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FINITE ELEMENT METHODS WITH NUMERICAL QUADRATURE FOR PARABOLIC AND PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH INTERFACES

Thesis Submitted in partial fulfillment of the requirements
for the award of the degree of Doctor of Philosophy

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Abstract

The purpose of the present work is to study finite element Galerkin methods for linear parabolic and parabolic integro-differential equations with interfaces. The emphasis is on the theoretical aspects of such methods.

An attempt is made in this thesis to extend known results for finite element Galerkin method for a parabolic differential equation to a parabolic equation with interfaces. Optimal $L^2(L^2)$ and $L^2(H^1)$ error estimates are shown to hold for both semidiscrete and fully discrete schemes with quadrature under minimum smoothness of the initial data. Due to low global regularity of the solutions, the error analysis of the standard finite element methods for parabolic problems is difficult to adopt for parabolic interface problems. In this work, we fill a theoretical gap between standard error analysis technique of finite element method for non interface problems and parabolic interface problems. Optimal $L^\infty(H^1)$ and $L^\infty(L^2)$ norms error estimates have been derived for the semidiscrete case under practical regularity assumptions of the true solution for fitted finite element method with straight interface triangles. Further, the fully discrete backward Euler scheme is also considered and optimal $L^\infty(L^2)$ norm error estimate is established. In this case, the initial data and interface function are assumed to be sufficiently smooth.

Although various FEM for parabolic interface problems have been proposed and studied in the literature, but FEM treatment to the integro-differential equations with interfaces is mostly missing. A priori error estimates are derived for integro-differential equations of parabolic type with interfaces. Continuous time Galerkin method for the spatially discrete scheme and backward difference scheme in time direction are discussed in $L^2(H^m)$ and $L^\infty(H^m)$ norms for fitted finite element method with straight interface triangles. More precisely, optimal error estimates are derived in $L^2(H^m)$ and $L^\infty(H^m)$ norms when initial data $u_0 \in H_0^1(\Omega)$ and $u_0 \in H^3 \cap H_0^1(\Omega)$, respectively. The achieved estimates are analogous to the case with a regular solution, however, due to low regularity, the proof requires a careful technical work coupled with a approximation result for the Ritz-Volterra projection under minimum regularity assumption.

Declaration

I, **Ram Charan Deka**, hereby declare that the subject matter in this thesis entitled **Finite Element Methods with Numerical Quadrature for Parabolic and Parabolic Integro-Differential Equations with Interfaces** is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/institute.

This thesis is being submitted to the Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

Place: Napaam
Date: 10-09-14

(Ram Charan Deka)

TEZPUR UNIVERSITY

Certificate

This is to certify that the thesis entitled **Finite Element Methods with Numerical Quadrature for Parabolic and Parabolic Integro-Differential Equations with Interfaces** submitted to the School of **Sciences** Tezpur University in partial fulfilment for the award of the degree of Doctor of Philosophy in **Mathematics** is a record of research work carried out by **Mr. Ram Charan Deka** under my supervision and guidance.

All help received by him from various sources have been dully acknowledged. No part of this thesis has been submitted elsewhere for award of any other degree.

April, 2014

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With Regards

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Contents

Chapter 1

Introduction

The purpose of this thesis is to present some results on finite element Galerkin methods for linear parabolic and parabolic integro-differential equations with discontinuous coefficients. This chapter introduces the problem and it contains the notations and preliminary materials to be used in the thesis. It also provides the survey for relevant literature and motivation for the present study. The chapter-wise description of the thesis is presented in the last section of this chapter.

1.1 Problem Description

Differential equations with discontinuous coefficients are often referred as interface problems. The discontinuity of the coefficients corresponds to the fact that the medium consists of two or more physically different materials. To begin with, we first introduce parabolic and parabolic integro-differential equations with interfaces.

Parabolic interface problems: Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Further, let $\Omega_1 \subset \Omega$ be an open domain with C^2 smooth boundary Γ and $\Omega_2 = \Omega \backslash \Omega_1$ (see, Figure 1.1). We now consider the following linear parabolic interface problems of the form

$$u_t(x,t) + \mathcal{L}u(x,t) = f(x,t) \text{ in } \Omega \times (0,T] \tag{1.1.1}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \text{ in } \Omega; \quad u(x.t) = 0 \text{ on } \partial\Omega \times (0,T] \tag{1.1.2}$$

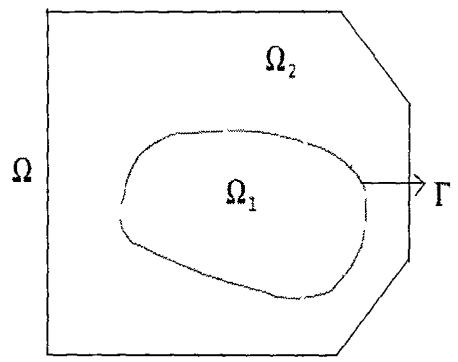

Figure 1.1: *Domain* Ω *and its sub domains* Ω_1, Ω_2 *with interface* Γ.

and interface conditions

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = g(x,t) \quad \text{along } \Gamma \times (0.T], \tag{1.1.3}$$

where $u(x,t)$ is a real-valued function of x and t, $u_t(x,t) = \frac{\partial u}{\partial t}(x,t)$ and $T < \infty$. The symbol $[v]$ is a jump of a quantity v across the interface Γ, i.e.. $[v](x) = v_1(x) - v_2(x), \ \ x \in \Gamma$, where $v_i(x) = v(x)|_{\Omega_i}$, $i = 1,2$ and $\mathbf{n}$ denotes the unit outward normal to the boundary $\partial\Omega_1$. Operator $\mathcal{L}$ is a second order elliptic partial differential operator of the form

$$\mathcal{L}v(x) = -\nabla.(\beta(x)\nabla v(x)).$$

We assume that the coefficient function β is positive and piecewise constant, i.e.,

$$\beta(x) = \beta_i \ \text{ in } \Omega_i, \ i = 1,2.$$

Further, $f = f(x,t)$ and $g = g(x,t)$ are real valued functions defined in $\Omega \times (0,T]$ and $\Gamma \times (0,T]$, respectively.

Parabolic integro-differential equations with interfaces: We shall also consider integro-differential equations of the form

$$u_t(x,t) + \mathcal{L}u(x,t) = f(x,t) + \int_0^t B(t,s)u(x,s)ds \quad \text{in } \Omega \times (0,T] \tag{1.1.4}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \ \text{ in } \Omega; \ \ u(x.t) = 0 \ \text{ on } \partial\Omega \times (0,T] \tag{1.1.5}$$

and interface conditions

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = 0 \quad \text{along } \Gamma \times (0, T]. \tag{1.1.6}$$

The domain Ω, operator $\mathcal{L}$, symbols $[v]$ and $\mathbf{n}$ are defined as before, and $T < \infty$. The operator $B(t, s)$ is a second order partial differential operator of the form

$$B(t, s) = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(b_{ij}(x; t, s)\frac{\partial}{\partial x_j}\right) + \sum_{j=1}^{2} b_j(x; t, s)\frac{\partial}{\partial x_j} + b_0(x; t, s)I.$$

The equations of the form (1.1.1)-(1.1.3) are often encountered in the theory of magnetic field, heat conduction theory, the theory of elasticity and in reaction diffusion problems. Many interface problems in material science and fluid dynamics are modeled after above problem when two or more distinct materials or fluids with different conductivities or densities or diffusions are involved. One interesting class of parabolic equations with discontinuous coefficients processes in heat conducting media with concentrated capacity in which the heat capacity coefficient contains a Dirac delta function, or equivalently, the jump of the heat flow at the singular point is proportional to the time derivative of the temperature (cf. [7]). For a detailed discussion on parabolic problems with discontinuous coefficients, see Dautray and Lions [18], Gilbarg and Trudinger [30], Ladyzhenskaya *et al.* [39], Li and Ito [40].

Equations (1.1.4) are often referred to as the parabolic partial differential equations with memory term or the Volterra integral term i.e. $\int_0^t B(t, s)u(x, s)ds$. Such problems and variants of them arise in several physical phenomena such as in models for heat conduction in rigid materials with memory, the compression of poro-viscoelastic media, reactor dynamics and epidemic models in biology. For a detailed discussion on models for heat conduction in materials with memory, see Belleni-Morante [6], Coleman and Gurtin [17], Gurtin and Pipkin [31], Miller [45], Nohel [47] and the references quoted therein. For the literature relating to other applications of the theory of parabolic integro-differential equations, one may refer to Habetler and Schiffman [32] for the models for the compression of poro-viscoelastic media, Pao [50]-[52] for reactor dynamics, Hornung and Showalter [35] for the compartment model of a double-porosity system and Capasso [11] for epidemic phenomena in biology. As a model for parabolic integro-differential equations (1.1.4) with discontinuous coefficients, we consider non-stationary heat conduction problems in two dimensions with memory and conduction coefficient β which is discontinuous across a smooth interface.

The presence of the Volterra integral term helps to accurately describe several physical phenomena, which causes some new difficulties in both theoretical analysis and numerical computation. Although various FEM for parabolic interface problems have been proposed and studied in the literature, but FEM treatment to the integro-differential equations with interfaces is mostly missing. An attempt has been made in this thesis to study the a priori error analysis for the parabolic integro-differential equations with discontinuous coefficients. In this process some new a priori error estimates are derived for parabolic interface problems.

1.2 Notation and Preliminaries

In this section, we shall introduce some standard notation and preliminaries to be used throughout of this work.

All functions considered here are real valued. Let Ω be a bounded domain in $\mathbb{R}^d$, $d-$dimensional Euclidian space and $\partial\Omega$ denote the boundary of Ω. Let $x = (x_1, x_2, \dots, x_d) \in \Omega$, and let $dx = dx_1 \dots dx_d$. Further, let $\alpha = (\alpha_1, \dots, \alpha_d)$ be a $d-$tuple with nonnegative integer components and denote order of α as $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$. Then, by $D^\alpha\phi$, we shall mean the αth derivative of ϕ defined by

$$D^\alpha\phi = \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}.$$

We shall make frequent reference to the following well-known function spaces. For $1 \le p < \infty$, $L^p(\Omega)$ denotes the linear space of equivalence classes of measurable functions ϕ in Ω such that $\int_\Omega |\phi(x)|^p dx$ exists and is finite. The norm on $L^p(\Omega)$ is given by

$$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |\phi(x)|^p dx\Big)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

For $p = \infty$, $L^\infty(\Omega)$ denotes the space of functions ϕ on Ω such that

$$\|\phi\|_{L^\infty(\Omega)} = \text{ess} \sup_{x\in\Omega} |\phi(x)| < \infty.$$

When $p = 2$, $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(\phi, \psi) = \int_\Omega \phi(x)\psi(x)dx.$$

By support of a function ϕ, supp ϕ, we mean the closure of all points x with $\phi(x) \neq 0$, i.e.,

$$\text{supp } \phi = \overline{\{x : \phi(x) \neq 0\}}.$$

For any nonnegative integer m, $C^m(\overline{\Omega})$ denotes the space of functions with continuous derivatives upto and including order m in $\overline{\Omega}$. $C_0^m(\Omega)$ is the space of all $C^m(\Omega)$ functions with compact support in Ω. Also, $C_0^\infty(\Omega)$ is the space of all infinitely differential functions with compact support in Ω.

We now introduce the notion of Sobolev spaces. Let $m \geq 0$ and real p with $1 \leq p < \infty$. The Sobolev space of order (m, p) on Ω, denoted by $W^{m,p}(\Omega)$, is defined as a linear space of functions (or equivalence class of functions) in $L^p(\Omega)$ whose distributional derivatives upto order m are also in $L^p(\Omega)$, i.e.,

$$W^{m,p}(\Omega) = \{\phi : \; D^\alpha \phi \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}.$$

The space $W^{m,p}(\Omega)$ is endowed with the norm

$$\begin{aligned} \|\phi\|_{m,p} &= \left(\int_\Omega \sum_{0 \leq |\alpha| \leq m} |D^\alpha \phi(x)|^p dx \right)^{\frac{1}{p}} \\ &= \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha \phi\|^p \right)^{\frac{1}{p}}, \; 1 \leq p < \infty. \end{aligned}$$

When $p = \infty$, the norm on the space $W^{m,\infty}(\Omega)$ is defined by

$$\|\phi\|_{m,\infty} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha \phi(x)\|_{L^\infty(\Omega)}.$$

For $p = 2$, these spaces will be denoted by $H^m(\Omega)$. The space $H^m(\Omega)$ is a Hilbert space with natural inner product defined by

$$(\phi, \psi) = \sum_{0 \leq |\alpha| \leq m} \int_\Omega D^\alpha \phi D^\alpha \psi dx, \quad \phi, \psi \in H^m(\Omega).$$

The sobolev space $H^m(\Omega)$ (respectively, $H_0^m(\Omega)$) is also defined as the closure of $C^m(\Omega)$ (respectively, $C_0^\infty(\Omega)$) with respect to the norm $\|\phi\|_m = \|\phi\|_{m,2}$. This result is true under some smoothness assumption on the boundary $\partial\Omega$. Clearly, $L^2(\Omega) = H^0(\Omega)$ and $H^m(\Omega) = W^{m,2}(\Omega)$. For a more complete discussion on Sobolev spaces, see Adams [1].

We shall also use the following spaces in our error analysis. For a given Banach space $\mathcal{B}$, we define, for $m = 0, 1$ and $1 \leq p < \infty$

$$W^{m,p}(0,T;\mathcal{B}) = \left\{ u(t) \in \mathcal{B} \text{ for a.e. } t \in (0,T) \text{ and } \sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j u(t)}{\partial t^j} \right\|_{\mathcal{B}}^p dt < \infty \right\}$$

equipped with the norm

$$\|u\|_{W^{m,p}(0,T;\mathcal{B})} = \left(\sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j u(t)}{\partial t^j} \right\|_{\mathcal{B}}^p dt \right)^{\frac{1}{p}}.$$

We write $H^m(0,T;\mathcal{B}) = W^{m,2}(0,T;\mathcal{B})$ and $L^2(0,T;\mathcal{B}) = H^0(0,T;\mathcal{B})$. When no risk of confusion exists we shall write $L^2(\mathcal{B})$ for $L^2(0,T;\mathcal{B})$ and $H^1(\mathcal{B})$ for $H^1(0,T;\mathcal{B})$.

Further, we denote $L^\infty(0,T;\mathcal{B})$ to be the collection of all functions $v \in \mathcal{B}$ such that

$$\operatorname*{ess\ sup}_{t\in(0,T]} \|v(x,t)\|_{\mathcal{B}} < \infty.$$

Below, we shall discuss some preliminary materials which will be of frequent use in error analysis in the subsequent chapters. The bilinear form $A(\cdot,\cdot)$ associated with the operator $\mathcal{L}$, given by

$$A(u,v) = \int_\Omega \beta(x)\nabla u \cdot \nabla v dx,$$

satisfies the following boundedness and coercive properties: For $\phi, \psi \in H^1(\Omega)$, there exists positive constants C and c such that

$$A(\phi,\psi) \leq C\|\phi\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)}$$

and

$$A(\phi,\phi) \geq c\|\phi\|^2_{H^1(\Omega)}.$$

From time to time we shall also use the following inequalities (see, Hardy *et al.* [34]):

(i) Young's inequality: For $a, b \geq 0$ and $\epsilon > 0$, the following inequality

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$$

holds.

(ii) Cauchy-Schwarz inequality: For $a, b \geq 0,\ 1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

In integral form , if ϕ and ψ are both real valued and $\phi \in L^p$ and $\psi \in L^q$, then

$$\int_\Omega \phi\psi \leq \|\phi\|_p \|\psi\|_q.$$

For $p = q = 2$, the above inequality is known as **Schwarz's inequality**. The discrete version of Schwarz's inequality may be stated as:

(iii) Let $\phi_j, \psi_j, j = 1, 2, \ldots, n$ be positive real numbers. Then

$$\sum_{j=1}^{n} \phi_j \psi_j \leq \left(\sum_{j=1}^{n} \phi_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \psi_j^2\right)^{\frac{1}{2}}.$$

Below, we state without proof, the following two versions of Grownwall's lemma. For a proof, see [55].

Lemma 1.2.1 (Continuous Gronwall's Lemma) *Let $G(t)$ be a continuous function and $H(t)$ a nonnegative continuous function on its interval $t_0 \leq t \leq t_0 + a$. If a continuous function $F(t)$ has the property*

$$F(t) \leq G(t) + \int_{t_0}^{t} F(s)H(s)ds \ \text{ for } t \in [t_0, t_0 + a],$$

then

$$F(t) \leq G(t) + \int_{t_0}^{t} G(s)H(s)exp\left[\int_{s}^{t} H(\tau)d\tau\right] ds \ \text{ for } t \in [t_0, t_0 + a].$$

In particular, when $G(t) = C$ a nonnegative constant, we have

$$F(t) \leq Cexp\left[\int_{t_0}^{t} H(s)ds\right] \ \text{ for } t \in [t_0, t_0 + a].$$

Lemma 1.2.2 (Discrete Gronwall's Lemma) *If $\langle y_n \rangle$, $\langle f_n \rangle$ and $\langle g_n \rangle$ are non-negative sequences and*

$$y_n \leq f_n + \sum_{0 \leq k < n} g_k y_k, \quad n \geq 0,$$

then

$$y_n \leq f_n + \sum_{0 \leq k < n} g_k f_k \exp\left(\sum_{k < j < n} g_j\right). \quad n \geq 0.$$

In addition, we shall also work on the following spaces:

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2) \quad \& \quad Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2).$$

For $w : [0, T] \to X$, $v : [0, T] \to Y$ and $t \in \times[0, T]$, we define

$$\begin{aligned} \|w(t)\|_X &= \|w(x,t)\|_{H^1(\Omega)} + \|w(x,t)\|_{H^2(\Omega_1)} + \|w(x,t)\|_{H^2(\Omega_2)} \\ &\equiv: \|w(t)\|_{H^1(\Omega)} + \|w(t)\|_{H^2(\Omega_1)} + \|w(t)\|_{H^2(\Omega_2)}. \end{aligned}$$

and

$$\begin{aligned} \|v(t)\|_Y &= \|v(x,t)\|_{L^2(\Omega)} + \|v(x,t)\|_{H^1(\Omega_1)} + \|v(x,t)\|_{H^1(\Omega_2)} \\ &\equiv: \|v(t)\|_{L^2(\Omega)} + \|v(t)\|_{H^1(\Omega_1)} + \|v(t)\|_{H^1(\Omega_2)}. \end{aligned}$$

Throughout this thesis, C is a positive generic constant independent of the mesh parameters $\{h, k\}$ and not necessarily be the same at each occurrence.

1.3 Background and Objectives

This section presents a brief survey of the relevant literature concerning the numerical solutions of interface problems by means of finite element method. It also elucidates the objectives for the present study.

Solving differential equations with discontinuous coefficients by means of classical finite element methods usually leads to the loss in accuracy. One major difficulty is that the solution has low global regularity and the elements do not fit with the interface of general shape. For non-interface problems, one can assume full regularities of the solutions (at least $H^2(\Omega)$) on whole physical domain. But for the interface problems, the global regularity of the solution is low. So the classical analysis is difficult to apply for the convergence analysis of the interface problems. Thus the numerical solution to the interface problem is challenging as well as interesting also.

Finite element methods for interface problems may be grouped into two categories: Fitted finite element method and Unfitted finite element method depending on the choice of the discretization. In fitted finite element method, the discretization is made in such a way that the grid line is either isoparametrically fitted to the interface or an approximation of the smooth interface. In unfitted finite element methods, the discretization is independent of the location of the interface.

In recent time, many new numerical methods have been developed to handle differential equations with singularity. Some of them are developed with the modifications in the standard methods, so that they can deal with the discontinuities and the singularities. We first give a brief account of the development of the finite element methods for elliptic interface problems. In [4], Babuška has studied the elliptic interface problem as an equivalent minimization problem. The finite element method is then applied to solve the minimization problem and sub-optimal H^1-norm error estimate is obtained. The algorithm in [4] requires the exact evaluation of line integrals on the boundary of the domain and on the interface, and exact integrals on the interface finite elements are also needed. In the absence of variational crimes, finite element approximation of interface problem has been studied by Barrett and Elliott in [5]. They have shown that the finite element solution converges to the true solution at optimal rate in L^2 and H^1 norms over any interior subdomain. In [5], it is assumed that the solution and the normal derivative of the solution are continuous along the interface, and fourth order differentiable on each subdomain. Bramble and King [8] have studied nonconforming finite element method for such problems. In their work, interior domains Ω_1 and Ω_2 are approximated by polygonal domains. Then the Dirichlet data and the interface function are transferred to the polygonal boundaries. Finally, discontinuous Galerkin finite element method has applied to the approximated problem and optimal order error estimates are derived for rough as well as smooth boundary data. Under the assumption that $f|_{\Omega_1} = 0$, Neilsen [46] has proved optimal order of convergence in H^1 norm. The algorithm in [46] requires that the interface triangles follow exactly the actual interface Γ. Conforming high order fitted finite element methods for elliptic interface problems can be found in Li *et al.* [41]. For finite element methods of order p, error estimates of $O(h^{\min\{p,(m+1)/2\}})$ and $O(h^{\min\{p,m\}+1})$ in the H^1 and L^2 norms, respectively, are obtained when the interface is approximated with splines of order m. Recently, a continuous finite element method for elliptic interface problems in a higher dimensional polyhedral domain is discussed by Duan *et al.* [28]. An error estimate of $O(h^r)$ in energy norm has been obtained between the analytical solution and the continuous finite element solution. The analytical solution is assumed to be in $\Pi_{l=1}^{L}(H^r(\Omega_l))^3$ for some $r \in (1/2, 1]$. Unfitted discontinuous Galerkin method, based on the symmetric interior penalty DG method, has been proposed to discretize elliptic interface problems in [43]. Optimal h-convergence of the method for arbitrary p in the energy and L^2 norms are obtained. This method can be treated as a generalization of the unfitted method given by Hansbo *et al.* [33] for elliptic interface problems. A comparative study on the existing numerical techniques to solve elliptic interface problems has been carried out in [38], which also includes extensive list of relevant literature.

We now turn to the finite element Galerkin approximation to parabolic interface problems (1.1.1)-(1.1.3). In the absence of memory term in (1.1.4), convergence analysis for parabolic interface problems via finite element procedure have been studied by several authors, see [3, 15, 21, 27, 53, 58, 59]. For the backward Euler time discretization, Chen and Zou ([15]) have studied the convergence of fully discrete solution to the exact solution using fitted finite element method with straight interface triangles. They have proved almost optimal error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms under practical regularity assumption of the solution. For similar finite element discretization, optimal error estimates in $L^2(H^1)$ norm have been derived in [59]. So, in order to maintain the best possible convergence rate in $L^2(L^2)$ norm, the authors of [58] have used a finite element discretization where interface triangles are assumed to be curved triangles instead of straight triangles like classical finite element methods. Optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are shown to hold for both semi discrete and fully discrete schemes. More recently, for similar triangulation, Deka and Sinha ([27]) have studied the pointwise-in-time convergence in finite element method for parabolic interface problems. Optimal error estimates have been obtained in $L^\infty(H^1)$ and $L^\infty(L^2)$ norms under the assumption that grid line exactly follow the actual interface. Similar results are also obtained by Attanayake and Senaratne in [3] for immersed finite element method. In [53], the author have analyzed the Lagrange multiplier method with penalty for parabolic initial boundary value problems using semi discrete and fully discrete schemes. For straight interface, sub-optimal order of estimates for both semi discrete and fully discrete schemes have been derived. Optimal order of convergence in fitted finite element method with straight interface triangles can be found in [21].

Numerical solutions by means of finite element Galerkin procedures for the parabolic integro-differential equation without interface can be found in [10. 12, 14, 42, 48, 64, 66, 67]. The first contribution in this direction is given by Yanik and Fairweather [66]. Assuming the exact solution is smooth, they derived optimal order a priori error estimates for fully discrete Crank-Nicolson scheme for nonlinear parabolic integro-differential equations (1.1.4) with $B(t, s)$ as a first-order partial differential operator. Subsequently, spatially semi-discrete scheme for (1.1.4) is thoroughly examined by Thomée and Zhang in [64]. They have obtained optimal order a priori error estimates in the L^2-norm for both smooth and non-smooth initial data by extending the spatially semidiscrete error analysis for linear parabolic equations [63] to parabolic integro-differential equations with an integral kernel consisting of a partial differential operator of order ≤ 2. The proof is based on the following decomposition of the main error $e = u - u_h$ as

$$e = (u - R_h u) + (R_h u - u_h),$$

where u_h and u denote the semidiscrete finite element solution and the exact solution of the parabolic integro-differential equation. respectively. Here, $R_h : H_0^1(\Omega) \to V_h$ is the Ritz projection introduced by Wheeler in [65]. A simple alternative approach is proposed by Cannon and Lin [10] and is further developed by Lin *et al.* in [42]. The key technical tool used in these works is a generalization of the Ritz projection operator R_h, namely the nonlocal projection or the Ritz-Volterra projection operator. In order to reduce the storage requirements during the time stepping of a general parabolic integro-differential equations, Sloan and Thomée [61] have first proposed the application of quadrature rules with relatively higher order truncation error. Later on, several researchers have given valuable contributions towards the convergence analysis of the finite element Galerkin solution to the solution of parabolic integro-differential equations and its variants in the a priori framework. We refer to Cannon and Lin [10], Le Roux and Thomée [57], Thomée and Zhang [64], Chen *et al.* [14], Pani *et al.* [48], Pani and Sinha [49], McLean and Thomée [44], Chen and Shih [12], Zhang [67] and Sinha *et al.* [60] for further works in this direction. Although various FEM for parabolic interface problems have been proposed and studied in the literature, but FEM treatment to the integro-differential equations with interfaces is mostly missing. For the finite element treatment of parabolic integro-differential equation with discontinuous coefficients. we refer to Pradhan *et al.* ([54]). In [54], authors have discussed a non-iterative domain decomposition procedure for parabolic integro-differential equation with interfaces and related a priori error estimates are derived.

In practice, the integrals appearing in finite element approximation are evaluated numerically by using some well known quadrature schemes. Quadrature based finite element method for elliptic interface problems have been discussed in [20, 36]. In [36], a mortar finite element method have been discussed for a finite element discretization where interface triangles are assumed to be curved triangles. Optimal L^2 norm and energy norm error estimates are achieved when the exact integration are replaced by quadrature. Author of [20] has obtained optimal order error estimates in L^2 and H^1 norms for conforming finite element method where the grid line need not follow the actual interface exactly. The previous work on finite element analysis with numerical quadrature for parabolic problems without interface can be found in [13], [56] and references therein.

The main objective is to study the convergence of fitted finite element solution to the exact solution of parabolic integro-differential equations with discontinuous coefficients. In this process some new a priori error estimates are derived for parabolic interface problems and those estimates are extended for integro-differential equations of parabolic type with interfaces. More precisely,

- **Quadrature Based Finite Element Methods for Linear Parabolic Interface Problems:** We have studied the effect of numerical quadrature in space on semidiscrete and fully discrete piecewise linear finite element methods for parabolic interface problems. Optimal $L^2(L^2)$ and $L^2(H^1)$ error estimates are shown to hold for semidiscrete problem under suitable regularity of the true solution in whole domain. Further, fully discrete scheme based on backward Euler method has also analyzed and optimal $L^2(L^2)$ norm error estimate is established (cf. [24]). Further, optimal $L^\infty(H^1)$ and $L^\infty(L^2)$ norms error estimates have been derived under the assumption that initial data is more regular(cf. [26]).

