbb{Z}$.

Again,

$$\begin{aligned}
[T_2^*, T_1]e_k &= T_2^*(T_1 e_k) - T_1(T_2^* e_k) \\
&= \begin{cases} T_2^* \alpha_k \, e_{k+\varepsilon_1} & \text{if } k_2 = 0 \\ T_2^* \alpha_k \, e_{k+\varepsilon_1} - T_1 \beta_{k-\varepsilon_2} e_{k-\varepsilon_2} & \text{if } k_2 > 0 \end{cases} \\
&= \begin{cases} 0 & \text{if } k_2 = 0 \\ (\alpha_k \beta_{k+\varepsilon_1-\varepsilon_2} - \alpha_{k-\varepsilon_2}\beta_{k-\varepsilon_2}) e_{k+\varepsilon_1-\varepsilon_2} & \text{if } k_2 > 0 \end{cases} \\
&= (\alpha_k \beta_{k+\varepsilon_1-\varepsilon_2} - \alpha_{k-\varepsilon_2}\beta_{k-\varepsilon_2}) e_{k+\varepsilon_1-\varepsilon_2},
\end{aligned}$$

$$\text{assuming } \beta_{(t_1,t_2)} = 0 \ \text{ for all } t_1 \in \mathbb{Z},\ t_2 < 0.$$

Similarly,

$[T_1^*, T_2]\, e_k = \big(\alpha_{k-\varepsilon_1+\varepsilon_2}\, \beta_k - \alpha_{k-\varepsilon_1}\beta_{k-\varepsilon_1}\big) e_{k-\varepsilon_1+\varepsilon_2}$ and $[T_2^*, T_2]\, e_k = \big(\beta_k^2 - \beta_{k-\varepsilon_2}^2\big)\, e_k.$

Let $x \in \ell^2(\mathbb{Z}_+^2)$ and $x = \sum_{k\in\mathbb{Z}_+^2} c_k \, e_k$.

Then

$$\begin{aligned}
\langle [T_1^*, T_1]x, x\rangle &= \left\langle \sum_{k=(k_1,k_2)} (\alpha_k^2 - \alpha_{k-\varepsilon_1}^2) c_k e_k, \sum_{t=(t_1,t_2)} c_t e_t \right\rangle \\
&= \sum_k (\alpha_k^2 - \alpha_{k-\varepsilon_1}^2) c_k \left\langle e_k, \sum_t c_t e_t \right\rangle \\
&= \sum_k (\alpha_k^2 - \alpha_{k-\varepsilon_1}^2) |c_k|^2 \\
&= \sum_{k_1=0} (\alpha_k^2 - \alpha_{k-\varepsilon_1}^2) |c_k|^2 + \sum_{k_1>0} (\alpha_k^2 - \alpha_{k-\varepsilon_1}^2) |c_k|^2 \\
&= \sum_{j=0}^{\infty} |c_{(0,j)}|^2 \alpha_{(0,j)}^2 + \sum_{k\in\mathbb{Z}_+^2} (\alpha_{k+\varepsilon_1}^2 - \alpha_k^2) |c_{k+\varepsilon_1}|^2 \qquad (6.1.2)
\end{aligned}$$

Similarly, we have

$$\langle [T_2^*, T_1]x, x\rangle = \sum_{k\in\mathbb{Z}_+^2} \bar{c}_{k+\varepsilon_1} c_{k+\varepsilon_2} (\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \alpha_k \beta_k) \qquad (6.1.3)$$

$$\langle [T_1^*, T_2]x, x\rangle = \sum_{k\in\mathbb{Z}_+^2} c_{k+\varepsilon_1} \bar{c}_{k+\varepsilon_2} (\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \alpha_k \beta_k) \qquad (6.1.4)$$

$$\langle [T_2^*, T_2]x, x\rangle = \sum_{i=0}^{\infty} |c_{(i,0)}|^2 \beta_{(i,0)}^2 + \sum_{k\in\mathbb{Z}_+^2} (\beta_{k+\varepsilon_2}^2 - \beta_k^2) |c_{k+\varepsilon_2}|^2 \qquad (6.1.5)$$

Using (6.1.2) to (6.1.5) in (6.1.1), we conclude that

T is weakly hyponormal

$$\Leftrightarrow \left\langle \begin{pmatrix} \sum_{j=0}^{\infty} |c_{(0,j)}|^2 \alpha_{(0,j)}^2 + \sum_k (\alpha_{k+\varepsilon_1}^2 - \alpha_k^2)|c_{k+\varepsilon_1}|^2 & \sum_k \bar{c}_{k+\varepsilon_1} c_{k+\varepsilon_2} (\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \alpha_k \beta_k) \\ \sum_k c_{k+\varepsilon_1} \bar{c}_{k+\varepsilon_2} (\alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \alpha_k \beta_k) & \sum_{i=0}^{\infty} |c_{(i,0)}|^2 \beta_{(i,0)}^2 + \sum_k (\beta_{k+\varepsilon_2}^2 - \beta_k^2)|c_{k+\varepsilon_2}|^2 \end{pmatrix} \right.$$

$$\left. \begin{pmatrix} 1 \\ \lambda \end{pmatrix}, \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \right\rangle \geq 0; \text{ for all } \lambda \in \mathbb{C} \text{ and } x = \sum_{k \in \mathbb{Z}_+^2} c_k \, e_k \in \ell^2(\mathbb{Z}_+^2)$$

$$\Leftrightarrow \sum_{j=0}^{\infty} |c_{(0,j)}|^2 \alpha_{(0,j)}^2 + |\lambda|^2 \sum_{i=0}^{\infty} |c_{(i,0)}|^2 \beta_{(i,0)}^2 + \sum_{k \in \mathbb{Z}_+^2} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda \, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda \, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle$$

$$\geq 0 \text{ for all } \lambda \in \mathbb{C} \text{ and } x = \sum_{k \in \mathbb{Z}_+^2} c_k \, e_k \in \ell^2(\mathbb{Z}_+^2).$$

□

Remark 6.1.1. If $T = (T_1, T_2)$ is hyponormal then $\Delta_k \geq 0$ for all $k \in \mathbb{Z}_+^2$, and hence by Theorem 6.1.1 it immediately follows that T is also weakly hyponormal.

6.2 Perturbations not affecting the core of T

6.2.1 Perturbation of the weight $\alpha_{(k_1,0)}$

For $k_1 \geq 0$, let $\alpha_{(k_1,0)}$ be slightly perturbed to a new weight x. For commutativity we change $\beta_{(k_1,0)}$ to $y = \frac{x\beta_{(k_1,0)}}{\alpha_{(k_1,0)}}$ and $\alpha_{(k_1-1,0)}$ to $z = \frac{\alpha_{(k_1-1,0)}\alpha_{(k_1,0)}}{x}$. The corresponding weight diagram is given in Figure 16.

Let $\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$ be the perturbed shift with weight sequences $\{\tilde{\alpha}_k\}_{k \in \mathbb{Z}_+^2}$ and $\{\tilde{\beta}_k\}_{k \in \mathbb{Z}_+^2}$.

Let

$$\tilde{\Delta}_k := \begin{pmatrix} \tilde{\alpha}_{k+\varepsilon_1}^2 - \tilde{\alpha}_k^2 & \tilde{\alpha}_{k+\varepsilon_2}\tilde{\beta}_{k+\varepsilon_1} - \tilde{\alpha}_k \tilde{\beta}_k \\ \tilde{\alpha}_{k+\varepsilon_2}\tilde{\beta}_{k+\varepsilon_1} - \tilde{\alpha}_k \tilde{\beta}_k & \tilde{\beta}_{k+\varepsilon_2}^2 - \tilde{\beta}_k^2 \end{pmatrix}.$$

Clearly, $\tilde{\Delta}_k = \Delta_k$ for all $k \in \mathbb{Z}_+^2$, except for $k = (k_1-2, 0), (k_1-1, 0)$ and $(k_1, 0)$.

Let $\mu_1 > 0$ and $\mu = (\mu_1, 0)$. Define $\bar{\Delta}_\mu = \begin{pmatrix} |\lambda|^2 \tilde{\beta}_{\mu+\varepsilon_1}^2 & 0 \\ 0 & 0 \end{pmatrix} + \tilde{\Delta}_\mu$

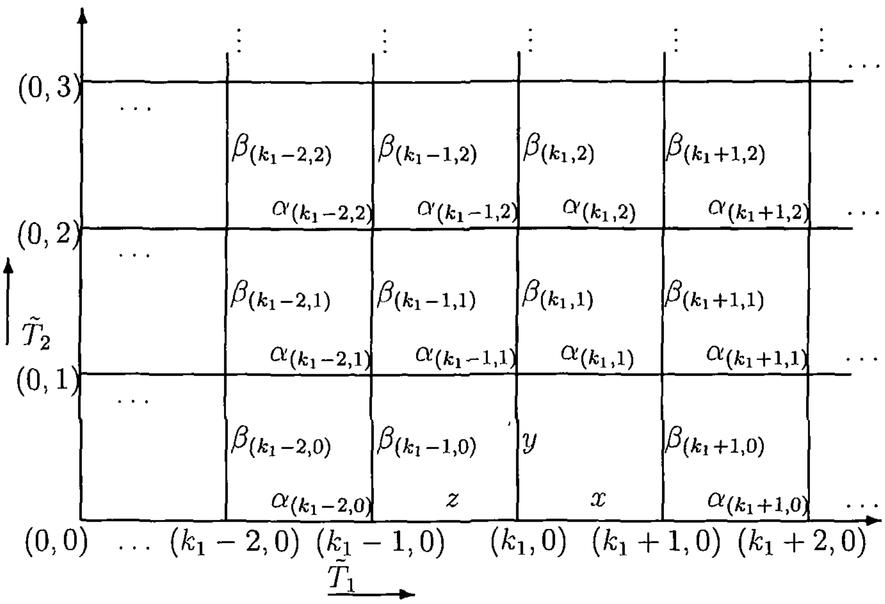


Figure 16

<u>Claim:</u> $\bar{\Delta}_\mu \geq 0$ for $\mu = (\mu_1, 0)$ and $\mu_1 = k_1 - 2, k_1 - 1, k_1$.

As

$$\bar{\Delta}_\mu = \begin{pmatrix} |\lambda|^2 \tilde{\beta}^2_{(\mu_1+1,0)} + \tilde{\alpha}^2_{(\mu_1+1,0)} - \tilde{\alpha}^2_{(\mu_1,0)} & \tilde{\alpha}_{(\mu_1,1)}\tilde{\beta}_{(\mu_1+1,0)} - \tilde{\alpha}_{(\mu_1,0)}\tilde{\beta}_{(\mu_1,0)} \\ \tilde{\alpha}_{(\mu_1,1)}\tilde{\beta}_{(\mu_1+1,0)} - \tilde{\alpha}_{(\mu_1,0)}\tilde{\beta}_{(\mu_1,0)} & \tilde{\beta}^2_{(\mu_1,1)} - \tilde{\beta}^2_{(\mu_1,0)} \end{pmatrix},$$

so

$$f(x) := det\bar{\Delta}_\mu = |\lambda|^2 \tilde{\beta}^2_{(\mu_1+1,0)}(\tilde{\beta}^2_{(\mu_1,1)} - \tilde{\beta}^2_{(\mu_1,0)}) + det\tilde{\Delta}_\mu.$$

Therefore,

$$\begin{aligned} f(\alpha_{(k_1,0)}) =& |\lambda|^2 \tilde{\beta}^2_{(\mu_1+1,0)}(\tilde{\beta}^2_{(\mu_1,1)} - \tilde{\beta}^2_{(\mu_1,0)}) + det\Delta_\mu > 0 \\ & (\because\ det\Delta_\mu \geq 0 \text{ and } \tilde{\beta}_{(\mu_1,1)} > \tilde{\beta}_{(\mu_1,0)}). \end{aligned}$$

Thus by continuity of f, there exists $\delta_\mu > 0$ such that $f(x) > 0$ for all $\lambda \in \mathbb{C}$ and for all $x \in (\alpha_{(k_1,0)} - \delta_\mu, \alpha_{(k_1,0)} + \delta_\mu)$.

Let $\delta = min\{\delta_{(k_1-2,0)}, \delta_{(k_1-1,0)}, \delta_{(k_1,0)}\}$. Then for $x \in (\alpha_{(k_1,0)} - \delta, \alpha_{(k_1,0)} + \delta)$, $\bar{\Delta}_\mu \geq 0$ for all $\lambda \in \mathbb{C}$, and the Claim is established.

Also, $\tilde{\Delta}_k = \Delta_k \geq 0 \;\; \forall \; k \in \mathbb{Z}_+^2$, except for $k = (k_1 - 2, 0), (k_1 - 1, 0)$ and $(k_1, 0)$. Therefore by Theorem 6.1.1 we conclude that there exists $\delta > 0$ such that for all $x \in (\alpha_{(k_1,0)} - \delta, \alpha_{(k_1,0)} + \delta)$ $\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$ is weakly hyponormal.

Thus for a hyponormal 2-variable weighted shift T, if $\alpha_{(k_1,0)}$ is slightly perturbed then, the perturbed shift $\tilde{T}$ still remains weakly hyponormal.

6.2.2 Perturbation of the weight $\alpha_{(0,k_2)}$

For any $k_2 > 0$, let $\alpha_{(0,k_2)}$ be slightly perturbed to a new weight x. To preserve commutativity, we change $\beta_{(0,k_2)}$ to $y = \frac{x\beta_{(1,k_2)}}{\alpha_{(0,k_2+1)}}$ and $\beta_{(0,k_2-1)}$ to $t = \frac{\alpha_{(0,k_2-1)}\beta_{(1,k_2-1)}}{x}$.

The weight diagram of $\tilde{T}$ is given in Figure 17.