- **Finite Element Galerkin Approximation for Parabolic Integro-Differential Equations with Discontinuous Coefficients:** In this work, convergence of continuous time Galerkin method for the spatially discrete scheme and backward difference scheme in time direction are discussed in $L^2(H^m)$ and $L^\infty(H^m)$ norms for fitted finite element method with straight interface triangles. Optimal error estimates are derived in $L^2(H^m)$ and $L^\infty(H^m)$ norms when initial data $u_0 \in H_0^1(\Omega)$ and $u_0 \in H^3 \cap H_0^1(\Omega)$, respectively (cf. [23], [25]).

1.4 Organization of the Thesis

This thesis consists of five chapters and is organized as follows. Chapter 1 introduces the problem and it contains the basic notations, and preliminary materials to be used throughout this thesis.

In Chapter 2, convergence of quadrature based finite element solution to the exact solution have been discussed in $L^2(L^2)$ and $L^2(H^1)$ norms. More precisely, optimal error estimates are derived for arbitrary shape but smooth interfaces with a practical finite element discretization. Further, optimal error estimates in $H^1(L^2)$ and $H^1(H^1)$ norms are derived under the high regularity of the initial conditions. The finite element discretization used in this work and a regularity result concerning parabolic interface problems are also introduced in this chapter.

Chapter 3 is devoted to the optimal $L^\infty(H^1)$ and $L^\infty(L^2)$ norms convergence of finite element method with quadrature for parabolic interface problems with straight interface triangles. The key to the analysis is the error estimates of elliptic projection under minimum smoothness of the solution.

Chapter 4 deals with the convergence of finite element method for a class of parabolic integro-differential equations with discontinuous coefficients. Under the assumption that $B(t, s)$ is a first order partial differential operator of the form

$$B(t, s)u(s) = \sum_{k=1}^{2} b_k(x; t, s)\frac{\partial u(x, s)}{\partial x_k} + u(x, s),$$

optimal $L^2(L^2)$ and $L^2(H^1)$ norms are shown to hold in this chapter. Further, existence and uniqueness of the solution for parabolic integro-differential equations with discontinuous coefficients is also discussed in this chapter.

Chapter 5 is concerned with a priori error estimates for interface problems (1.1.4)-(1.1.6). Optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$ norms are established for continuous time discretization. Further, the fully discrete scheme based on a symmetric difference approximation is considered and optimal order convergence in H^1 norm is established. The crucial fact used in this work is the newly established approximation result for the Ritz-Volterra projection under minimum regularity assumption.

For clarity of presentation we have repeatedly given equations (1.1.1) – (1.1.3) or (1.1.4) – (1.1.6) at the beginning of subsequent chapters.

Chapter 2

Quadrature based Finite Element Methods for Linear Parabolic Interface Problems: $L^2(L^2)$ and $L^2(H^1)$ Error Estimates

In this chapter, we study the effect of numerical quadrature in space on semidiscrete and fully discrete piecewise linear finite element methods for parabolic interface problems. Optimal $L^2(L^2)$ and $L^2(H^1)$ error estimates are shown to hold for semidiscrete problem under suitable regularity of the true solution in whole domain. Further, fully discrete scheme based on backward Euler method has also analyzed and optimal $L^2(L^2)$ norm error estimate is established. The error estimates are obtained for fitted finite element discretization based on straight interface triangles.

2.1 Introduction

In this chapter, we consider a linear parabolic equation of the form

$$u_t + \mathcal{L}u = f(x, t) \quad \text{in } \Omega \times (0, T] \tag{2.1.1}$$

with initial and boundary conditions

$$u(x, 0) = u_0 \ \text{ in } \Omega \ \ \& \ \ u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T] \tag{2.1.2}$$

and interface conditions

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = g(x,t) \quad \text{along } \Gamma \times (0,T]. \tag{2.1.3}$$

Here, $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ is a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ is an open domain with C^2 smooth boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega \backslash \Omega_1$ (see, Figure 1.1). Here, $f = f(x,t)$ and $g = g(x,t)$ are real valued functions defined in $\Omega \times (0,T]$ and $\Gamma \times (0,T]$, respectively. The operator $\mathcal{L}$, symbols $[v]$ and $\mathbf{n}$ are defined as in Chapter 1.

For our subsequent analysis, we now recall the bilinear form $A(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by

$$A(u,v) = \int_\Omega \beta(x) \nabla u \cdot \nabla v dx \quad \forall u, v \in H^1(\Omega).$$

Then the weak formulation of the interface problem (2.1.1)-(2.1.3) is stated as follows: Find $u \in H_0^1(\Omega)$ such that

$$(u_t, v) + A(u,v) = (f,v) + \langle g, v\rangle_\Gamma \quad \forall v \in H_0^1(\Omega), \quad t \in (0,T] \tag{2.1.4}$$

with $u(0) = u_0$. Here, $(\cdot,\cdot)$ and $\langle \cdot,\cdot\rangle_\Gamma$ are used to denote the inner products of $L^2(\Omega)$ space and $L^2(\Gamma)$ space, respectively.

Convergence of the quadrature based finite element solution to the exact solution have been discussed in $L^2(L^2)$ and $L^2(H^1)$ norms. More precisely, optimal error estimates are derived for arbitrary shape but smooth interfaces with a practical finite element discretization. The key to the present analysis is the introduction of some auxiliary projections, duality arguments and some newly established convergence results in $H^1(L^2)$ and $H^1(H^1)$ norms for parabolic interface problems without quadrature. To the best of our knowledge, the effect of numerical quadrature in finite element methods for the parabolic interface problems have not been studied earlier. The previous work on finite element analysis with numerical quadrature for parabolic problems without interface can be found in [13], [56] and references therein.

The rest of the chapter is organized as follows. In Section 2.2, we introduce the triangulation and recall some basic results from the literature. While Section 2.3 is devoted to the error analysis for the semidiscrete finite element approximation. error estimates for the fully discrete backward Euler time stepping scheme are derived in Section 2.4.

2.2 Preliminaries

Due to the presence of discontinuous coefficients the solution u, in general, does not belong to $H^2(\Omega)$. Regarding the regularity for the solution of the interface problem (2.1.1)-(2.1.3), we have the following result (cf. [15, 39, 58]).

Theorem 2.2.1 *Let $f \in H^1(0,T;L^2(\Omega))$, $g = 0$ and $u_0 \in H_0^1(\Omega)$. Then the problem (2.1.1)-(2.1.3) has a unique solution $u \in L^2(0,T;X \cap H_0^1(\Omega)) \cap H^1(0,T;Y)$. Further, for $u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$ and $f \in H^1(0,T;H^1(\Omega))$, solution u satisfies the following a priori estimate*

$$\int_0^t \{\|u_t\|^2_{H^2(\Omega_1)} + \|u_t\|^2_{H^2(\Omega_2)}\}ds \le C\{\|u_t(0)\|^2_{H^1(\Omega)} + \int_0^t \|f_t\|^2_{L^2(\Omega)}ds\}. \tag{2.2.1}$$

Proof. The existence of unique solution can be found in [15, 39].

Next, to obtain the a priori estimate we first transform the problem (2.1.1)-(2.1.3) to the following equivalent problem:

For a.e. $t \in (0,T]$, $u_t(x,t) \in H^2(\Omega_1) \cap H^2(\Omega_2)$ satisfies the following elliptic interface problem

$$-\nabla \cdot (\beta(x)\nabla u_t) = f_t - u_{tt} \quad \text{in } \Omega_i,\ i = 1,2 \tag{2.2.2}$$

along with boundary condition

$$u_t(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T] \tag{2.2.3}$$

and jump conditions (cf. [37])

$$[u_t] = 0 \quad \text{and} \quad \left[\beta\frac{\partial u_t}{\partial \mathbf{n}}\right] = 0 \ \text{ along } \Gamma. \tag{2.2.4}$$

From the a priori estimate for elliptic interface problem (cf. [15]), it follows that

$$\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)} \le C\{\|u_{tt}\|_{L^2(\Omega)} + \|f_t\|_{L^2(\Omega)}\}. \tag{2.2.5}$$

For any

$$v \in Y \cap \{\psi : \psi = 0 \ \text{ on } \partial\Omega\} \ \& \ [v] = 0 \text{ along } \Gamma,$$

we obtain

$$
\begin{aligned}
&-\int_{\Omega_1} \nabla \cdot (\beta_1 \nabla u) v dx - \int_{\Omega_2} \nabla \cdot (\beta_2 \nabla u) v dx \\
&= -\int_\Gamma \beta_1 \frac{\partial u}{\partial \mathbf{n}} v ds + \int_{\Omega_1} \beta_1 \nabla u \cdot \nabla v dx \\
&\quad + \int_\Gamma \beta_2 \frac{\partial u}{\partial \mathbf{n}} v ds + \int_{\Omega_2} \beta_2 \nabla u \cdot \nabla v dx \\
&= \int_{\Omega_1} \beta_1 \nabla u \cdot \nabla v dx + \int_{\Omega_2} \beta_2 \nabla u \cdot \nabla v dx + \int_\Gamma \left[\beta \frac{\partial u}{\partial \mathbf{n}} v\right] ds \\
&= A^1(u,v) + A^2(u,v). \qquad (2.2.6)
\end{aligned}
$$

Since $[v] = 0$ and $[\beta \partial u / \partial \mathbf{n}] = 0$ along Γ. Here. $A^l(.,.) : H^1(\Omega_l) \times H^1(\Omega_l) \to \mathbb{R}$ are local bilinear map given by

$$
A^l(w,v) = \int_{\Omega_l} \beta_l \nabla w \cdot \nabla v dx, \quad l = 1, 2.
$$

Then multiplying (2.2.2) by such v and integrating over Ω, we have

$$
(u_{tt}, v) + A^1(u_t, v) + A^2(u_t, v) = (f_t, v). \qquad (2.2.7)
$$

Again it follows from the arguments of [37] that $[u_{tt}] = 0$ along Γ and $u_{tt} = 0$ on $\partial\Omega$, and hence equation (2.2.7) leads to

$$
(u_{tt}, u_{tt}) + A^1(u_t, u_{tt}) + A^2(u_t, u_{tt}) = (f_t, u_{tt}) \qquad (2.2.8)
$$

so that

$$
\begin{aligned}
&\int_0^t \|u_{tt}\|^2_{L^2(\Omega)} ds + \frac{1}{2} A^1(u_t, u_t) + \frac{1}{2} A^2(u_t, u_t) \\
&\le \frac{1}{2} A^1(u_t(0), u_t(0)) + \frac{1}{2} A^2(u_t(0), u_t(0)) + C \int_0^t \|f_t\|^2_{L^2(\Omega)} ds.
\end{aligned}
$$

Under the assumption that $u_0 \in H^3(\Omega)$ and $f(x,0) \in H^1(\Omega)$, we have $u_t(0) \in H^1(\Omega)$. Therefore, u_{tt} satisfies the following a priori estimate

$$
\int_0^t \|u_{tt}\|^2_{L^2(\Omega)} ds \le C\{\|u_t(0)\|^2_{H^1(\Omega)} + \int_0^t \|f_t\|^2_{L^2(\Omega)} ds\}.
$$

Finally, using above estimate in (2.2.5) we obtain

$$
\int_0^t \{\|u_t\|^2_{H^2(\Omega_1)} + \|u_t\|^2_{H^2(\Omega_2)}\} ds \le C\{\|u_t(0)\|^2_{H^1(\Omega)} + \int_0^t \|f_t\|^2_{L^2(\Omega)} ds\}. \quad \square
$$

Remark 2.2.1 *Consider the following interface problems*

$$\xi_t - \nabla \cdot (\beta(x)\nabla\xi) = f(x,t) \quad in\, \Omega \times (0,T]$$
$$\xi(x,0) = \frac{1}{2}u_0 \quad in\, \Omega; \quad \xi(x,t) = 0 \quad on\ \partial\Omega \times (0,T]$$
$$[\xi] = 0, \quad \left[\beta\frac{\partial \xi}{\partial \mathbf{n}}\right] = 0 \quad along\, \Gamma,$$

and

$$\psi_t - \nabla \cdot (\beta(x)\nabla\psi) = 0 \quad in\, \Omega \times (0,T]$$
$$\psi(x,0) = \frac{1}{2}u_0 \quad in\, \Omega; \quad \psi(x,t) = 0 \quad on\ \partial\Omega \times (0,T]$$
$$[\psi] = 0, \quad \left[\beta\frac{\partial \psi}{\partial \mathbf{n}}\right] = g(x,t) \quad along\, \Gamma.$$

Then, $\xi + \psi$ satisfies the following weak formulation

$$(\xi_t + \psi_t, v) + A(\xi + \psi, v) = (f, v) + \langle g, v\rangle_\Gamma \quad \forall v \in H_0^1(\Omega). \tag{2.2.9}$$

Subtracting (2.2.9) from (2.1.4), we obtain

$$(u_t - \xi_t - \psi_t, v) + A(u - \xi - \psi, v) = 0. \tag{2.2.10}$$

Setting $v = u - \xi - \psi$ in (2.2.10) and coercivity of $A(.,.)$ leads to

$$\|u - \xi - \psi\|^2_{L^2(\Omega)} \le C\|u(0) - \xi(0) - \psi(0)\|^2_{L^2(\Omega)}.$$

Finally, use the fact $u(0) = \xi(0) + \psi(0)$ to have $u = \xi + \psi$ for a.e. $(x,t) \in \Omega \times (0,T]$.

For $g \in H^2(0,T; H^2(\Gamma))$, we assume that

$$\psi \in L^2(0,T; X \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2))$$

so that $u \in H^1(0,T; L^2(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2))$.

Thus, under the assumptions $u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$, $f \in H^1(0,T; L^2(\Omega))$, $f(x,0) \in H^1(\Omega)$ and $g \in H^2(0,T; H^2(\Gamma))$, solution u for the interface problem (2.1.1)-(2.1.3) is unique and $u \in L^2(0,T; X \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2))$. □

We now describe the triangulation $\mathcal{T}_h$ of Ω. We first approximate the domain Ω_1 by a domain Ω_1^h with the polygonal boundary Γ_h whose vertices all lie on the interface

Γ. Let Ω_2^h be the approximation for the domain Ω_2 with polygonal exterior and interior boundaries as $\partial\Omega$ and Γ_h, respectively. The triangles with one or two vertices on Γ are called the interface triangles, the set of all interface triangles is denoted by $\mathcal{T}_\Gamma^*$ and we write $\Omega_\Gamma^* = \cup_{K\in\mathcal{T}_\Gamma^*} K$.

We assume that the triangulation $\mathcal{T}_h$ of the domain Ω satisfy the following conditions:

($\mathcal{A}1$) $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$.

($\mathcal{A}2$) If K_1, $K_2 \in \mathcal{T}_h$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is a common vertex or edge of both triangles.

($\mathcal{A}3$) Each triangle $K \in \mathcal{T}_h$ is either in Ω_1^h or Ω_2^h and has at most two vertices lying on Γ_h.

($\mathcal{A}4$) For each triangle $K \in \mathcal{T}_h$, let $r_K, \overline{r}_K$ be the radii of its inscribed and circumscribed circles, respectively. Let $h = \max\{\overline{r}_K : K \in \mathcal{T}_h\}$.

Let V_h be a family of finite dimensional subspaces of $H_0^1(\Omega)$ defined on $\mathcal{T}_h$ consisting of piecewise linear functions vanishing on the boundary $\partial\Omega$ and satisfying the following approximation properties

$$\inf_{v_h\in V_h}\{\|v - v_h\|_{L^2(\Omega)} + h\|\nabla(v - v_h)\|_{L^2(\Omega)}\} \leq Ch^s\|v\|_{H^s(\Omega)}, \quad 1 \leq s \leq 2, \tag{2.2.11}$$

when $v \in H^s(\Omega) \cap H_0^1(\Omega)$. Examples of such finite element spaces can be found in [9] and [16]. Further, we assume the following inverse estimate

$$\|\phi\|_{H^1(\Omega)} \leq Ch^{-1}\|\phi\|_{L^2(\Omega)} \;\; \forall\, \phi \in V_h. \tag{2.2.12}$$

In order to study the effect of numerical quadrature we need to define approximation of the original bilinear form $A(.,.)$. For this purpose, we define the approximation $\beta_h(x)$ of the coefficient $\beta(x)$ as follows: For each triangle $K \in \mathcal{T}_h$, let $\beta_K(x) = \beta_i$ if $K \subset \Omega_i^h$, i=1 or 2. Then β_h is defined as

$$\beta_h(x) = \beta_K(x) \quad \forall K \in \mathcal{T}_h.$$

Then the approximation $A_h(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ to $A(.,.)$ can be defined as

$$A_h(w, v) = \sum_{K\in\mathcal{T}_h} \int_K \beta_K(x)\nabla w \cdot \nabla v dx \;\; \forall w, v \in H^1(\Omega).$$

To handle the L^2 inner product, we define the approximation on V_h and its induced norm by

$$(w,v)_h = \sum_{K\in\mathcal{T}_h} \left\{ \frac{1}{3}\text{meas(K)} \sum_{j=1}^{3} w(P_j^K)v(P_j^K) \right\}, \tag{2.2.13}$$

and $\|\phi\|_h = (\phi,\phi)_h^{\frac{1}{2}}$, where P_j^K are the vertices for the triangle K.

Let $\Pi_h : X \to V_h$ be the linear interpolation operator defined in [15]. For any $v \in X$, let v_i be the restriction of v on Ω_i for $i = 1, 2$. As the interface is of class C^2, we can extend the function $v_i \in H^2(\Omega_i)$ on to the whole Ω and obtain the function $\tilde{v}_i \in H^2(\Omega)$ such that $\tilde{v}_i = v_i$ on Ω_i and

$$\|\tilde{v}_i\|_{H^2(\Omega)} \le C\|v_i\|_{H^2(\Omega_i)}, i = 1, 2. \tag{2.2.14}$$

For the existence of such extensions, we refer to Stein [62]. Then, for $K \in \mathcal{T}_h$, we now define

$$\Pi_h u = \begin{cases} \Pi_h\tilde{u}_1 & \text{if } K \subseteq \Omega_1^h \\ \Pi_h\tilde{u}_2 & \text{if } K \subseteq \Omega_2^h. \end{cases}$$

The following optimal approximation of Π_h operator is borrowed from [20].

Lemma 2.2.1 *For $v \in X$ with $[v] = 0$ along Γ, then the following approximation properties*

$$\|v - \Pi_h v\|_{H^m(\Omega)} \le Ch^{2-m}\|v\|_X, \ m = 0, 1,$$

holds true. □

We now recall some existing results on the approximation A_h and the inner product which will be frequently used in our analysis. For a proof, we refer to [16, 59].

Lemma 2.2.2 *On V_h the norms $\|.\|_{L^2(\Omega)}$ and $\|.\|_h$ are equivalent. Further, for $w, v \in V_h$ and $f \in X$, we have*

$$\begin{aligned} |A_h(w,v) - A(w,v)| &\le Ch \sum_{K\in\mathcal{T}_\Gamma^*} \|\nabla v\|_{L^2(K)}\|\nabla w\|_{L^2(K)}, \\ |(w,v) - (w,v)_h| &\le Ch^2\|w\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \\ |(\Pi_h f, v)_h - (f,v)| &\le Ch^2\|f\|_X\|v\|_{H^1(\Omega)}. \ \square \end{aligned}$$

We denote $\mathcal{X}$ to be the collection of all $v \in \{\psi \in L^2(\Omega) : \psi = 0 \text{ on } \partial\Omega\} \cap H^2(\Omega_1) \cap H^2(\Omega_2)$ with $[v] = 0$ and $[\beta \partial v/\partial \mathbf{n}] = 0$ along Γ. For any $v \in \mathcal{X}$, we define

$$f^* = \begin{cases} -\nabla \cdot (\beta_1 \nabla v) & \text{in } \Omega_1 \\ -\nabla \cdot (\beta_2 \nabla v) & \text{in } \Omega_2. \end{cases}$$

Clearly $f^* \in L^2(\Omega)$. Then define $P_h : \mathcal{X} \to V_h$ by

$$A_h(P_h v, v_h) = (f^*, v_h) \quad \forall v_h \in V_h.$$

Again

$$\begin{aligned} (f^*, v_h) &= -\int_{\Omega_1} \nabla \cdot (\beta_1 \nabla v) v_h dx - \int_{\Omega_2} \nabla \cdot (\beta_2 \nabla v) v_h dx \\ &= -\int_\Gamma \beta_1 \frac{\partial v}{\partial \mathbf{n}} v_h ds + \int_{\Omega_1} \beta_1 \nabla v \cdot \nabla v_h dx \\ &\quad + \int_\Gamma \beta_2 \frac{\partial v}{\partial \mathbf{n}} v_h ds + \int_{\Omega_2} \beta_2 \nabla v \cdot \nabla v_h dx \\ &= \int_{\Omega_1} \beta_1 \nabla v \cdot \nabla v_h dx + \int_{\Omega_2} \beta_2 \nabla v \cdot \nabla v_h dx + \int_\Gamma \left[\beta \frac{\partial v}{\partial \mathbf{n}}\right] v_h ds \\ &=: A^1(v, v_h) + A^2(v, v_h). \end{aligned}$$

Thus, we have

$$A_h(P_h v, v_h) = A^1(v, v_h) + A^2(v, v_h) \quad \forall v_h \in V_h. \tag{2.2.15}$$

Regarding the approximation properties of P_h operator defined by (2.2.15), we have the following result (cf. [2])

Lemma 2.2.3 *Let P_h be defined by (2.2.15), then for any $v \in \mathcal{X}$ there exists a positive constant C independent of the mesh parameter h such that*

$$\|P_h v - v\|_{H^1(\Omega_1)} + \|P_h v - v\|_{H^1(\Omega_2)} \le Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}),$$
$$\|P_h v - v\|_{L^2(\Omega)} \le Ch^2(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}). \quad \square$$

Let $L_h : L^2(\Omega) \to V_h$ be the standard L^2 projection defined by

$$(L_h v, \phi) = (v, \phi), \quad v \in L^2(\Omega) \;\; \forall \, \phi \in V_h. \tag{2.2.16}$$

A simple application of Lemma 2.2.3 and inverse inequality (2.2.12) leads to the following optimal error estimates for L^2 projection.

Lemma 2.2.4 *Let L_h be defined by (2.2.16). Then, for $v \in \mathcal{X}$, there exists a positive constant C independent of the mesh parameter h such that*

$$\|v - L_h v\|_{L^2(\Omega)} \le Ch^2\Big(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\Big),$$
$$\|v - L_h v\|_{H^1(\Omega_1)} + \|v - L_h v\|_{H^1(\Omega_2)} \le Ch\Big(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\Big). \quad \square$$

2.3 Error Estimates for the Semidiscrete Problem

This section deals with the error analysis for the spatially discrete scheme. For $f \in X$ and $g = 0$, the semidiscrete finite element method with quadrature is defined as: Find $u_h^*(t) \in V_h$ such that

$$(u_{ht}^*, v_h)_h + A_h(u_h^*, v_h) = (\Pi_h f, v_h)_h \quad \forall v_h \in V_h, \tag{2.3.1}$$

with $u_h^*(0) = P_h u_0$.

In order to discuss the error analysis of finite element method with quadrature, we consider the following auxiliary approximation $u_h \in V_h$ given by

$$(u_{ht}, v_h) + A_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h, \quad t \in (0, T], \tag{2.3.2}$$

with $u_h(0) = P_h u_0$.

Now, define the error $e(t) = u(t) - u_h^*(t)$ as

$$e(t) = u(t) - u_h^*(t) = u(t) - u_h(t) + u_h(t) - u_h^*(t) = e_1(t) + e_2(t),$$

where $e_1(t) = u(t) - u_h(t),\ e_2(t) = u_h(t) - u_h^*(t)$.

For the quadrature free error $e_1(t)$, we have the following error estimates (see, Theorems 3.1-3.2 in [21])

Theorem 2.3.1 *Let u and u_h be the solutions of (2.1.1)-(2.1.3) and* (2.3.2)*, respectively. Then, for $u_0 \in H_0^1(\Omega)$, $g = 0$ and $f \in H^1(0,T;L^2(\Omega))$, there is a positive constant C independent of h such that*

$$\|u - u_h\|_{L^2(0,T;L^2(\Omega))} + h\|u - u_h\|_{L^2(0,T;H^1(\Omega))}$$
$$\le Ch^2\left(\|u_0\|^2_{H^1(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))} + \|u(T)\|^2_X + \|u\|^2_{L^2(0,T;X)}\right). \quad \square$$

Further, splitting e_1 in terms of standard ρ and θ as

$$e_1 = (u - P_h u) + (P_h u - u_h) = \rho + \theta,$$

where $\rho = u - P_h u$ and $\theta = P_h u - u_h$, we note that (cf. [63])

$$(\theta_t, v_h) + A_h(\theta, v_h) = -(\rho_t, v_h) \tag{2.3.3}$$

For $v_h = \theta_t$, we have

$$\begin{aligned}(\theta_t, \theta_t) + \frac{1}{2}\frac{d}{dt}A_h(\theta, \theta) &\leq \|\rho_t\|_{L^2(\Omega)}\|\theta_t\|_{L^2(\Omega)} \\ &\leq C_\epsilon\|\rho_t\|^2_{L^2(\Omega)} + \frac{\epsilon}{2}\|\theta_t\|^2_{L^2(\Omega)}.\end{aligned}$$

Integrating the above equation form 0 to t and using Lemma 2.2.3, we obtain

$$\begin{aligned}\int_0^t \|\theta_t\|^2_{L^2(\Omega)}ds + A_h(\theta, \theta) &\leq C\int_0^t \|\rho_t\|^2_{L^2(\Omega)}ds \\ &\leq Ch^4\sum_{i=1}^{2}\int_0^t \|u_t\|^2_{H^2(\Omega_i)}ds. \end{aligned}\tag{2.3.4}$$

Again inverse estimate (2.2.12) leads to

$$\int_0^t \|\theta_t\|^2_{H^1(\Omega)}ds \leq Ch^{-2}\int_0^t \|\theta_t\|^2_{L^2(\Omega)}ds \leq Ch^2\sum_{i=1}^{2}\int_0^t \|u_t\|^2_{H^2(\Omega_i)}ds. \tag{2.3.5}$$

Finally, Lemma 2.2.3 together with estimates (2.3.4)-(2.3.5) leads to the following $H^1(L^2)$ and $H^1(H^1)$ norms error estimate

Theorem 2.3.2 *Let u and u_h be the solutions of (2.1.1)-(2.1.3) and* (2.3.2), *respectively. Then, for $u_0 \in H_0^1(\Omega) \cap H^3(\Omega)$, $g = 0$ and $f \in H^1(0, T; H^1(\Omega))$, there is a positive constant C independent of h such that*

$$\int_0^t \|e_1'(t)\|^2_{L^2(\Omega)}ds + h^2\sum_{i=1}^{2}\int_0^t \|e_1'(t)\|^2_{H^1(\Omega_i)}ds \leq Ch^4\sum_{i=1}^{2}\int_0^t \|u_t\|^2_{H^2(\Omega_i)}ds. \quad \square$$

Remark 2.3.1 *The optimal error estimates in $H^1(L^2)$ and $H^1(H^1)$ norms are derived for high regularity of the initial conditions. Under low regularity assumptions of the initial data, solution $u \in H^1(0, T; Y)$ and for which $P_h u_t$ is not well defined. The initial data is assumed to be very regular, so that a solution exists and belongs to the necessary Sobolev spaces satisfying a priori estimate (2.2.1). To the best of our knowledge, convergence of finite element method in $H^1(L^2)$ and $H^1(H^1)$ norms for the parabolic interface problems have not been studied earlier.*

Next, for the term e_2, we have

$$\begin{aligned}
&C\|u_h - u_h^*\|^2_{H^1(\Omega)} \\
&\le A_h(u_h - u_h^*, u_h - u_h^*) = A_h(u_h, u_h - u_h^*) - A_h(u_h^*, u_h - u_h^*) \\
&= (f, u_h - u_h^*) - (u_{ht}, u_h - u_h^*) + (u_{ht}^*, u_h - u_h^*)_h - (\Pi_h f, u_h - u_h^*)_h \\
&= \{(f, u_h - u_h^*) - (\Pi_h f, u_h - u_h^*)\} + \{(\Pi_h f, u_h - u_h^*) - (\Pi_h f, u_h - u_h^*)_h\} \\
&\quad + \{(u_{ht}^*, u_h - u_h^*)_h - (u_{ht}^*, u_h - u_h^*)\} - (u_{ht} - u_{ht}^*, u_h - u_h^*) \\
&=: I_1 + I_2 + I_3 - \frac{1}{2}\frac{d}{dt}\|u_h - u_h^*\|^2_{L^2(\Omega)}.
\end{aligned}$$

Integrating from 0 to T and assuming $u_h(0) = u_h^*(0)$, we have

$$\int_0^T \|e_2\|^2_{H^1(\Omega)} ds \le \int_0^T (I_1 + I_2 + I_3) ds. \tag{2.3.6}$$

By Lemma 2.2.1 and Cauchy-Schwarz inequality it follows that

$$\int_0^T I_1 ds \le Ch^2 \left(\int_0^T \|f\|_X^2 ds\right)^{\frac{1}{2}} \left(\int_0^T \|e_2\|^2_{L^2(\Omega)} ds\right)^{\frac{1}{2}}. \tag{2.3.7}$$

Applying Lemma 2.2.2 for I_2, we have

$$\int_0^T I_2 ds \le Ch^2 \left(\int_0^T \|f\|_X^2 ds\right)^{\frac{1}{2}} \left(\int_0^T \|e_2\|^2_{H^1(\Omega)} ds\right)^{\frac{1}{2}}. \tag{2.3.8}$$

Similarly for I_3, we have

$$\int_0^T I_3 ds \le Ch^2 \left(\int_0^T \|u_{ht}^*\|^2_{H^1(\Omega)} ds\right)^{\frac{1}{2}} \left(\int_0^T \|e_2\|^2_{H^1(\Omega)} ds\right)^{\frac{1}{2}}.$$

Then apply inverse inequality (2.2.12) to have

$$\begin{aligned}
\int_0^T I_3 ds &\le Ch \left(\int_0^T \|u_{ht}^*\|^2_{L^2(\Omega)} ds\right)^{\frac{1}{2}} \left(\int_0^T \|e_2\|^2_{H^1(\Omega)} ds\right)^{\frac{1}{2}} \\
&\le Ch \left(\int_0^T \|f\|_X^2 ds + \|u_0\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}} \left(\int_0^T \|e_2\|^2_{H^1(\Omega)} ds\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.3.9}$$

Estimates (2.3.6)-(2.3.9) yields

$$\left(\int_0^T \|e_2\|^2_{H^1(\Omega)} ds\right)^{\frac{1}{2}} \le Ch\left(\|f\|^2_{L^2(0,T;X)} + \|u_0\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}. \tag{2.3.10}$$

This together with Theorem 2.3.1 leads to the following optimal $L^2(H^1)$ norm estimate.