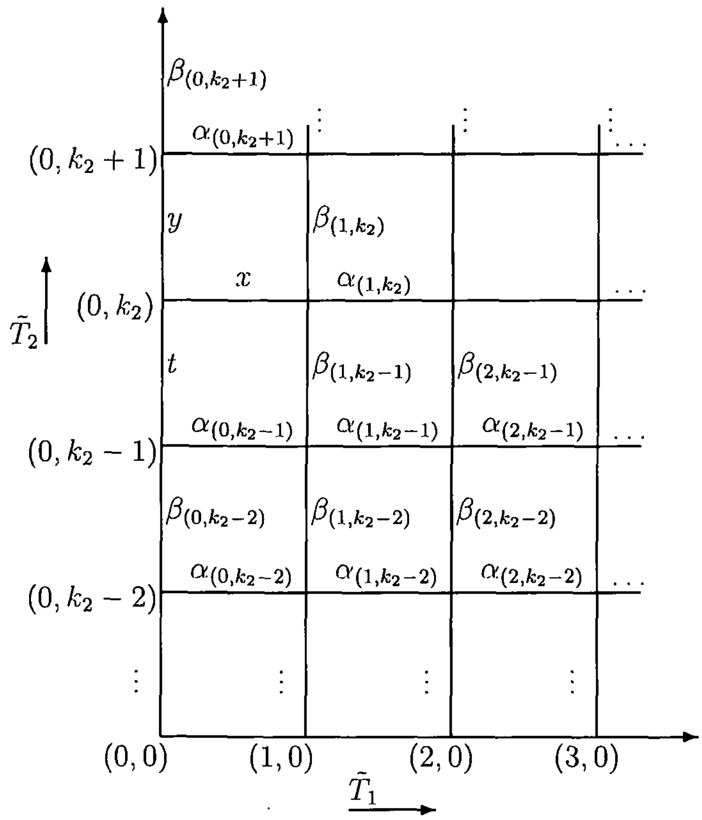


Figure 17

We have $\tilde{\Delta}_k = \Delta_k$ for all $k \in \mathbb{Z}_+^2$, except for $k = (0, k_2-2), (0, k_2-1)$ and $(0, k_2)$ Also as T is hyponormal, so $\Delta_k \geq 0$ for all k. Thus, we have $\tilde{\Delta}_k \geq 0$ for all k except for $k = (0, k_2 - 2), (0, k_2 - 1)$ and $(0, k_2)$.

For $\mu = (0, \mu_2)$, $\mu_2 \in \mathbb{Z}_+$. Define $\bar{\bar{\Delta}}_\mu := \begin{pmatrix} 0 & 0 \\ 0 & \frac{\tilde{\alpha}^2_{\mu+\varepsilon_2}}{|\lambda|^2} \end{pmatrix} + \tilde{\Delta}_\mu$. Clearly, the positivity of $\bar{\bar{\Delta}}_\mu$ implies the positivity of $f(c_{ij}, \lambda)$. Now we will show that $\bar{\bar{\Delta}}_\mu \geq 0$ for $\mu = (0, k_2 - 2), (0, k_2 - 1)$ and $(0, k_2)$.

Consider $\mu_2 \in \{k_2 - 2, k_2 - 1, k_2\}$ and $\mu = (0, \mu_2)$.

$$\bar{\bar{\Delta}}_\mu = \begin{pmatrix} \tilde{\alpha}^2_{\mu+\varepsilon_1} - \tilde{\alpha}^2_\mu & \tilde{\alpha}_{\mu+\varepsilon_2}\tilde{\beta}_{\mu+\varepsilon_1} - \tilde{\alpha}_\mu\tilde{\beta}^2_\mu \\ \tilde{\alpha}_{\mu+\varepsilon_2}\tilde{\beta}_{\mu+\varepsilon_1} - \tilde{\alpha}_\mu\tilde{\beta}^2_\mu & \tilde{\beta}^2_{\mu+\varepsilon_2} - \tilde{\beta}^2_\mu + \frac{\tilde{\alpha}^2_{\mu+\varepsilon_2}}{|\lambda|^2} \end{pmatrix}$$

and

$$\begin{aligned} g(x) &:= det\bar{\bar{\Delta}}_\mu \\ &= \frac{\tilde{\alpha}^2_{\mu+\varepsilon_2}}{|\lambda|^2}(\tilde{\alpha}^2_{\mu+\varepsilon_1} - \tilde{\alpha}^2_\mu) + det\tilde{\Delta}_\mu \end{aligned}$$

$$g(\alpha_{(0,t)}) = \left[\frac{\tilde{\alpha}^2_{\mu+\varepsilon_2}}{|\lambda|^2}(\tilde{\alpha}^2_{\mu+\varepsilon_1} - \tilde{\alpha}^2_\mu)\right]_{x=\alpha_{(0,t)}} + det\Delta_\mu > 0$$
$$(\because\ det\Delta_\mu \geq 0 \text{ and } \tilde{\alpha}_{\mu+\varepsilon_1} > \tilde{\alpha}_\mu).$$

So by continuity of g, there exists $\delta_\mu > 0$ such that $g(x) > 0$ for all $\lambda \in \mathbb{C}$ and for all $x \in (\alpha_{(0,k_2)} - \delta_\mu, \alpha_{(0,k_2)} + \delta_\mu)$.

Let $\delta = min\{\delta_{(0,k_2-2)}, \delta_{(0,k_2-1)}, \delta_{(0,k_2)}\}$. Then for $x \in (\alpha_{(0,k_2)} - \delta_\mu, \alpha_{(0,k_2)} + \delta_\mu)$, $\bar{\bar{\Delta}}_\mu \geq 0$ for all $\lambda \in \mathbb{C}$ and for all $\mu = (0, k_2-2), (0, k_2-1)$ and $(0, k_2)$ Therefore, we conclude that the perturbed shift is weakly hyponormal.

6.3 Reformulation of weak hyponormality

Theorem 6.3.1. *Let $\mathcal{A} = \{(k_1, 0) : k_1 \in \mathbb{Z}_+\}$. For $\lambda \in \mathbb{C}$ and $\mu \in \mathcal{A}$, define $\mathcal{M}_\mu := \begin{pmatrix} |\lambda|^2\beta^2_{\mu+\varepsilon_1} & 0 \\ 0 & 0 \end{pmatrix} + \Delta_\mu$. Then T is weakly hyponormal if and only if*

$$|\lambda|^2|c_{(0,0)}|^2\beta^2_{(0,0)} + \sum_{j=0}^{\infty}|c_{(0,j)}|^2\alpha^2_{(0,j)} + \sum_{\mu\in\mathcal{A}}\left\langle \mathcal{M}_\mu \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle$$
$$+ \sum_{k\in\mathbb{Z}^2_+\setminus\mathcal{A}}\left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle \geq 0$$

for all $\lambda \in \mathbb{C}$ and $\sum c_k e_k \in \ell^2(\mathbb{Z}^2_+)$

Proof. For $\mu \in \mathcal{A}$,

$$\left\langle \mathcal{M}_\mu \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle$$
$$= \left\langle \begin{pmatrix} |\lambda|^2\beta^2_{\mu+\varepsilon_1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle + \left\langle \Delta_\mu \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle$$
$$= |\lambda|^2|c_{\mu+\varepsilon_1}|^2\beta^2_{\mu+\varepsilon_1} + \left\langle \Delta_\mu \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle$$

Therefore,

$$|\lambda|^2|c_{(0,0)}|^2\beta^2_{(0,0)} + \sum_{j=0}^{\infty}|c_{(0,j)}|^2\alpha^2_{(0,j)} + \sum_{\mu\in\mathcal{A}}\left\langle \mathcal{M}_\mu \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle$$
$$+ \sum_{k\in\mathbb{Z}^2_+\setminus\mathcal{A}}\left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle$$
$$= |\lambda|^2|c_{(0,0)}|^2\beta^2_{(0,0)} + \sum_{j=0}^{\infty}|c_{(0,j)}|^2\alpha^2_{(0,j)} + \sum_{\mu\in\mathcal{A}}|\lambda|^2|c_{\mu+\varepsilon_1}|^2\beta^2_{\mu+\varepsilon_1}$$
$$+ \sum_{\mu\in\mathcal{A}}\left\langle \Delta_\mu \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle + \sum_{k\in\mathbb{Z}^2_+\setminus\mathcal{A}}\left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle$$
$$= \sum_{j=0}^{\infty}|c_{(0,j)}|^2\alpha^2_{(0,j)} + |\lambda|^2\sum_{i=0}^{\infty}|c_{(i,0)}|^2\beta^2_{(i,0)} + \sum_{k\in\mathbb{Z}^2_+}\left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle$$

Hence the result follows from Theorem 6.1.1. □

Theorem 6.3.2. *Let $\mathcal{B} = \{(0, k_2) : k_2 \in \mathbb{Z}_+\}$. For $\lambda \in \mathbb{C}$ and $\mu \in \mathcal{B}$, define $\mathcal{N}_\mu := \begin{pmatrix} 0 & 0 \\ 0 & \frac{\alpha^2_{\mu+\varepsilon_2}}{|\lambda|^2} \end{pmatrix} + \Delta_\mu$. Then T is weakly hyponormal if and only if $\forall\ \lambda \in \mathbb{C}$ and $\sum c_k e_k \in \ell^2(\mathbb{Z}^2_+)$,*

$$|c_{(0,0)}|^2 \alpha^2_{(0,0)} + |\lambda|^2 \sum_{i=0}^{\infty} |c_{(i,0)}|^2 \beta^2_{(i,0)} + \sum_{\mu \in \mathcal{B}} \left\langle \mathcal{N}_\mu \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle$$
$$+ \sum_{k \in \mathbb{Z}^2_+ \setminus \mathcal{B}} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle \geq 0.$$

Proof. For $\mu \in \mathcal{B}$,

$$\left\langle \mathcal{N}_\mu \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle$$
$$= \left\langle \begin{pmatrix} 0 & 0 \\ 0 & \frac{\alpha^2_{\mu+\varepsilon_2}}{|\lambda|^2} \end{pmatrix} \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle + \left\langle \Delta_\mu \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle$$
$$= |c_{\mu+\varepsilon_2}|^2 \alpha^2_{\mu+\varepsilon_2} + \left\langle \Delta_\mu \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle$$

Therefore,

$$|c_{(0,0)}|^2 \alpha^2_{(0,0)} + \sum_{i=0}^{\infty} |c_{(i,0)}|^2 \beta^2_{(i,0)} + \sum_{\mu \in \mathcal{B}} \left\langle \mathcal{N}_\mu \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle$$
$$+ \sum_{k \in \mathbb{Z}^2_+ \setminus \mathcal{B}} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle$$
$$= |c_{(0,0)}|^2 \alpha^2_{(0,0)} + \sum_{i=0}^{\infty} |c_{(i,0)}|^2 \beta^2_{(i,0)} + \sum_{\mu \in \mathcal{B}} |c_{\mu+\varepsilon_2}|^2 \alpha^2_{\mu+\varepsilon_2}$$
$$+ \sum_{\mu \in \mathcal{B}} \left\langle \Delta_\mu \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{\mu+\varepsilon_1} \\ \lambda\, c_{\mu+\varepsilon_2} \end{pmatrix} \right\rangle + \sum_{k \in \mathbb{Z}^2_+ \setminus \mathcal{B}} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle$$
$$= \sum_{j=0}^{\infty} |c_{(0,j)}|^2 \alpha^2_{(0,j)} + |\lambda|^2 \sum_{i=0}^{\infty} |c_{(i,0)}|^2 \beta^2_{(i,0)} + \sum_{k \in \mathbb{Z}^2_+} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle$$

Hence the result follows from Theorem 6.1.1. □

Let $k = (k_1, k_2)$ and $|k| = k_1 + k_2$. Also (for convenience of notation) let us denote by a_k; b_k, d_k the following $a_k = \alpha^2_{k+\varepsilon_1} - \alpha^2_k$, $b_k = \alpha_{k+\varepsilon_2}\beta_{k+\varepsilon_1} - \alpha_k\beta_k$; $d_k = \beta^2_{k+\varepsilon_2} - \beta^2_k$. Then $\Delta_k = \begin{pmatrix} a_k & b_k \\ b_k & d_k \end{pmatrix}$

Let

$$L_0 = \begin{pmatrix} a_{(0,0)} + |\lambda|^2\beta^2_{(1,0)} & b_{(0,0)} \\ b_{(0,0)} & d_{(0,0)} + \frac{\alpha^2_{(0,1)}}{|\lambda|^2} \end{pmatrix}$$

$$L_1 = \begin{pmatrix} a_{(1,0)} + |\lambda|^2\beta^2_{(2,0)} & b_{(1,0)} & 0 \\ b_{(1,0)} & d_{(1,0)} + \frac{a_{(0,1)}}{|\lambda|^2} & \frac{b_{(0,1)}}{|\lambda|^2} \\ 0 & \frac{b_{(0,1)}}{|\lambda|^2} & \frac{d_{(0,1)}}{|\lambda|^2} + \frac{\alpha^2_{(0,2)}}{|\lambda|^4} \end{pmatrix}$$

$$L_2 = \begin{pmatrix} a_{(2,0)} + |\lambda|^2\beta^2_{(3,0)} & b_{(2,0)} & 0 & 0 \\ b_{(2,0)} & d_{(2,0)} + \frac{a_{(1,1)}}{|\lambda|^2} & \frac{b_{(1,1)}}{|\lambda|^2} & 0 \\ 0 & \frac{b_{(1,1)}}{|\lambda|^2} & \frac{d_{(1,1)}}{|\lambda|^2} + \frac{a_{(0,2)}}{|\lambda|^4} & \frac{b_{(0,2)}}{|\lambda|^4} \\ 0 & 0 & \frac{b_{(0,2)}}{|\lambda|^4} & \frac{d_{(0,2)}}{|\lambda|^4} + \frac{\alpha^2_{(0,3)}}{|\lambda|^6} \end{pmatrix}$$

So, in general L_n is a matrix $(A_{(i,j)})$ of size $(n+2)\times(n+2)$, with $A_{(i,j)}$ defined as follows:

1. $A_{(1,1)} = a_{(n,0)} + |\lambda|^2\beta^2_{(n+1,0)}$.
2. $A_{(n+2,n+2)} = \frac{d_{(0,n)}}{|\lambda|^{2n}} + \frac{\alpha^2_{(0,n+1)}}{|\lambda|^{2(n+1)}}$
3. $A_{(i,j)} = A_{(j,i)}$
4. $A_{(i,j)} = 0$ if $j > i+1$ for all i.
5. $A_{(i,i+1)} = \frac{b_{(n+1-i,i-1)}}{|\lambda|^{2(i-1)}}$ for $i = 1, 2, \ldots, n+1$
6. $A_{(i,j)} = \frac{d_{(n+2-i,i-2)}}{|\lambda|^{2(i-2)}} + \frac{a_{(n+1-i,i-1)}}{|\lambda|^{2(i-1)}}$ for $i = 2, 3, \ldots, n+1$

Also for $\sum_{k\in\mathbb{Z}^2_+} c_k\, e_k \in \ell^2(\mathbb{Z}^2_+)$ and $\lambda \in \mathbb{C}$, let

$$X_0 = \begin{pmatrix} c_{(1,0)} \\ \lambda c_{(0,1)} \end{pmatrix},\ X_1 = \begin{pmatrix} c_{(2,0)} \\ \lambda c_{(1,1)} \\ \lambda^2 c_{(0,2)} \end{pmatrix},\ X_2 = \begin{pmatrix} c_{(3,0)} \\ \lambda c_{(2,1)} \\ \lambda^2 c_{(1,2)} \\ \lambda^3 c_{(0,3)} \end{pmatrix}$$

In general, $X_n = \begin{pmatrix} c_{(n+1,0)} \\ \lambda c_{(n,1)} \\ \lambda^2 c_{(n-1,2)} \\ \vdots \\ \lambda^{n+1} c_{(0,n+1)} \end{pmatrix}$

That is X_n is a column matrix $(B_{(i,1)})$, where $B_{(i,1)} = \lambda^{i-1} c_{(n+2-i,\, i-1)}$ for $i = 1, 2, \ldots, n+2$.