Theorem 2.3.3 *Let u and u_h^* be the solutions of (2.1.1)-(2.1.3) and* (2.3.1), *respectively. Then, for $f \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;X)$, $g=0$ and $u_0 \in H_0^1(\Omega)$, the following $L^2(H^1)$ norm error estimate holds*

$$\|u - u_h^*\|_{L^2(0,T;H^1(\Omega))}$$
$$\leq Ch\left(\|u_0\|^2_{H^1(\Omega)} + \|f\|^2_{L^2(0,T;X)} + \|u(T)\|^2_X + \|u\|^2_{L^2(0,T;X)} \right)^{\frac{1}{2}}. \quad \square$$

Next, for L^2-norm error estimate, we shall use the elliptic duality argument. For this purpose, we now consider the following auxiliary problem: Find $w \in H_0^1(\Omega)$ such that

$$A(w,v) = (u_h - u_h^*, v) \quad \forall v \in H_0^1(\Omega), \ \ t \in (0,T] \tag{2.3.11}$$

with $[w] = 0$ & $\left[\beta \frac{\partial w}{\partial n}\right] = 0$ across the interface Γ. Then its finite element approximation with quadrature is defined to be a function $w_h \in V_h$ satisfying

$$A_h(w_h, v_h) = (u_h - u_h^*, v_h)_h \quad \forall v_h \in V_h, \ \ t \in (0,T]. \tag{2.3.12}$$

Then following the arguments of Deka ([20]), we have

$$\|w - w_h\|_{L^2(\Omega)} + h\|\nabla(w - w_h)\|_{L^2(\Omega)} \leq Ch^2 \|u_h - u_h^*\|_{H^1(\Omega)}. \tag{2.3.13}$$

Again subtracting (2.3.1) from (2.3.2), we obtain

$$\begin{aligned}(u_{ht} - u_{ht}^*, v_h)_h + A_h(u_h - u_h^*, v_h) &= (f, v_h) - (\Pi_h f, v_h)_h \\ &\quad + (u_{ht}, v_h)_h - (u_{ht}, v_h). \end{aligned} \tag{2.3.14}$$

Setting $v = u_h - u_h^*$ in (2.3.11), we have

$$\begin{aligned}\|u_h - u_h^*\|^2_{L^2(\Omega)} &= A(w, u_h - u_h^*) \\ &= A(w - w_h, u_h - u_h^*) + A(w_h, u_h - u_h^*) \\ &= A(w - w_h, u_h - u_h^*) + A(u_h, w_h) - A(u_h^*, w_h) \\ &= A(w - w_h, u_h - u_h^*) + A(u_h, w_h) - A_h(u_h, w_h) \\ &\quad + A_h(u_h, w_h) - A_h(u_h^*, w_h) + A_h(u_h^*, w_h) - A(u_h^*, w_h).\end{aligned}$$

Further equation (2.3.14) leads to

$$\begin{aligned}\|u_h - u_h^*\|^2_{L^2(\Omega)} &= \{A(w - w_h, u_h - u_h^*)\} + \{A_h(u_h^*, w_h) - A(u_h^*, w_h)\} \\ &\quad + \{A(u_h, w_h) - A_h(u_h, w_h)\} \\ &\quad + (f, w_h) - (u_{ht}, w_h) + (u_{ht}^*, w_h)_h - (\Pi_h f, w_h)_h \\ &= \{A(w - w_h, u_h - u_h^*)\} + \{A_h(u_h^*, w_h) - A(u_h^*, w_h)\} \\ &\quad + \{A(u_h, w_h) - A_h(u_h, w_h)\} \\ &\quad + \{(f, w_h) - (\Pi_h f, w_h)_h\} + \{(u_{ht}, w_h)_h - (u_{ht}, w_h)\} \\ &\quad - (u_{ht} - u_{ht}^*, w_h)_h \\ &=: J_1 + J_2 + J_3 + J_4 + J_5 - (u_{ht} - u_{ht}^*, w_h)_h. \end{aligned} \tag{2.3.15}$$

Differentiating (2.3.12) with respect to t, we obtain

$$A_h(w_{ht}, v_h) = (u_{ht} - u_{ht}^*, v_h)_h.$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}A_h(w_h, w_h) = A_h(w_{ht}, w_h) = (u_{ht} - u_{ht}^*, w_h)_h,$$

and hence, integrating (2.3.15) from 0 to T we obtain

$$\|u_h - u_h^*\|^2_{L^2(0,T;L^2(\Omega))} \le C\int_0^T (|J_1| + |J_2| + |J_3| + |J_4| + |J_5|)ds. \tag{2.3.16}$$

Here, we have used $A_h(w_h(0), w_h(0)) = 0$. Now, we estimate each term separately. For the term J_1, use (2.3.10) and (2.3.13) to have

$$\begin{aligned}\int_0^T |J_1|ds &\le C\left(\int_0^T \|w - w_h\|^2_{H^1(\Omega)}ds\right)^{\frac{1}{2}}\left(\int_0^T \|u_h - u_h^*\|^2_{H^1(\Omega)}ds\right)^{\frac{1}{2}} \\ &\le Ch^2\|e_2\|_{L^2(0,T;L^2(\Omega))}\left(\|f\|^2_{L^2(0,T;X)} + \|u_0\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}. \end{aligned} \tag{2.3.17}$$

Using Lemma 2.2.2, estimate (2.3.13) and Theorem 2.3.3, we have

$$\begin{aligned}\int_0^T |J_2|ds &\le Ch\left(\int_0^T \|u_h^*\|^2_{H^1(\Omega_\Gamma^*)}ds\right)^{\frac{1}{2}}\left(\int_0^T \|w_h\|^2_{H^1(\Omega_\Gamma^*)}ds\right)^{\frac{1}{2}} \\ &\le Ch\big(\|u_h^* - u\|_{L^2(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^1(\Omega_\Gamma^*))}\big) \\ &\quad \times \big(\|w_h - w\|_{L^2(0,T;H^1(\Omega))} + \|w\|_{L^2(0,T;H^1(\Omega_\Gamma^*))}\big) \\ &\le Ch^2\left(\|f\|^2_{L^2(0,T;X)} + \|u_0\|^2_{H^1(\Omega)} + \|u\|^2_{L^2(0,T;X)}\right)^{\frac{1}{2}} \\ &\quad \times \|e_2\|_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{2.3.18}$$

Here, we have used the fact that (cf. Deka and Sinha [59], page 260)

$$\|u\|_{H^1(\Omega_\Gamma^*)} \le Ch^{\frac{1}{2}}\|u\|_X, \qquad \|w\|_{H^1(\Omega_\Gamma^*)} \le Ch^{\frac{1}{2}}\|w\|_X \le Ch^{\frac{1}{2}}\|u_h - u_h^*\|_{L^2(\Omega)}.$$

Similarly, for the term J_3, we have

$$\begin{aligned}\int_0^T |J_3|ds &\le Ch^2\Big(\|f\|^2_{L^2(0,T;X)} + \|u_0\|^2_{H^1(\Omega)} + \|u\|^2_{L^2(0,T;X)}\Big)^{\frac{1}{2}} \\ &\quad \times \|e_2\|_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{2.3.19}$$

Arguing as in I_1 and I_2, we obtain

$$\begin{aligned}\int_0^T |J_4|ds &\le Ch^2\|f\|_{L^2(0,T;X)}\|w_h\|_{L^2(0,T;H^1(\Omega))} \\ &\le Ch^2\|f\|_{L^2(0,T;X)}\|e_2\|_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{2.3.20}$$

Here, we have used the fact that $\|w_h\|_{H^1(\Omega)} \le C\|u_h - u_h^*\|_{L^2(\Omega)}$.

For the term J_5, we again recall Lemma 2.2.2 along with Theorem 2.3.2 to have

$$\begin{aligned}\int_0^T |J_5|ds &\le Ch^2\Big(\int_0^T \|u_{ht}\|^2_{H^1(\Omega)}ds\Big)^{\frac{1}{2}}\Big(\int_0^T \|w_h\|^2_{H^1(\Omega)}ds\Big)^{\frac{1}{2}} \\ &\le Ch^2\Big(\sum_{i=1}^2\int_0^T \|u_t\|^2_{H^2(\Omega_i)}ds\Big)^{\frac{1}{2}}\|e_2\|_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{2.3.21}$$

Then combine (2.3.16)-(2.3.21) to have

$$\begin{aligned}\|e_2\|_{L^2(0,T;L^2(\Omega))} &\le Ch^2\Big(\|u_0\|^2_{H^1(\Omega)} + \|f\|^2_{L^2(0,T;X)} \\ &\quad + \|u\|^2_{L^2(0,T;X)} + \sum_{i=1}^2 \|u_t\|^2_{L^2(0,T;H^2(\Omega_i))}\Big)^{\frac{1}{2}}, \end{aligned}$$

which together with Theorem 2.3.1 leads to the following optimal error estimate

Theorem 2.3.4 *Let u and u_h^* be the solutions of (2.1.1)-(2.1.3) and* (2.3.1), *respectively. Then, for $u_0 \in H_0^1(\Omega)\cap H^3(\Omega)$, $g = 0$ and $f \in H^1(0,T;H^1(\Omega))\cap L^2(0.T;X)$, there is a positive constant C independent of h such that*

$$\begin{aligned}\|u - u_h^*\|_{L^2(0,T;L^2(\Omega))} &\le Ch^2\Big(\int_0^T \|f\|^2_X ds + \|u(T)\|^2_X + \int_0^T \|u\|^2_X ds \\ &\quad + \sum_{i=1}^2 \int_0^T \|u_t\|^2_{H^2(\Omega_i)}ds\Big)^{\frac{1}{2}}. \quad \square \end{aligned}$$

2.4 Error Estimate for Fully Discrete case

In this section, we give error estimates for the fully discrete scheme with quadrature. Optimal order error estimate in $L^2(L^2)$ norm is derived.

In order to discretize (2.3.1) in time. we first divide the interval $[0, T]$ into M equally spaced subintervals by the following points

$$0 = t^0 \leq t^1 \leq ... \leq t^M = T$$

with $t^n = nk$, $k = T/M$ the time step. Let $I_n = (t_{n-1}, t_n]$ be the n-th sub interval. We shall use the finite dimensional space

$$S_{kh} = \{\phi : [0, T] \to V_h \ : \phi|_{I_n} \in V_h \text{ is constant in time}\}.$$

For $\phi \in S_{kh}$, we denote by ϕ^n the value of ϕ at t_n and write S_{kh}^n for the restriction to I_n of the functions in S_{kh}. Now we introduce the backward difference quotient

$$\Delta_k \phi^n = \frac{\phi^n - \phi^{n-1}}{k},$$

for a given sequence $\{\phi^n\}_{n=0}^M \subset L^2(\Omega)$. For a given Banach space $\mathcal{B}$ and some function $\xi \in L^2(0, T; \mathcal{B})$, we write

$$\overline{\xi}^n = k^{-1} \int_{I_n} \xi(x, t) dt. \tag{2.4.1}$$

Then, we consider the following fully discrete Galerkin method with quadrature: For $1 \leq n \leq M$, find $w_h^n \in S_{kh}$ such that

$$(\Delta_k w_h^n, v_h)_h + A_h(w_h^n, v_h) = (\overline{f}^n, v_h) \quad \forall v_h \in S_{kh}^n, \tag{2.4.2}$$

with $w_h^0 = L_h u_0$.

Before proceeding further, we introduce the following auxiliary discrete problem: For $n = M, M-1, ..., 1$ find $z_h^{n-1} \in V_h$ such that

$$(-\Delta_k z_h^n, v_h)_h + A_h(z_h^{n-1}, v_h) = (\overline{u}_I^n - w_h^n, v_h)_h \quad \forall v_h \in V_h \tag{2.4.3}$$

with $z_h^M = 0$ and

$$\overline{u}_I^n = k^{-1} \int_{I_n} \Pi_h u dt.$$

We shall need the following stability result for z_h^{n-1} satisfying (2.4.3).

Lemma 2.4.1 *For z_h^{n-1}, we have*

$$\|z_h^0\|^2_{H^1(\Omega)} + \sum_{n=1}^{M} k\|\Delta_k z_h^n\|^2_{L^2(\Omega)} \le \sum_{n=1}^{M} k\|\overline{u}_I^n - w_h^n\|^2_{L^2(\Omega)}.$$

Proof. The lemma can be proved by setting $v_h = -k\Delta_k z_h^n$ in (2.4.3) and applying the argument of [58]. We omit the details. $\square$

We need the following interface approximation estimate for z_h^{n-1}, which is crucial to study the L^2-norm error estimate.

Lemma 2.4.2 *For z_h^{n-1}, we have*

$$\sum_{n=1}^{M} k\|z_h^{n-1}\|^2_{H^1(\Omega_\Gamma^*)} \le Ch\left(\sum_{n=1}^{M} k\|\overline{u}_I^n - w_h^n\|^2_{L^2(\Omega)}\right).$$

Proof. Let $z^{n-1} \in X \cap H_0^1(\Omega)$ be the solution of the following auxiliary problem

$$A(z^{n-1}, v) = (\overline{u}_I^n - w_h^n + \Delta_k z_h^n, v) \quad \forall v \in H_0^1(\Omega). \tag{2.4.4}$$

Then applying elliptic regularity estimate (cf. [15]), we have

$$\|z^{n-1}\|_X \le C(\|\overline{u}_I^n - w_h^n\|_{L^2(\Omega)} + \|\Delta_k z_h^n\|_{L^2(\Omega)}). \tag{2.4.5}$$

We know from (2.4.3) that z_h^{n-1} is the finite element approximation of z^{n-1} with quadrature. Then arguing as in Theorem 3.1 of [20], we have

$$\begin{aligned}
\|\Pi_h z^{n-1} - z_h^{n-1}\|^2_{H^1(\Omega)} &\le Ch\|z^{n-1}\|_X\|\Pi_h z^{n-1} - z_h^{n-1}\|_{H^1(\Omega)} \\
&+ Ch^2\|\overline{u}_I^n - w_h^n + \Delta_k z_h^n\|_{H^1(\Omega)}\|\Pi_h z^{n-1} - z_h^{n-1}\|_{H^1(\Omega)} \\
&\le Ch\|z^{n-1}\|_X\|\Pi_h z^{n-1} - z_h^{n-1}\|_{H^1(\Omega)} \\
&+ Ch\|\overline{u}_I^n - w_h^n + \Delta_k z_h^n\|_{L^2(\Omega)}\|\Pi_h z^{n-1} - z_h^{n-1}\|_{H^1(\Omega)}.
\end{aligned}$$

Then apply Lemma 2.2.1 and (2.4.5) to have

$$\|z^{n-1} - z_h^{n-1}\|_{H^1(\Omega)} \le Ch(\|\overline{u}_I^n - w_h^n\|_{L^2(\Omega)} + \|\Delta_k z_h^n\|_{L^2(\Omega)}).$$

Summing over n from $n = 1$ to $n = M$ and applying Lemma 2.4.1, we obtain

$$\sum_{n=1}^{M} k\|z^{n-1} - z_h^{n-1}\|^2_{H^1(\Omega)} \le Ch^2\sum_{n=1}^{M} k\|\overline{u}_I^n - w_h^n\|^2_{L^2(\Omega)}. \tag{2.4.6}$$

Again using the fact $\|z^{n-1}\|_{H^1(\Omega_\Gamma^*)} \leq Ch^{\frac{1}{2}}\|z^{n-1}\|_X$ and (2.4.6), we have

$$\begin{aligned}\sum_{n=1}^{M} k\|z_h^{n-1}\|^2_{H^1(\Omega_\Gamma^*)} &\leq Ch^2\sum_{n=1}^{M} k\|\overline{u}_I^{\,n} - w_h^n\|^2_{L^2(\Omega)} + Ch\sum_{n=1}^{M} k\|z^{n-1}\|^2_X \\ &\leq Ch\sum_{n=1}^{M} k\|\overline{u}_I^{\,n} - w_h^n\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.4.7}$$

In the last inequality we have used (2.4.5) and Lemma 2.4.1. □

Next, we introduce the interpolant $P_k \in S_{kh}$ of u defined by

$$\overline{P_k}^{\,n} = \frac{1}{k}\int_{I_n} P_h u ds.$$

Then, for $n = 1, 2. ..., M$ it follows from (2.2.15) that

$$A_h(z_h^{n-1}, \overline{P_k}^{\,n}) = A(z_h^{n-1}, \overline{u}^{\,n}). \tag{2.4.8}$$

By setting $v_h = k(\overline{P_k}^{\,n} - w_h^n)$ in (2.4.3), we obtain

$$\begin{aligned} Ck\|\overline{u}_I^{\,n} - w_h^n\|^2_{L^2(\Omega)} &\leq k(\overline{u}_I^{\,n} - w_h^n, \overline{u}_I^{\,n} - \overline{P_k}^{\,n})_h + k(-\Delta_k z_h^n, \overline{P_k}^{\,n} - w_h^n)_h \\ &\quad + kA_h(z_h^{n-1}, \overline{P_k}^{\,n}) - kA_h(z_h^{n-1}, w_h^n) \end{aligned}$$

which together with (2.4.8) yields

$$\begin{aligned} Ck\|\overline{u}_I^{\,n} - w_h^n\|^2_{L^2(\Omega)} &\leq k(\overline{u}_I^{\,n} - w_h^n, \overline{u}_I^{\,n} - \overline{P_k}^{\,n})_h + k(-\Delta_k z_h^n, \overline{P_k}^{\,n} - w_h^n)_h \\ &\quad + kA(z_h^{n-1}, \overline{u}^{\,n}) - kA_h(z_h^{n-1}, w_h^n). \end{aligned} \tag{2.4.9}$$

Again, note that for all $v \in H_0^1(\Omega)$, we have

$$(\Delta_k u^n, v) + A(\overline{u}^{\,n}, v) = (\overline{f}^{\,n}, v), \quad 1 \leq n \leq M. \tag{2.4.10}$$

Then it is easy to verify from the estimates (2.4.2) and (2.4.10) that

$$\begin{aligned} A(z_h^{n-1}, \overline{u}^{\,n}) - A_h(z_h^{n-1}, w_h^n) &= (\Delta_k u^n, z_h^{n-1})_{\bar{h}} - (\Delta_k u^n, z_h^{n-1}) \\ &\quad + (-\Delta_k(u^n - w_h^n), z_h^{n-1})_{\bar{h}}. \end{aligned} \tag{2.4.11}$$

Since the solutions concerned are only on $H^1(\Omega)$ globally, it is not meaningful to use the definition (2.2.13) for evaluation of the term $(v, \phi_h)_h$ for $v \in X$ and $\phi_h \in V_h$. Therefore, notations $(v, \phi_h)_{\tilde{h}}$ and $(\phi_h, v)_{\tilde{h}}$ have been introduced and are evaluated by the following formulae

$$(v, \phi_h)_{\tilde{h}} = (L_h v, \phi_h)_h \ \ \& \ \ (\phi_h, v)_{\tilde{h}} = (\phi_h, L_h v)_h.$$

Then, for any $v \in H_0^1(\Omega)$ and $\phi_h \in V_h$, it is easy to verify the following facts

$$\begin{aligned}
&(v_h, \phi_h)_{\tilde{h}} = (v_h, \phi_h)_h, \quad v_h \in V_h \ \ \& \ \ (v, \phi_h)_{\tilde{h}} = (\phi_h, v)_{\tilde{h}}, \\
&(\phi + \psi, \phi_h)_{\tilde{h}} = (\phi, \phi_h)_{\tilde{h}} + (\psi, \phi_h)_{\tilde{h}}, \quad \phi, \ \psi \in H_0^1(\Omega), \\
&|(v, \phi_h)_{\tilde{h}}| \leq C\|v\|_{L^2(\Omega)}\|\phi_h\|_{L^2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
|(v, \phi_h)_{\tilde{h}} - (v, \phi_h)| &= |(L_h v, \phi_h)_h - (L_h v, \phi_h)| \\
&\leq Ch^2\|L_h v\|_{H^1(\Omega)}\|\phi_h\|_{H^1(\Omega)} \\
&\leq Ch^2\|v\|_{H^1(\Omega)}\|\phi_h\|_{H^1(\Omega)}. \qquad (2.4.12)
\end{aligned}$$

Now, estimate (2.4.11) together with (2.4.9) leads to

$$\begin{aligned}
Ck\|\overline{u}_I^n - w_h^n\|_{L^2(\Omega)}^2 &\leq k(\overline{u}_I^n - w_h^n, \overline{u}_I^n - \overline{P}_k^{\,n})_h + k(-\Delta_k z_h^n, \overline{P}_k^{\,n} - w_h^n)_{\tilde{h}} \\
&\quad + k(\Delta_k u^n, z_h^{n-1})_{\tilde{h}} - k(\Delta_k u^n, z_h^{n-1}) \\
&\quad + k(-\Delta_k(u^n - w_h^n), z_h^{n-1})_{\tilde{h}} \\
&\leq k(\overline{u}_I^n - w_h^n, \overline{u}_I^n - \overline{P}_k^{\,n})_h + k(-\Delta_k z_h^n, \overline{P}_k^{\,n} - u^n)_{\tilde{h}} \\
&\quad + k(-\Delta_k z_h^n, u^n - w_h^n)_{\tilde{h}} + k(\Delta_k u^n, z_h^{n-1})_{\tilde{h}} \\
&\quad - k(\Delta_k u^n, z_h^{n-1}) + k(-\Delta_k(u^n - w_h^n), z_h^{n-1})_{\tilde{h}}. \qquad (2.4.13)
\end{aligned}$$

Summing over n, we have

$$\begin{aligned}
&C\sum_{n=1}^{M} k\|\overline{u}_I^n - w_h^n\|_{L^2(\Omega)}^2 \\
&\leq \sum_{n=1}^{M} k(\overline{u}_I^n - w_h^n, \overline{u}_I^n - \overline{P}_k^{\,n})_h + \sum_{n=1}^{M} k(-\Delta_k z_h^n, \overline{P}_k^{\,n} - u^n)_{\tilde{h}} \\
&\quad + \sum_{n=1}^{M} k\left\{(\Delta_k u^n, z_h^{n-1})_{\tilde{h}} - (\Delta_k u^n, z_h^{n-1})\right\} + \sum_{n=1}^{M} k\Big\{(-\Delta_k z_h^n, u^n - w_h^n)_{\tilde{h}} \\
&\qquad + (-\Delta_k(u^n - w_h^n), z_h^{n-1})_{\tilde{h}}\Big\} \\
&=: IV_1 + IV_2 + IV_3 + IV_4. \qquad (2.4.14)
\end{aligned}$$

Before estimating the four terms appearing in (2.4.14) we first rewrite IV_4. Using the fact that $z_h^M = 0$ and applying the identity

$$\sum_{n=1}^{M}(a_n - a_{n-1})b_n = a_M b_M - a_0 b_0 - \sum_{n=1}^{M} a_{n-1}(b_n - b_{n-1})$$

to IV_4 with $a_n = z_h^n$ and $b_n = u^n - w_h^n$, we obtain

$$IV_4 = (z_h^0, u_0 - w_h(0))_{\tilde{h}} = (z_h^0, u_0 - L_h u_0)_{\tilde{h}} \le C\|z_h^0\|_{L^2(\Omega)}\|u_0 - L_h u_0\|_{L^2(\Omega)}.$$

Then Lemma 2.2.4 and Lemma 2.4.1 leads to

$$|IV_4| \le Ch^2\|u_0\|_{H^2(\Omega)}\left(\sum_{n=1}^{M} k\|\overline{u}_I^n - w_h^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}. \tag{2.4.15}$$

Again it is easy to verify from Lemma 2.2.1 and Lemma 2.2.3 that

$$\sum_{n=1}^{M} k\|\overline{u}_I^n - \overline{P_k}^n\|_{H^m(\Omega)}^2 \le Ch^{4-2m}\|u\|_{L^2(0,T,X)}^2, \quad m = 0, 1. \tag{2.4.16}$$

Applying (2.4.16) for IV_1, we get

$$\begin{aligned} |IV_1| &\le C\left(\sum_{n=1}^{M} k\|\overline{u}_I^n - \overline{P_k}^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{M} k\|\overline{u}_I^n - w_h^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\ &\le Ch^2\|u\|_{L^2(0,T,X)}\left(\sum_{n=1}^{M} k\|\overline{u}_I^n - w_h^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.4.17}$$