Following the notations introduced above, Theorem 6.1.1 can be reformulated as follows.

Theorem 6.3.3. *A 2-variable weighted shift $T = (T_1, T_2)$ with weight sequences $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}_+^2}$ and $\beta = \{\beta_k\}_{k\in\mathbb{Z}_+^2}$ is weakly hyponormal if and only if for all $\lambda \in \mathbb{C}$ and $X = \sum_{k\in\mathbb{Z}_+^2} c_k\, e_k \in \ell^2(\mathbb{Z}_+^2)$, we have*

$$|c_{(0,0)}|^2(\alpha_{(0,0)}^2 + |\lambda|^2\beta_{(0,0)}^2) + \sum_{n=0}^{\infty} \langle L_n X_n, X_n\rangle \geq 0.$$

Proof. Direct calculation shows that

$$\begin{aligned} \langle L_0 X_0, X_0\rangle &= |\lambda|^2 |c_{(1,0)}|^2 \beta_{(1,0)}^2 + \left\langle \Delta_{(0,0)} \begin{pmatrix} c_{(0,0)+\varepsilon_1} \\ \lambda\, c_{(0,0)+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{(0,0)+\varepsilon_1} \\ \lambda\, c_{(0,0)+\varepsilon_2} \end{pmatrix} \right\rangle \\ \langle L_1 X_1, X_1\rangle &= |\lambda|^2 |c_{(2,0)}|^2 \beta_{(2,0)}^2 + |c_{(0,2)}|^2 \alpha_{(0,2)}^2 \\ &\quad + \sum_{|k|=1} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle \end{aligned}$$

Similarly,

$$\begin{aligned} \langle L_n X_n, X_n\rangle &= |\lambda|^2 |c_{(n+1,0)}|^2 \beta_{(n+1,0)}^2 + |c_{(0,n+1)}|^2 \alpha_{(0,n+1)}^2 \\ &\quad + \sum_{|k|=n} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle \end{aligned}$$

Therefore,

$$\begin{aligned} &|c_{(0,0)}|^2(\alpha_{(0,0)}^2 + |\lambda|^2\beta_{(0,0)}^2) + \sum_{n=0}^{\infty} \langle L_n X_n, X_n\rangle \\ &= \sum_{j=0}^{\infty} |c_{(0,j)}|^2 \alpha_{(0,j)}^2 + |\lambda|^2 \sum_{i=0}^{\infty} |c_{(i,0)}|^2 \beta_{(i,0)}^2 + \sum_{k\in\mathbb{Z}_+^2} \left\langle \Delta_k \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix}, \begin{pmatrix} c_{k+\varepsilon_1} \\ \lambda\, c_{k+\varepsilon_2} \end{pmatrix} \right\rangle \end{aligned}$$

The result now follows from Theorem 6.1.1. □

6.4 Perturbation of the weight $\alpha_{(k_1,k_2)}$

For $s \geq 0$ define L_s as follows:

$$\begin{pmatrix} a_{(s,0)} + |\lambda|^2\beta^2_{(s+1,0)} & b_{(s,0)} & 0 & \cdots & 0 \\ b_{(s,0)} & d_{(s,0)} + \frac{a_{(s-1,1)}}{|\lambda|^2} & \frac{b_{(s-1,1)}}{|\lambda|^2} & \cdots & 0 \\ 0 & \frac{b_{(s-1,1)}}{|\lambda|^2} & \frac{d_{(s-1,1)}}{|\lambda|^2} + \frac{a_{(s-2,2)}}{|\lambda|^4} & \cdots & 0 \\ 0 & 0 & \frac{b_{(s-2,2)}}{|\lambda|^4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{b_{(0,s)}}{|\lambda|^{2s}} \\ 0 & 0 & 0 & \cdots & \frac{d_{(0,s)}}{|\lambda|^{2s}} + \frac{\alpha^2_{(0,s+1)}}{|\lambda|^{2(s+1)}} \end{pmatrix}$$

Choose $k = (k_1, k_2)$ arbitrarily and fix it. Let α_k be perturbed to the weight to x. For commutativity, β_k is changed to $y = \frac{\beta_k x}{\alpha_k}$, $\alpha_{k-\varepsilon_1}$ is changed to $z = \frac{\alpha_{k-\varepsilon_1}\alpha_k}{x}$ and $\beta_{k-\varepsilon_2}$ is changed to $t = \frac{\beta_{k-\varepsilon_2}\alpha_k}{x}$.

Let $\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$ be the perturbed shift with weight sequences $\{\tilde{\alpha}_\tau\}_{\tau\in\mathbb{Z}^2_+}$ and $\{\tilde{\beta}_\tau\}_{\tau\in\mathbb{Z}^2_+}$ as defined in section 6.2.1. Also just as Δ_τ and L_s are defined with respect to T, in a similar way $\tilde{\Delta}_\tau$ and $\tilde{L}_s$ are defined for $\tilde{T}$. As T is hyponormal so $L_s \geq 0$ for all $s \in \mathbb{Z}_+$. Also $\tilde{L}_s = L_s$ for $s < |k| - 2$ and $s > |k|$. So if we can show that $L_s \geq 0$ for $|k| - 2 \leq s \leq |k|$, then by Theorem 6.3.3 we can conclude that $\tilde{T}$ is weakly hyponormal.

For example if $k = (2, 2)$ then $\tilde{L}_2, \tilde{L}_3, \tilde{L}_4$ can be represented by the following weight diagram:

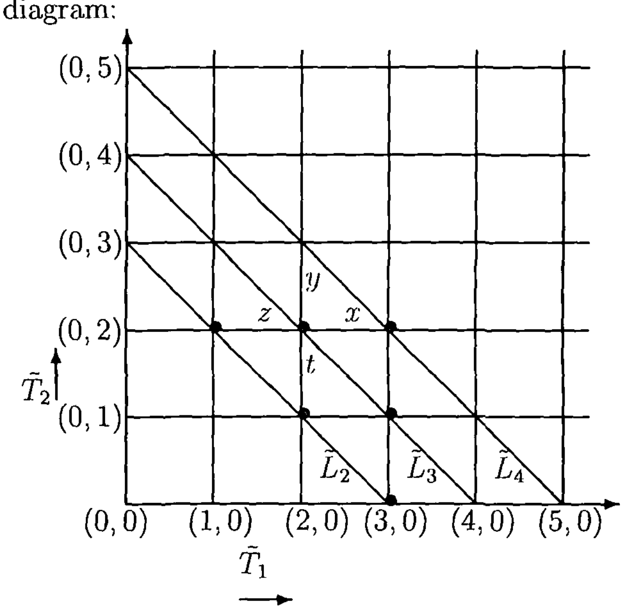


Figure 18

Theorem 6.4.1. *Let $T = (T_1, T_2)$ be hyponormal with weight sequences $\{\alpha_\tau\}_{\tau \in Z_+^2}$ and $\{\beta_\tau\}_{\tau \in Z_+^2}$. Then for any $k \in \mathbb{Z}_+^2$, a slight perturbation of the weight α_k makes the perturbed shift $\tilde{T}$ weakly hyponormal (assuming that β_k, $\alpha_{k-\epsilon_1}$ and $\beta_{k-\epsilon_2}$ are also necessarily perturbed to preserve commutativity).*

Proof. Let

$$g_0(x) := \tilde{a}_{(s,0)} + |\lambda|^2 \tilde{\beta}_{(s+1,0)}^2 > 0.$$

$$\begin{aligned} g_1(x) &:= det \begin{pmatrix} \tilde{a}_{(s,0)} + |\lambda|^2 \tilde{\beta}_{(s+1,0)}^2 & \tilde{b}_{(s,0)} \\ \tilde{b}_{(s,0)} & \tilde{d}_{(s,0)} + \frac{\tilde{a}_{(s-1,1)}}{|\lambda|^2} \end{pmatrix} \\ &= \frac{\tilde{a}_{(s-1,1)}}{|\lambda|^2} g_0(x) + det\tilde{\Delta}_{(s,0)} + |\lambda|^2 \tilde{\beta}_{(s+1,0)}^2 \tilde{d}_{(s,0)} \\ &> det\tilde{\Delta}_{(s,0)}. \end{aligned}$$

$$\begin{aligned} g_2(x) &:= det \begin{pmatrix} \tilde{a}_{(s,0)} + |\lambda|^2 \tilde{\beta}_{(s+1,0)}^2 & \tilde{b}_{(s,0)} & 0 \\ \tilde{b}_{(s,0)} & \tilde{d}_{(s,0)} + \frac{\tilde{a}_{(s-1,1)}}{|\lambda|^2} & \frac{\tilde{b}_{(s-1,1)}}{|\lambda|^2} \\ 0 & \frac{\tilde{b}_{(s-1,1)}}{|\lambda|^2} & \frac{\tilde{d}_{(s-1,1)}}{|\lambda|^2} + \frac{\tilde{a}_{(s-2,2)}}{|\lambda|^4} \end{pmatrix} \\ &= \frac{\tilde{\alpha}_{(s-2,2)}}{|\lambda|^4} g_1(x) + \frac{\tilde{d}_{(s-1,1)}}{|\lambda|^2} g_1(x) - \frac{\tilde{b}_{(s-1,1)}^2}{|\lambda|^4} g_0(x) \\ &= \frac{\tilde{a}_{(s-2,2)}}{|\lambda|^4} g_1(x) + det\tilde{\Delta}_{(s-1,1)} \frac{g_0(x)}{|\lambda|^4} + \frac{\tilde{d}_{(s-1,1)}}{|\lambda|^2} \left(det\tilde{\Delta}_{(s,0)} + |\lambda|^2 \tilde{\beta}_{(s+1,0)}^2 \tilde{d}_{(s,0)} \right) \\ &= \frac{\tilde{a}_{(s-2,2)}}{|\lambda|^4} g_1(x) + det\tilde{\Delta}_{(s-1,1)} \frac{g_0(x)}{|\lambda|^4} + \frac{\tilde{d}_{(s-1,1)}}{|\lambda|^2} det\tilde{\Delta}_{(s,0)} + \tilde{d}_{(s-1,1)} \tilde{d}_{(s,0)} \tilde{\beta}_{(s+1,0)}^2 \end{aligned}$$

$$g_3(x) := det \begin{pmatrix} \tilde{a}_{(s,0)} + |\lambda|^2 \tilde{\beta}_{(s+1,0)}^2 & \tilde{b}_{(s,0)} & 0 & 0 \\ \tilde{b}_{(s,0)} & \tilde{d}_{(s,0)} + \frac{\tilde{a}_{(s-1,1)}}{|\lambda|^2} & \frac{\tilde{b}_{(s-1,1)}}{|\lambda|^2} & 0 \\ 0 & \frac{\tilde{b}_{(s-1,1)}}{|\lambda|^2} & \frac{\tilde{d}_{(s-1,1)}}{|\lambda|^2} + \frac{\tilde{a}_{(s-2,2)}}{|\lambda|^4} & \frac{\tilde{b}_{(s-2,2)}}{|\lambda|^4} \\ 0 & 0 & \frac{\tilde{b}_{(s-2,2)}}{|\lambda|^4} & \frac{\tilde{d}_{(s-2,2)}}{|\lambda|^4} + \frac{\tilde{a}_{(s-3,3)}}{|\lambda|^6} \end{pmatrix}$$

$$= \frac{\tilde{a}_{(s-3,3)}}{|\lambda|^6} g_2(x) + det\tilde{\Delta}_{(s-2,2)} \frac{g_1(x)}{|\lambda|^8} + det\tilde{\Delta}_{(s-1,1)} \tilde{d}_{(s-2,2)} \frac{g_0(x)}{|\lambda|^8}$$
$$+ det\tilde{\Delta}_{(s,0)} \frac{\tilde{d}_{(s-1,1)}\tilde{d}_{(s-2,2)}}{|\lambda|^6} + \tilde{\beta}^2_{(s+1,0)} \frac{\tilde{d}_{(s,0)}\tilde{d}_{(s-1,1)}\tilde{d}_{(s-2,2)}}{|\lambda|^4}.$$

Similarly,

$$g_4(x) := \frac{\tilde{a}_{(s-4,4)}}{|\lambda|^8} g_3(x) + det\tilde{\Delta}_{(s-3,3)} \frac{g_2(x)}{|\lambda|^{12}} + det\tilde{\Delta}_{(s-2,2)} \tilde{d}_{(s-3,3)} \frac{g_1(x)}{|\lambda|^{14}}$$
$$+ det\tilde{\Delta}_{(s-1,1)} \tilde{d}_{(s-2,2)} \tilde{d}_{(s-3,3)} \frac{g_0(x)}{|\lambda|^{14}} + det\tilde{\Delta}_{(s,0)} \frac{\tilde{d}_{(s-1,1)}\tilde{d}_{(s-2,2)}\tilde{d}_{(s-3,3)}}{|\lambda|^{12}}$$
$$+ \tilde{\beta}^2_{(s+1,0)} \frac{\tilde{d}_{(s,0)}\tilde{d}_{(s-1,1)}\tilde{d}_{(s-2,2)}\tilde{d}_{(s-3,3)}}{|\lambda|^{10}}.$$

For $j = 0, 1, \ldots, s+1$, let M_j denote the $(j+1) \times (j+1)$ leading submatrix of $\tilde{L}_s$. If $g_j(x) := detM_j$ then

$$g_0(x) = \tilde{a}_{(s,0)} + |\lambda|^2 \tilde{\beta}^2_{(s+1,0)},$$
$$g_1(x) = \frac{\tilde{a}_{(s-1,1)}}{|\lambda|^2} g_0(x) + det\tilde{\Delta}_{(s,0)} + |\lambda|^2 \tilde{\beta}^2_{(s+1,0)} \tilde{d}_{(s,0)}$$

For $j = 2, 3, \ldots, s$, $g_j(x)$ is

$$g_j(x) := \frac{\tilde{a}_{(s-j,j)}}{|\lambda|^{2j}} g_{j-1}(x) + det\tilde{\Delta}_{(s-j+1,s-1)} \frac{g_{j-2}(x)}{|\lambda|^{4(j-1)}} + \sum_{l=2}^{j-1} \left(det\tilde{\Delta}_{(s-j+l,j-l)} \right)$$
$$\frac{\prod_{r=1}^{l-1} \tilde{d}_{(s-j+r,j-r)} g_{(j-l-1)}(x)}{|\lambda|^{4(j-l)+(l-1)(2j+l)}} + det\tilde{\Delta}_{(s,0)} \frac{\prod_{r=1}^{j-1} \tilde{d}_{(s-r,r)}}{|\lambda|^{j(j-1)}} + |\lambda|^2 \tilde{\beta}^2_{(s+1,0)} \frac{\prod_{r=0}^{j-1} \tilde{d}_{(s-r,r)}}{|\lambda|^{j(j-1)}}$$

and

$$g_{s+1}(x) := \frac{\tilde{\alpha}^2_{(0,s+1)}}{|\lambda|^{2(s+1)}} g_s(x) + det\tilde{\Delta}_{(0,s)} \frac{g_{s-1}(x)}{|\lambda|^{4s}} + \frac{\tilde{d}_{(0,s)}}{|\lambda|^{2s}} \Bigg(det\tilde{\Delta}_{(1,s-1)} \frac{g_{s-2}(x)}{|\lambda|^{4(s-1)}}$$
$$+ \sum_{l=2}^{s-1} \left(det\tilde{\Delta}_{(l,s-l)} \right) \frac{\prod_{r=1}^{l-1} \tilde{d}_{(r,s-r)} g_{(s-l-1)}(x)}{|\lambda|^{4(s-l)+(l-1)(2s+l)}} + det\tilde{\Delta}_{(s,0)} \frac{\prod_{r=1}^{s-1} \tilde{d}_{(s-r,r)}}{|\lambda|^{s(s-1)}}$$
$$+ |\lambda|^2 \tilde{\beta}^2_{(s+1,0)} \frac{\prod_{r=0}^{s-1} \tilde{d}_{(s-r,r)}}{|\lambda|^{s(s-1)}} \Bigg).$$

At $x = \alpha_k$, we have $det\tilde{\Delta}_\tau = det\Delta_\tau \geq 0$ for all $\tau \in \mathbb{Z}_+^2$. Also as $g_0(\alpha_k) > 0$, hence for all $j = 1, \ldots, s$, we have

$$g_j(\alpha_k) \geq \frac{a_{(s-j,j)}}{|\lambda|^{2j}} g_{j-1}(\alpha_k) > 0.$$

Similarly

$$g_{s+1}(\alpha_k) = \frac{\alpha^2_{(0,s+1)}}{|\lambda|^{2(s+1)}} g_s(\alpha_k) > 0.$$

Thus by continuity of g_j there exists $\delta_k > 0$ such that $g_j(x) > 0$ for all $x \in \big(\alpha_k - \delta_k, \alpha_k + \delta_k\big)$, which implies that $\tilde{L}_s \geq 0$. So by Theorem 6.3.3, $\tilde{T}$ is weakly hyponormal for any slight perturbation of α_k.

□

Chapter 7

Back-step extension of weighted shifts

7.1 Introduction

If W_α is a hyponormal weighted shift with weight sequence α, then for any subsequence β of α, W_β is again a hyponormal shift on $\ell^2(\mathbb{Z}_+)$. We ask the question if this property carries over to a quadratic hyponormal or a positive quadratic hyponormal or a subnormal weighted shift W_α. In response, we come up with the following examples.

Example 7.1.1. *Consider the weight sequence* $\alpha : \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{43}{80}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots,$ *with a subsequence* $\beta : \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots,$ *which in turn has a subsequence* $\gamma : \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$ *Here W_α is q.h., W_β is not q.h. and W_γ is again q.h.*

Example 7.1.2. *Consider the weight sequence* $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots,$ *and a subsequence* $\beta : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{6}{7}}, \sqrt{\frac{8}{9}}, \ldots$ *Then W_α is p.q.h. but W_β is not p.q.h.*

However, for the case of a subnormal weighted shift W_α with weight sequence $\alpha := \{\alpha_n\}_{n=0}^{\infty}$, Curto in [12, Proposion 8] proposed a concrete set of conditions under which x can be suitably chosen so that for the weight sequence

$\beta : x, \alpha_0, \alpha_1, \ldots, W_\beta$ is again subnormal. This result is as follows:

Theorem 7.1.1. *[12] Let T be a weighted shift whose restriction to $\bigvee\{e_1, e_2, \ldots\}$ is subnormal, with associated measure μ. Then T is subnormal if and only if*

1. $\frac{1}{t} \in L^1(\mu)$ *and*
2. $\alpha_0^2 \leq \left(\|\frac{1}{t}\|_{L^1(\mu)}\right)^{-1}$

In particular, T is never subnormal if $\mu\{0\} > 0$.

This is referred to as the one-step backward extension of a one-variable subnormal weighted shift. Later an improved version of this result was given by Curto and Yoon [37, Proposition 1.5]. In the same paper they have also given the NASC for subnormal backward extension of a 2-variable weighted shift [37, Proposition 2.9]. However these results only deal with one-step backward extension. In this chapter we try to extend this idea to formulate conditions for existence of n-step backward extension of a subnormal weighted shift. We do this for both single variable weighted shifts as well as for two variable weighted shifts. We first derive our results using a technique similar to that of [12, 37]. However, in the last section of the chapter we show how these results can also be derived by using Schur product technique.

7.2 Backward extension for one variable weighted shifts

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$, a bounded sequence of positive real numbers, let W_α be the associated unilateral weighted shift on $\ell^2(\mathbb{Z}_+)$. The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0 \\ \alpha_0^2 \ldots \alpha_{k-1}^2 & \text{if } k > 0 \end{cases}$$

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger [8]: "W_α is subnormal if and only if there exists a probability measure ξ supported in $[0, \|W_\alpha\|^2]$ such that $\gamma_k(\alpha) := \alpha_0^2 \ldots \alpha_{k-1}^2 = \int t^k d\xi(t)$ $(\forall k \geq 1)$ ". ξ is called the Berger measure of W_α.
For instance, the Berger measures of U_+ and S_a are δ_1 and $(1-a^2)\delta_0 + a^2\delta_1$, respectively, where δ_x denotes the point mass probability measure with support the singleton $\{x\}$. Also we denote by $U_+ = shift(1, 1, 1, \ldots)$ the (unweighted) unilateral shift, and for $0 < a < 1$ we let $S_a := shift(a, 1, 1, \ldots)$.
Again, if W_α is subnormal, and if for $h \geq 1$ we let $M_h := \bigvee\{e_n : n \geq h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{M_h}$ is $\frac{1}{\gamma_h} t^h d\xi(t)$.
Consider $\gamma_k(W_\alpha)$ and $\gamma_k(W_\alpha|_{M_h})$ are as moments of the weighted shifts W_α and $W_\alpha|_{M_h}$ respectively. The moments are related as

$$\begin{aligned}\gamma_k(W_\alpha|_{M_h}) \equiv \alpha_h^2\alpha_{h+1}^2 \ldots \alpha_{h+k-1}^2 &= \frac{\gamma_{k+h}(W_\alpha)}{\alpha_0^2\alpha_1^2 \ldots \alpha_{h-1}^2} \\ &= \frac{\gamma_{k+h}(W_\alpha)}{\gamma_h(W_\alpha)},\end{aligned}$$

so that for all $k \geq 0$,

$$\int t^k d\eta_h(t) = \frac{1}{\gamma_h} \int t^{k+h} d\xi(t),$$

where η_h and ξ are the Berger's measure for the weighted shifts $W_\alpha|_{M_h}$ and W_α respectively.
Therefore

$$\eta_h(t) = \frac{1}{\gamma_h} t^h d\xi(t).$$

We begin by stating the one-step subnormal backward extension of a one-variable weighted shift.

Theorem 7.2.1. *(1-step backward extension) [37] Let T be a weighted shift whose restriction $T_M := T|_M$ to $M := \bigvee\{e_1, e_2, \ldots\}$ is subnormal, with associated measure μ_M. Then T is subnormal (with associated measure μ) if and only if*

1. $\frac{1}{t} \in L^1(\mu_M)$
2. $\alpha_0^2 \leq \left(\|\frac{1}{t}\|_{L^1(\mu_M)}\right)^{-1}$

In this case, $d\mu(t) = \frac{\alpha_0^2}{t} d\mu_M(t) + \left(1 - \alpha_0^2 \|\frac{1}{t}\|_{L^1(\mu_M)}\right) d\delta_0(t)$ where δ_0 denotes Dirac measure at 0. In particular, T is never subnormal when $\mu_M(\{0\}) > 0$.

Theorem 7.2.2. *(2-step backward extension) Let T be a weighted shift whose restriction $T|_{M_2}$ to $M_2 := \bigvee\{e_2, e_3, \ldots\}$ is subnormal, with associate measure η_2. Then T is subnormal (with associate measure η) if and only if*

1. $\frac{1}{t^2} \in L^1(\eta_2)$
2. $\alpha_0^2\alpha_1^2 \leq \left(\|\frac{1}{t^2}\|_{L^1(\eta_2)}\right)^{-1}$
3. $\alpha_1^2 = \left(\|\frac{1}{t}\|_{L^1(\eta_2)}\right)^{-1}$

In this case, $d\eta(t) = \left(1 - \alpha_0^2\alpha_1^2 \|\frac{1}{t^2}\|_{L^1(\eta_2)}\right) d\delta_0(t) + \frac{\alpha_0^2\alpha_1^2}{t^2} d\eta_2(t)$, where δ_0 denotes the Dirac measure at 0. In particular, T is never subnormal if $\eta_2(\{0\}) > 0$

Proof $\Longrightarrow$) Assume that T is subnormal, so clearly $T|_{M_2}$ is subnormal The moments of T and $T|_{M_2}$ are related by the equation

$$\gamma_k(T|_{M_2}) \equiv \alpha_2^2\alpha_3^2 \ldots \alpha_{k+1}^2 = \frac{\gamma_{k+2}(T)}{\alpha_0^2\alpha_1^2}$$

so that for all $k \geq 0$,

$$\int t^k d\eta_2(t) = \frac{1}{\alpha_0^2\alpha_1^2} \int t^{k+2} d\eta(t)$$

that is, $d\eta_2(t) = \frac{t^2}{\alpha_0^2\alpha_1^2} d\eta(t)$. Let $\eta(0) = \lambda$, $(\lambda \geq 0)$, so it follows at once that

$$\begin{aligned} d\eta(t) &= \lambda \, d\delta_0(t) + \frac{\alpha_0^2\alpha_1^2}{t^2} d\eta_2(t) \qquad (7.2.1)\\ \Rightarrow \int d\eta(t) &= \lambda \int d\delta_0(t) + \alpha_0^2\alpha_1^2 \int \frac{1}{t^2} d\eta_2(t) \\ \Rightarrow \quad 1 &= \lambda + \alpha_0^2\alpha_1^2 \left\| \frac{1}{t^2} \right\|_{L^1(\eta_2)} \end{aligned}$$

that is $\alpha_0^2\alpha_1^2 \|\frac{1}{t^2}\|_{L^1(\eta_2)} = 1 - \lambda \leq 1$, also $\frac{1}{t^2} \in L^1(\eta_2)$. Also, substituting the value of λ in (7.2.1), we have

$$d\eta(t) = \left(1 - \alpha_0^2\alpha_1^2 \left\| \frac{1}{t^2} \right\|_{L^1(\eta_2)}\right) d\delta_0(t) + \frac{\alpha_0^2\alpha_1^2}{t^2} d\eta_2(t).$$

Again, suppose η_1 is the measure associated with the shift $T|_{M_1}$, where $M_1 := \bigvee\{e_1, e_2, \dots\}$. Then by Theorem 7.2.1, subnormality of $T|_{M_1}$ and $T|_{M_2}$ will imply that

$$\frac{1}{t} \in L^1(\eta_2), \; \alpha_1^2 \leq \left(\left\| \frac{1}{t} \right\|_{L^1(\eta_2)} \right)^{-1}$$

and

$$d\eta_1(t) = \eta_1(0) \, d\delta_0(t) + \frac{\alpha_1^2}{t} d\eta_2(t), \text{ where } \eta_1(0) = \left(1 - \alpha_1^2 \left\| \frac{1}{t} \right\|_{L^1(\eta_2)}\right).$$

Now, suppose $\alpha_1^2 < (\|\frac{1}{t}\|_{L^1(\eta_2)})^{-1} \;\Rightarrow\; \eta_1(0) > 0$.