Similarly, for IV_2, use of (2.4.16) and Lemma 2.4.1 leads to

$$\begin{aligned} |IV_2| &\le C\left(\sum_{n=1}^{M} k\|\Delta_k z_h^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\ &\quad\times\left[\left(\sum_{n=1}^{M} k\|\overline{P_k}^n - \overline{u}^n\|_{L^2(\Omega)}^2\right) + \left(\sum_{n=1}^{M} k\|\overline{u}^n - u^n\|_{L^2(\Omega)}^2\right)\right]^{\frac{1}{2}} \\ &\le C(k+h^2)\left(\|u\|_{L^2(0,T,X)}^2 + \|u_t\|_{L^2(0,T,L^2(\Omega))}^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{M} k\|\Delta_k z_h^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\ &\le C(k+h^2)\left(\|u\|_{L^2(0,T,X)}^2 + \|u_t\|_{L^2(0,T,L^2(\Omega))}^2\right)^{\frac{1}{2}} \\ &\quad\times\left(\sum_{n=1}^{M} k\|\overline{u}_I^n - w_h^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.4.18}$$

Finally, for the term IV_3, we use (2.4.12) to have

$$|IV_3| \leq Ch^2 \sum_{n=1}^{M} k\|\Delta_k u^n\|_{H^1(\Omega)}\|z_h^{n-1}\|_{H^1(\Omega)}. \qquad (2.4.19)$$

Again, it is easy to see that

$$\sum_{n=1}^{M} k\|\Delta_k u^n\|_{H^1(\Omega)}^2 \leq C\left(\|u_t\|_{L^2(0,T;H^2(\Omega_1))}^2 + \|u_t\|_{L^2(0,T;H^2(\Omega_2))}^2\right).$$

Then apply Lemma 2.4.2 and estimate (2.4.19) to have

$$\begin{aligned} |IV_3| &\leq Ch^2\left(\|u_t\|_{L^2(0,T;H^2(\Omega_1))}^2 + \|u_t\|_{L^2(0,T;H^2(\Omega_2))}^2\right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{n=1}^{M} k\|\overline{u}_I^n - w_h^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}. \end{aligned} \qquad (2.4.20)$$

By a simple calculation it follows that

$$\begin{aligned} \|u - w_h\|_{L^2(0,T;L^2(\Omega))} &\leq \|u - \overline{u}^n\|_{L^2(0,T;L^2(\Omega))} + \|\overline{u}^n - \overline{u}_I^n\|_{L^2(0,T;L^2(\Omega))} \\ &\quad + \left(\sum_{n=1}^{M} k\|\overline{u}_I^n - w_h^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\ &\leq Ck\|u_t\|_{L^2(0,T;L^2(\Omega))} + Ch^2\|u\|_{L^2(0,T;X)} \\ &\quad + \left(\sum_{n=1}^{M} k\|\overline{u}_I^n - w_h^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}. \end{aligned} \qquad (2.4.21)$$

Then, estimates (2.4.14)-(2.4.15) and (2.4.17)-(2.4.21) yields the following convergence result

Theorem 2.4.1 *Let u and w_h be the solutions of the problem (2.1.1)-(2.1.3) and (2.4.2), respectively. Then, for $f \in H^1(0,T;H^1(\Omega))$, $g = 0$ and $u_0 \in H_0^1(\Omega) \cap H^3(\Omega)$, the following $L^2(L^2)$ norm estimate holds*

$$\|u - w_h\|_{L^2(0,T;L^2(\Omega))} \leq C(k + h^2)\tilde{B}(f, u_0, u, u_t)$$

where $\tilde{B}(f, u_0, u, u_t)$ is a function of f, u_0, u, u_t. □

Chapter 3

Finite Element Method with Quadrature for Parabolic Interface Problems: $L^\infty(L^2)$ and $L^\infty(H^1)$ Error Estimates

The purpose of this chapter is to establish some new a priori pointwise-in-time error estimates in finite element method with quadrature for parabolic interface problems. Due to low global regularity of the solutions, the error analysis of the standard finite element methods for parabolic problems is difficult to adopt for parabolic interface problems. In this work, we fill a theoretical gap between standard error analysis technique of finite element method for non interface problems and parabolic interface problems. Optimal $L^\infty(H^1)$ and $L^\infty(L^2)$ norms error estimates have been derived for the semidiscrete case under practical regularity assumptions of the true solution for fitted finite element method with straight interface triangles. Further, the fully discrete backward Euler scheme is also considered and optimal $L^\infty(L^2)$ norm error estimate is established. The interface is assumed to be smooth for our purpose.

3.1 Introduction

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ be an open domain with C^2 smooth boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega \backslash \Omega_1$ be an another open domain contained in Ω with boundary $\Gamma \cup \partial\Omega$ (see. Figure 1.1). In $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, we consider the following parabolic interface problem

$$u_t - \nabla \cdot (\beta(x)\nabla u) = f(x,t) \quad \text{in } \Omega \times (0,T] \tag{3.1.1}$$

with initial and boundary conditions

$$u(x,0) = u_0 \quad \text{in } \Omega; \quad u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T] \tag{3.1.2}$$

and jump conditions on the interface

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = g(x,t) \quad \text{along } \Gamma \times (0,T], \tag{3.1.3}$$

where the symbol $[v]$ is a jump of a quantity v across the interface Γ and $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega_1$. The coefficient function β is positive and piecewise constant, i. e.

$$\beta(x) = \beta_i \quad \text{for } x \in \Omega_i, \quad i = 1,2.$$

Here, $f = f(x,t)$ and $g = g(x,t)$ are real valued functions defined in $\Omega \times (0,T]$ and $\Gamma \times (0,T]$, respectively. Throughout this chapter, we assume $u_0 \in H_0^1(\Omega) \cap H^3(\Omega)$.

Although a good number of articles is devoted to the convergence of finite element solution of parabolic interface problems in $L^2(L^2)$ and $L^2(H^1)$ norms, but pointwise-in-time error analysis is mostly missing. More recently, Deka and Sinha ([27]) have studied the pointwise-in-time convergence in finite element method for parabolic interface problems. They have shown optimal error estimates in $L^\infty(H^1)$ and $L^\infty(L^2)$ norms under the assumption that grid line exactly follow the actual interface. This may causes some technical difficulties in practice for the evaluation of the integrals over those curved elements near the interface. Further, it may be computationally inconvenient to fit the mesh to an arbitrary interface exactly. a finite element discretization based on previous chapter is considered. In this work, we are able to show that the standard error analysis technique of finite element method can be extended to parabolic interface problems. Optimal order pointwise-in-time error estimates in the L^2 and H^1 norms are established for the semidiscrete scheme. In addition, a fully discrete method based on backward Euler time-stepping scheme is analyzed and related optimal pointwise-in-time error bounds are derived. To the best of our knowledge, optimal pointwise-in-time error estimates for a finite element discretization based on [15] have not been established earlier for the parabolic interface problem. The achieved estimates are analogous to the case with a regular solution, however, due to low regularity, the proof requires a careful technical work coupled with a approximation result for the linear interpolant. Other technical tools used in this work are Sobolev embedding inequality, approximation properties for elliptic projection, duality arguments and some known results on elliptic interface problems.

A brief outline of this chapter chapter is as follows. In Section 3.2, we introduce some notation, recall some basic results from the literature and prove some approximation properties related to the auxiliary projection used in our analysis. While Section 3.3 is devoted to the error analysis for the semidiscrete finite element approximation, error estimates for the fully discrete backward Euler time stepping scheme are derived in Section 3.4.

3.2 Notations and Preliminaries

In order to introduce the weak formulation of the problem, we now recall the local bilinear form $A^l(.,.) : H^1(\Omega_l) \times H^1(\Omega_l) \to \mathbb{R}$ by

$$A^l(w,v) = \int_{\Omega_l} \beta_l \nabla w \cdot \nabla v dx. \quad l = 1,2.$$

Then the global bilinear map $A(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is defined by

$$\begin{aligned} A(w,v) &= \int_{\Omega} \beta(x) \nabla w \cdot \nabla v dx \\ &= A^1(w,v) + A^2(w,v) \ \ \forall\, w,\ v \in H_0^1(\Omega). \end{aligned} \tag{3.2.1}$$

The weak form for the problem (3.1.1)-(3.1.3) is defined as follows: Find $u : (0,T] \to H_0^1(\Omega)$ such that

$$(u_t, v) + A(u,v) = (f,v) + \langle g, v\rangle_\Gamma \ \ \forall v \in H_0^1(\Omega),\ a.e.\ t \in (0,T] \tag{3.2.2}$$

with $u(x,0) = u_0(x)$. Here, $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_\Gamma$ are used to denote the inner products of $L^2(\Omega)$ and $L^2(\Gamma)$ spaces, respectively.

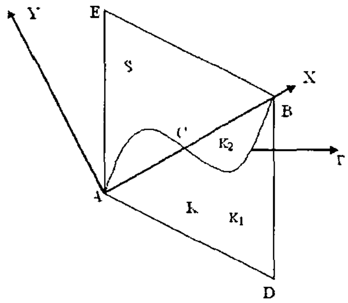

Figure 3.1: Interface triangles K and S along with interface Γ

Regarding the regularity for the solution of the interface problem (3.1.1)-(3.1.3), we have borrowed the following result from previous chapter.

Theorem 3.2.1 *Let $f \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, $g \in H^2(0,T;H^2(\Gamma))$ and $u_0 \in H^3(\Omega) \cap H^1_0(\Omega)$. Then solution $u \in L^2(0,T;X \cap H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2))$.* $\Box$

In this chapter, the convergence analysis has been carried out for $g(x,t) \neq 0$ on $\Gamma \times [0,T)$ and accordingly we need some relevant notations. Further, notations $A_h(\cdot,\cdot)$, $(\cdot,\cdot)_h$ and finite dimensional space V_h are with same meaning as in previous chapter.

Let $X^\star$ be the collection of all $v \in L^2(\Omega)$ with the property that $v \in H^2(\Omega_1) \cap H^2(\Omega_2) \cap \{\psi : \psi = 0 \text{ on } \partial\Omega\}$ and $[v] = 0$ along Γ. Since Γ is of class C^2, thus $v_i = v|_{\Omega_i}$, $i = 1, 2$ can be extended to $\tilde{v}_i \in H^2(\Omega)$ such that

$$\|\tilde{v}_i\|_{H^2(\Omega)} \leq C\|v_i\|_{H^2(\Omega_i)}.$$

For the existence of such extensions, we refer to Stein [62]. Further, we have a C^2 function ϕ in $[C, B]$ (see, Figure 3.1) such that (c.f. [29])

$$|\phi(x)| \leq Ch^2$$

and hence

$$\text{meas}(K_2) = \int_C^B |\phi(x)|dx \leq Ch^2 \int_C^B dx \leq Ch^3.$$

Let $\Pi_h : C(\overline{\Omega}) \to V_h$ be the Lagrange interpolation operator corresponding to the space V_h. Then, for $K \in \mathcal{T}_h$ and $v \in X^\star$, we now define

$$v_I = \begin{cases} \Pi_h \tilde{v}_1 & \text{if } K \subseteq \Omega_1^h \\ \Pi_h \tilde{v}_2 & \text{if } K \subseteq \Omega_2^h. \end{cases} \tag{3.2.3}$$

Following the lines of proof for Lemma 2.2.3 in [2], it is possible to obtain the following optimal error bounds for linear interpolant v_I in $X^\star$. We include the proof for the completeness of this work.

Lemma 3.2.1 *For any $v \in X^\star$, we have*

$$\|v - v_I\|_{H^1(\Omega_1)} + \|v - v_I\|_{H^1(\Omega_2)} \leq Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

Proof. For H^1 norm estimate, we have

$$\begin{aligned} &\|v - v_I\|_{H^1(\Omega_1)} + \|v - v_I\|_{H^1(\Omega_2)} \\ &\leq \sum_{K \in \mathcal{T}_h \setminus \mathcal{T}_\Gamma^*} \|v - v_I\|_{H^1(K)} + \sum_{K \in \mathcal{T}_\Gamma^*} \{\|v - v_I\|_{H^1(K_1)} + \|v - v_I\|_{H^1(K_2)}\} \\ &\leq Ch\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\} \\ &\quad + \sum_{K \in \mathcal{T}_\Gamma^*} \{\|v - v_I\|_{H^1(K_1)} + \|v - v_I\|_{H^1(K_2)}\}. \end{aligned} \tag{3.2.4}$$

Here, $K_1 = K \cap \Omega_1$ and $K_2 = K \cap \Omega_2$. Again, for any $K \in \mathcal{T}_h$ either $K \subseteq \Omega_1^h$ or $K \subseteq \Omega_2^h$. Let $K \subseteq \Omega_1^h$, then $v_I = \Pi_h \tilde{v}_1$ and hence, we have

$$\begin{aligned} \|v - v_I\|_{H^1(K_1)} &= \|\tilde{v}_1 - \Pi_h \tilde{v}_1\|_{H^1(K_1)} \leq \|\tilde{v}_1 - \Pi_h \tilde{v}_1\|_{H^1(K)} \\ &\leq Ch\|\tilde{v}_1\|_{H^2(K)} \leq Ch\|v_1\|_{H^2(\Omega_1)}. \end{aligned} \tag{3.2.5}$$

Again, since $v \in H^2(\Omega_2)$ and $K_2 \subseteq \Omega_2$ with $\text{meas}(K_2) \leq Ch^3$, we have

$$\begin{aligned} \|v - v_I\|_{H^1(K_2)} &\leq Ch^{\frac{3(p-2)}{2p}} \|v - v_I\|_{W^{1,p}(K_2)} \;\; \forall p > 2 \\ &= Ch\|v - v_I\|_{W^{1,6}(K_2)} = Ch\|v_2 - \Pi_h \tilde{v}_1\|_{W^{1,6}(K_2)} \\ &\leq Ch\|\tilde{v}_2 - \tilde{v}_1\|_{W^{1,6}(K_2)} + Ch\|\tilde{v}_1 - \Pi_h \tilde{v}_1\|_{W^{1,6}(K_2)} \\ &\leq Ch\|\tilde{v}_2 - \tilde{v}_1\|_{W^{1,6}(K)} + Ch\|\tilde{v}_1 - \Pi_h \tilde{v}_1\|_{W^{1,6}(K)} \\ &\leq Ch\|\tilde{v}_2 - \tilde{v}_1\|_{H^2(\Omega)} + Ch\|\tilde{v}_1\|_{H^2(K)} \\ &\leq Ch\|\tilde{v}_1\|_{H^2(\Omega)} + Ch\|\tilde{v}_2\|_{H^2(\Omega)} \\ &\leq Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}). \end{aligned} \tag{3.2.6}$$

Then Lemma 3.2.1 follows immediately from the estimates (3.2.4)-(3.2.6). $\square$

Let $Y^\star$ be the collection of all $v \in L^2(\Omega)$ such that $v \in H^1(\Omega_1) \cap H^1(\Omega_2) \cap \{\psi : \psi = 0 \text{ on } \partial\Omega\}$ with $[v] = 0$ along Γ. For any $v \in Y^\star$, we define

$$A_h(R_h v, v_h) = A^1(v, v_h) + A^2(v, v_h) \ \ \forall v_h \in V_h. \tag{3.2.7}$$

Remark 3.2.1 *Elliptic projection R_h defined by (3.2.7) is analogous to the projection P_h defined by (2.2.15) in Chapter 2. Only difference is the domain of definition. While P_h is defined on $\mathcal{X} = \{\psi \in L^2(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2) : \ \psi = 0$ on $\partial\Omega$ & $[\psi] = 0 = [\beta\partial v/\partial\mathbf{n}] = 0$ on $\Gamma\}$, operator R_h is defined on a more general space $Y^\star$. Further, existence of operator R_h can be verified by Lax-Milgram lemma.* $\square$

The following lemma shows that optimal approximation of R_h can be derived for $v \in X^\star$.

Lemma 3.2.2 *Let R_h be defined by (3.2.7), then for any $v \in X^\star$ there is a positive constant C independent of the mesh parameter h such that*

(a) $\|R_h v - v\|_{H^1(\Omega_1)} + \|R_h v - v\|_{H^1(\Omega_2)} \le Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)})$,

(b) $\|R_h v - v\|_{L^2(\Omega)} \le Ch^2(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)})$.

Proof. Coercivity of each local bilinear map and the definition of R_h projection leads to

$$\begin{aligned}
&\|v - R_h v\|^2_{H^1(\Omega_1)} + \|v - R_h v\|^2_{H^1(\Omega_2)} \\
&\le C\{A^1(v - R_h v, v - v_h) + A^2(v - R_h v, v - v_h)\} \\
&\quad + CA^1(v, v_h - R_h v) - CA^1(R_h v, v_h - R_h v) \\
&\quad + CA^2(v, v_h - R_h v) - CA^2(R_h v, v_h - R_h v) \\
&= C\{A^1(v - R_h v, v - v_h) + A^2(v - R_h v, v - v_h)\} \\
&\quad + C\{A^1_h(R_h v, v_h - R_h v) - A^1(R_h v, v_h - R_h v)\} \\
&\quad + C\{A^2_h(R_h v, v_h - R_h v) - A^2(R_h v, v_h - R_h v)\} \\
&= C\{A^1(v - R_h v, v - v_h) + A^2(v - R_h v, v - v_h)\} \\
&\quad + C\{A_h(R_h v, v_h - R_h v) - A(R_h v, v_h - R_h v)\}.
\end{aligned}$$

Then it follows from Lemma 2.2.2 and Young's inequality that

$$\begin{aligned}
&\|v - R_h v\|^2_{H^1(\Omega_1)} + \|v - R_h v\|^2_{H^1(\Omega_2)} \\
&\leq C\|v - R_h v\|_{H^1(\Omega_1)}\|v - v_h\|_{H^1(\Omega_1)} + C\|v - R_h v\|_{H^1(\Omega_2)}\|v - v_h\|_{H^1(\Omega_2)} \\
&\quad + Ch\|R_h v\|_{H^1(\Omega)}\|v_h - R_h v\|_{H^1(\Omega)} \\
&\leq \epsilon\|v - R_h v\|^2_{H^1(\Omega_1)} + \frac{C}{\epsilon}\|v - v_h\|^2_{H^1(\Omega_1)} + \epsilon\|v - R_h v\|^2_{H^1(\Omega_2)} \\
&\quad + \frac{C}{\epsilon}\|v - v_h\|^2_{H^1(\Omega_2)} + \frac{Ch^2}{\epsilon}\|R_h v\|^2_{H^1(\Omega)} + \epsilon\|v_h - R_h v\|^2_{H^1(\Omega)}.
\end{aligned}$$

Again applying the fact $\|R_h v\|_{H^1(\Omega)} \leq C(\|v\|_{H^1(\Omega_1)} + \|v\|_{H^1(\Omega_2)})$ and for suitable $\epsilon > 0$, we have

$$\begin{aligned}
\|v - R_h v\|^2_{H^1(\Omega_1)} + \|v - R_h v\|^2_{H^1(\Omega_2)} \;\leq\; & C\|v - v_h\|^2_{H^1(\Omega_1)} + C\|v - v_h\|^2_{H^1(\Omega_2)} \\
& + Ch^2\{\|v\|^2_{H^1(\Omega_1)} + \|v\|^2_{H^1(\Omega_2)}\}.
\end{aligned}$$

Now, setting $v_h = v_I$ and then using Lemma 3.2.1, we have

$$\|v - R_h v\|_{H^1(\Omega_1)} + \|v - R_h v\|_{H^1(\Omega_2)} \leq Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

This completes the proof of part (a) of Lemma 3.2.2.

For L^2 norm error estimate, we will use the duality argument. For this purpose, we consider the following interface problem

$$-\nabla \cdot (\beta \nabla \phi) = v - R_h v$$

with the boundary condition $\phi = 0$ on $\partial\Omega$ and interface conditions $[\phi] = 0$, $[\beta \frac{\partial \phi}{\partial \mathbf{n}}] = 0$ along Γ.

Now, multiply the above equation by $w \in Y^\star$ and then integrate over Ω to have

$$A^1(\phi, w) + A^2(\phi, w) = (v - R_h v, w). \tag{3.2.8}$$

Let $\phi_h \in V_h$ be the finite element approximation to ϕ defined as: Find $\phi_h \in V_h$ such that

$$A_h(\phi_h, w_h) = (v - R_h v, w_h) \;\; \forall w_h \in V_h. \tag{3.2.9}$$

Arguing as in part (a), it can be concluded that

$$\begin{aligned}
\|\phi - \phi_h\|_{H^1(\Omega_1)} \;+\; & \|\phi - \phi_h\|_{H^1(\Omega_2)} \\
& \leq C(\|\phi - w_h\|_{H^1(\Omega_1)} + \|\phi - w_h\|_{H^1(\Omega_2)}) \\
& \quad + Ch(\|\phi\|_{H^2(\Omega_1)} + \|\phi\|_{H^2(\Omega_2)}) \;\; \forall w_h \in V_h.
\end{aligned}$$

Let ϕ_I be defined as in (3.2.3) and then set $w_h = \phi_I$ to have

$$\begin{aligned}\|\phi - \phi_h\|_{H^1(\Omega_1)} + \|\phi - \phi_h\|_{H^1(\Omega_2)} &\le Ch(\|\phi\|_{H^2(\Omega_1)} + \|\phi\|_{H^2(\Omega_2)}) \\ &\le Ch\|v - R_h v\|_{L^2(\Omega)}.\end{aligned}$$

In the last inequality, we used the elliptic regularity estimate $\|\phi\|_X \le C\|v - R_h v\|_{L^2(\Omega)}$ (cf. [15]). Thus, we have

$$\|\phi - \phi_h\|_{H^1(\Omega)} \le Ch\|v - R_h v\|_{L^2(\Omega)}. \tag{3.2.10}$$

Since $[v - R_h v] = 0$ along Γ and $v - R_h v \in L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2) \cap \{\psi : \psi = 0 \text{ on } \partial\Omega\}$, therefore, set $w = v - R_h v$ in (3.2.8) to have

$$\begin{aligned}\|v - R_h v\|^2_{L^2(\Omega)} &= A^1(\phi, v - R_h v) + A^2(\phi, v - R_h v) \\ &= A^1(\phi - \phi_h, v - R_h v) + A^2(\phi - \phi_h, v - R_h v) \\ &\quad + \{A^1(\phi_h, v - R_h v) + A^2(\phi_h, v - R_h v)\} \\ &\le C\|\phi - \phi_h\|_{H^1(\Omega_1)}\|v - R_h v\|_{H^1(\Omega_1)} \\ &\quad + C\|\phi - \phi_h\|_{H^1(\Omega_2)}\|v - R_h v\|_{H^1(\Omega_2)} \\ &\quad + \{A^1(\phi_h, v) + A^2(\phi_h, v)\} - \{A^1(\phi_h, R_h v) + A^2(\phi_h, R_h v)\} \\ &\le Ch\|v - R_h v\|_{L^2(\Omega)} \cdot Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}) \\ &\quad + A_h(R_h v, \phi_h) - A(R_h v, \phi_h) \\ &= Ch^2\|v - R_h v\|_{L^2(\Omega)}(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}) \\ &\quad + \{A_h(R_h v, \phi_h) - A(R_h v, \phi_h)\} \\ &=: Ch^2\|v - R_h v\|_{L^2(\Omega)}(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}) + (J).\end{aligned} \tag{3.2.11}$$

Now, we apply Lemma 2.2.2 to have

$$\begin{aligned}|(J)| &\le Ch \sum_{K \in \mathcal{T}^*_\Gamma} \|R_h v\|_{H^1(K)}\|\phi_h\|_{H^1(K)} \\ &\le Ch \sum_{K_1} \|R_h v\|_{H^1(K_1)}\|\phi_h\|_{H^1(K_1)} \\ &\quad + Ch \sum_{K_2} \|R_h v\|_{H^1(K_2)}\|\phi_h\|_{H^1(K_2)} \\ &= (J)_1 + (J)_2.\end{aligned} \tag{3.2.12}$$

Again, using part (a) and estimate (3.2.10), we have

$$\begin{aligned}
&\|R_h v\|_{H^1(K_2)}\|\phi_h\|_{H^1(K_2)} \\
&\le \{\|R_h v - v\|_{H^1(K_2)} + \|v\|_{H^1(K_2)}\}\{\|\phi_h - \phi\|_{H^1(K_2)} + \|\phi\|_{H^1(K_2)}\} \\
&\le \{\|R_h v - v\|_{H^1(\Omega_2)} + \|\tilde{v}_2\|_{H^1(K_2)}\}\{\|\phi_h - \phi\|_{H^1(\Omega_2)} + \|\phi\|_{H^1(K_2)}\} \\
&\le C\{h\|v\|_{H^2(\Omega_1)} + h\|v\|_{H^2(\Omega_2)} + \|\tilde{v}_2\|_{H^1(K)}\} \\
&\quad \times\{h\|v - R_h v\|_{L^2(\Omega)} + \|\phi\|_{H^1(K)}\}.
\end{aligned} \tag{3.2.13}$$

Setting $p = 4$ in the Sobolev embedding inequality (cf. [62, 63])

$$\|v\|_{L^p(K_2)} \le Cp^{\frac{1}{2}}\|v\|_{H^1(K_2)} \;\; \forall v \in H^1(K_2), \;\; p > 2 \tag{3.2.14}$$

and further, using Hölder's inequality, we obtain

$$\begin{aligned}
\|\tilde{v}_2\|_{H^1(K)} &= \|\tilde{v}_2\|_{L^2(K)} + \|\nabla\tilde{v}_2\|_{L^2(K)} \\
&\le Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{L^4(K)} + Ch^{\frac{1}{2}}\|\nabla\tilde{v}_2\|_{L^4(K)} \\
&\le Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{H^1(K)} + Ch^{\frac{1}{2}}\|\nabla\tilde{v}_2\|_{H^1(K)} \\
&\le Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{H^2(K)} \le Ch^{\frac{1}{2}}\|v_2\|_{H^2(\Omega_2)},
\end{aligned} \tag{3.2.15}$$

where we have used the fact that $\text{meas}(K) \le Ch^2$. Similarly, for $\|\phi\|_{H^1(K)}$, we have

$$\|\phi\|_{H^1(K)} \le Ch^{\frac{1}{2}}\|\phi\|_X \le Ch^{\frac{1}{2}}\|v - R_h v\|_{L^2(\Omega)}. \tag{3.2.16}$$

Combining (3.2.13)-(3.2.16), we have

$$\begin{aligned}
&\|R_h v\|_{H^1(K_2)}\|\phi_h\|_{H^1(K_2)} \\
&\le Ch\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\}\|v - R_h v\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore, for $(J)_2$, we have

$$(J)_2 \le Ch^2\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\}\|v - R_h v\|_{L^2(\Omega)}. \tag{3.2.17}$$

Similarly, for $(J)_1$, we have

$$(J)_1 \le Ch^2\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\}\|v - R_h v\|_{L^2(\Omega)}. \tag{3.2.18}$$

Then, using the estimates (3.2.17) and (3.2.18) in (3.2.12), we have

$$|(J)| \le Ch^2\|v - R_h v\|_{L^2(\Omega)}(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}). \tag{3.2.19}$$

Finally, (3.2.11) and (3.2.19) leads to the following optimal L^2 norm estimate

$$\|v - R_h v\|_{L^2(\Omega)} \leq Ch^2(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

This completes the rest of the proof. □

Let $g_h \in V_h$ be the linear interpolant of g given by

$$g_h = \sum_{j=1}^{m_h} g(P_j)\Phi_j^h,$$

where $\{\Phi_j^h\}_{j=1}^{m_h}$ is the set of standard nodal basis functions corresponding to the nodes $\{P_j\}_{j=1}^{m_h}$ on the interface Γ. Following the argument of [15] it is possible to obtain the following approximation property of g_h to the interface function g.