Which is a contradiction to the initial assumption that T is subnormal. Therefore, $\alpha_1^2 = (\|\frac{1}{t}\|_{L^1(\eta_2)})^{-1}$.

$\Longleftarrow$) Let conditions 1, 2 and 3 hold and

$$d\eta(t) = \left(1 - \alpha_0^2\alpha_1^2 \left\| \frac{1}{t^2} \right\|_{L^1(\eta_2)}\right) d\delta_0(t) + \frac{\alpha_0^2\alpha_1^2}{t^2} \, d\eta_2(t) \qquad (7.2.2)$$

For $k = 0$,

$$\int d\eta(t) = \left(1 - \alpha_0^2\alpha_1^2 \left\|\frac{1}{t^2}\right\|_{L^1(\eta_2)}\right) \int d\delta_0(t) + \alpha_0^2\alpha_1^2 \int \frac{1}{t^2}\, d\eta_2(t)$$

$$\Rightarrow \int d\eta(t) = \left(1 - \alpha_0^2\alpha_1^2 \left\|\frac{1}{t^2}\right\|_{L^1(\eta_2)}\right) + \alpha_0^2\alpha_1^2 \left\|\frac{1}{t^2}\right\|_{L^1(\eta_2)}$$

$$\Rightarrow \int d\eta(t) = 1 = \gamma_0(T)$$

For $k = 1$, using (7.2.2) we have

$$\int t\, d\eta(t) = \int \frac{\alpha_0^2\alpha_1^2}{t}\, d\eta_2(t) = \alpha_0^2\, \alpha_1^2 \left\|\frac{1}{t}\right\|_{L^1(\eta_2)} = \alpha_0^2 \qquad \left(\because\ \alpha_1^2 \left\|\frac{1}{t}\right\|_{L^1(\eta_2)} = 1\right)$$
$$= \gamma_1(T)$$

For $k \geq 2$,

$$\int t^k\, d\eta(t) = \alpha_0^2\alpha_1^2 \int t^{k-2} d\eta_2(t) = \alpha_0^2\alpha_1^2\gamma_{k-2}(T|_{M_2}) = \gamma_k(T)$$

Thus T is subnormal with Berger measure η.

Also if $\eta_2(0) > 0$ will imply that $T|_{M_1}$ is not subnormal, therefore T is not subnormal. □

A similar argument will yield the NASC for 3-step backward extension, and in general, the n-step subnormal backward extension of a 1-variable weighted shift will be as follows:

Theorem 7.2.3. *(n-step backward extension) For $n \geq 2$, let T be a weighted shift whose restriction $T|_{M_n}$ to $M_n := \bigvee\{e_n, e_{n+1}, \ldots\}$ is subnormal, with associate measure η_n. Then T is subnormal (with associate measure η) if and only if*

1. $\frac{1}{t^n} \in L^1(\eta_n)$

2. $\alpha_0^2\alpha_1^2\dots\alpha_{n-1}^2 \leq \left(\|\frac{1}{t^n}\|_{L^1(\eta_n)}\right)^{-1}$

3. $\alpha_i^2\alpha_{i+1}^2\dots\alpha_{n-1}^2 = \left(\|\frac{1}{t^{n-i}}\|_{L^1(\eta_n)}\right)^{-1}$ *for* $1 \leq i \leq n-1$.

In this case,

$$d\eta(t) = \left(1 - \alpha_0^2\alpha_1^2\dots\alpha_{n-1}^2 \left\|\frac{1}{t^n}\right\|_{L^1(\eta_n)}\right) d\delta_0(t) + \frac{\alpha_0^2\alpha_1^2\dots\alpha_{n-1}^2}{t^n}\, d\eta_n(t),$$

where δ_0 denotes the Dirac measure at 0. In particular, T is never subnormal if $\eta_n(\{0\}) > 0$.

Corollary 7.2.4. *Let T be a subnormal weighted shift and for $j \geq 2$, let $M_j := \bigvee\{e_j, e_{j+1}, \dots\}$. Let η_j denote the Berger measure of $T|_{M_j}$. Then $\alpha_1, \alpha_2, \dots, \alpha_{j-1}$ is completely determined by η_j that is, $\alpha_{j-1}^2 = \left(\|\frac{1}{t}\|_{L^1(\eta_j)}\right)^{-1}$.*
Also, if T is subnormal then condition 3 of Theorem 7.2.3 imply that

$$\left\|\frac{1}{t^{n-i}}\right\|_{L^1(\eta_n)} = \left\|\frac{1}{t}\right\|_{L^1(\eta_{i+1})} \left\|\frac{1}{t}\right\|_{L^1(\eta_{i+2})} \cdots \left\|\frac{1}{t}\right\|_{L^1(\eta_n)} \quad \textit{for } 1 \leq i \leq n.$$

7.3 Backstep extension of 2-variable weighted shifts

A 2-variable weighted shift $T = (T_1, T_2)$ is said to be subnormal if it admits a commuting normal extension. Equivalently, $T = (T_1, T_2)$ is subnormal if there exist normal operators N_1 and N_2 such that N_i is a normal extension of T_i $(i = 1, 2)$ and N_1, N_2 commute. Clearly, each component T_i of a subnormal 2-variable weighted shift $T = (T_1, T_2)$ must be subnormal.

Theorem 7.3.1. *[67](Berger's theorem for 2-variable case) A 2-variable weighted shift $T = (T_1, T_2)$ admits a commuting normal extension if and only if there is a probability measure μ defined on the 2-dimensional rectangle $R =$*

$[0, a_1] \times [0, a_2]$, $(a_i := \|T_i\|^2)$ *such that*

$$\gamma_k = \int\int_R t^k d\mu(t) := \int\int_R t_1^{k_1} t_2^{k_2} d\mu(t_1, t_2) (\forall\, k \in \mathbb{Z}_+^2).$$

We also include a few more definitions and results that are to be used in the sequel.

Definition 7.3.1. [37] Let μ and ν be two positive measures on $\mathbb{R}_+$. We say that $\mu \le \nu$ on $X := \mathbb{R}_+$ if $\mu(E) \le \nu(E)$ for all Borel subset $E \subseteq \mathbb{R}_+$; equivalently, $\mu \le \nu$ if and only if $\int f d\mu \le \int f d\nu$ for all $f \in C(X)$ such that $f \ge 0$ on $\mathbb{R}_+$.

Definition 7.3.2. [37] Let μ be the positive measures on $X \times Y \equiv \mathbb{R}_+ \times \mathbb{R}_+$, and assume $\frac{1}{t} \in L^1(\mu)$. The extremal measure μ_{ext} (which is a probability measure) on $X \times Y$ is given by $d\mu_{ext}(s,t) := (1 - \delta_0(t)) \frac{1}{t\|\frac{1}{t}\|_{L^1(\mu)}} d\mu(s,t)$

Definition 7.3.3. [37] Given a measure μ on $X \times Y$, the marginal measure μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection on X. Thus $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$. If μ is a probability measure, then so is μ^X.

Lemma 7.3.2. *[37] Let μ be the Berger measure of 2-variable weighted shift T and let ξ be the Berger measure of the shift$(\alpha_{(0,0)}, \alpha_{(1,0)}, \cdots)$. Then $\xi = \mu^X$. As a consequence $\int\int f(s) d\mu(s,t) = \int f(s) d\mu^X(s)$ for all $f \in C(X)$.*

Corollary 7.3.3. *[37] Let μ be the Berger measure of a 2-variable weighted shift T. For $j \ge 1$, let $d\mu_j(s,t) = \frac{1}{\gamma_{(0,j)}} t^j d\mu(s,t)$. Then the Berger measure of the shift$(\alpha_{(0,j)}, \alpha_{(1,j)}, \cdots)$ is $\xi_j = \mu_j^X$.*

Lemma 7.3.4. *[37] Let μ and ω be two measures on $X \times Y$, and assume that $\mu \le \omega$. Then $\mu^X \le \omega^X$.*

Lemma 7.3.5. *Let μ be a positive measure on $\mathbb{R}_+ \times \mathbb{R}_+$ such that $\mu(E \times \{0\}) = 0$ for all Borel sets $E \subseteq \mathbb{R}_+$. For $n \geq 1$, let $\frac{1}{t^n} \in L^1(\mu)$. Then the extremal measure $\mu_{(ext)^n}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ is given by*

$$d\mu_{(ext)^n}(s,t) := \frac{1-\delta_0(t)}{t^n \left\|\frac{1}{t^n}\right\|_{L^1(\mu)}} d\mu(s,t).$$

Proof. For $n = 1$, $\frac{1}{t} \in L^1(\mu)$ and we have

$$d\mu_{(ext)}(s,t) := \frac{1-\delta_0(t)}{t \left\|\frac{1}{t}\right\|_{L^1(\mu)}} d\mu(s,t) \text{ (by Definition 7.3.2)}$$

Suppose result is true for n i.e.,

$$d\mu_{(ext)^n}(s,t) := \frac{1-\delta_0(t)}{t^n \left\|\frac{1}{t^n}\right\|_{L^1(\mu)}} d\mu(s,t).$$

Let $\frac{1}{t^{n+1}} \in L^1(\mu)$. Then,

$$\begin{aligned}
\iint \frac{1}{t}\, d\mu_{(ext)^n}(s,t) &= \iint \frac{1-\delta_0(t)}{t^{n+1}\|\frac{1}{t^n}\|_{L^1(\mu)}}\, d\mu(s,t) \\
&= \iint \frac{1}{t^{n+1}\|\frac{1}{t^n}\|_{L^1(\mu)}}\, d\mu(s,t) \\
&\qquad (\because\ \mu(E \times \{0\}) = 0,\ \forall E \subseteq \mathbb{R}_+) \\
&= \frac{\|\frac{1}{t^{n+1}}\|_{L^1(\mu)}}{\|\frac{1}{t^n}\|_{L^1(\mu)}} < \infty
\end{aligned}$$

Therefore

$$\frac{1}{t} \in L^1(\mu)_{(ext)^n} \text{ and } \left\|\frac{1}{t}\right\|_{L^1(\mu)_{(ext)^n}} \left\|\frac{1}{t^n}\right\|_{L^1(\mu)} = \left\|\frac{1}{t^{n+1}}\right\|_{L^1(\mu)}.$$

Now as $\frac{1}{t} \in L^1(\mu)_{(ext)^n}$, so by Definition 7.3.2,

$$\begin{aligned}
d\mu_{(ext)^{n+1}}(s,t) &:= \frac{1-\delta_0(t)}{t \left\|\frac{1}{t}\right\|_{L^1(\mu)_{(ext)^n}}} d(\mu)_{(ext)^n}(s,t) \\
&= \frac{1-\delta_0(t)}{t^{n+1} \left\|\frac{1}{t}\right\|_{L^1(\mu)_{(ext)^n}} \left\|\frac{1}{t^n}\right\|_{L^1(\mu)}} d\mu(s,t) \\
&= \frac{1-\delta_0(t)}{t^{n+1} \left\|\frac{1}{t^{n+1}}\right\|_{L^1(\mu)}} d\mu(s,t)
\end{aligned}$$

Thus the result hold (by induction) for all $n = 1, 2, \ldots$ □

Theorem 7.3.6. *(1-step backward extension) [37] Let $T = (T_1, T_2)$ be a 2-variable weighted shift and M be the subspace of $\ell^2(\mathbb{Z}_+^2)$ associated to indices $k = (k_1, k_2)$ with $k_2 \geq 1$. Let $T_M := T|_M$ be subnormal with associated measure μ_M and let $W_0 := shift(\alpha_{(0,0)}, \alpha_{(1,0)}, \cdots)$ is subnormal with associated measure ν. Then T is subnormal if and only if*

1. $\frac{1}{t} \in L^1(\mu_M)$
2. $\beta^2_{(0,0)} \leq \left(\left\|\frac{1}{t}\right\|_{L^1(\mu_M)}\right)^{-1}$
3. $\beta^2_{(0,0)} \left\|\frac{1}{t}\right\|_{L^1(\mu_M)} (\mu_M)^X_{ext} \leq \nu$

Moreover, if $\beta^2_{(0,0)}\|\frac{1}{t}\|_{L^1(\mu_M)} = 1$, then $(\mu_M)^X_{ext} = \nu$. In the case when T is subnormal, the Berger measure μ of T is given by

$$d\mu(s,t) = \beta^2_{(0,0)} \left\|\frac{1}{t}\right\|_{L^1(\mu_M)} d(\mu_M)_{ext}(s,t) + \left(d\nu(s) - \beta^2_{(0,0)} \left\|\frac{1}{t}\right\|_{L^1(\mu_M)} d(\mu_m)^X_{ext}(s)\right) d\delta_0(t).$$

Theorem 7.3.7. *(2-step backward extension) Let $T = (T_1, T_2)$ be a 2-variable weighted shift with the weight sequences α and β. Assume that $T|_{M_2}$ the restriction of T to $M_2 := \bigvee\{e_{(k_1,k_2)} : k_2 \geq 2\}$ is subnormal with associated measure μ_2. Let $W_0 := shift(\alpha_{(0,0)}, \alpha_{(1,0)}, \cdots)$ and $W_1 := shift(\alpha_{(0,1)}, \alpha_{(1,1)}, \cdots)$ be subnormal with associated measures ξ_0 and ξ_1 respectively. Then T is subnormal with associated measure μ if and only if*

1. $\frac{1}{t^2} \in L^1(\mu_2)$
2. $\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \leq 1$
3. $\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} (\mu_2)^X_{(ext)^2} \leq \xi_0$
4. $\beta^2_{(0,1)} \left\|\frac{1}{t}\right\|_{L^1(\mu_2)} = 1$

5. $(\mu_2)^X_{ext} = \xi_1$

Moreover, if $\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} = 1$, *then* $(\mu_2)^X_{(ext)^2} = \xi_0$. *In the case when* T *is subnormal, the Berger measure* μ *of* T *is given by,*

$$\mu = \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} (\mu_2)_{(ext)^2} + \left(\xi_0 - \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} d(\mu_2)^X_{(ext)^2}\right) \times \delta_0$$

Proof. $\Longrightarrow$) Let T be subnormal. Then $T|_{M_1}$ and $T|_{M_2}$ are also subnormal with the corresponding Berger measures μ_1 and μ_2 respectively. The moments are related as follows:

$$\gamma_{(k_1,\,k_2+1)}(T) = \beta^2_{(0,0)}\gamma_{(k_1,k_2)}(T|_{M_1})$$

$$\gamma_{(k_1,\,k_2+2)}(T) = \beta^2_{(0,0)}\beta^2_{(0,1)}\gamma_{(k_1,k_2)}(T|_{M_2})$$

Therefore, the subnormality of T, $T|_{M_1}$ and $T|_{M_2}$ imply that

$$t\,d\mu(s,t) = \beta^2_{(0,0)}d\mu_1(s,t) \tag{7.3.1}$$

$$t^2\,d\mu(s,t) = \beta^2_{(0,0)}\beta^2_{(0,1)}d\mu_2(s,t) \tag{7.3.2}$$

Therefore, $\mu_1(E\times\{0\}) = 0$, $\mu_2(E\times\{0\}) = 0$, $\forall\, E \subseteq \mathbb{R}_+$.

Now,

$$\begin{aligned}\iint \frac{1}{t^2}\,d\mu_2(s,t) = \iint_{t>0} \frac{1}{t^2}\,d\mu_2(s,t) &= \frac{1}{\beta^2_{(0,0)}\beta^2_{(0,1)}}\iint_{t>0} d\mu(s,t)\\ &= \frac{1}{\beta^2_{(0,0)}\beta^2_{(0,1)}}\,\mu(t>0) \qquad (7.3.3)\\ &\le \frac{1}{\beta^2_{(0,0)}\beta^2_{(0,1)}}\end{aligned}$$

So, $\frac{1}{t^2} \in L^1(\mu_2)$ and $\beta^2_{(0,0)}\beta^2_{(0,1)}\|\frac{1}{t^2}\|_{L^1(\mu_2)} \le 1$, which establishes 1 and 2.

For arbitrary Borel sets $E \subseteq \mathbb{R}_+$ and $F \subseteq \mathbb{R}_+$, we have

$$\begin{aligned}
\beta^2_{(0,0)}\beta^2_{(0,1)}&\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)}(\mu_2)_{(ext)^2}(E\times F)\\
&=\beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)}\iint_{E\times F} d(\mu_2)_{(ext)^2}(s,t)\\
&=\beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)}\iint_{E\times F}(1-\delta_0(t))\frac{1}{t^2\,\|\frac{1}{t^2}\|_{L^1(\mu_2)}}\,d\mu_2(s,t)\\
&=\iint_{E\times(F\setminus\{0\})}\beta^2_{(0,0)}\beta^2_{(0,1)}\frac{1}{t^2}\,d\mu_2(s,t)\\
&=\iint_{E\times(F\setminus\{0\})} d\mu(s,t)\\
&=\mu(E\times(F\setminus\{0\}))\le\mu\,(E\times F) \qquad (7.3.4)
\end{aligned}$$

and by Lemmas 7.3.4 and 7.3.2, $\beta^2_{(0,0)}\beta^2_{(0,1)}\|\frac{1}{t^2}\|_{L^1(\mu_2)}(\mu_2)^X_{(ext)^2}\le\mu^X=\xi_0$.

If $\beta^2_{(0,0)}\beta^2_{(0,1)}\|\frac{1}{t^2}\|_{L^1(\mu_2)}=1$ then by (7.3.3) $\mu(t>0)=1$, and so $\mu(E\times(F\setminus\{0\}))=\mu(E\times F)$. Therefore, from (7.3.4) we get $(\mu_2)_{(ext)^2}=\mu\Rightarrow(\mu_2)^X_{(ext)^2}=\xi_0$.

Again,

$$\begin{aligned}
\left\|\frac{1}{t}\right\|_{L^1(\mu_2)}&=\iint\frac{1}{t}\,d\mu_2(s,t)=\frac{1}{\beta^2_{(0,0)}\beta^2_{(0,1)}}\iint t\,d\mu(s,t)\\
&=\frac{\gamma_{(0,1)}(T)}{\beta^2_{(0,0)}\beta^2_{(0,1)}}=\frac{\beta^2_{(0,0)}}{\beta^2_{(0,0)}\beta^2_{(0,1)}}=\frac{1}{\beta^2_{(0,1)}}
\end{aligned}$$

which gives $\beta^2_{(0,1)}\|\frac{1}{t}\|_{L^1(\mu_2)}=1$, proving 4.

Since $T|_{M_1}$ is a 1-step subnormal extension of $T|_{M_2}$, and also $\beta^2_{(0,1)}\left\|\frac{1}{t}\right\|_{L^1(\mu_2)}=1$, so by Theorem 7.3.6, we have $\xi_1=(\mu_2)^X_{ext}$.

Finally from (7.3.2) we have $t^2\,d\mu(s,t)=\beta^2_{(0,0)}\beta^2_{(0,1)}d\mu_2(s,t)$. So if $\mu(s,0)=\lambda(s)$ then

$$d\mu(s,t)=d\lambda(s)\,d\delta_0(t)+\frac{\beta^2_{(0,0)}\beta^2_{(0,1)}}{t^2}d\mu_2(s,t)$$

$$\Rightarrow d\mu(s,t) = d\lambda(s)\, d\delta_0(t) + \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} d(\mu_2)_{(ext)^2}(s,t)$$

$$\Rightarrow \iint d\mu(s,t) = \int d\lambda(s) \int d\delta_0(t) + \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \iint d(\mu_2)_{(ext)^2}(s,t)$$

$$\Rightarrow \int d\mu^X(s) = \int d\lambda(s) + \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \int d(\mu_2)^X_{(ext)^2}(s)$$

$$\Rightarrow \int d\xi_0(s) = \int d\lambda(s) + \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \int d(\mu_2)^X_{(ext)^2}(s)$$

$$\Rightarrow d\xi_0(s) = d\lambda(s) + \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} d(\mu_2)^X_{(ext)^2}(s)$$

Therefore,

$$\begin{aligned} d\mu(s,t) =& \left(d\xi_0(s) - \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} d(\mu_2)^X_{(ext)^2}(s)\right) d\delta_0(t) \\ &+ \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} d(\mu_2)_{(ext)^2}(s,t). \end{aligned}$$

$\Longleftarrow$) Conditions 4 and 5 imply that T_{M_1} is subnormal with measure μ_1 such that $\mu_1(E \times \{0\}) = 0$ for all Borel sets $E \subseteq \mathbb{R}_+$.

Given conditions 1 to 2, let

$$\begin{aligned} \mu(s,t) :=& \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} (\mu_2)_{(ext)^2}(s,t) \\ &+ \left(\xi_0(s) - \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} (\mu_2)^X_{(ext)^2}(s)\right)\delta_0(t). \end{aligned}$$

If $\beta^2_{(0,0)}\beta^2_{(0,1)}\|\frac{1}{t^2}\|_{L^1(\mu_2)} = 1$ then total mass of the second summand is zero, and so $\mu := (\mu_2)_{(ext)^2}$.

For $j = 0$,

$$\begin{aligned} \iint s^i\, d(\mu)(s,t) =& \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \iint s^i\, d(\mu_2)_{(ext)^2}(s,t) \\ &+ \int s^i d\xi_0(s) - \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \int d(\mu_2)^X_{(ext)^2}(s) \end{aligned}$$

$$= \int s^i d\xi_0(s) \text{ (using Lemma 7.3.2)}$$
$$= \gamma_{(i,0)}(T)$$

For $j = 1$,

$$\begin{aligned}
\iint s^i t\, d(\mu)(s,t) &= \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \iint s^i t\, d(\mu_2)_{(ext)^2}(s,t) \\
&= \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \iint s^i t \frac{(1-\delta_0(t))}{t\, \|\frac{1}{t}\|_{L^1(\mu_2)_{ext}}}\, d(\mu_2)_{ext}(s,t) \\
&= \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t}\right\|_{L^1(\mu_2)} \iint s^i(1-\delta_0(t))\, d(\mu_2)_{ext}(s,t) \\
&= \beta^2_{(0,0)} \int s^i\, d(\mu_2)^X_{ext}(s) \text{ (using 4)} \\
&= \beta^2_{(0,0)} \int s^i\, d\xi_1(s) = \beta^2_{(0,0)}\alpha^2_{(0,1)} \cdots \alpha^2_{(i-1,1)} \\
&= \gamma_{(i,1)}(T)
\end{aligned}$$

For $j > 1$,

$$\begin{aligned}
\iint s^i t^j d(\mu)(s,t) &= \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \iint s^i t^j d(\mu_2)_{(ext)^2}(s,t) \\
&= \beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \iint s^i t^j (1-\delta_0(t)) \frac{1}{t^2\, \|\frac{1}{t^2}\|_{L^1(\mu_2)}}\, d\mu_2(s,t) \\
&= \beta^2_{(0,0)}\beta^2_{(0,1)} \iint s^i t^{j-2} d\mu_2(s,t) \\
&= \beta^2_{(0,0)}\beta^2_{(0,1)}\gamma_{(i,j-2)}(T|_{M_2}) = \gamma_{(i,j)}(T)
\end{aligned}$$

Hence, it follows that T is subnormal with Berger measure μ.

□

Theorem 7.3.8. *(n-step subnormal backward extension of a 2-variable weighted shift) Let $T = (T_1, T_2)$ be a 2-variable weighted shift with double indexed weight sequences $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}^2_+}$ and $\beta = \{\beta_k\}_{k\in\mathbb{Z}^2_+}$. For $n \geq 1$, let M_n be the subspace associated to the indices $k = (k_1, k_2)$ with $k_2 \geq n$. Assume* *that $T|_{M_n}$ is subnormal with associated measure μ_n. For $0 \le i \le n-1$ let $W_i := shift\{\alpha_{(0,i)}, \alpha_{(1,i)}, \ldots\}$ be subnormal with associated measures ξ_i respectively. Then T is subnormal if and only if*

1. $\frac{1}{t^n} \in L^1(\mu_n)$
2. $\prod_{j=0}^{n-1} \beta^2_{(0,j)} \left\|\frac{1}{t^n}\right\|_{L^1(\mu_n)} \le 1$
3. $\prod_{j=0}^{n-1} \beta^2_{(0,j)} \left\|\frac{1}{t^n}\right\|_{L^1(\mu_n)} (\mu_n)^X_{(ext)^n} \le \xi_0$
4. $\prod_{j=i}^{n-1} \beta^2_{(0,j)} \left\|\frac{1}{t^{n-i}}\right\|_{L^1(\mu_n)} = 1$ *for* $1 \le i \le n-1$
5. $(\mu_n)^X_{(ext)^i} = \xi_{n-i}$ *for* $1 \le i \le n-1$

Moreover, if $\prod_{j=0}^{n-1} \beta^2_{(0,j)} \left\|\frac{1}{t^n}\right\|_{L^1(\mu_n)} = 1$, then $(\mu_n)^X_{(ext)^n} = \xi_0$. In the case when T is subnormal, the Berger measure μ of T is given by,

$$\mu = \prod_{j=0}^{n-1} \beta^2_{(0,j)} \left\|\frac{1}{t^n}\right\|_{L^1(\mu_n)} (\mu_n)_{(ext)^n} + \left(\xi_0 - \prod_{j=0}^{n-1} \beta^2_{(0,j)} \left\|\frac{1}{t^n}\right\|_{L^1(\mu_n)} (\mu_n)^X_{(ext)^n}\right) \times \delta_0$$

The proof being similar to that of Theorem 7.3.7 is omitted.

7.4 Derivation of above results using Schur product technique

In this section we show that the above results can also be derived using Schur product technique.

Definition 7.4.1. $T = (T_1, \ldots, T_N)$, where each T_j acts on a Hilbert space H, is said to be unitarily equivalent to $S = (S_1, \ldots, S_N)$, where each S_j acts on a Hilbert space K, if there exists a unitary operator $\mathcal{U} : H \to K$ such that $\mathcal{U}^* S_j \mathcal{U} = T_j$ for $1 \le j \le N$.

For $L = (l, m)$ and $I = (i, j)$ in $\mathbb{Z}_+^2$, let $H_I := \bigvee_{(k_1, k_2) \in \mathbb{Z}_+^2} \left\{e_{(i+lk_1, j+mk_2)}\right\}$.

In the sequel, we choose $l, m \geq 1$ and $0 \leq i \leq l-1$, $0 \leq j \leq m-1$.

Explanation:

If $L = (1,1)$ then $i = j = 0$ and so $H_I = H_{(0,0)} = \bigvee_{(k_1, k_2) \in \mathbb{Z}_+^2} \{e_{(k_1, k_2)}\} = \ell^2(\mathbb{Z}_+^2)$.