Lemma 3.2.3 *Let $g \in H^2(\Gamma)$. If Ω_Γ^* is the union of all interface triangles then we have*

$$\left| \int_\Gamma g v_h ds - \int_{\Gamma_h} g_h v_h ds \right| \leq Ch^2\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega_\Gamma^*)} \quad \forall v_h \in V_h.$$

Proof. It follows from [15] (see, page 186) that

$$\left| \int_\Gamma g v_h ds - \int_{\Gamma_h} g_h v_h ds \right|$$
$$\leq Ch^2\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega_\Gamma^*)} + Ch^{3/2}\|g\|_{H^2(\Gamma)}\|v_h\|_{L^2(\Omega_\Gamma^*)} \quad \forall v_h \in V_h.$$

Arguing as in (3.2.15), we obtain

$$\begin{aligned} \|v_h\|_{L^2(\Omega_\Gamma^*)} &= \sum_{K \in \mathcal{T}_\Gamma^*} \|v_h\|_{L^2(K)} \\ &\leq Ch^{1/2} \sum_{K \in \mathcal{T}_\Gamma^*} \|v_h\|_{L^4(K)} \leq Ch^{1/2}\|v_h\|_{H^1(\Omega_\Gamma^*)}. \end{aligned}$$

The desire result follows immediately from the above two estimates. □

3.3 Error Analysis for the Semidiscrete Scheme

In this section, we discuss the semidiscrete finite element method for the problem (3.1.1)-(3.1.3) and derive optimal error estimates in L^2 and H^1 norms.

The continuous-time Galerkin finite element approximation to (3.2.2) is stated as follows: Find $u_h : [0,T] \to V_h$ such that $u_h(0) = R_h u_0$ and

$$(u_{ht}, v_h)_h + A_h(u_h, v_h) = (f, v_h)_h + \langle g_h, v_h \rangle_{\Gamma_h} \quad \forall v_h \in V_h, \quad t \in (0,T]. \tag{3.3.1}$$

Write the error $e(t) = u - u_h = u - R_h u + R_h u - u_h = \rho + \theta$, with $\rho = u - R_h u$ and $\theta = R_h u - u_h$. Again, using (3.2.7) for $v = u \in X^\star$ and further differentiating with respect to t, we have

$$A_h((R_h u)_t, v_h) = A^1(u_t, v_h) + A^2(u_t, v_h).$$

Also,

$$A_h(R_h u_t, v_h) = A^1(u_t, v_h) + A^2(u_t, v_h).$$

From the above two equations, we have

$$A_h((R_h u)_t - R_h u_t, v_h) = 0 \;\; \forall v_h \in V_h.$$

Setting $v_h = (R_h u)_t - R_h u_t$ in the above equation, we obtain $(R_h u)_t = R_h u_t$.

Now, by the definition R_h operator, (3.2.2) and (3.3.1), we obtain

$$\begin{aligned}
(\theta_t, v_h)_h + A_h(\theta, v_h) &= ((R_h u)_t - u_{ht}, v_h)_h + A_h(R_h u - u_h, v_h) \\
&= (R_h u_t, v_h)_h + A_h(R_h u, v_h) - (u_{ht}, v_h)_h - A_h(u_h. v_h) \\
&= (R_h u_t, v_h)_h + A(u, v_h) - (f, v_h)_h - \langle g_h, v_h \rangle_{\Gamma_h} \\
&= \{(R_h u_t, v_h)_h - (R_h u_t, v_h)\} + \{(f, v_h) - (f, v_h)_h\} \\
&\quad + \{\langle g, v_h \rangle_\Gamma - \langle g_h, v_h \rangle_{\Gamma_h}\} + (-\rho_t, v_h).
\end{aligned}$$

For $v_h = \theta$, we have

$$\begin{aligned}
(\theta_t, \theta)_h + C\|\theta\|^2_{H^1(\Omega)} &\le Ch^2 \|R_h u_t\|_{H^1(\Omega)} \|\theta\|_{H^1(\Omega)} + Ch^2 \|f\|_{H^2(\Omega)} \|\theta\|_{H^1(\Omega)} \\
&\quad + Ch^2 \|g\|_{H^2(\Gamma)} \|\theta\|_{H^1(\Omega^\star_\Gamma)} + C\|\rho_t\|_{L^2(\Omega)} \|\theta\|_{L^2(\Omega)} \\
&\le C_\epsilon \bigg(\|\rho_t\|^2_{L^2(\Omega)} + h^4 \{ \|R_h u_t\|^2_{H^1(\Omega)} + \|f\|^2_{H^2(\Omega)} \\
&\quad + \|g\|^2_{H^2(\Gamma)} \} \bigg) + C(\epsilon) \|\theta\|^2_{H^1(\Omega)}.
\end{aligned}$$

Here, we have used Lemma 2.2.2 and Lemma 3.2.3. Integrating the above equation form 0 to t and using Lemma 3.2.2, we obtain

$$\|\theta(t)\|^2_{L^2(\Omega)} \le Ch^4 \int_0^t \left(\sum_{i=1}^2 \|u_t\|^2_{H^2(\Omega_i)} + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)} \right) ds. \tag{3.3.2}$$

Now, combining Lemma 3.2.2 and (3.3.2), we have the following optimal pointwise-in-time L^2-norm error estimates.

Theorem 3.3.1 *Let u and u_h be the solutions of the problem (3.1.1)-(3.1.3) and (3.3.1), respectively. Assume that $u_h(0) = R_h u_0$. Then there exists a constant C independent of h such that*

$$\begin{aligned}\|e(t)\|_{L^2(\Omega)} &\leq Ch^2\Bigg[\|u\|_X + \Big(\int_0^t \Big\{\sum_{i=1}^{2}\|u_t\|^2_{H^2(\Omega_i)} \\ &\qquad +\|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\Big\}ds\Big)^{\frac{1}{2}}\Bigg]. \quad \square\end{aligned}$$

For H^1-norm estimate, we first use Lemma 3.2.2 to have

$$\sum_{i=1}^{2}\|\rho(t)\|_{H^1(\Omega_i)} \leq Ch\sum_{i=1}^{2}\|u\|_{H^2(\Omega_i)}. \tag{3.3.3}$$

Applying inverse estimate (2.2.12), we obtain

$$\begin{aligned}\|\theta(t)\|_{H^1(\Omega)} &\leq Ch^{-1}\|\theta(t)\|_{L^2(\Omega)} \\ &\leq Ch^{-1}h^2\left[\int_0^t\Big(\sum_{i=1}^{2}\|u_t\|^2_{H^2(\Omega_i)} + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\Big)ds\right]^{\frac{1}{2}} \\ &= Ch\left[\int_0^t\Big(\sum_{i=1}^{2}\|u_t\|^2_{H^2(\Omega_i)} + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\Big)ds\right]^{\frac{1}{2}}. \qquad (3.3.4)\end{aligned}$$

Combining (3.3.3) and (3.3.4), we have the following optimal pointwise-in-time H^1-norm error estimates.

Theorem 3.3.2 *Let u and u_h be the solutions of the problem (3.1.1)-(3.1.3) and (3.3.1), respectively. Assume that $u_h(0) = R_h u_0$. Then there exists a constant C independent of h such that*

$$\begin{aligned}\|e(t)\|_{H^1(\Omega)} &\leq Ch\Bigg[\|u\|_X + \Big(\int_0^t \{\sum_{i=1}^{2}\|u_t\|^2_{H^2(\Omega_i)} \\ &\qquad +\|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\}ds\Big)^{\frac{1}{2}}\Bigg]. \quad \square\end{aligned}$$

3.4 Error Analysis for the Fully Discrete Scheme

A fully discrete scheme based on backward Euler method is proposed and analyzed in this section. Optimal L^2 norm error estimate is obtained for fully discrete scheme.

We first partition the interval [0. T] into M equally spaced subintervals by the following points

$$0 = t_0 < t_1 < \ldots < t_M = T$$

with $t_n = nk,\ k = \frac{T}{M}$, be the time step. Let $I_n = (t_{n-1}, t_n]$ be the n-th subinterval. Now we introduce the backward difference quotient

$$\Delta_k \phi^n = \frac{\phi^n - \phi^{n-1}}{k},$$

for a given sequence $\{\phi^n\}_{n=0}^{M} \subset L^2(\Omega)$. For $\phi(t) \in V_h$, we denote ϕ^n be the value of ϕ at $t = t_n$.

The fully discrete finite element approximation to the problem (3.2.2) is defined as follows: For $n = 1, \ldots, M$, find $U^n \in V_h$ such that

$$(\Delta_k U^n, v_h)_h + A_h(U^n, v_h) = (f^n, v_h) + \langle g_h^n, v_h\rangle_{\Gamma_h} \ \ \forall v_h \in V_h \tag{3.4.1}$$

with $U^0 = R_h u_0$. For each $n = 1, \ldots, M$. the existence of a unique solution to (3.4.1) can be found in [15]. We then define the fully discrete solution to be a piecewise constant function $U_h(x, t)$ in time and is given by

$$U_h(x,t) = U^n(x) \ \ \forall t \in I_n, \ \ 1 \le n \le M.$$

We now prove the main result of this section in the following theorem.

Theorem 3.4.1 *Let u and U be the solutions of the problem (3.1.1)-(3.1.3) and (3.4.1), respectively. Assume that $U^0 = R_h u_0$. Then there exists a constant C independent of h and k such that*

$$\begin{aligned}&\|U(t_n) - u(t_n)\|_{L^2(\Omega)}\\ &\le C(h^2 + k)\Big\{\|u^0\|_{H^2(\Omega)} + |||g^n||| + \|u_{tt}\|_{L^2(0,T;L^2(\Omega))} + \sum_{i=1}^{2}\|u_t\|_{L^2(0,T;H^2(\Omega_i))}\Big\}.\end{aligned}$$

Proof. We write the error $U^n - u^n$ at time t_n as

$$U^n - u^n = (U^n - R_h u^n) + (R_h u^n - u^n) \equiv: \theta^n + \rho^n$$

where $\theta^n = U^n - R_h u^n$ and $\rho^n = R_h u^n - u^n$.

For θ^n, we have the following error equation

$$\begin{aligned}
&(\Delta_k \theta^n, v_h)_h + A_h(\theta^n, v_h) \\
&= (-\Delta_k R_h u^n + \Delta_k U^n, v_h)_h + A_h(-R_h u^n + U^n, v_h) \\
&= (\Delta_k U^n, v_h)_h + A_h(U^n, v_h) - (\Delta_k R_h u^n, v_h)_h - A_h(R_h u^n, v_h) \\
&= (f^n, v_h) + \langle g_h^n, v_h\rangle_{\Gamma_h} - (\Delta_k R_h u^n, v_h)_h - A(u^n, v_h) \\
&= (f^n, v_h) + \langle g_h^n, v_h\rangle_{\Gamma_h} - (\Delta_k R_h u^n, v_h)_h \\
&\quad +(u_t^n, v_h) - (f^n, v_h) - \langle g^n, v_h\rangle_\Gamma \\
&\equiv: -(w^n, v_h) + \{(\Delta_k R_h u^n, v_h) - (\Delta_k R_h u^n, v_h)_h\} \\
&\quad +\{\langle g_h^n, v_h\rangle_{\Gamma_h} - \langle g^n, v_h\rangle_\Gamma\}, \qquad (3.4.2)
\end{aligned}$$

where $w^n = \Delta_k R_h u^n - u_t^n$. For simplicity of the exposition, we write $w^n = w_1^n + w_2^n$, where $w_1^n = R_h \Delta_k u^n - \Delta_k u^n$ and $w_2^n = \Delta_k u^n - u_t^n$.

Now, setting $v_h = \theta^n$ in (3.4.2), we have

$$\begin{aligned}
(\Delta_k \theta^n, \theta^n)_h + A_h(\theta^n, \theta^n) &= -(w^n, \theta^n) + \{(\Delta_k R_h u^n, \theta^n) - (\Delta_k R_h u^n, \theta^n)_h\} \\
&\quad +\{\langle g_h^n, \theta^n\rangle_{\Gamma_h} - \langle g^n, \theta^n\rangle_\Gamma\}. \qquad (3.4.3)
\end{aligned}$$

Since $A_h(\theta^n, \theta^n) \geq C\|\theta^n\|^2_{H^1(\Omega)}$, we have

$$\begin{aligned}
\|\theta^n\|_{L^2(\Omega)} &\leq k\|w^n\|_{L^2(\Omega)} + \|\theta^{n-1}\|_{L^2(\Omega)} + Ch^2 k^{\frac{1}{2}} \|R_h \Delta_k u^n\|_{H^1(\Omega)} \\
&\quad +Ch^2 k^{\frac{1}{2}} \|g^n\|_{H^2(\Gamma)} \\
&\leq \|\theta^0\|_{L^2(\Omega)} + k\sum_{j=1}^n \|w_1^j\|_{L^2(\Omega)} + k\sum_{j=1}^n \|w_2^j\|_{L^2(\Omega)} \\
&\quad +Ch^2 k^{\frac{1}{2}} \sum_{j=1}^n \|w_1^j\|_{H^1(\Omega)} + Ch^2 k^{\frac{1}{2}} \sum_{j=1}^n \|\Delta_k u^j\|_{H^1(\Omega)} \\
&\quad +Ch^2 k^{\frac{1}{2}} |||g^n|||, \qquad (3.4.4)
\end{aligned}$$

with $|||g^n||| = \max_{1\leq j\leq n} |||g^j|||_{H^2(\Gamma)}$.

In Ω_1, the term w_1^j can be expressed as

$$\begin{aligned} w_1^j &= R_h\Delta_k u_1^j - \Delta_k u_1^j = (R_h - I)(\Delta_k u_1^j) \\ &= (R_h - I)\frac{1}{k}\int_{t_{j-1}}^{t^j} u_{1,t}dt = \frac{1}{k}\int_{t_{j-1}}^{t^j} (R_h u_{1,t} - u_{1,t})dt, \end{aligned}$$

where u_i, $i = 1, 2$ is the restriction of u in Ω_i and $u_{i,t} = \frac{\partial u_i}{\partial t}$.

An application of Lemma 3.2.2 leads to

$$k\|w_1^j\|_{L^2(\Omega_1)} \le Ch^2\int_{t_{j-1}}^{t^j} \Big\{\sum_{i=1}^{2} \|u_t\|_{H^2(\Omega_i)}\Big\}dt.$$

Similarly, we obtain

$$k\|w_1^j\|_{L^2(\Omega_2)} \le Ch^2\int_{t_{j-1}}^{t^j} \Big\{\sum_{i=1}^{2} \|u_t\|_{H^2(\Omega_i)}\Big\}dt.$$

Using above two estimates, we have

$$k\sum_{j=1}^{n} \|w_1^j\|_{L^2(\Omega)} \le Ch^2\int_{0}^{t_n} \Big\{\sum_{i=1}^{2} \|u_t\|_{H^2(\Omega_i)}\Big\}dt. \tag{3.4.5}$$

Similarly, for the term w_2^n, we have

$$kw_2^j = u^j - u^{j-1} - ku_t^j = -\int_{t_{j-1}}^{t_j} (s - t_{j-1})u_{tt}ds$$

and hence

$$k\|w_2^j\|_{L^2(\Omega_i)} \le k\int_{t_{j-1}}^{t_j} \|u_{tt}\|_{L^2(\Omega_i)}ds.$$

Summing over j from $j = 1$ to $j = n$, we obtain

$$k\sum_{j=1}^{n} \|w_2^j\|_{L^2(\Omega)} \le Ck\int_{0}^{t_n} \Big\{\sum_{i=1}^{2} \|u_{tt}\|_{L^2(\Omega_i)}\Big\}dt. \tag{3.4.6}$$

Arguing as in (3.4.5), we obtain

$$k\sum_{j=1}^{n} \|w_1^j\|_{H^1(\Omega)} \le Ch\int_{0}^{t_n} \Big\{\sum_{i=1}^{2} \|u_t\|_{H^2(\Omega_i)}\Big\}dt. \tag{3.4.7}$$

Combining (3.4.4) - (3.4.7) and using the fact that

$$k\sum_{j=1}^{n} \|\Delta_k u^j\|_{H^1(\Omega)}^2 \le C\int_{0}^{t_n} \Big\{\sum_{i=1}^{2} \|u_t\|_{H^1(\Omega_i)}^2\Big\}dt,$$

we obtain

$$\begin{aligned}\|\theta^n\|_{L^2(\Omega)} &\le C(h^2+k)\\ &\times\left[\sum_{i=1}^{2}\left\{\|u_t\|_{L^2(0,T;H^2(\Omega_i))}+\|u_{tt}\|_{L^2(0,T;L^2(\Omega_i))}\right\}+|||g^n|||\right].\end{aligned} \tag{3.4.8}$$

An application of Lemma 3.2.2 for ρ^n yields

$$\|\rho^n\|_{L^2(\Omega)} \le Ch^2\sum_{i=1}^{2}\|u^n\|_{H^2(\Omega_i)}.$$

Again, it is easy to verify that

$$\|u^n\|_{H^2(\Omega_i)} \le \|u^0\|_{H^2(\Omega_i)} + \int_0^{t_n}\|u_t\|_{H^2(\Omega_i)}dt.$$

Thus, we have

$$\|\rho^n\|_{L^2(\Omega)} \le Ch^2\left\{\|u^0\|_{H^2(\Omega)}+\sum_{i=1}^{2}\|u_t\|_{L^2(0,T;H^2(\Omega_i))}\right\}. \tag{3.4.9}$$

Combining (3.4.8) and (3.4.9) the desired estimate is easily obtained. This completes the proof. □

Chapter 4

FEM for Parabolic Integro-Differential Equations with Interfaces: $L^2(L^2)$ and $L^2(H^1)$ Error Estimates

In this chapter, convergence of finite element method for a class of parabolic integro-differential equations with discontinuous coefficients are analyzed. Optimal $L^2(L^2)$ and $L^2(H^1)$ norms are shown to hold when the finite element space consists of piecewise linear functions on a mesh that do not require to fit exactly to the interface. Both continuous time and discrete time Galerkin methods are discussed for arbitrary shape but smooth interfaces.

4.1 Introduction

In this work, we consider the following parabolic integro-differential equation

$$u_t(x,t) - \nabla \cdot (\beta \nabla u(x,t)) = f(x,t) + \int_0^t B(t,s)u(s)ds \quad \text{in } \Omega \times (0,T] \tag{4.1.1}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \;\; \text{in } \Omega \;\&\;\; u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T] \tag{4.1.2}$$

where $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ is a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ is an open domain with C^2 smooth boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega \backslash \Omega_1$ (see, Figure 1.1). Coefficient $\beta(x)$ is positive and piecewise constant. We write

$$\beta(x) = \beta_i \quad \text{for} \quad x \in \Omega_i, \; i = 1, 2,$$

and $B(t, s)$ is a first order partial differential operator of the form

$$B(t,s)u(s) = \sum_{k=1}^{2} b_k(x;t,s)\frac{\partial u(x,s)}{\partial x_k} + u(x,s).$$

For compatibility of the problem (4.1.1)-(4.1.2), we assume that the solution $u(x,t)$ satisfies the following jump conditions on the interface Γ

$$[u] = 0, \quad \left[\beta(x)\frac{\partial u}{\partial \mathbf{n}}\right] = 0 \quad \text{along } \Gamma \times (0, T]. \tag{4.1.3}$$

The symbol $[v]$ is a jump of a quantity v across the interface Γ and $\mathbf{n}$ denotes the unit outward normal to the boundary $\partial\Omega_1$.

Coefficients of $B(t, s)$ satisfy the following assumption: there exists positive constant K_1 such that

$$|b_k(x;t,s)|, \; \left|\frac{\partial b_k(x;t,s)}{\partial x_k}\right|, \; |b_k'(x;t,s)| \leq K_1 \;\text{ in }\; \Omega \times (0,T], \; k = 1, \; 2, \tag{4.1.4}$$

$b_k'(x;t,s)$, $k = 1, \; 2$, is the partial derivative of b_k with respect to s. The non-homogeneous term $f = f(x,t)$ and initial data $u_0(x)$ are given functions.

For the finite element treatment of parabolic integro-differential equation with discontinuous coefficients, we refer to Pradhan *et. al.* ([54]). They have discussed a non-iterative domain decomposition procedure for parabolic integro-differential equation with interfaces and related a priori error estimates are derived. Numerical solutions by means of finite element Galerkin procedures for the parabolic integro-differential equation without interface can be found in [10, 12, 42, 48, 64, 66, 67].

The organization of this chapter is as follows: While Section 4.2 introduces the regularity of the problem, finite element discretization and approximation properties of some auxiliary projection, Section 4.3 is concerned on the convergence of semi discrete finite element solution to the exact solution. Section 4.4 is devoted to the fully discrete error analysis.

4.2 Preliminaries

In this section, we shall study the regularity and the finite element approximation to the solution of the interface problems (4.1.1)-(4.1.3) under the appropriate regularity conditions on f and u_0.

Since we limit ourselves to finite element analysis, we only concern about the regularity of the weak solution u for the interface problem (4.1.1)-(4.1.3). Let $A(.,.)$ and $B(t,s;.,.)$ be the bilinear forms on $H^1(\Omega) \times H^1(\Omega)$ corresponding to the operators $\mathcal{L}$ and $B(t,s)$, respectively *i.e.*,

$$A(w,v) = \int_\Omega \beta(x)\nabla w.\nabla v dx,$$

and

$$B(t,s;u(s),\phi) = \int_\Omega \left\{ \sum_{k=1}^{2} b_k(x;t,s)\frac{\partial u(x,s)}{\partial x_k} + u(x,s) \right\}\phi \, dx.$$

Under the assumption (4.1.4), for $\phi \in L^2(0,T;H_0^1(\Omega))$ and $\psi \in L^2(0,T;L^2(\Omega))$, it is easy to see that

$$|B(t,s;\phi(s),\psi(t))| \le C\|\phi(x,s)\|_{H^1(\Omega)}\|\psi(x,t)\|_{L^2(\Omega)}.$$

For $\phi \in L^2(0,T;Y)$ with $[\phi] = 0$ along $\Gamma \times (0,T]$ and $\phi = 0$ on $\partial\Omega \times (0,T]$, and $\psi \in L^2(0,T;H^1(\Omega))$, we have

$$\begin{aligned}
\int_\Omega \mathbf{b}\cdot\nabla\phi\psi dx &= \int_{\Omega_1} \mathbf{b}\cdot\nabla\phi\psi dx + \int_{\Omega_2} \mathbf{b}\cdot\nabla\phi\psi dx \\
&= \int_\Gamma \mathbf{b}\psi\cdot\mathbf{n}\phi_1 ds - \int_\Gamma \mathbf{b}\psi\cdot\mathbf{n}\phi_2 ds \\
&\quad - \int_{\Omega_1} \nabla\cdot(\mathbf{b}\psi)\phi dx - \int_{\Omega_2} \nabla\cdot(\mathbf{b}\psi)\phi dx \\
&= -\int_\Omega \nabla\cdot(\mathbf{b}\psi)\phi dx.
\end{aligned}$$

This together with assumption (4.1.4) leads to

$$|B(t,s;\phi(s),\psi(t))| \le C\|\phi(x,s)\|_{L^2(\Omega)}\|\psi(x,t)\|_{H^1(\Omega)}$$

and hence

$$|B_s(t,s;\phi(s),\psi(t))| \le C\Big\{\|\phi(x.s)\|_{H^1(\Omega)} + \|\phi_s(x.s)\|_{L^2(\Omega)}\Big\}\|\psi(x,t)\|_{H^1(\Omega)}.$$

Here, we have assumed that $\phi \in H^1(0,T;Y)$ with $[\phi_s] = 0$ along $\Gamma \times (0,T]$ and $\phi_s = 0$ on $\partial\Omega \times (0,T]$.

Then the weak formulation is defined as: Find $u : [0,T] \to H_0^1(\Omega)$ such that

$$(u_t, v) + A(u,v) = (f,v) + \int_0^t B(t,s;u(s),v)ds \quad \forall v \in H_0^1(\Omega), \;\; t \in (0,T] \tag{4.2.1}$$

with $u(0) = u_0$.

Clearly the problem (4.2.1) has a unique solution $u \in L^2(0,T;H_0^1(\Omega))$. Regarding the regularity for the solution of the problem (4.2.1), we have the following result.

Theorem 4.2.1 *Let $f \in H^1(0,T;L^2(\Omega))$ and $u_0 \in H_0^1(\Omega)$. Then the problem (4.2.1) has a unique solution $u \in L^2(0,T;X \cap H_0^1(\Omega)) \cap H^1(0,T;Y)$.*

Proof. We consider the following parabolic interface problem: Find $\tilde{u} : [0,T] \to H_0^1(\Omega)$ such that

$$(\tilde{u}_t, v) + A(\tilde{u},v) = \left(f + \int_0^t B(t,s)u(s)ds, v\right) \quad \forall v \in H_0^1(\Omega), \; t \in (0,T] \tag{4.2.2}$$

with $\tilde{u}(0) = u_0$ and $[\tilde{u}] = 0 = \left[\beta\frac{\partial\tilde{u}}{\partial n}\right]$ along $\Gamma \times (0,T]$. Then using the regularity result for the parabolic interface problems (cf. [15], [39]), we have

$$\tilde{u} \in L^2(0,T;X \cap H_0^1(\Omega)) \cap H^1(0,T;Y).$$

Now, subtracting (4.2.2) from (4.2.1), we have

$$(u_t - \tilde{u}_t, v) + A(u - \tilde{u}, v) = 0 \quad \forall v \in H_0^1(\Omega), \; t \in (0,T]. \tag{4.2.3}$$

Setting $v = u - \tilde{u} \in H_0^1(\Omega)$ in (4.2.3), we have

$$\frac{1}{2}\frac{d}{dt}\|u - \tilde{u}\|^2_{L^2(\Omega)} + A(u - \tilde{u}, u - \tilde{u}) = 0.$$

Integrating from 0 to t and using the fact $u(0) = u_0 = \tilde{u}(0)$, we obtain

$$\frac{1}{2}\|u - \tilde{u}\|^2_{L^2(\Omega)} + \int_0^t A(u - \tilde{u}, u - \tilde{u})ds = 0$$

which implies $u(x,t) = \tilde{u}(x,t)$ in $\Omega \times [0,T]$ and this completes the rest of the proof. □

Remark 4.2.1 *From (4.2.2), it is clear that* $\tilde{u}$ *satisfies the following equation*

$$\tilde{u}_t + \mathcal{L}\tilde{u} = h(x,t) \quad in\, \Omega \times (0,T]$$

with $h(x,t) = f(x,t) + \int_0^t B(t,s)u(s)ds$. *Then it follows from Theorem 2.2.1 in Chapter 2 that* $\tilde{u} \in H^2(0,T;L^2(\Omega))$ *provided* $f \in H^2(0,T;L^2(\Omega))$, $f(x,0) \in H^2(\Omega)$ *and* $u_0 \in H_0^1(\Omega) \cap H^3(\Omega)$. □

Let $\mathcal{T}_h$ be a triangulation of domain Ω as defined in Chapter 2 and V_h be a family of finite dimensional subspaces of $H_0^1(\Omega)$ based on $\mathcal{T}_h$ consisting of piecewise linear functions vanishing on the boundary $\partial\Omega$. For a triangulation $\mathcal{T}_h$, triangles with one or two vertices on Γ are called the interface triangles.