If $L = (2,1)$ then $0 \leq i \leq 1$ and $j = 0$. As $H_{(0,0)} = \bigvee_{(k_1, k_2) \in \mathbb{Z}_+^2} \{e_{(2k_1, k_2)}\}$ and $H_{(1,0)} = \bigvee_{(k_1, k_2) \in \mathbb{Z}_+^2} \{e_{(1+2k_1, k_2)}\}$. So, $\ell^2(\mathbb{Z}_+^2) = H_{(0,0)} \bigoplus H_{(1,0)}$. Thus,

$$\ell^2(\mathbb{Z}_+^2) = \bigoplus_{j=0}^{m-1} \bigoplus_{i=0}^{l-1} H_{(i,j)}.$$

Definition 7.4.2. For $\delta = (\delta_{(k_1, k_2)}) \in \ell^\infty(\mathbb{Z}_+^2)$ define $P_{(L:I)} : \ell^\infty(\mathbb{Z}_+^2) \to \ell^\infty(\mathbb{Z}_+^2)$ as

$$P_{(L:I)}(\delta) = \left\{ \prod_{p=0}^{l-1} \delta_{(i+k_1 l+p, j+k_2 m)} \right\}_{(k_1, k_2) \in \mathbb{Z}_+^2}$$

and $Q_{(L:I)} : \ell^\infty(\mathbb{Z}_+^2) \to \ell^\infty(\mathbb{Z}_+^2)$ as

$$Q_{(L:I)}(\delta) = \left\{ \prod_{p=0}^{m-1} \delta_{(i+k_1 l, j+k_2 m+p)} \right\}_{(k_1, k_2) \in \mathbb{Z}_+^2}.$$

Definition 7.4.3. Define S_1 and S_2 on $\ell^\infty(\mathbb{Z}_+^2)$ as

$$(S_1\gamma)(k_1, k_2) = \gamma(k_1 + 1, k_2)$$

$$(S_2\gamma)(k_1, k_2) = \gamma(k_1, k_2 + 1)$$

for $\gamma = (\gamma_{(k_1, k_2)}) \in \ell^\infty(\mathbb{Z}_+^2)$. Note $S_1 S_2 = S_2 S_1$.

Proposition 7.4.1. *$P_{(L:(0,0))}\, S_1^i S_2^j = P_{(L:I)}$ and $Q_{(L:(0,0))}\, S_1^i S_2^j = Q_{(L:I)}$.*

Proof. $S_1^i S_2^j(\delta)(k_1, k_2) = \delta(k_1 + i, k_2 + j) = \tilde{\delta}(k_1, k_2)$ (say). Then

$$\begin{aligned} P_{(L:(0,0))}\, S_1^i S_2^j(\delta)(k_1, k_2) &= P_{(L:(0,0))}\tilde{\delta}(k_1, k_2) \\ &= \prod_{p=0}^{l-1} \tilde{\delta}(k_1 l + p,\ k_2 m) \end{aligned}$$

$$
\begin{aligned}
&= \prod_{p=0}^{l-1} \delta(i + k_1 l + p,\ j + k_2 m) \\
&= P_{(L\,I)}(\delta)(k_1, k_2)
\end{aligned}
$$

Similarly, $Q_{(L\,(0,0))}\, S_1^i S_2^j = Q_{(L\,I)}$. ◻

Given, $\alpha = \{\alpha_{(k_1,k_2)}\} \in \ell^\infty(\mathbb{Z}_+^2)$ and $\beta = \{\beta_{(k_1,k_2)}\} \in \ell^\infty(\mathbb{Z}_+^2)$, let $T = (T_1, T_2)$ be 2-variable weighted shift with weight sequences α and β, defined as

$$
\begin{aligned}
T_1\, e_{(k_1,k_2)} &= \alpha_{(k_1,k_2)} e_{(k_1+1,\, k_2)} \\
T_2\, e_{(k_1,k_2)} &= \beta_{(k_1,k_2)} e_{(k_1,\, k_2+1)}
\end{aligned}
$$

Let $T_{(L\,I)} = \big((T_{(L\,I)})_1, (T_{(L\,I)})_2\big)$ be 2-variable weighted shift with weight sequences $P_{(L\,I)}(\alpha)$ and $Q_{(L\,I)}(\beta)$, defined as

$$
\begin{aligned}
\big(T_{(L\,I)}\big)_1\, e_{(k_1,k_2)} &= \left\{\prod_{p=0}^{l-1} \alpha_{(i+k_1 l+p,\, j+k_2 m)}\right\} e_{(k_1+1,\, k_2)} \\
\big(T_{(L\,I)}\big)_2\, e_{(k_1,k_2)} &= \left\{\prod_{p=0}^{m-1} \beta_{(i+k_1 l,\, j+k_2 m+p)}\right\} e_{(k_1,\, k_2+1)}
\end{aligned}
$$

Now, $T^L := (T_1^l, T_2^m)$ and $T^L|_{H_I} := \big(T_1^l|_{H_I}, T_2^m|_{H_I}\big)$

Proposition 7.4.2. *T^L is unitarily equivalent to $\bigoplus_{j=0}^{m-1} \bigoplus_{i=0}^{l-1} T_{(L\,I)}$.*

Proof. Define $\mathcal{U} : \ell^2(\mathbb{Z}_+^2) \to H_I$ as $\mathcal{U}\, e_{(k_1,k_2)} = e_{(i+k_1 l,\, j+k_2 m)}$ Then for $e_{(k_1,k_2)} \in H_I$, $\mathcal{U}^*\, e_{(k_1,k_2)} = e_{\left(\frac{k_1-i}{l}, \frac{k_2-j}{m}\right)}$ and so $\mathcal{U}\mathcal{U}^* = I = \mathcal{U}^*\mathcal{U}$
Now, $T^{(l,m)} = (T_1^l, T_2^m)$ and $T^{(l,m)}|_{H_I} = (T_1^l|_{H_I}, T_2^m|_{H_I})$
As

$$
\mathcal{U}^* T_1^l|_{H_I} \mathcal{U}\, e_{(k_1,k_2)} = \left\{\prod_{p=0}^{l-1} \alpha_{(i+k_1 l+p,\, j+k_2 m)}\right\} e_{(k_1+1,\, k_2)} = (T_{(L\,I)})_1\, e_{(k_1,k_2)},
$$

so similarly,

$$
\mathcal{U}^* T_2^l|_{H_I} \mathcal{U}\, e_{(k_1,k_2)} = (T_{(L\,I)})_2\, e_{(k_1,k_2)}
$$

Thus by Definition 7.4.1, $\left(T_1^l|_{H_I}, T_2^m|_{H_I}\right) \cong \left((T_{(L\,I)})_1, (T_{(L\,I)})_2\right)$. That is, $T^L|_{H_I} \cong T_{(L\,I)}$.

As

$$\ell^2(\mathbb{Z}_+^2) = \bigoplus_{j=0}^{m-1}\bigoplus_{i=0}^{l-1} H_I,$$

so

$$T^L = \bigoplus_{j=0}^{m-1}\bigoplus_{i=0}^{l-1} T^L|_{H_I} \cong \bigoplus_{j=0}^{m-1}\bigoplus_{i=0}^{l-1} T_{(L\,I)}.$$

□

Corollary 7.4.3. **a)** *T^L is k-hyponormal if and only if $T_{(L\,I)}$ is k-hyponormal for all $0 \le i \le l-1$, $0 \le j \le m-1$.*

b) *T^L is subnormal if and only if $T_{(L\,I)}$ is subnormal for all $0 \le i \le l-1$, $0 \le j \le m-1$*

c)

$$\begin{aligned} T \text{ is subnormal} &\Rightarrow T^L = (T_1^l, T_2^m) \text{ is subnormal} \\ &\Rightarrow T^L|_{H_I} \text{ is subnormal for } 0 \le i \le l-1,\ 0 \le j \le m-1 \\ &\Rightarrow T_{(L\,I)} \text{ is subnormal for } 0 \le i \le l-1,\ 0 \le j \le m-1. \end{aligned}$$

We are now seek to identify the Berger measure $\mu_{(L\,I)}$ corresponding to $T_{(L\,I)}$

Theorem 7.4.4. *$d\mu_{(L\,I)}(s,t) = \frac{s^{i/l}\,t^{j/m}}{\gamma_{(i,j)}(T)}\, d\mu(s^{1/l}, t^{1/m}) = \frac{s^{i/l}\,t^{j/m}}{\gamma_{(i,j)}(T)}\, d\mu_{(L\,(0,0))}(s, t)$ for $0 \le i \le l-1$, $0 \le j \le m-1$. If $\mu(s,t) = \nu(s,t) + \rho(s)\delta_0(t)$, where $\nu(E \times \{0\}) = 0\ \forall\ E \subseteq \mathbb{R}_+$, then*

(a) $d\mu_{(L\,(i,0))}(s,t) = \dfrac{s^{i/l}}{\gamma_{(i,0)}(T)}\, d\mu(s^{1/l}, t^{1/m})$

(b) For $1 \le j \le m-1, d\mu_{(L\,I)}(s,t) = \dfrac{s^{i/l}\,t^{j/m}}{\gamma_{(i,j)}(T)}\, d\nu(s^{1/l}, t^{1/m})$

Proof. Let $\gamma_{(k_1,k_2)}(T)$ and $\gamma_{(k_1,k_2)}(T_{(L\,I)})$ denote the moment sequences related to T and $T_{(L\,I)}$ respectively.

Then

$$\begin{aligned}
\gamma_{(k_1,\,k_2)}(T_{(L\,I)}) &= \frac{\gamma_{(i+k_1l,\,j+k_2m)}(T)}{\gamma_{(i,j)}(T)} \\
\Rightarrow \iint s^{k_1}\, t^{k_2} d\mu_{(L\,I)}(s,t) &= \frac{1}{\gamma_{(i,j)}(T)} \iint s^{i+k_1l}\, t^{j+k_2m} d\mu(s,t) \\
&= \frac{1}{\gamma_{(i,j)}(T)} \iint s^{i/l}\, s^{k_1}\, t^{j/m} t^{k_2} d\mu\left(s^{1/l},\, t^{1/m}\right) \\
\Rightarrow \qquad d\mu_{(L\,I)}(s,t) &= \frac{s^{i/l}\, t^{j/m}}{\gamma_{(i,j)}(T)}\, d\mu\left(s^{1/l},\, t^{1/m}\right)
\end{aligned} \tag{7.4.1}$$

Also, $d\mu_{(L\,(0,0))}(s,\,t) = d\mu\left(s^{1/l},\, t^{1/m}\right)$. Therefore

$$d\mu_{(L\,I)}(s,t) = \frac{s^{i/l}\, t^{j/m}}{\gamma_{(i,j)}(T)}\, d\mu\left(s^{1/l},\, t^{1/m}\right) = \frac{s^{i/l}\, t^{j/m}}{\gamma_{(i,j)}(T)}\, d\mu_{(L\,(0,0))}(s,\,t)$$

for $0 \le i \le l-1,\ 0 \le j \le m-1$.

If $\mu(s,t) = \nu(s,t) + \rho(s)\delta_0(t)$, then from (7.4.1), we get

$$d\mu_{(L\,(i,0))}(s,t) = \frac{s^{i/l}}{\gamma_{(i,0)}(T)}\, d\mu\left(s^{1/l},\, t^{1/m}\right)$$

For $1 \le j \le m-1$,

$$\begin{aligned}
\iint s^{k_1}\, t^{k_2} d\mu_{(L\,I)}(s,t) &= \frac{1}{\gamma_{(i,j)}(T)} \iint s^{i+k_1l}\, t^{j+k_2m} d\mu(s,t) \\
&= \frac{1}{\gamma_{(i,j)}(T)} \iint s^{i+k_1l}\, t^{j+k_2m} d\nu(s,\,t)\ (\because\ j+k_2m>0,\ \forall\, k_2) \\
\Rightarrow \qquad d\mu_{(L\,I)}(s,t) &= \frac{s^{i/l}\, t^{j/m}}{\gamma_{(i,j)}(T)}\, d\nu(s^{1/l},\, t^{1/m})
\end{aligned}$$

$\square$

Theorem 7.4.5. *Let $T = (T_1, T_2)$ be 2-variable weighted shift with weight sequences α and β, and $M = \bigvee_{k_2 \geq 1} e_{(k_1,k_2)}$. If $T_M := T|_M$ is subnormal, then for $L = (l, m)$, with $l \geq 1, m \geq 1$, the following are equivalent:*

(a) T^L is k-hyponormal.

(b) $T_{(L\,(i,0))}$ is k-hyponormal for $0 \leq i \leq l-1$

Proof. $(a) \Rightarrow (b)$ is obvious from Corollary 7.4.3.

$(b) \Rightarrow (a)$ · Here

$$T^L \cong \bigoplus_{j=0}^{m-1} \bigoplus_{i=0}^{l-1} T_{(L\,I)}.$$

Given that $T_{(L\,(i,0))}$ is k-hyponormal for $0 \leq i \leq l-1$. To show $T_{(L\,(i,j))}$ is k-hyponormal for $0 \leq i \leq l-1$ and $1 \leq j \leq m-1$.

Define $\tilde{\alpha}_{(k_1,k_2)} = \alpha_{(k_1,k_2+1)}$ and $\tilde{\beta}_{(k_1,k_2)} = \beta_{(k_1,k_2+1)}$.