For our convenience, we also recall the elliptic projection $P_h : \mathcal{X} \to V_h$ defined as

$$A_h(P_h v, v_h) = A^1(v, v_h) + A^2(v, v_h) \;\; \forall v_h \in V_h, \; v \in \mathcal{X} \tag{4.2.4}$$

and standard L^2 projection $L_h : L^2(\Omega) \to V_h$ defined by

$$(L_h v, v_h) = (v, v_h) \, \forall v_h \in V_h, \; v \in L^2(\Omega). \tag{4.2.5}$$

The space $\mathcal{X}$ is as defined in Chapter 2.

The following result plays a crucial role in our subsequent analysis. For a proof, we refer to Lemma 3.3 of [59]

Lemma 4.2.1 *If* Ω_Γ^* *is the union of all interface triangles, then we have*

$$\|v\|_{H^1(\Omega_\Gamma^*)} \leq Ch^{\frac{1}{2}} \|v\|_X \; \forall v \in X. \;\; \square$$

Further, we need the following approximation properties

Lemma 4.2.2 *If* $\mathcal{T}_\Gamma^*$ *is the collection of all interface triangles, then*

$$\sum_{K \in \mathcal{T}_\Gamma^*} \|\nabla v_h\|^2_{L^2(\tilde{K})} \leq Ch \sum_{K \in \mathcal{T}_\Gamma^*} \|\nabla v_h\|^2_{L^2(K)} \; \forall v_h \in V_h.$$

Proof. Suppose $K \in \mathcal{T}_\Gamma^*$ and $\tilde{K}$ is either K_1 or K_2, $K_i = K \cap \Omega_i$ for $i = 1,2$. More precisely, $\tilde{K} = K_1$ if $K \subset \Omega_2^h$ and $\tilde{K} = K_2$ if $K \subset \Omega_1^h$. Assume $K \subset \Omega_1^h$, as shown in Figure 3.1.

Since, $\forall v_h \in V_h$, $|\nabla v_h|$ is constant in $K \in \mathcal{T}_h$, thus we have

$$\begin{aligned}
\|\nabla v_h\|^2_{L^2(\tilde{K})} &= \int_{\tilde{K}} |\nabla v_h|^2 dx \\
&= C^2 \int_{\tilde{K}} dx, \quad C = |\nabla v_h| = \text{constant} \\
&= C^2 \operatorname{meas}(\tilde{K}).
\end{aligned}$$

Again integrating over K and using the fact $\operatorname{meas}(\tilde{K}) \leq Ch_K^3$, we have

$$\begin{aligned}
\operatorname{meas}(K)\|\nabla v_h\|^2_{L^2(\tilde{K})} &= \operatorname{meas}(\tilde{K})\|\nabla v_h\|^2_{L^2(K)} \\
&\leq Ch_K^3 \|\nabla v_h\|^2_{L^2(K)}.
\end{aligned}$$

Further, apply the fact that $\operatorname{meas}(K) \geq Ch_K^2$ and summing over $K \in \mathcal{T}_\Gamma^*$, we have

$$\sum_{K \in \mathcal{T}_\Gamma^*} \|\nabla v_h\|^2_{L^2(\tilde{K})} \leq Ch \sum_{K \in \mathcal{T}_\Gamma^*} \|\nabla v_h\|^2_{L^2(K)}.$$

This completes the proof of the lemma 4.2.2. □

4.3 Continuous Time Galerkin Finite Element

In this section, an attempt is made to carry over known results for semidiscrete finite element Galerkin method for a parabolic equation to an integro-differential equation of parabolic type. Optimal order convergence results are obtained in $L^2(L^2)$ and $L^2(H^1)$ norms.

The continuous time Galerkin finite element approximation to (4.2.1) is stated as: Find $u_h : [0, T] \to V_h$ such that

$$(u_{ht}, v_h) + A_h(u_h, v_h) = (f, v_h) + \int_0^t B(t, s; u_h(s), v_h) ds \quad \forall v_h \in V_h \tag{4.3.1}$$

with $u_h(0) = L_h u_0$. Subtracting (4.3.1) from (4.2.1), we have

$$\begin{aligned}
(u_t - u_{ht}, v_h) + A(u - u_h, v_h) &= A_h(u_h, v_h) - A(u_h, v_h) \\
&+ \int_0^t B(t, s; (u - u_h)(s), v_h) ds \quad \forall v_h \in V_h.
\end{aligned} \tag{4.3.2}$$

Define the error $e(t)$ as $e(t) = u(t) - u_h(t)$. Setting $v_h = L_h u$ in (4.3.2) and using (4.2.5), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|e\|^2_{L^2(\Omega)} &+ A(e,e) \\
&= \{A_h(u_h, L_h u - u_h) - A(u_h, L_h u - u_h)\} + \frac{1}{2}\frac{d}{dt}\|u - L_h u\|^2_{L^2(\Omega)} \\
&\quad + A(e, u - L_h u) + \int_0^t B(t,s;e(s), L_h u - u)ds \\
&\quad + \int_0^t B(t,s;e(s), u - u_h)ds.
\end{aligned}$$

Then use coercivity and continuity of $A(.,.)$ to have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e\|^2_{L^2(\Omega)} + C\|e(t)\|^2_{H^1(\Omega)} \\
&\leq |A_h(u_h, L_h u - u_h) - A(u_h, L_h u - u_h)| + \frac{1}{2}\frac{d}{dt}\|u - L_h u\|^2_{L^2(\Omega)} \\
&+ C\|e(t)\|_{H^1(\Omega)}\|u - L_h u\|_{H^1(\Omega)} + C\Big(\int_0^t \|e(s)\|^2_{H^1(\Omega)}ds\Big)^{\frac{1}{2}}\|u - L_h u\|_{L^2(\Omega)} \\
&+ C\Big(\int_0^t \|e(s)\|^2_{H^1(\Omega)}ds\Big)^{\frac{1}{2}}\|e(t)\|_{L^2(\Omega)} \\
&\leq |A_h(u_h, L_h u - u_h) - A(u_h, L_h u - u_h)| + \frac{1}{2}\frac{d}{dt}\|u - L_h u\|^2_{L^2(\Omega)} \\
&+ C(\epsilon)\|e(t)\|^2_{H^1(\Omega)} + C_\epsilon\|u - L_h u\|^2_{H^1(\Omega)} + C_\epsilon \int_0^t \|e(s)\|^2_{H^1(\Omega)}ds \\
&+ C(\epsilon)\|u - L_h u\|^2_{L^2(\Omega)} + C_\epsilon \int_0^t \|e(s)\|^2_{H^1(\Omega)}ds + C(\epsilon)\|e(t)\|^2_{H^1(\Omega)}.
\end{aligned}$$

Now, integrating from 0 to t and setting suitable ϵ, we obtain

$$\begin{aligned}
\int_0^t \|e(s)\|^2_{H^1(\Omega)}ds &\leq \int_0^t |A_h(u_h, L_h u - u_h) - A(u_h, L_h u - u_h)|ds \\
&\quad + \frac{1}{2}\|u(t) - L_h u(t)\|^2_{L^2(\Omega)} + C\int_0^t \|u - L_h u\|^2_{H^1(\Omega)}ds \\
&\quad + C\int_0^t\int_0^\tau \|e(s)\|^2_{H^1(\Omega)}dsd\tau \\
&=: (I)_1 + (I)_2 + (I)_3 + C\int_0^t\int_0^\tau \|e(s)\|^2_{H^1(\Omega)}dsd\tau. \qquad (4.3.3)
\end{aligned}$$

For the term $(I)_1$, use Lemma 2.2.2 and Lemma 2.2.4 to have

$$
\begin{aligned}
&|A_h(u_h, L_h u - u_h) - A(u_h, L_h u - u_h)| \\
&\leq Ch\|u_h\|_{H^1(\Omega)}\|L_h u - u_h\|_{H^1(\Omega)} \\
&\leq Ch\|u_h\|_{H^1(\Omega)}\|L_h u - u\|_{H^1(\Omega)} + Ch\|u_h\|_{H^1(\Omega)}\|e(t)\|_{H^1(\Omega)} \\
&\leq C_\epsilon h^2\|u_h\|^2_{H^1(\Omega)} + C(\epsilon)\|L_h u - u\|^2_{H^1(\Omega)} + C(\epsilon)\|e(t)\|^2_{H^1(\Omega)} \\
&\leq C_\epsilon h^2\|u_h\|^2_{H^1(\Omega)} + C(\epsilon)h^2\|u(x,t)\|^2_X + C(\epsilon)\|e(t)\|^2_{H^1(\Omega)}
\end{aligned}
$$

and hence

$$
\begin{aligned}
(I)_1 \;\leq\; & C_\epsilon h^2 \int_0^t \|u_h\|^2_{H^1(\Omega)} ds + C(\epsilon)h^2 \int_0^t \|u(x,s)\|^2_X ds \\
& + C(\epsilon)\int_0^t \|e(s)\|^2_{H^1(\Omega)} ds. \qquad (4.3.4)
\end{aligned}
$$

Similarly for the terms $(I)_2$ & $(I)_3$, we have

$$
(I)_2 \leq Ch^4\|u(x,t)\|^2_X \quad \& \quad (I)_3 \leq Ch^2\int_0^t \|u(x,s)\|^2_X ds. \qquad (4.3.5)
$$

Then combining the estimates (4.3.3)-(4.3.5) and using the fact

$$
\int_0^t \|u_h\|^2_{H^1(\Omega)} ds \leq C\Big(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f\|^2_{L^2(\Omega)} ds\Big),
$$

we have

$$
\begin{aligned}
\int_0^t \|e(s)\|^2_{H^1(\Omega)} ds \;\leq\; & Ch^2\Big(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f\|^2_{L^2(\Omega)} ds + \|u(x,t)\|^2_X \\
& + \int_0^T \|u(x,s)\|^2_X ds\Big) + \int_0^t C\Big(\int_0^\tau \|e(s)\|^2_{H^1(\Omega)} ds\Big) d\tau.
\end{aligned}
$$

Then a simple application of Grownwall's Lemma leads to the following optimal $L^2(H^1)$ norm error estimate

Theorem 4.3.1 *Let u and u_h be the solutions of the problem (4.2.1)and (4.3.1), respectively. Then, for $f \in H^1(0,T;L^2(\Omega))$ and $u_0 \in H^1_0(\Omega)$, there exist a constant C independent of h such that*

$$
\begin{aligned}
\|e(s)\|_{L^2(0,T;H^1(\Omega))} \;\leq\; & Ch\Big(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f\|^2_{L^2(\Omega)} ds \\
& + \|u(x,T)\|^2_X + \int_0^T \|u(x,s)\|^2_X ds\Big)^{\frac{1}{2}}. \quad \square
\end{aligned}
$$

For the L^2 norm error estimate we shall use the duality trick. For this purpose we consider the following interface problem: Find $w \in H_0^1(\Omega)$ such that

$$A(w.v) = (u - u_h, v) \;\; \forall v \in H_0^1(\Omega) \tag{4.3.6}$$

and its finite element approximation is defined to be the function $w_h \in V_h$ such that

$$A_h(w_h, v_h) = (u - u_h, v_h) \;\; \forall v_h \in V_h. \tag{4.3.7}$$

Note that $w \in X \cap H_0^1(\Omega)$ is the solution of the elliptic interface problem (4.3.6) with the jump conditions $[w] = 0$, $\left[\beta(x)\frac{\partial w}{\partial n}\right] = 0$ along Γ. Further, w satisfies the a priori estimate

$$\|w\|_X \le C\|u - u_h\|_{L^2(\Omega)}. \tag{4.3.8}$$

Then it follows from [22] (see, Theorem 3.1) that

$$\|w - w_h\|_{H^1(\Omega)} \le Ch\|u - u_h\|_{L^2(\Omega)}. \tag{4.3.9}$$

Setting $v = u - u_h \in H_0^1(\Omega)$ in (4.3.6) and using (4.3.2), we obtain

$$\begin{aligned}
\|e(t)\|^2_{L^2(\Omega)} &= A(w - w_h, u - u_h) + A(w_h, u - u_h) \\
&= A(w - w_h, u - u_h) + A_h(u_h. w_h) - A(u_h, w_h) \\
&\quad -(e_t, w_h) + \int_0^t B(t, s; e(s), w_h) ds \\
&\le C\|w - w_h\|_{H^1(\Omega)}\|u - u_h\|_{H^1(\Omega)} + A_h(u_h, w_h) - A(u_h, w_h) \\
&\quad -(e_t, w_h) + \int_0^t B(t, s; e(s), w_h) ds.
\end{aligned} \tag{4.3.10}$$

Again from the equation (4.3.7), we note that

$$\frac{1}{2}\frac{d}{dt}A_h(w_h. w_h) = A_h(w_{ht}, w_h) = (u_t - u_{ht}, w_h)$$

and hence, estimate (4.3.10) reduces to

$$\begin{aligned}
\|e(t)\|^2_{L^2(\Omega)} &\le C\|w - w_h\|_{H^1(\Omega)}\|u - u_h\|_{H^1(\Omega)} + A_h(u_h, w_h) - A(u_h, w_h) \\
&\quad -\frac{1}{2}\frac{d}{dt}A_h(w_h, w_h) + \int_0^t B(t, s; e(s), w_h) ds \\
&\le C\{h\|e(t)\|_{L^2(\Omega)}\|e(t)\|_{H^1(\Omega)}\} + C\{h\|e(t)\|_{H^1(\Omega)}\|e(t)\|_{L^2(\Omega)} \\
&\quad +h^2\|u(x,t)\|_X\|e(t)\|_{L^2(\Omega)}\} - \frac{1}{2}\frac{d}{dt}A_h(w_h, w_h) \\
&\quad +C\|e(s)\|_{L^2(0,t,L^2(\Omega))}\|e(t)\|_{L^2(\Omega)}.
\end{aligned} \tag{4.3.11}$$

Here, we have used the fact that $\|w_h\|_{H^1(\Omega)} \leq C\|u - u_h\|_{L^2(\Omega)}$ and the estimate for the term $(II)_2$ in [22] (see, page 216). Further, a simple application of Young's inequality leads to

$$\begin{aligned} \|e(t)\|^2_{L^2(\Omega)} &\leq C_\epsilon h^2\|e(t)\|^2_{H^1(\Omega)} + C_\epsilon h^4\|u(x,t)\|^2_X + C_\epsilon\|e(s)\|^2_{L^2(0,t;L^2(\Omega))} \\ &+C(\epsilon)\|e(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\frac{d}{dt}A_h(w_h, w_h). \end{aligned} \quad (4.3.12)$$

Therefore, for suitable $\epsilon > 0$ and integrating from 0 to t, we have

$$\begin{aligned} \int_0^t \|e(s)\|^2_{L^2(\Omega)}ds &\leq Ch^2\int_0^t \|e(s)\|^2_{H^1(\Omega)}ds + Ch^4\int_0^t \|u(x,s)\|^2_X ds \\ &+C\int_0^t\int_0^\tau \|e(s)\|^2_{L^2(\Omega)}dsd\tau + \frac{1}{2}A_h(w_h(0), w_h(0)). \end{aligned} \quad (4.3.13)$$

Taking $t \to 0$, it now follows from (4.3.7) that

$$A_h(w_h(0), w_h(0)) = (u_0 - L_h u_0, w_h(0)) = 0.$$

This together with (4.3.13), Gronwall's inequality and Theorem 4.3.1 leads to the following optimal $L^2(L^2)$ norm error estimate

Theorem 4.3.2 *Let u and u_h be the solutions of the problem (4.2.1)and (4.3.1), respectively. Then, for $f \in H^1(0,T;L^2(\Omega))$ and $u_0 \in H^1_0(\Omega)$, there exist a constant C independent of h such that*

$$\begin{aligned} \|e(s)\|_{L^2(0,T;L^2(\Omega))} &\leq Ch^2\Big(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f\|^2_{L^2(\Omega)}ds \\ &+ \|u(x,T)\|^2_X + \int_0^T \|u(x,s)\|^2_X ds\Big)^{\frac{1}{2}}. \quad \square \end{aligned}$$

4.4 Discrete Time Galerkin Method

In this section, we shall discretize the equation (4.3.1) in time direction. We shall make use of backward difference scheme to discretize the problem in time direction and the piecewise linear finite element method in space. Optimal error estimate in $L^2(H^1)$ norm is derived for smooth initial function.

We first divide the interval $[0,T]$ into M equally spaced subintervals by the following points

$$0 = t^0 < t^1 < \cdots < t^M = T,$$

with $t^n = nk$, $k = T/M$ be the time step. Let $I_n = (t_{n-1}, t_n]$ be the n-th sub interval. For a given sequence $\{\phi^n\}_{n=1}^M \subset L^2(\Omega)$, we introduce the backward difference quotient

$$\Delta_k \phi^n = \frac{\phi^n - \phi^{n-1}}{k}.$$

The fully discrete finite element approximation to the problem (4.3.1) is defined as follows: For $1 \le n \le M$, find $U^n \in V_h$ such that

$$(\Delta_k U^n, v_h) + A_h(U^n, v_h) = (f^n, v_h) + k\sum_{j=0}^{n-1} B(t_n, t_j; U^j, v_h) \quad \forall v_h \in V_h \tag{4.4.1}$$

with $U^0 = L_h u_0$ and the integral term in (4.3.1) has been approximated by the rectangle rule

$$\int_0^{t_n} \phi(s)ds \approx k\sum_{j=0}^{n-1} \phi^j = Q_1^n \phi, \; 0 < t_n \le T.$$

Note that the quadrature error in $I_n = (t_{n-1}, t_n)$ is estimated as

$$\int_{I_n} \phi(s)ds - k\phi^{n-1} = \int_{I_n}\int_{t_{n-1}}^{s} \phi'(\tau)d\tau ds = \int_{I_n} (t_n - \tau)\phi'(\tau)d\tau$$

and hence

$$|Q_1^n \phi - \int_0^{t_n} \phi(s)ds| \le k\int_0^{t_n} |\phi'(\tau)|d\tau. \tag{4.4.2}$$

Regarding the stability of the fully discrete solution, we have the following result.

Lemma 4.4.1 *Let U^n be the solution for the fully discrete scheme defined by (4.4.1), then we have*

$$\begin{aligned} \|U^M\|^2_{L^2(\Omega)} + k\sum_{n=1}^{M} \|U^n\|^2_{H^1(\Omega)} \;&\le\; Ck\sum_{n=1}^{M} \|f^n\|^2_{L^2(\Omega)} + Ck\int_0^T \|u_t\|^2_{L^2(\Omega)}ds \\ &\quad + Ck\int_0^T \|u\|^2_{L^2(\Omega)}ds + C\|u_0\|^2_{L^2(\Omega)}. \end{aligned}$$

Proof. The lemma can be proved by setting $v_h = kU^n$ in (4.4.1) and using (4.4.2). We omit the details. □

For the convenience, let us define the piecewise constant function $U_{h,k}$ in time by $U_{h,k}(x,t) = U^n(x)$, $\forall t \in I_n$, $n = 1, 2, 3, \ldots, M$. Then, regarding the convergence of $U_{h,k}$, we have the following result.

Theorem 4.4.1 *Let u and U_{hk} be the solutions of the problems (4.2.1) and (4.4.1), respectively. Assume that $u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$, $f \in H^2(0,T;L^2(\Omega))$ and $f(x,0) \in H^2(\Omega)$. Then there exist a positive constant C, independent of h and k such that*

$$\|u - U_{hk}\|_{L^2(0,T,H^1(\Omega))} \le C(u_0, f, u, u_t, u_{tt})(k+h).$$

Proof. At $t = t_n$, (4.2.1) reduces to

$$(u_t^n, v_h) + A(u^n, v_h) = (f^n, v_h) + \int_0^{t_n} B(t_n, s; u(s), v_h)ds \;\; \forall v \in H_0^1(\Omega). \tag{4.4.3}$$

For simplicity of the exposition, we write $u^n = u(x, nk)$, $e^n = u^n - U^n$ and $w^n = u^n - P_h u^n$. Using (4.4.1) and (4.4.3), it follows that

$$\begin{aligned}
&(\Delta_k e^n, e^n) + A(e^n, e^n) \\
&= (\Delta_k e^n, w^n) + A(e^n, w^n) + (\Delta_k u^n - u_t^n, P_h u^n - U^n) \\
&\quad + \{A_h(U^n, P_h u^n - U^n) - A(U^n, P_h u^n - U^n)\} \\
&\quad + \Big\{ \int_0^{t_n} B(t_n, s; u(s), U^n - P_h u^n)ds - k\sum_{j=0}^{n-1} B(t_n, t_j; u^j, U^n - P_h u^n) \Big\} \\
&\quad + k\sum_{j=0}^{n-1} B(t_n, t_j; e^j, U^n - P_h u^n) \\
&=: \sum_{j=1}^{6} (II)_j.
\end{aligned} \tag{4.4.4}$$

where

$$(II)_1 = (\Delta_k e^n, w^n), \quad (II)_2 = A(e^n, w^n), \quad (II)_3 = (\Delta_k u^n - u_t^n, P_h u^n - U^n),$$

$$(II)_4 = \{A_h(U^n, P_h u^n - U^n) - A(U^n, P_h u^n - U^n)\},$$

$$(II)_5 = \int_0^{t_n} B(t_n, s; u(s), U^n - P_h u^n)ds - k\sum_{j=0}^{n-1} B(t_n, t_j; u^j, U^n - P_h u^n),$$

$$(II)_6 = k\sum_{j=0}^{n-1} B(t_n, t_j; e^j, U^n - P_h u^n).$$

Summing (4.4.4) over n from $n = 1$ to $n = M$, we have

$$\frac{1}{2}\|e^M\|^2_{L^2(\Omega)} + k\sum_{n=1}^{M} A(e^n, e^n) + \frac{k}{2}\sum_{n=1}^{M} \|\Delta_k e^n\|^2_{L^2(\Omega)} \le \frac{1}{2}\|e^0\|^2_{L^2(\Omega)} + Ck\sum_{n=1}^{M}\sum_{j=1}^{6} (II)_j. \tag{4.4.5}$$

Using Lemma 2.2.3 and Young's inequality, we obtain

$$k\sum_{n=1}^{M}(II)_1 \leq C_\epsilon h^4 k\sum_{n=1}^{M}\|u^n\|_X^2 + C(\epsilon)k\sum_{n=1}^{M}\|\Delta_k e^n\|_{L^2(\Omega)}^2. \tag{4.4.6}$$

Similarly,

$$k\sum_{n=1}^{M}(II)_2 \leq C_\epsilon h^2 k\sum_{n=1}^{M}\|u^n\|_X^2 + C(\epsilon)k\sum_{n=1}^{M}\|e^n\|_{H^1(\Omega)}^2. \tag{4.4.7}$$

To estimate $k\sum_{n=1}^{M}(II)_3$, we first note that

$$\Delta_k u^n - \frac{\partial u^n}{\partial t} = -\frac{1}{k}\int_{t_{n-1}}^{t_n}(s-t_{n-1})u_{ss}(s)ds$$

and hence using Lemma 2.2.3, we obtain

$$\begin{aligned} k\sum_{n=1}^{M}(II)_3 \;&\leq\; C_\epsilon k^2\|u_{tt}\|_{L^2(0,T;L^2(\Omega))}^2 + C(\epsilon)h^2 k\sum_{n=1}^{M}\|u^n\|_X^2 \\ &\quad + C(\epsilon)k\sum_{n=1}^{M}\|e^n\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.4.8}$$

Using Lemma 2.2.2, we obtain

$$\begin{aligned} k\sum_{n=1}^{M}(II)_4 \;&\leq\; Chk\sum_{n=1}^{M}\Big\{\|U^n\|_{H^1(\Omega)}\|P_h u^n - U^n\|_{H^1(\Omega)}\Big\} \\ &\leq\; hk\sum_{n=1}^{M}\Big\{C_\epsilon\|U^n\|_{H^1(\Omega)}^2 + C(\epsilon)\|P_h u^n - U^n\|_{H^1(\Omega)}^2\Big\} \\ &\leq\; C(\epsilon)hk\sum_{n=1}^{M}\|e^n\|_{H^1(\Omega)}^2 + C(\epsilon)h^3 k\sum_{n=1}^{M}\|u^n\|_X^2 \\ &\quad + Chk\sum_{n=1}^{M}\|f^n\|_{L^2(\Omega)}^2 + Chk\int_0^T\|u_t\|_{L^2(\Omega)}^2 ds \\ &\quad + Chk\int_0^T\|u\|_{L^2(\Omega)}^2 ds + Ch\|u_0\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.4.9}$$

In the last inequality, we have used Lemma 2.2.3 and Lemma 4.4.1.

Again, setting $p=4$ in the Sobolev embedding inequality (cf. [62, 63])

$$\|v\|_{L^p(K)} \leq Cp^{\frac{1}{2}}\|v\|_{H^1(K)} \;\; \forall v \in H^1(K), \;\; p > 2$$

and using Hölder's inequality, we obtain

$$\begin{aligned}
\|u_0\|_{L^2(\Omega)} &= \sum_{K\in\mathcal{T}_h} \|u_0\|_{L^2(K)} \\
&\le Ch^{\frac{1}{2}} \sum_{K\in\mathcal{T}_h} \|u_0\|_{L^4(K)} \\
&\le Ch^{\frac{1}{2}} \sum_{K\in\mathcal{T}_h} \|u_0\|_{H^1(K)} = Ch^{\frac{1}{2}}\|u_0\|_{H^1(\Omega)}
\end{aligned}$$

where we have used the fact that $\text{meas}(K) \le Ch^2,\ K \in \mathcal{T}_h$. Using this fact in (4.4.9), we have

$$\begin{aligned}
k\sum_{n=1}^{M}(II)_4 &\le C(\epsilon)hk\sum_{n=1}^{M}\|e^n\|^2_{H^1(\Omega)} + C(\epsilon)h^3k\sum_{n=1}^{M}\|u^n\|^2_X \\
&\quad + Chk\sum_{n=1}^{M}\|f^n\|^2_{L^2(\Omega)} + Chk\int_0^T \|u_t\|^2_{L^2(\Omega)}ds \\
&\quad + Chk\int_0^T \|u\|^2_{L^2(\Omega)}ds + Ch^2\|u_0\|^2_{H^1(\Omega)}. \qquad (4.4.10)
\end{aligned}$$

Finally, (4.4.2) leads to

$$\begin{aligned}
k\sum_{n=1}^{M}(II)_5 &\le \sum_{n=1}^{M}k^2\int_{t_{n-1}}^{t_n}\{\|u\|_{H^1(\Omega)} + \|u_s\|_{L^2(\Omega)}\}ds\|w^n\|_{H^1(\Omega)} \\
&\quad + \sum_{n=1}^{M}k^2\int_{t_{n-1}}^{t_n}\{\|u\|_{H^1(\Omega)} + \|u_s\|_{L^2(\Omega)}\}ds\|e^n\|_{H^1(\Omega)} \\
&\le C_\epsilon k^2\sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\{\|u\|_{H^1(\Omega)} + \|u_s\|_{L^2(\Omega)}\}^2ds + C(\epsilon)k^2\sum_{n=1}^{M}\|w^n\|^2_{H^1(\Omega)} \\
&\quad + C(\epsilon)k^2\sum_{n=1}^{M}\|e^n\|^2_{H^1(\Omega)} \\
&\le C_\epsilon k^2\{\|u\|_{L^2(0,T,H^1(\Omega))} + \|u_t\|_{L^2(0,T,L^2(\Omega))}\}^2 + C(\epsilon)kh^2\sum_{n=1}^{M}k\|u^n\|^2_X \\
&\quad + C(\epsilon)k^2\sum_{n=1}^{M}\|e^n\|^2_{H^1(\Omega)}. \qquad (4.4.11)
\end{aligned}$$

Then, for $k = O(h)$ and suitable $\epsilon > 0$, it follows from the estimates (4.4.5)-(4.4.11) that

$$\begin{aligned}\sum_{n=1}^{M} k\|e^n\|^2_{H^1(\Omega)} &\leq Ch^2(\|u_0\|^2_X + \sum_{n=1}^{M}\|u^n\|^2_X + \|u\|^2_{L^2(0,T;L^2(\Omega))} \\ &\quad + \|u_t\|^2_{L^2(0,T;L^2(\Omega))} + \|u_{tt}\|^2_{L^2(0,T;L^2(\Omega))} \\ &\quad + \sum_{n=1}^{M}\|f^n\|^2_{L^2(\Omega)}) + Ck^2\sum_{n=1}^{M}\sum_{j=0}^{n-1} B(t_n, t_j; e^j, U^n - P_h u^n) \\ &=: \tilde{C} + Ck^2 \sum_{n=1}^{M}\sum_{j=0}^{n-1} B(t_n, t_j; e^j, U^n - P_h u^n). \qquad (4.4.12)\end{aligned}$$

Then it follows from [14] (see, Lemma 7 therein) that

$$\begin{aligned}\sum_{n=1}^{M} k\|e^n\|^2_{H^1(\Omega)} &\leq \tilde{C} + C(\epsilon)k^2\sum_{n=1}^{M}\|e^n\|^2_{H^1(\Omega)} + C_\epsilon k^2\sum_{n=1}^{M-1}\sum_{j=0}^{n-1}\|e^j\|^2_{H^1(\Omega)} \\ &\quad + C(\epsilon)k^2\sum_{n=1}^{M}\|w^n\|^2_{H^1(\Omega)}.\end{aligned}$$

Thus, for suitable $\epsilon > 0$, we have

$$\sum_{n=1}^{M} k\|e^n\|^2_{H^1(\Omega)} \leq \tilde{C} + Ckh^2\sum_{n=1}^{M}k\|u^n\|^2_X + Ck^2\sum_{n=1}^{M-1}\sum_{j=0}^{n-1}\|e^j\|^2_{H^1(\Omega)}. \qquad (4.4.13)$$

Setting $\xi_l = \sum_{n=1}^{l} k\|e^n\|^2_{H^1(\Omega)}$ in (4.4.13), we obtain

$$\xi_M \leq \tilde{C} + Ck\sum_{n=1}^{M-1}\xi_n.$$

Then a simple application of discrete Grownwall's lemma leads to

$$\begin{aligned}\xi_M &\leq Ch^2(\|u_0\|^2_X + \sum_{n=1}^{M}\|u^n\|^2_X + \|u\|^2_{L^2(0,T;L^2(\Omega))} + \|u_t\|^2_{L^2(0,T;L^2(\Omega))} \\ &\quad + \|u_{tt}\|^2_{L^2(0,T;L^2(\Omega))} + \sum_{n=1}^{M}\|f^n\|^2_{L^2(\Omega)}). \qquad (4.4.14)\end{aligned}$$

Again it follows from Chen and Zou [15] that

$$\|u - U_{hk}\|_{L^2(0,T,H^1(\Omega))} \leq Ck\|u_t\|_{L^2(0,T;Y)} + C\left(\sum_{n=1}^{M} k\|e^n\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}. \qquad (4.4.15)$$

Then Theorem 4.4.1 follows immediately from (4.4.14)-(4.4.15). □

Chapter 5

FEM for Parabolic Integro-Differential Equations with Interfaces: $L^\infty(L^2)$ and $L^\infty(H^1)$ Error Estimates

In the previous chapter, we have considered a interface problem of parabolic-integro type with first order memory term. Finite element treatment for parabolic integro-differential equations with discontinuous coefficients and second order memory term are presented in this work. Convergence of continuous time Galerkin method for the spatially discrete scheme and backward difference scheme in time direction are discussed in $L^2(H^m)$ and $L^\infty(H^m)$ norms for fitted finite element method with straight interface triangles. Optimal error estimates are derived in $L^2(H^m)$ and $L^\infty(H^m)$ norms when initial data $u_0 \in H_0^1(\Omega)$ and $u_0 \in H^3 \cap H_0^1(\Omega)$, respectively.

5.1 Introduction

The aim of this chapter is to analyze finite element methods for solving initial-boundary value problems of the form

$$u_t(x,t) - \nabla \cdot (\beta \nabla u(x,t)) = f(x,t) + \int_0^t B(t,s)u(s)ds \quad \text{in } \Omega \times (0,T] \tag{5.1.1}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \text{ in } \Omega \; \& \; u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T] \tag{5.1.2}$$

where $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ is a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ is an open domain with C^2 smooth boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega \backslash \bar{\Omega}_1$ (see, Figure 1.1). Information between both the domains are transferred via jump conditions

$$[u] = 0, \quad \left[\beta(x)\frac{\partial u}{\partial \mathbf{n}}\right] = 0 \quad \text{along } \Gamma \times (0,T]. \tag{5.1.3}$$

The symbol $[v]$ is a jump of a quantity v across the interface Γ and $\mathbf{n}$ denotes the unit outward normal to the boundary $\partial\Omega_1$. We write

$$\beta(x) = \beta_i \quad \text{for} \quad x \in \Omega_i, \; i = 1,2.$$

Further, $B(t,s)$ is a second order partial differential operator of the form

$$B(t,s)u(s) = -\nabla \cdot (b(x;t,s)\nabla u) + b_0(x;t,s)u(x,s).$$

Coefficients of $B(t,s)$ are assumed to be smooth and satisfy the following assumption: there exists a positive constant K_1 such that

$$|b(x;t,s)|, \; |b_0(x;t,s)| \; \& \; |b'(x;t,s)| \leq K_1 \quad \text{in } \Omega \times (0,T], \tag{5.1.4}$$

$b'(x;t,s)$ is the partial derivative of b with respect to s. The non-homogeneous term $f = f(x,t)$ and initial data $u_0(x)$ are given functions.

In this chapter, an attempt is made to carry over known results of finite element Galerkin method for non interface parabolic integro-differential equation to integro-differential equation of parabolic type with discontinuous coefficients. A priori error estimates are derived for minimum smooth and sufficiently regular initial data. More precisely, optimal error estimates are derived in $L^2(H^m)$ and $L^\infty(H^m)$ norms when initial data $u_0 \in H_0^1(\Omega)$ and $u_0 \in H^3 \cap H_0^1(\Omega)$, respectively. The achieved estimates are analogous to the case with a regular solution, however, due to low regularity, the proof requires a careful technical work coupled with a approximation result for the Ritz-Volterra projection under minimum regularity assumption. Other technical tools used in this work are Sobolev embedding inequality, approximation properties for elliptic projection, duality arguments and some known results on elliptic interface problems. The main emphasis of this work is on the theoretical aspect of convergence of finite element method under the low global regularity of the true solution. Numerical solutions by means of finite element Galerkin procedures for the parabolic integro-differential equation without interface can be found in [10, 12, 14, 42, 48, 64, 66, 67].

For the purpose of finite element Galerkin procedure, we need bilinear forms associated with the operators in (5.1.1). Let $A(.,.)$ and $B(t,s;.,.)$ be the bilinear forms on $H_0^1 \times H_0^1$ corresponding to operators $\mathcal{L}$ and $B(t,s)$ *i.e.*,

$$\begin{aligned} A(w,v) &= \int_\Omega \beta(x)\nabla w \cdot \nabla v dx \quad \text{and} \\ B(t,s;w(s),v) &= \int_\Omega \big(b(x;t,s)\nabla w(x,s) \cdot \nabla v + b_0(x;t,s)w(x,s)v\big)dx. \end{aligned}$$

The organization of this chapter is as follows: While section 5.2 introduces the regularity of the problem, finite element discretization and approximation properties of some auxiliary projection, section 5.3 is concerned with the convergence of semidiscrete finite element solution to the exact solution in $L^2(L^2)$ and $L^2(H^1)$ norms. section 5.4 is devoted to the point wise in time error analysis in L^2 and H^1 norms for the semidiscrete case. Finally, backward difference scheme has been used to discretize the problem in time direction and related error estimates are derived in section 5.5.

5.2 Preliminaries

In this section, we shall study the regularity and the finite element approximation to the solution of the interface problems (5.1.1)-(5.1.3).

The weak formulation of the problem (5.1.1)-(5.1.3) may be stated as: Find $u : [0,T] \to H_0^1$ such that

$$(u_t,\phi) + A(u,\phi) = \int_0^t B(t,s;u(s),\phi)ds + (f,\phi) \ \ \forall \phi \in H_0^1(\Omega), t \in (0,T] \qquad (5.2.1)$$

with $u(x,0) = u_0$.

Clearly, under the assumptions (5.1.4), the problem (5.2.1) has a unique solution $u \in L^2(0,T;H_0^1(\Omega))$ (cf. [64]). Again it follows from the analysis of previous chapter that the solution u can be characterized as a solution of parabolic interface problem. For the regularity results of parabolic interface problems, we refer to [15, 39, 58]. Therefore, we assume the following regularity result for the weak solution u.

Theorem 5.2.1 *Let $f \in H^1(0,T;L^2(\Omega))$ and $u_0 \in H_0^1(\Omega)$. Then the problem (5.2.1) has a unique solution $u \in L^2(0,T;X \cap H_0^1(\Omega)) \cap H^1(0,T;Y)$.* $\square$

Remark 5.2.1 *It is observed from the regularity result that $u_0 \in H_0^1(\Omega)$ is the minimum regularity assumption for the existence of solution in $L^2(0,T;X \cap H_0^1(\Omega)) \cap H^1(0,T;Y)$ (cf. [15, 39]). For more regular initial data $u_0 \in H_0^1(\Omega) \cap H^3(\Omega)$ and $f \in H^1(0,T;H^1(\Omega))$, it follows from Chapter 3 that $u \in L^2(0,T;X \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2))$.* $\square$

Central to the analysis of finite element methods for integro-differential equations has been the Ritz-Volterra projection introduced in [42]. Before proceeding further, let us recall some notations from Chapter 3. Let $Y^\star$ be the collection of all $v \in L^2(\Omega)$ such that $v \in H^1(\Omega_1) \cap H^1(\Omega_2) \cap \{\psi : \psi = 0 \text{ on } \partial\Omega\}$ with $[v] = 0$ along Γ. For any $v \in Y^\star$, we define

$$A_h(R_h v, v_h) = A^1(v, v_h) + A^2(v, v_h) \quad \forall v_h \in V_h. \tag{5.2.2}$$

The Ritz-Volterra projection $W_h : Y^\star \to V_h$ is defined as

$$\begin{aligned} A_h(W_h v, v_h) &= A_h(R_h v, v_h) \\ &\quad + \int_0^t \tilde{B}(t,s;(W_h v - v)(s), v_h)ds \quad \forall\, v_h \in V_h,\ v \in Y^\star. \end{aligned} \tag{5.2.3}$$

Here, bilinear map $\tilde{B}(t,s;.,.)$ is defined as

$$\tilde{B}(t,s;w(s),z) = \sum_{l=1}^{2} \int_{\Omega_l} \big(b(x;t,s)\nabla w(x,s) \cdot \nabla z + b_0(x;t,s)w(x,s)z\big)dx.$$

Note that, for $v \in X \cap H_0^1(\Omega)$, $W_h v$ satisfies the following identity

$$\begin{aligned} A_h(W_h v, v_h) &= A_h(R_h v, v_h) \\ &\quad + \int_0^t B(t,s;(W_h v - v)(s), v_h)ds \quad \forall\, v_h \in V_h. \end{aligned}$$

The approximation properties of Ritz-Volterra projection are well known (c.f. [12], [42]) for sufficiently smooth functions. Here, indeed we will show the same optimal error estimates for H^1 and L^2 norms, even if the solution u does not belong to H^2 globally.

By setting $v_h = W_h v(t) - R_h v(t)$ in (5.2.3) and using Lemma 3.2.2, we obtain

$$
\begin{aligned}
&\|W_h v(t) - R_h v(t)\|^2_{H^1(\Omega)} \\
&\leq C\|W_h v(t) - R_h v(t)\|_{H^1(\Omega)} \int_0^t \sum_{l=1}^{2} \|W_h v(s) - v(s)\|_{H^1(\Omega_l)} ds \\
&\leq C_\epsilon \|W_h v(t) - R_h v(t)\|^2_{H^1(\Omega)} \\
&\quad + C(\epsilon) \int_0^t \{\|W_h v(s) - R_h v(s)\|^2_{H^1(\Omega)} + \sum_{l=1}^{2} \|R_h v(s) - v(s)\|^2_{H^1(\Omega_l)}\} ds \\
&\leq C_\epsilon \|W_h v(t) - R_h v(t)\|^2_{H^1(\Omega)} + C(\epsilon) \int_0^t \|W_h v(s) - R_h v(s)\|^2_{H^1(\Omega)} ds \\
&\quad + C(\epsilon) h^2 \int_0^t \{\|v(s)\|^2_{H^2(\Omega_1)} + \|v(s)\|^2_{H^2(\Omega_2)}\} ds.
\end{aligned}
$$

Hence, for suitable $\epsilon > 0$, we obtain

$$
\begin{aligned}
\|W_h v(t) - R_h v(t)\|^2_{H^1(\Omega)} \leq\ & Ch^2 \int_0^t \{\|v(s)\|^2_{H^2(\Omega_1)} + \|v(s)\|^2_{H^2(\Omega_2)}\} ds \\
& + C \int_0^t \|W_h v(s) - R_h v(s)\|^2_{H^1(\Omega)} ds.
\end{aligned}
$$

Then Grownwall's inequality leads to

$$
\|W_h v(t) - R_h v(t)\|^2_{H^1(\Omega)} \leq Ch^2 \int_0^t \{\|v(s)\|^2_{H^2(\Omega_1)} + \|v(s)\|^2_{H^2(\Omega_2)}\} ds
$$

and hence

$$
\begin{aligned}
\sum_{l=1}^{2} \|W_h v(t) - v(t)\|^2_{H^1(\Omega_l)} \leq\ & Ch^2 \sum_{l=1}^{2} \|v(t)\|^2_{H^2(\Omega_l)} \\
& + Ch^2 \int_0^t \sum_{l=1}^{2} \|v(s)\|^2_{H^2(\Omega_l)} ds. \qquad (5.2.4)
\end{aligned}
$$

For the L^2 norm error estimate, we consider the following interface problem: For fixed $t \in [0, T]$, find $z(t) \in X \cap H^1_0(\Omega)$ such that

$$
-\nabla \cdot (\beta(x) \nabla z(t)) = W_h v(t) - R_h v(t) \text{ in } \Omega
$$

along with interface conditions $[z] = 0 = [\partial z/\partial \mathbf{n}]$ along Γ. Then z satisfies the following a priori estimate

$$\|z(t)\|_{H^2(\Omega_1)} + \|z(t)\|_{H^2(\Omega_2)} \leq C\|W_h v(t) - R_h v(t)\|_{L^2(\Omega)}.$$

For $\phi \in H_0^1(\Omega)$, we obtain

$$\begin{aligned} -\int_\Omega \nabla \cdot (\beta(x)\nabla z(t))\phi dx &= -\int_{\partial\Omega} \beta(x)\nabla z(t) \cdot \mathbf{n}\phi ds + \int_\Omega \beta(x)\nabla z(t) \cdot \nabla\phi dx \\ &= \int_\Omega \beta(x)\nabla z(t) \cdot \nabla\phi dx = A(z(t), \phi). \end{aligned}$$

Thus, weak formulation may be defined as : Find $z(t) \in H_0^1(\Omega)$ such that

$$A(z(t), \phi) = (W_h v(t) - R_h v(t), \phi) \quad \forall\, \phi \in H_0^1(\Omega) \tag{5.2.5}$$

and finite element approximation $z_h(t) \in V_h$ satisfying

$$A_h(z_h(t), \phi_h) = (W_h v(t) - R_h v(t), \phi_h) \;\; \forall\, \phi_h \in V_h. \tag{5.2.6}$$

Next, apply Theorem 3.1 in [22] to have

$$\|z(t) - z_h(t)\|_{H^1(\Omega)} \leq Ch\|W_h v(t) - R_h v(t)\|_{L^2(\Omega)}.$$

Setting $\phi_h = W_h v(t) - R_h v(t)$ in (5.2.6), we have

$$\begin{aligned} \|W_h v(t) - R_h v(t)\|^2_{L^2(\Omega)} &= A_h(z_h(t), W_h v(t) - R_h v(t)) \\ &= \int_0^t \tilde{B}(t, s; (W_h v - v)(s), z_h(t)) ds \\ &\equiv: T_1 + T_2, \end{aligned} \tag{5.2.7}$$

with

$$\begin{aligned} T_1 &= \int_0^t \tilde{B}(t, s; (W_h v - v)(s), (z_h - z)(t)) ds, \\ T_2 &= \int_0^t \tilde{B}(t, s; (W_h v - v)(s), z(t)) ds. \end{aligned}$$

For the term T_1, we use (5.2.4) to have

$$\begin{aligned}
|T_1| &\le C\|z_h(t)-z(t)\|_{H^1(\Omega)}\int_0^t\sum_{l=1}^2\|W_hv(s)-v(s)\|_{H^1(\Omega_l)}ds \\
&\le Ch\|z(t)\|_X Ch\int_0^t\{\|v(s)\|_{H^2(\Omega_1)}+\|v(s)\|_{H^2(\Omega_2)}\}ds \\
&\le Ch^2\|W_hv(t)-R_hv(t)\|_{L^2(\Omega)}\int_0^t\{\|v(s)\|_{H^2(\Omega_1)}+\|v(s)\|_{H^2(\Omega_2)}\}ds \\
&\le C_\epsilon\|W_hv(t)-R_hv(t)\|^2_{L^2(\Omega)} \\
&\quad +C(\epsilon)h^4\int_0^t\{\|v(s)\|^2_{H^2(\Omega_1)}+\|v(s)\|^2_{H^2(\Omega_2)}\}ds. \qquad (5.2.8)
\end{aligned}$$

To estimate T_2, we need some preparation. For $\phi\in L^2(0,T;Y)$ with $[\phi]=0$ along Γ, we have

$$\begin{aligned}
&\int_{\Omega_1} b(x;t,s)\nabla\phi\nabla z dx+\int_{\Omega_2} b(x;t,s)\nabla\phi\nabla z dx \\
&=\int_\Gamma b(x;t,s)\frac{\partial z_1}{\partial \mathbf{n}}\phi ds-\int_\Gamma b(x;t,s)\frac{\partial z_2}{\partial \mathbf{n}}\phi ds \\
&\quad -\int_{\Omega_1}\nabla\cdot(b(x;t,s)\nabla z)\phi dx-\int_{\Omega_2}\nabla\cdot(b(x;t,s)\nabla z)\phi dx.
\end{aligned}$$

Using the fact $[\frac{\partial z}{\partial \mathbf{n}}\phi]=0$ along Γ, we obtain

$$\begin{aligned}
\tilde{B}(t,s;W_hv(t)-v(t),z(t)) &= -\sum_{l=1}^2\int_{\Omega_l}\nabla\cdot(b(x;t,s)\nabla z(t))(W_hv-v)(t)dx \\
&\quad +\sum_{l=1}^2\int_{\Omega_l} b_0(x;t,s)z(t)(W_hv-v)(t)dx,
\end{aligned}$$

so that

$$|\tilde{B}(t,s;W_hv(t)-v(t),z(t))|\le C\|W_hv(t)-v(t)\|_{L^2(\Omega)}\sum_{l=1}^2\|z(t)\|_{H^2(\Omega_l)}.$$

Hence

$$\begin{aligned}
|T_2| &\le C\|z(t)\|_X\int_0^t\|W_hv(s)-v(s)\|_{L^2(\Omega)}ds \\
&\le C_\epsilon\|z(t)\|^2_X+C(\epsilon)\int_0^t\|W_hv(s)-v(s)\|^2_{L^2(\Omega)}ds.
\end{aligned}$$

This together with Lemma 3.2.2 leads to

$$
\begin{aligned}
|T_2| &\leq C\|z(t)\|_X \int_0^t \|W_h v(s) - v(s)\|_{L^2(\Omega)} ds \\
&\leq C_\epsilon \|z(t)\|_X^2 + C(\epsilon) \int_0^t \|W_h v(s) - v(s)\|_{L^2(\Omega)}^2 ds \\
&\leq C_\epsilon \|W_h v(t) - R_h v(t)\|_{L^2(\Omega)}^2 + C(\epsilon) h^4 \int_0^t \{\|v(s)\|_{H^2(\Omega_1)}^2 + \|v(s)\|_{H^2(\Omega_2)}^2\} ds \\
&\quad + C(\epsilon) \int_0^t \|W_h v(s) - R_h v(s)\|_{L^2(\Omega)}^2 ds. \qquad (5.2.9)
\end{aligned}
$$

Combining (5.2.7)-(5.2.9) and setting suitable $\epsilon > 0$, we obtain

$$
\begin{aligned}
\|W_h v(t) - R_h v(t)\|_{L^2(\Omega)}^2 &\leq Ch^4 \int_0^t \{\|v(s)\|_{H^2(\Omega_1)}^2 + \|v(s)\|_{H^2(\Omega_2)}^2\} ds \\
&\quad + C \int_0^t \|W_h v(s) - R_h v(s)\|_{L^2(\Omega)}^2 ds.
\end{aligned}
$$

Finally, Grownwall's Lemma yields

$$
\|W_h v(t) - R_h v(t)\|_{L^2(\Omega)}^2 \leq Ch^4 \int_0^t \{\|v(s)\|_{H^2(\Omega_1)}^2 + \|v(s)\|_{H^2(\Omega_2)}^2\} ds.
$$

Hence, Lemma 3.2.2 leads to

$$
\begin{aligned}
\|W_h v(t) - v(t)\|_{L^2(\Omega)}^2 &\leq Ch^4 \{\|v(t)\|_{H^2(\Omega_1)}^2 + \|v(t)\|_{H^2(\Omega_2)}^2\} \\
&\quad + Ch^4 \int_0^t \{\|v(s)\|_{H^2(\Omega_1)}^2 + \|v(s)\|_{H^2(\Omega_2)}^2\} ds. \qquad (5.2.10)
\end{aligned}
$$

5.3 $L^2(L^2)$ and $L^2(H^1)$ norms Error Estimates

In this section, optimal order convergence results are obtained in $L^2(L^2)$ and $L^2(H^1)$ norms for semidiscrete finite element Galerkin method. Here, we have assumed $u_0 \in H_0^1(\Omega)$ and $f \in H^1(0, T; L^2(\Omega))$.

The continuous time Galerkin finite element approximation to (5.2.1) is stated as: Find $u_h : [0, T] \to V_h$ such that

$$
(u_{ht}, v_h) + A_h(u_h, v_h) = (f, v_h) + \int_0^t B(t, s; u_h(s), v_h) ds \quad \forall v_h \in V_h \qquad (5.3.1)
$$

with $u_h(0) = L_h u_0$. Subtracting (5.3.1) from (5.2.1), we have

$$
\begin{aligned}
(u_t - u_{ht}, v_h) + A(u - u_h, v_h) &= A_h(u_h, v_h) - A(u_h, v_h) \\
&+ \int_0^t B(t, s; (u - u_h)(s), v_h) ds \ \ \forall v_h \in V_h. \qquad (5.3.2)
\end{aligned}
$$

Define the error $e(t)$ as $e(t) = u(t) - u_h(t)$. Then following the lines of proof for Theorem 4.3.1 in Chapter 4, it is possible to obtain the following optimal error estimate in $L^2(H^1)$ norm. For $f \in H^1(0, T; L^2(\Omega))$ and $u_0 \in H_0^1(\Omega)$, there exists a constant C independent of h such that

$$
\begin{aligned}
\|e(s)\|_{L^2(0,t;H^1(\Omega))} &\leq Ch\Big(\|u_0\|^2_{H^1(\Omega)} + \int_0^t \|f(s)\|^2_{L^2(\Omega)} ds \\
&\qquad + \|u(t)\|^2_X + \int_0^t \|u(s)\|^2_X ds\Big)^{\frac{1}{2}} \\
&=: C(u_0, f, u)h. \qquad (5.3.3)
\end{aligned}
$$

Here, $C(u_0, f, u_t)$ is a positive constant, independent of h, such that

$$
C(u_0, f, u) = C\Big(\|u_0\|^2_{H^1(\Omega)} + \int_0^t \|f(s)\|^2_{L^2(\Omega)} ds + \|u(t)\|^2_X + \int_0^t \|u(s)\|^2_X ds\Big)^{\frac{1}{2}}
$$

for some positive constant C.

The memory term considered in Chapter 4 involve only a first order partial differential equation and hence Theorem 4.3.2, therein, can not be easily extended for the equation (5.1.1) containing second order equation as memory. For the L^2 norm error estimate, we again recall the duality trick: For fixed $t \in [0, T]$, find $w(t) \in H_0^1(\Omega)$ such that

$$
A(w(t), v) = (u(t) - u_h(t), v) \ \ \forall v \in H_0^1(\Omega) \qquad (5.3.4)
$$

and its finite element approximation is defined to be the function $w_h(t) \in V_h$ such that

$$
A_h(w_h(t). v_h) = (u(t) - u_h(t), v_h) \ \ \forall v_h \in V_h. \qquad (5.3.5)
$$

Note that solution $w(t)$ to the problem (5.3.4) belongs to $X \cap H_0^1(\Omega)$ and satisfies the jump conditions $[w] = 0$, $\left[\beta(x)\frac{\partial w}{\partial n}\right] = 0$ along Γ. Further, w satisfies the *a priori* estimate

$$
\|w(t)\|_X \leq C\|u(t) - u_h(t)\|_{L^2(\Omega)}. \qquad (5.3.6)
$$

Regarding the convergence of w_h, we have (see, Theorem 3.1 in [22])

$$\|w(t) - w_h(t)\|_{H^1(\Omega)} \le Ch\|u(t) - u_h(t)\|_{L^2(\Omega)}. \tag{5.3.7}$$

Then it follows from [23] that

$$\begin{aligned} \|e(t)\|^2_{L^2(\Omega)} &\le C\{h\|e(t)\|_{L^2(\Omega)}\|e(t)\|_{H^1(\Omega)}\} + C\{h\|e(t)\|_{H^1(\Omega)}\|e(t)\|_{L^2(\Omega)} \\ &\quad + h^2\|u(t)\|_X\|e(t)\|_{L^2(\Omega)}\} - \frac{1}{2}\frac{d}{dt}A_h(w_h(t), w_h(t)) + (J), \end{aligned} \tag{5.3.8}$$

with $(J) = \int_0^t B(t, s; e(s), w_h(t))ds$.