$(T_M)_{(L\,I)}$ is a 2-variable weighted shift with weight sequences

$$P_{(L\,I)}(\tilde{\alpha}) = \left\{ \prod_{p=0}^{l-1} \tilde{\alpha}_{(i+k_1 l+p,\, j+k_2 m)} \right\} = \left\{ \prod_{p=0}^{l-1} \alpha_{(i+k_1 l+p,\, j+k_2 m+1)} \right\} = P_{(L\,(i,j+1))}(\alpha)$$

and

$$Q_{(L\,I)}(\tilde{\beta}) = Q_{(L\,(i,j+1))}(\beta)$$

Thus $(T_M)_{(L\,(i,j))} = T_{(L\,(i,j+1))}$ for $0 \leq i \leq l-1$ and $0 \leq j \leq m-1$

That is, $(T_M)_{(L\,(i,j-1))} = T_{(L\,(i,j))}$ for $0 \leq i \leq l-1$ and $1 \leq j \leq m$

Now

T_M is subnormal $\Rightarrow T_M^L$ is subnormal

$\Rightarrow (T_M)_{(L\,I)}$ is subnormal and hence k-hyponormal

for $0 \leq i \leq l-1$, $0 \leq j \leq m-1$ (by Corollary 7 4 3)

$\Rightarrow (T_M)_{(L\,(i,j-1))}$ is k-hyponormal for $0 \leq i \leq l-1$, $1 \leq j \leq m$

$\Rightarrow (T)_{(L\,I)}$ is k -hyponormal for $0 \leq i \leq l-1$, $1 \leq j \leq m-1$

$\square$

Theorem 7.4.6. *Let $T = (T_1, T_2)$ be 2-variable weighted shift with weight sequences α and β. Let $M_n = \bigvee_{k_2 \geq n} e_{(k_1,k_2)}$ and $T_{M_n} := T|_{M_n}$ be subnormal. For $L = (l, m)$ with $l \geq 1, m \geq 1, I = (i, j)$ with $0 \leq i \leq l-1,\ 0 \leq j \leq m-1$, and $k \geq 1$. then the following are equivalent*

(a) T^L is k-hyponormal.

(b) $T_{(L\,(i,0))}, T_{(L\,(i,1))}, \ldots, T_{(L\,(i,n-1))}$ are k-hyponormal for all $0 \leq i \leq l-1$

Theorem 7.4.7. *Let $T = (T_1, T_2)$ be 2-variable weighted shift with weight sequences α and β. Let $M_1 = \bigvee_{k_2 \geq 1} e_{(k_1,k_2)}$ and $T_{M_1} := T|_{M_1}$ be subnormal with the Berger measure $\mu_1(s,t) = \nu_1(s,t) + \rho(s)\,\delta_0(t)$ and $W_0 := shift(\alpha_{(0,0)}, \alpha_{(1,0)}, \ )$ be subnormal with associated measure ξ_0 Then $T^{(1,2)}$ is subnormal if and only if*

$$\beta^2_{(0,0)} \leq \left(\left\| \frac{1}{t} \right\|_{L^1(\nu_1)} \right)^{-1}$$

$$\text{and} \qquad \beta^2_{(0,0)} \left\| \frac{1}{t} \right\|_{L^1(\nu_1)} (\nu_1)^X_{ext} \leq \xi_0$$

If $\rho(s) = 0$, then $T^{(1,2)}$ is subnormal if and only if T is subnormal.

Proof. By the Theorem 7 4.5, if T_{M_1} is subnormal, then $T^{(1,2)}$ is subnormal if and only if $T_{((1,2)\,(0,0))}$ is subnormal So, it suffices to check for $T_{((1,2)\,(0,0))}$ Again $T_{((1,2)\,(0,0))}$ is the 1-step back extension of $(T_{M_1})_{((1,2)\,(0,1))}$

Since T_{M_1} is subnormal with measure μ_1, so by Corollary 7.4.3 (c) $(T_{M_1})_{((1,2)\,(0,1))}$ is also subnormal with measure $(\mu_1)_{((1,2)\,(0,1))}$. Therefore by Theorem 7.3.6, $T_{((1,2)\,(0,0))}$ is subnormal if and only if

$$\beta^2_{(0,0)}\beta^2_{(0,1)} \le \left(\left\|\frac{1}{t}\right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}\right)}\right)^{-1} \tag{7.4.2}$$

$$\beta^2_{(0,0)}\beta^2_{(0,1)}\left\|\frac{1}{t}\right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}\right)}\left((\mu_1)_{((1,2):(0,1))}\right)^X_{ext} \le \xi_0 \tag{7.4.3}$$

Now,

$$d(\mu_1)_{\left((1,2):(0,1)\right)}(s,t) = \frac{t^{1/2}}{\gamma_{(0,1)}(T_{M_1})}\, d\nu_1(s,t^{1/2}) \tag{7.4.4}$$

and

$$\begin{aligned}
d(\mu_1)_{\left((1,2):(0,1)\right)_{ext}}(s,t) &= \frac{(1-\delta_0(t))}{t\|\frac{1}{t}\|_{L^1\left((\mu_1)_{((1,2):(0,1))}\right)}}\, d(\mu_1)_{\left((1,2):(0,1)\right)}(s,t)\\
&= \frac{(1-\delta_0(t))}{t^{1/2}\int \frac{1}{t^{1/2}}\, d\nu_1(s,t^{1/2})}\, d\nu_1(s,t^{1/2}) \quad \text{(using (7.4.4))}\\
&= \frac{(1-\delta_0(t))}{t\int \frac{1}{t}\, d\nu_1(s,t)}\, d\nu_1(s,t)\\
&= d(\nu_1)_{ext}(s,t)
\end{aligned} \tag{7.4.5}$$

Now,

$$\begin{aligned}
(7.4.2) &\Rightarrow \beta^2_{(0,0)}\beta^2_{(0,1)}\int \frac{1}{t}\, d(\mu_1)_{\left((1,2):(0,1)\right)}(s,t) \le 1\\
&\Rightarrow \beta^2_{(0,0)}\int \frac{1}{t^{1/2}}\, d\nu_1(s,t^{1/2}) \le 1\\
&\Rightarrow \beta^2_{(0,0)}\int \frac{1}{t}\, d\nu_1(s,t) \le 1\\
&\Rightarrow \beta^2_{(0,0)} \le \left(\left\|\frac{1}{t}\right\|_{L^1(\nu_1)}\right)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
(7.4.3) &\Rightarrow \beta^2_{(0,0)}\beta^2_{(0,1)}\int \frac{1}{t}\, d((\mu_1)_{((1,2):(0,1))}(s,t))\ \left((\mu_1)_{((1,2):(0,1))}(s,t)\right)^X_{ext} \le \xi_0(s)\\
&\Rightarrow \beta^2_{(0,0)}\int \frac{1}{t^{1/2}}\, d\nu_1(s,t^{1/2})\ \left(\nu_1(s,t)\right)^X_{ext} \le \xi_0(s)\\
&\Rightarrow \beta^2_{(0,0)}\int \frac{1}{t}\, d\nu_1(s,t)\ \left(\nu_1(s,t)\right)^X_{ext} \le \xi_0(s)\\
&\Rightarrow \beta^2_{(0,0)}\left\|\frac{1}{t}\right\|_{L^1(\nu_1)}(\nu_1)^X_{ext} \le \xi_0
\end{aligned}$$

If $\rho(s) = 0$, then $\mu_1(s,t) = \nu_1(s,t)$. Therefore by Theorem 7.3.6, $T^{(1,2)}$ is subnormal if and only if T is subnormal.

$\square$

Theorem 7.4.8. *Let T be a 2-variable weighted shift with the weight sequences α and β. Assume that $T_{M_2} := T|_{M_2}$ the restriction of T to $M_2 := \bigvee\{e_{(k_1,k_2)} : k_2 \geq 2\}$ is subnormal with associated measure μ_2. Let $W_0 := shift(\alpha_{(0,0)}, \alpha_{(1,0),\ldots})$ and $W_1 := shift(\alpha_{(0,1)}, \alpha_{(1,1),\ldots})$ be subnormal with associated measures ξ_0 and ξ_1 respectively. Then T is subnormal with associated measure μ if and only if*

(i) $\frac{1}{t^2} \in L^1(\mu_2)$

(ii) $\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} \leq 1$

(iii) $\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t^2}\right\|_{L^1(\mu_2)} (\mu_2)^X_{(ext)^2} \leq \xi_0$

(iv) $\beta^2_{(0,1)} \left\|\frac{1}{t}\right\|_{L^1(\mu_2)} = 1$

(v) $(\mu_2)^X_{ext} = \xi_1$

Proof. Assume that T be subnormal. Since T_{M_1} is a subnormal weighted shift possessing a subnormal extension T, so $\beta^2_{(0,1)} = \left(\left\|\frac{1}{t}\right\|_{L^1(\mu_2)}\right)^{-1}$ and $(\mu_2)^X_{ext} = \xi_1$, Moreover, if μ_1 is a Berger measure of T_{M_1}, then $\mu_1 = (\mu_2)_{ext}$. Since T is subnormal so by Corollary 7.4.3 (c), $T_{((1,2)\cdot(0,0))}$ is also subnormal. Again $T_{((1,2):(0,0))}$ is the 1-step extension of $(T_{M_1})_{((1,2):(0,1))}$. Therefore by Theorem 7.3.6, $T_{((1,2):(0,0))}$ is subnormal if and only if

$$\frac{1}{t} \in L^1\left((\mu_1)_{((1,2)\ (0,1))}\right) \tag{7.4.6}$$

$$\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t}\right\|_{L^1\left((\mu_1)_{((1,2)\ (0,1))}\right)} \leq 1 \tag{7.4.7}$$

$$\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t}\right\|_{L^1\left((\mu_1)_{((1,2)\ (0,1))}\right)} \left((\mu_1)_{((1,2)\ (0,1))}\right)^X_{ext} \leq \xi_0 \tag{7.4.8}$$

Now,

$$d(\mu_1)_{\left((1,2)\ (0,1)\right)}(s,t) = \frac{t^{1/2}}{\gamma_{(0,1)}(T_{M_1})}\, d\mu_1(s, t^{1/2}) = d\mu_2(s, t^{1/2}). \tag{7.4.9}$$

So, (7.4.6) implies that $\frac{1}{t^2} \in L^1(\mu_2(s,t))$ and so also $\frac{1}{t} \in L^1(\mu_2(s,t))$. Also, $\mu_1(E \times \{0\}) = 0$, $\mu_2(E \times \{0\}) = 0\ \forall E \subseteq \mathbb{R}_+$.

and

$$\begin{aligned}
d(\mu_1)_{\left((1,2)\ (0,1)\right)_{ext}}(s,t) &= \frac{(1-\delta_0(t))}{t\|\frac{1}{t}\|_{L^1\left((\mu_1)_{((1,2)\ (0,1))}(s,t)\right)}}\, d(\mu_1)_{\left((1,2)\ (0,1)\right)}(s,t) \\
&= \frac{(1-\delta_0(t))}{t\|\frac{1}{t}\|_{L^1\left(\mu_2(s,t^{1/2})\right)}}\, d\mu_2(s,t^{1/2}) \text{ (using (7.4.9))} \\
&= \frac{(1-\delta_0(t))}{t^2\|\frac{1}{t^2}\|_{L^1\left(\mu_2(s,t)\right)}}\, d\mu_2(s,t) \\
&= d(\mu_2)_{(ext)^2}(s,t)
\end{aligned} \tag{7.4.10}$$

Again from (7.4.7), we get

$$\begin{aligned}
&\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t}\right\|_{L^1\left(\mu_2(s,t^{1/2})\right)} \leq 1 \\
\Rightarrow\ &\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t^2}\right\|_{L^1\left(\mu_2(s,t)\right)} \leq 1
\end{aligned}$$

and from (7.4.8), we get

$$\beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t}\right\|_{L^1\left(\mu_2(s,t^{1/2})\right)} (\mu_2(s,t))^X_{(ext)^2} \leq \xi_0(s) \text{ (using(7.4.9) and (7.4.10))}$$

$$\Rightarrow\ \beta^2_{(0,0)}\beta^2_{(0,1)} \left\|\frac{1}{t^2}\right\|_{L^1\left(\mu_2(s,t)\right)} (\mu_2(s,t))^X_{(ext)^2} \leq \xi_0(s)$$

($\Longleftarrow$) Suppose all the conditions are hold. To show T is subnormal. From conditions (i), (iv) and since T_{M_2} is subnormal so by Theorem 7.3.6, T_{M_1} is subnormal with the Berger measure μ_1 such that $\mu_1(E \times \{0\}) = 0$ for all $E \subseteq \mathbb{R}_+$ and $\mu_1 = (\mu_2)_{ext}$. So by Theorem 7.4.7 to check the subnormality of T, it suffices to check the subnormality of $T^{(1,2)}$ and by Theorem 7.4.5 this reduces to verifying the subnormality of $T_{((1,2):(0,0))}$. Again $T_{((1,2):(0,0))}$ is the 1-step extension of $(T_{M_1})_{((1,2):(0,1))}$ (which is subnormal).

Now, since $(T_{M_1})_{((1,2):(0,1))}$, T_{M_1} and T_{M_2} are subnormal with measures $(\mu_1)_{((1,2):(0,1))}$, μ_1 and μ_2 respectively. So, we can establish as above that $d(\mu_1)_{\left((1,2):(0,1)\right)}(s,t) = d\mu_2(s, t^{1/2})$ and $d(\mu_1)_{((1,2):(0,1))_{ext}}(s,t) = d(\mu_2)_{(ext)^2}(s,t)$.

So, condition (i) implies that $\frac{1}{t} \in L^1((\mu_1)_{((1,2).(0,1))})$. From condition (ii) we will get

$$\begin{aligned} &\beta^2_{(0,0)}\beta^2_{(0,1)} \left\| \frac{1}{t^2} \right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}(s,t^2)\right)} \leq 1 \\ \Rightarrow\ &\beta^2_{(0,0)}\beta^2_{(0,1)} \left\| \frac{1}{t} \right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}(s,t)\right)} \leq 1 \end{aligned}$$

and condition (iii) will give,

$$\begin{aligned} &\beta^2_{(0,0)}\beta^2_{(0,1)} \left\| \frac{1}{t^2} \right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}(s,t^2)\right)} \left((\mu_1)_{((1,2):(0,1))}(s,t)\right)^X_{ext} \leq \xi_0(s) \\ \Rightarrow\ &\beta^2_{(0,0)}\beta^2_{(0,1)} \left\| \frac{1}{t} \right\|_{L^1\left((\mu_1)_{((1,2):(0,1))}(s,t)\right)} \left((\mu_1)_{((1,2):(0,1))}(s,t)\right)^X_{ext} \leq \xi_0(s) \end{aligned}$$

Thus by Theorem 7.3.6, $T_{((1,2):(0,0))}$ is subnormal and hence T is subnormal. $\square$

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