Term (J) can be rewritten as

$$\begin{aligned} (J) &= \int_0^t B(t, s; e(s), w_h(t))ds \\ &= \int_0^t B(t, s; e(s), (w_h - w)(t))ds + \int_0^t B(t, s; e(s), w(t))ds \\ &\equiv: (J)_1 + (J)_2, \end{aligned}$$

where $(J)_i$, $i = 1, 2$, are defined as

$$(J)_1 = \int_0^t B(t, s; e(s), (w_h - w)(t))ds, \quad (J)_2 = \int_0^t B(t, s; e(s), w(t))ds.$$

For the term $(J)_1$, apply (5.3.3) and (5.3.7) to have

$$\begin{aligned} |(J)_1| &\le C\|e(s)\|_{L^2(0,t;H^1(\Omega))}\|w_h(t) - w(t)\|_{H^1(\Omega)} \\ &\le C(u_0, f, u)h^2\|e(t)\|_{L^2(\Omega)}. \end{aligned} \tag{5.3.9}$$

Before estimating $(J)_2$. we need some preparation. For fixed $t \in [0, T]$, we define

$$f^*(s) = -\nabla \cdot (b(x; t, s)\nabla w_k(t)) + b_0(x; t, s)w_k(t), \quad (x, s) \in \Omega_k \times (0, t), \ k = 1, 2.$$

Clearly $f^*(s) \in L^2(\Omega)$ and assumptions (5.1.4) leads to

$$\|f^*(s)\|_{L^2(\Omega)} \le C\{\|w(t)\|_{H^2(\Omega_1)} + \|w(t)\|_{H^2(\Omega_2)}\} \ \forall\, s, \quad s < t.$$

Further,

$$\begin{aligned} (f^*(s), e(s)) &= -\int_\Omega \nabla \cdot (b(x; t. s)\nabla w(t))e(s)dx + \int_\Omega b_0(x; t, s)w(t)e(s)dx \\ &= \int_\Omega b(x; t, s)\nabla w(t) \cdot \nabla e(s)dx + \int_\Omega b_0(x; t, s)w(t)e(s)dx \\ &= B(t, s; e(s), w(t)) \end{aligned}$$

and hence

$$\begin{aligned}(J)_2 &= \int_0^t B(t,s;e(s),w(t))ds = \int_0^t (f^*(s),e(s))ds \\ &\le C\int_0^t \|f^*(s)\|_{L^2(\Omega)}\|e(s)\|_{L^2(\Omega)}ds \\ &\le C\{\|w(t)\|_{H^2(\Omega_1)} + \|w(t)\|_{H^2(\Omega_2)}\}\int_0^t \|e(s)\|_{L^2(\Omega)}ds \\ &\le C\|e(t)\|_{L^2(\Omega)}\|e(s)\|_{L^2(0,t;L^2(\Omega))}. \end{aligned} \tag{5.3.10}$$

Combining the estimates (5.3.8)-(5.3.10), we obtain

$$\begin{aligned}\|e(t)\|^2_{L^2(\Omega)} &\le C\{h\|e(t)\|_{L^2(\Omega)}\|e(t)\|_{H^1(\Omega)}\} + C\{h\|e(t)\|_{H^1(\Omega)}\|e(t)\|_{L^2(\Omega)} \\ &\quad +h^2\|u(t)\|_X\|e(t)\|_{L^2(\Omega)}\} - \frac{1}{2}\frac{d}{dt}A_h(w_h(t),w_h(t)) \\ &\quad +C(u_0,f,u)h^2\|e(t)\|_{L^2(\Omega)} \\ &\quad +C\|e(t)\|_{L^2(\Omega)}\|e(s)\|_{L^2(0,t;L^2(\Omega))}.\end{aligned}$$

Further, a simple application of Young's inequality leads to

$$\begin{aligned}\|e(t)\|^2_{L^2(\Omega)} &\le C_\epsilon h^2\|e(t)\|^2_{H^1(\Omega)} + C_\epsilon h^4\|u(t)\|^2_X + C_\epsilon\|e(s)\|^2_{L^2(0,t;L^2(\Omega))} \\ &\quad +C_\epsilon(u_0,f,u)h^4 + C(\epsilon)\|e(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\frac{d}{dt}A_h(w_h,w_h).\end{aligned} \tag{5.3.11}$$

Therefore, for suitable $\epsilon > 0$ and integrating from 0 to t, we have

$$\begin{aligned}\int_0^t \|e(s)\|^2_{L^2(\Omega)}ds &\le Ch^2\int_0^t \|e(s)\|^2_{H^1(\Omega)}ds + Ch^4\int_0^t \|u(s)\|^2_X ds \\ &\quad + h^4\int_0^t C_\epsilon(u_0,f,u)ds \\ &\quad + C\int_0^t\int_0^\tau \|e(s)\|^2_{L^2(\Omega)}dst\tau + \frac{1}{2}A_h(w_h(0),w_h(0)).\end{aligned} \tag{5.3.12}$$

Taking $t \to 0$, it now follows from (5.3.5) that

$$A_h(w_h(0),w_h(0)) = (u_0 - L_h u_0, w_h(0)) = 0.$$

This together with (5.3.12), Gronwall's inequality and (5.3.3) leads to the following optimal $L^2(L^2)$ norm error estimate.

Theorem 5.3.1 *Let u and u_h be the solutions of the problem (5.2.1)and (5.3.1), respectively. Then, for $f \in H^1(0,T;L^2(\Omega))$ and $u_0 \in H_0^1(\Omega)$, there exists a constant $\tilde{C}$ independent of h such that*

$$\|e(s)\|_{L^2(0,t;L^2(\Omega))} \le \tilde{C}(u_0,f,u)h^2. \quad \square$$

5.4 $L^\infty(L^2)$ and $L^\infty(H^1)$ norms Error Estimates

In this section, optimal order convergence results are obtained in $L^\infty(L^2)$ and $L^\infty(H^1)$ norms. We have assumed that initial data $u_0 \in H_0^1(\Omega) \cap H^3(\Omega)$ and $u_h(0) = W_h u_0$. For the simplicity of the exposition, we have used symbol $C(u, u_t)$, depends on u and u_t, to denote a positive term such that

$$\|u(t)\|_X^2 + \|u\|_{L^2(0,t,X)}^2 + \int_0^t \sum_{i=1}^2 \|u_t(s)\|_{H^2(\Omega_i)}^2 ds \le C(u, u_t).$$

Setting $u(t) - u_h(t) = u(t) - W_h u(t) + W_h u(t) - u_h(t) = \rho(t) + \theta(t)$, we obtain

$$\begin{aligned}(\theta_t, v_h) + A(\theta, v_h) &= -(\rho_t, v_h) + \int_0^t B(t, s; \rho(s), v_h) ds + \int_0^t B(t, s; \theta(s), v_h) ds \\ &\quad + \{A_h(u_h - W_h u, v_h) - A(u_h - W_h u, v_h)\} \\ &\quad + A_h(W_h u - R_h u, v_h). \end{aligned} \tag{5.4.1}$$

It follows from the definitions of R_h and W_h operators that

$$A_h(W_h u(t) - R_h u(t), v_h) = \int_0^t B(t, s; (W_h u - u)(s), v_h) ds.$$

This together with (5.4.1), we obtain the following error equation in θ

$$(\theta_t, v_h) + A_h(\theta, v_h) = -(\rho_t, v_h) + \int_0^t B(t, s; \theta(s), v_h) ds. \tag{5.4.2}$$

Set $v_h = \theta_t$ in (5.4.2) to have

$$\begin{aligned}(\theta_t, \theta_t) + \frac{1}{2}\frac{d}{dt} A_h(\theta, \theta) &\le C_\epsilon \|\rho_t\|_{L^2(\Omega)}^2 + C(\epsilon)\|\theta_t\|_{L^2(\Omega)}^2 \\ &\quad + Ch^{-1}\|\theta_t\|_{L^2(\Omega)} \int_0^t \|\theta(s)\|_{H^1(\Omega)} ds \\ &\le C_\epsilon \|\rho_t\|_{L^2(\Omega)}^2 + C(\epsilon)\|\theta_t\|_{L^2(\Omega)}^2 \\ &\quad + C_\epsilon h^{-2} \int_0^t \|\theta(s)\|_{H^1(\Omega)}^2 ds + C(\epsilon)\|\theta_t\|_{L^2(\Omega)}^2. \end{aligned} \tag{5.4.3}$$

Here, we have used Young's inequality and inverse estimate (2.2.12). Thus, for suitable $\epsilon > 0$, we get

$$\|\theta_t\|_{L^2(\Omega)}^2 + \frac{1}{2}\frac{d}{dt} A_h(\theta, \theta) \le C\|\rho_t\|_{L^2(\Omega)}^2 + Ch^{-2} \int_0^t \|\theta(s)\|_{H^1(\Omega)}^2 ds. \tag{5.4.4}$$

Then integrating (5.4.4) from 0 to t and applying estimate (5.2.10), we obtain

$$\begin{aligned}\int_0^t \|\theta_t\|^2_{L^2(\Omega)}ds + \|\theta\|^2_{H^1(\Omega)} &\leq C\int_0^t \|\rho_t\|^2_{L^2(\Omega)}ds + Ch^{-2}\int_0^t\int_0^\tau \|\theta(s)\|^2_{H^1(\Omega)}dsd\tau \\ &\leq C(u,u_t)h^4 + Ch^{-2}\int_0^t (t-s)\|\theta(s)\|^2_{H^1(\Omega)}ds \\ &\leq C(u,u_t)h^4 + Ch^{-2}\int_0^t \|\theta(s)\|^2_{H^1(\Omega)}ds. \qquad (5.4.5)\end{aligned}$$

Then a simple application of Grownwall's Lemma leads to

$$\|\theta\|^2_{H^1(\Omega)} \leq G(t)h^4 + Ch^4\int_0^t G(s)H(s)e^{-Ch^2(t-s)}ds,$$

with $G(t) = C(u, u_t)$ and $H(s) = Ch^{-2}$. Further using the fact that $e^{-x} \leq 1,\ x > 0$, we obtain

$$\|\theta\|^2_{H^1(\Omega)} \leq C(u,u_t)h^4 + Ch^2\int_0^t C(u,u_s)ds. \qquad (5.4.6)$$

Now, combining (5.2.4) and (5.4.6), we obtain the following optimal H^1-norm error estimate.

Theorem 5.4.1 *Let u and u_h be the solutions of the problem (5.2.1)and (5.3.1), respectively. Then, for $u_0 \in H^1_0(\Omega)\cap H^3(\Omega)$ and $f \in H^1(0,T;H^1(\Omega))$, we have*

$$\|e(t)\|_{H^1(\Omega)} \leq Ch\Big(C(u,u_t) + \int_0^t C(u,u_s)ds\Big)^{\frac{1}{2}}. \quad \square$$

Next, set $v_h = \theta(t)$ in (5.4.2) to have

$$\begin{aligned}(\theta_t,\theta) + A_h(\theta,\theta) &\leq C_\epsilon\|\rho_t\|^2_{L^2(\Omega)} + C(\epsilon)\|\theta\|^2_{L^2(\Omega)} \\ &\quad + C_\epsilon\int_0^t \|\theta\|^2_{H^1(\Omega)}ds + C(\epsilon)\|\theta\|^2_{H^1(\Omega)}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(\Omega)} + C\|\theta\|^2_{H^1(\Omega)} &\leq C_\epsilon\|\rho_t\|^2_{L^2(\Omega)} + C(\epsilon)\|\theta\|^2_{L^2(\Omega)} \\ &\quad + C_\epsilon\int_0^t \|\theta\|^2_{H^1(\Omega)}ds + C(\epsilon)\|\theta\|^2_{H^1(\Omega)}. \qquad (5.4.7)\end{aligned}$$

Then integrating (5.4.7) from 0 to t, we obtain

$$\begin{aligned}\frac{1}{2}\|\theta\|^2_{L^2(\Omega)} + C\int_0^t \|\theta\|^2_{H^1(\Omega)}ds &\leq C_\epsilon \int_0^t \|\rho_s\|^2_{L^2(\Omega)}ds + C(\epsilon)\int_0^t \|\theta\|^2_{L^2(\Omega)}ds \\ &\quad + C_\epsilon \int_0^t\int_0^\tau \|\theta\|^2_{H^1(\Omega)}dsd\tau \\ &\quad + C(\epsilon)\int_0^t \|\theta\|^2_{H^1(\Omega)}ds. \qquad (5.4.8)\end{aligned}$$

Hence, for suitable $\epsilon > 0$, we have

$$\begin{aligned}\|\theta\|^2_{L^2(\Omega)} + \int_0^t \|\theta\|^2_{H^1(\Omega)}ds &\leq C\int_0^t \|\rho_s\|^2_{L^2(\Omega)}ds + C\int_0^t\int_0^\tau \|\theta\|^2_{H^1(\Omega)}dsd\tau \\ &\leq C(u, u_t)h^4 + C\int_0^t\int_0^\tau \|\theta\|^2_{H^1(\Omega)}dsd\tau. \qquad (5.4.9)\end{aligned}$$

Here we have used estimate (5.2.10). Splitting (5.4.9) into two parts, we obtain

$$\|\theta\|^2_{L^2(\Omega)} \leq C(u, u_t)h^4 + C\int_0^t\int_0^\tau \|\theta\|^2_{H^1(\Omega)}dsd\tau, \qquad (5.4.10)$$

$$\int_0^t \|\theta\|^2_{H^1(\Omega)}ds \leq C(u, u_t)h^4 + C\int_0^t\int_0^\tau \|\theta\|^2_{H^1(\Omega)}dsd\tau. \qquad (5.4.11)$$

For the term $\int_0^t \|\theta\|^2_{H^1(\Omega)}ds$, we use Grownwall's Lemma in (5.4.11) to have

$$\int_0^t \|\theta\|^2_{H^1(\Omega)}ds \leq Ch^4\Big(C(u, u_t) + \int_0^t C(u, u_s)ds\Big).$$

This together with (5.4.10) leads to

$$\|\theta(t)\|^2_{L^2(\Omega)}ds \leq Ch^4\Big(C(u, u_t) + \int_0^t C(u, u_s)ds\Big). \qquad (5.4.12)$$

Finally, approximation result (5.2.10) together with (5.4.12) yields the following optimal L^2-norm error estimate.

Theorem 5.4.2 *Let u and u_h be the solutions of the problem (5.2.1)and (5.3.1), respectively. Then, for $u_0 \in H^1_0(\Omega) \cap H^3(\Omega)$ and $f \in H^1(0, T; H^1(\Omega))$, we have*

$$\|e(t)\|_{L^2(\Omega)} \leq Ch^2\Big(C(u.u_t) + \int_0^t C(u, u_s)ds\Big)^{\frac{1}{2}}. \quad \square$$

5.5 Discrete time Galerkin Method

In this section, we shall consider the completely discrete scheme for the problem (5.3.1). Backward difference scheme has been used to discretize the problem in time direction and the piecewise linear finite element method in space. Optimal error estimate is shown in L^2 norm for sufficiently smooth initial data. For the simplicity, we have assumed that $f = 0$ in Ω.

We first divide the interval $[0, T]$ into N equally spaced subintervals by the following points

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

with $t_n = nk$, $k = T/N$ be the time step. Let $I_n = (t_{n-1}, t_n]$ be the n-th sub interval. For a given sequence $\{\phi^n\}_{n=1}^N \subset L^2(\Omega)$, we introduce the backward difference quotient

$$\Delta_k \phi^n = \frac{\phi^n - \phi^{n-1}}{k}.$$

For $\phi(t) \in V_h$, we denote ϕ^n be the value of ϕ at $t = t_n$.

The complete discrete finite element approximation to the problem (5.3.1) is defined as follows: For $1 \leq n \leq N$, find $U^n \in V_h$ such that

$$(\Delta_k U^n, v_h) + A_h(U^n, v_h) = k \sum_{j=0}^{n-1} B(t_n, t_j; U^j, v_h) \quad \forall v_h \in V_h \tag{5.5.1}$$

with $U^0 = W_h u_0$.

Integral term in (5.3.1) has been approximated by the rectangle rule

$$\int_0^{t_n} \phi(s)ds \approx k \sum_{j=0}^{n-1} \phi^j = Q_1^n \phi, \; 0 < t_n \leq T.$$

Note that the quadrature error in $I_n = (t_{n-1}, t_n]$ is estimated as

$$\int_{I_n} \phi(s)ds - k\phi^{n-1} = \int_{I_n} \int_{t_{n-1}}^{s} \phi'(\tau)d\tau ds = \int_{I_n} (t_n - \tau)\phi'(\tau)d\tau$$

and hence

$$|Q_1^n \phi - \int_0^{t_n} \phi(s)ds| \leq k \int_0^{t_n} |\phi'(\tau)|d\tau. \tag{5.5.2}$$

At $t = t_n$, (5.2.1) reduces to

$$(u_t^n, v_h) + A(u^n, v_h) = \int_0^{t_n} B(t_n, s; u(s), v_h)ds \;\; \forall v \in H_0^1(\Omega). \tag{5.5.3}$$

We write the error $U^n - u^n$ at time t_n as

$$U^n - u^n = (U^n - W_h u^n) + (W_h u^n - u^n) \equiv: \theta^n + \rho^n$$

where $\theta^n = U^n - W_h u^n$ and $\rho^n = W_h u^n - u^n$.

Combining (5.5.1) and (5.5 3), we obtain

$$\begin{aligned}
&(\Delta_k \theta^n, v_h) + A_h(\theta^n, v_h) \\
&= k \sum_{j=0}^{n-1} B(t_n, t_j; \theta^j, v_h) - (w^n, v_h) + k \sum_{j=0}^{n-1} B(t_n, t_j; W_h u^j, v_h) \\
&\quad - \int_0^{t_n} B(t_n, s; W_h u(s), v_h) ds. \qquad (5.5.4)
\end{aligned}$$

Here, $w^n = \Delta_k W_h u^n - u_t^n$. For simplicity of the exposition, we write $w^n = w_1^n + w_2^n$, where $w_1^n = W_h \Delta_k u^n - \Delta_k u^n$ and $w_2^n = \Delta_k u^n - u_t^n$.

Now, setting $v_h = \theta^n$ in (5.5.4), we have

$$\begin{aligned}
\frac{1}{2}\|\theta^n\|^2_{L^2(\Omega)} + k\|\theta^n\|^2_{H^1(\Omega)} \leq\ & k^2 \sum_{j=0}^{n-1} B(t_n, t_j; \theta^j, \theta^n) + C(\epsilon) k \|w^n\|^2_{L^2(\Omega)} \\
& + C_\epsilon k \|\theta^n\|^2_{L^2(\Omega)} + k \Big[k \sum_{j=0}^{n-1} B(t_n, t_j; W_h u^j, \theta^n) \\
& - \int_0^{t_n} B(t_n, s; W_h u(s), \theta^n) ds \Big] \\
& + \frac{1}{2}\|\theta^{n-1}\|^2_{L^2(\Omega)}. \qquad (5.5.5)
\end{aligned}$$

Thus, for suitable $\epsilon > 0$ and summing (5.5.5) over n from $n = 1$ to $n = M$, we have

$$\begin{aligned}
\|\theta^M\|^2_{L^2(\Omega)} + k \sum_{n=1}^{M} \|\theta^n\|^2_{H^1(\Omega)} \leq\ & k^2 \sum_{n=1}^{M} \sum_{j=0}^{n-1} B(t_n, t_j; \theta^j, \theta^n) + Ck \sum_{n=1}^{M} \|w^n\|^2_{L^2(\Omega)} \\
& + k \sum_{n=1}^{M} \Big[k \sum_{j=0}^{n-1} B(t_n, t_j; W_h u^j, \theta^n) \\
& - \int_0^{t_n} B(t_n, s; W_h u(s), \theta^n) ds \Big] \\
\equiv:\ & k^2 \sum_{n=1}^{M} \sum_{j=0}^{n-1} B(t_n, t_j; \theta^j, \theta^n) + (I)_1 + (I)_2. \qquad (5.5.6)
\end{aligned}$$

Terms $(I)_1$ and $(I)_2$ are given by

$$(I)_1 = Ck\sum_{n=1}^{M}\|w^n\|^2_{L^2(\Omega)} = Ck\sum_{n=1}^{M}\|w_1^n + w_2^n\|^2_{L^2(\Omega)} \ \&$$

$$(I)_2 = k\sum_{n=1}^{M}\Big[k\sum_{j=0}^{n-1} B(t_n, t_j; W_h u^j, \theta^n) - \int_0^{t_n} B(t_n, s; W_h u(s), \theta^n)ds\Big].$$

Now, we proceed to estimate both the terms separately. In Ω_1, the term w_1^n can be expressed as

$$\begin{aligned} w_1^n &= W_h\Delta_k u_1^n - \Delta_k u_1^n = (W_h - I)(\Delta_k u_1^n) \\ &= (W_h - I)\frac{1}{k}\int_{t_{n-1}}^{t^n} u_{1,t}dt = \frac{1}{k}\int_{t_{n-1}}^{t^n}(W_h u_{1,t} - u_{1,t})dt, \end{aligned}$$

where u_i, $i = 1, 2$, is the restriction of u in Ω_i and $u_{i,t} = \frac{\partial u_i}{\partial t}$.

An application of estimate (5.2.10) leads to

$$\begin{aligned} k\|w_1^n\|_{L^2(\Omega_1)} &\leq Ch^2\int_{t_{n-1}}^{t^n}\sum_{i=1}^{2}\|u_t\|_{H^2(\Omega_i)}dt \\ &\leq Ch^2k^{\frac{1}{2}}\left(\int_{t_{n-1}}^{t^n}\left(\sum_{i=1}^{2}\|u_t\|_{H^2(\Omega_i)}\right)^2 dt\right)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$k\|w_1^n\|^2_{L^2(\Omega_1)} \leq Ch^4\int_{t_{n-1}}^{t^n}\sum_{i=1}^{2}\|u_t\|^2_{H^2(\Omega_i)}dt.$$

Similarly, we obtain

$$k\|w_1^n\|^2_{L^2(\Omega_2)} \leq Ch^4\int_{t_{n-1}}^{t^n}\sum_{i=1}^{2}\|u_t\|^2_{H^2(\Omega_i)}dt.$$

Using above two estimates, we have

$$k\sum_{n=1}^{M}\|w_1^n\|^2_{L^2(\Omega)} \leq Ch^4\int_0^{t_n}\sum_{i=1}^{2}\|u_t\|^2_{H^2(\Omega_i)}dt. \tag{5.5.7}$$

For the term w_2^n, we have

$$kw_2^n = u^n - u^{n-1} - ku_t^n = -\int_{t_{n-1}}^{t_n}(s - t_{n-1})u_{tt}ds$$

and hence

$$k\|w_2^n\|_{L^2(\Omega_i)} \le k\int_{t_{n-1}}^{t_n} \|u_{tt}\|_{L^2(\Omega_i)}ds \le kk^{\frac{1}{2}}\left(\int_{t_{n-1}}^{t_n} \|u_{tt}\|^2_{L^2(\Omega_i)}dt\right)^{\frac{1}{2}}.$$

Summing over n from $n = 1$ to $n = M$, we obtain

$$k\sum_{n=1}^{M} \|w_2^n\|^2_{L^2(\Omega)} \le Ck^2\int_0^{t_n} \|u_{tt}\|^2_{L^2(\Omega)}dt. \tag{5.5.8}$$

In view of estimates (5.5.7)-(5.5.8), the following estimate holds for $(I)_1$

$$(I)_1 \le Ch^4\int_0^{t_n}\sum_{i=1}^{2} \|u_t\|^2_{H^2(\Omega_i)}dt + Ck^2\int_0^{t_n} \|u_{tt}\|^2_{L^2(\Omega)}dt. \tag{5.5.9}$$

Next, we write $\phi(s) = B(t_n, s; W_hu(s), \theta^n)$ so that estimate (5.5.2) leads to

$$\begin{aligned}
(I)_2 &= k\sum_{n=1}^{M}\left(k\sum_{j=0}^{n-1}\phi(t_j) - \int_0^{t_n}\phi(s)ds\right) \\
&\le k\sum_{n=1}^{M}\left(k\int_0^{t_n}\left|\frac{\partial\phi(s)}{\partial s}\right|ds\right).
\end{aligned} \tag{5.5.10}$$

Then apply assumptions (5.1.4) to have

$$\left|\frac{\partial\phi(s)}{\partial s}\right| \le C\{\|W_hu(s)\|_{H^1(\Omega)} + \|W_hu_s(s)\|_{H^1(\Omega)}\}\|\theta^n\|_{H^1(\Omega)}.$$

This together with (5.5.10) yields

$$\begin{aligned}
(I)_2 &\le Ck^2\sum_{n=1}^{M}\int_0^{t_n}\{\|W_hu(s)\|_{H^1(\Omega)} + \|W_hu_s(s)\|_{H^1(\Omega)}\}\|\theta^n\|_{H^1(\Omega)}ds \\
&\le C(\epsilon)k^2\sum_{n=1}^{M}\int_0^{t_n}\{\|W_hu(s)\|^2_{H^1(\Omega)} + \|W_hu_s(s)\|^2_{H^1(\Omega)}\}ds \\
&\quad +C_\epsilon k^2\sum_{n=1}^{M}\|\theta^n\|^2_{H^1(\Omega)}.
\end{aligned} \tag{5.5.11}$$

Finally, use estimates (5.5.9) and (5.5.11) in (5.5.6) to have

$$\begin{aligned}
&\|\theta^M\|^2_{L^2(\Omega)} + k\sum_{n=1}^{M}\|\theta^n\|^2_{H^1(\Omega)} \\
&\le Ch^4\int_0^{t_N}\sum_{i=1}^{2}\|u_s\|^2_{H^2(\Omega_i)}ds + Ck^2\int_0^{t_N}\|u_{ss}\|^2_{L^2(\Omega)}ds \\
&\quad + Ck^2\sum_{n=1}^{M}\int_0^{t_N}\{\|W_hu(s)\|^2_{H^1(\Omega)} + \|W_hu_s(s)\|^2_{H^1(\Omega)}\}ds \\
&\quad + k^2\sum_{n=1}^{M}\sum_{j=0}^{n-1}B(t_n,t_j;\theta^j,\theta^n) \\
&\le \tilde{C}_N(h^4+k^2) + k^2\sum_{n=1}^{M}\sum_{j=0}^{n-1}B(t_n,t_j;\theta^j,\theta^n). \qquad (5.5.12)
\end{aligned}$$

Here, $\tilde{C}_N > 0$ is a constant independent of M such that

$$C\{\|u_{tt}\|^2_{L^2(0,T,L^2(\Omega))} + \|u\|^2_{L^2(0,T,X)} + \sum_{i=1}^{2}\|u_t\|^2_{L^2(0,T,H^2(\Omega_i))}\} \le \tilde{C}_N$$

Then it follows from [14] (see, Lemma 7 therein) that

$$\begin{aligned}
\|\theta^M\|^2_{L^2(\Omega)} + k\sum_{n=1}^{M}\|\theta^n\|^2_{H^1(\Omega)} &\le \tilde{C}_N(h^4+k^2) + C(\epsilon)k^2\sum_{n=1}^{M}\|\theta^n\|^2_{H^1(\Omega)} \\
&\quad + C_\epsilon k^2\sum_{n=1}^{M-1}\sum_{j=0}^{n-1}\|\theta^j\|^2_{H^1(\Omega)}
\end{aligned}$$

and hence

$$\|\theta^M\|^2_{L^2(\Omega)} + k\sum_{n=1}^{M}\|\theta^n\|^2_{H^1(\Omega)} \le \tilde{C}_N(h^4+k^2) + Ck^2\sum_{n=1}^{M-1}\sum_{j=0}^{n-1}\|\theta^j\|^2_{H^1(\Omega)}. \qquad (5.5.13)$$

Setting $\xi_l = \sum_{n=1}^{l} k\|\theta^n\|^2_{H^1(\Omega)}$ in (5.5.13), we obtain

$$\xi_M \le \tilde{C}_N(h^4+k^2) + Ck\sum_{n=1}^{M-1}\xi_n.$$

Then a simple application of discrete Grownwall's lemma leads to

$$k\sum_{n=1}^{M}\|\theta^n\|^2_{H^1(\Omega)} = \xi_M \le \tilde{C}_N(h^4+k^2)\sum_{n=1}^{M-1}Ck \le \tilde{C}_N(h^4+k^2)kN. \qquad (5.5.14)$$

In combination (5.2.10) leads to the following optimal L^2 norm error estimate.

Theorem 5.5.1 *Assume that* $u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$. *Then there exist a positive constant* C_N, *independent of* h *and* k, *such that*

$$\|U^M - u(t_M)\|_{L^2(\Omega)} \leq C_N(h^2 + k), \quad 1 \leq M \leq N. \quad \square$$

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