ARITHMETIC IDENTITIES OF THE COEFFICIENTS OF SOME THETA FUNCTIONS AND COLORED PARTITION IDENTITIES

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Dedicated to my family

(Maa and Baa)

Abstract

First of all, we find arithmetic identities for the number of representations of a positive integer as a sum of three squares and a sum of three triangular numbers by employing two beautiful theta function identities of Ramanujan. We also use several Lambert series identities to establish infinite families of congruences modulo 3 and modulo 7 of the coefficients of some theta functions.

Recently, Sandon and Zanello conjectured 29 highly non-trivial colored partition identities. In this thesis, we establish 17 of them, find analogous colored partition identities of the remaining 12 and also find some new colored partition identities of the same type by using the theory of Ramanujan's theta functions and modular equations. Finally, we interpret several modular equations of Ramanujan involving multipliers in terms of partitions, overpartitions, overpartition pairs and regular partitions.

DECLARATION BY THE CANDIDATE

I, Bidyut Boruah, hereby declare that the subject matter in this thesis entitled, **"Arithmetic Identities of the Coefficients of Some Theta Functions and Colored Partition Identities"**, is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/institute.

This thesis is being submitted to the Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

Place: Tezpur.

Date: 12.05.2014

Signature of the candidate

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CERTIFICATE OF THE SUPERVISOR

This is to certify that the thesis entitled **"Arithmetic Identities of the Coefficients of Some Theta Functions and Colored Partition Identities"** submitted to the School of Sciences of Tezpur University in partial fulfilment for the award of the degree of Doctor of Philosophy in Mathematical Sciences is a record of research work carried out by **Mr. Bidyut Boruah** under my supervision and guidance.

All help received by him from various sources have been duly acknowledged.

No part of this thesis have been submitted elsewhere for award of any other degree.

Place: Tezpur
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Chapter 1

Introduction

The thesis consists of five chapters including this introductory chapter. We briefly introduce the basic concepts and terminology in the following few subsections.

1.1 Ramanujan's theta functions and modular equations

Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{1.1.1}$$

Jacobi's famous triple product identity [13, p. 35, Entry 19] takes the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{1.1.2}$$

where, as customary, we use the standard notation for q-products

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n \geq 1,$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \quad |q| < 1.$$

It can be easily shown that [13, p. 34, Entry 18] $f(a,b)$ satisfies

$$f(a,b) = f(b,a),$$

$$f(1,a) = 2f(a,a^3),$$

$$f(-1,a) = 0,$$

and, if n is an integer,

$$f(a,b) = a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n, b(ab)^{-n}).$$

The three special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{1.1.3}$$

$$\psi(q) := f(q,q^3) = \frac{1}{2}f(1,q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.1.4}$$

$$f(-q) := f(-q,-q^2) = \sum_{k=0}^{\infty}(-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)/2} = (q;q)_\infty, \tag{1.1.5}$$

where here and throughout the sequel, we assume that $|q| < 1$. The product representations in (1.1.3)–(1.1.5) arise from (1.1.2) and the last equality in (1.1.5) is Euler's famous pentagonal number theorem.

Next, we give the definition of a modular equation as understood by Ramanujan. First of all, the complete elliptic integral of the first kind $K(k)$ is defined by

$$\begin{aligned} K(k) :&= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^2}{(n!)^2}k^{2n} \\ &= \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \quad (0 < k < 1), \end{aligned} \tag{1.1.6}$$

where the series representation is found by expanding the integrand in a binomial series and integrating termwise and ${}_2F_1$ is one of the hypergeometric functions ${}_pF_{p-1}$, $p \geq 1$, which are defined by

$${}_pF_{p-1}(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_{p-1}; x) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_{p-1})_n}\frac{x^n}{n!}, \quad |x| < 1,$$

where b_i's are not nonpositive integers. The number k is called the modulus of $K := K(k)$, and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', ℓ and ℓ', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.1.7}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and ℓ which is induced by (1.1.7). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = \ell^2$. It is then said that β has degree n over α. For example, we recall from [13, Entry 5(ii), p. 230] that if β has degree 3 over α, then

$$(\alpha\beta)^{1/4} + ((1-\alpha)(1-\beta))^{1/4} = 1.$$

The corresponding multiplier m is defined by

$$m = \frac{K}{L}.$$

If $q = \exp(-\pi K'/K)$, then one of the fundamental results in the theory of elliptic functions [13, Entry 6, p. 101] is given by

$$\varphi^2(q) = \frac{2}{\pi}K(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

where $\varphi(q)$ is as defined in (1.1.3).

The above identity enables one to derive formulas for φ, ψ and f at different arguments in terms of α, q, and $z := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)$. In particular, Ramanujan recorded the following identities.

Lemma 1.1.1. [13, pp. 122–124, Entries 10, 11, 12] *If*

$$q = \exp\left(-\pi\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha)}\right)$$

and z is as defined above, then

$$\begin{aligned}
\varphi(q) &= \sqrt{z}, \\
\varphi(-q) &= \sqrt{z}(1-\alpha)^{1/4}, \\
\varphi(-q^2) &= \sqrt{z}(1-\alpha)^{1/8}, \\
\psi(q) &= \sqrt{\frac{z}{2}}\left(\frac{\alpha}{q}\right)^{1/8} \\
\psi(-q) &= \sqrt{\frac{z}{2}}\left(\frac{\alpha(1-\alpha)}{q}\right)^{1/8} \\
\psi(q^2) &= \frac{1}{2}\sqrt{z}\left(\frac{\alpha}{q}\right)^{1/4} \\
f(q) &= \sqrt{z}\left(\frac{\alpha(1-\alpha)}{16q}\right)^{1/24}
\end{aligned}$$

$$f(-q) = \sqrt{z}(1-\alpha)^{1/24}\left(\frac{\alpha}{16q}\right)^{1/24}$$

and

$$f(-q^2) = \sqrt{z}\left(\frac{\alpha(1-\alpha)}{16q}\right)^{1/12}.$$

Suppose that β has degree n over α. If we replace q by q^n above, then the same evaluations hold with α replaced by β and z replaced by $z_n := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)$.

Ramanujan also studied "mixed" modular equations, in which four distinct moduli appear. We now define such a modular equation. Let K, K', L_1, L_1', L_2, L_2', L_3, and L_3' denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, and $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2, and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1\frac{K'}{K} = \frac{L_1'}{L_1}, \quad n_2\frac{K'}{K} = \frac{L_2'}{L_2}, \quad \text{and} \quad n_3\frac{K'}{K} = \frac{L_3'}{L_3} \tag{1.1.8}$$

hold. Then a "mixed" modular equation is a relation among the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, and $\sqrt{\delta}$ that is induced by (1.1.8). It is then said that β, γ, and δ are of degrees n_1, n_2, and n_3, respectively, over α. The multiplier m connecting α and β and the multiplier m' connecting γ and δ are then defined by

$$m = \frac{z_1}{z_{n_1}} \quad \text{and} \quad m' = \frac{z_{n_2}}{z_{n_3}},$$

where $z_r = \varphi^2(q^r)$. For example, we recall from [13, p. 384, Entry 11(x)] that

$$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4} - \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} - 4\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/6} = mm',$$

where, β, γ and δ have degrees 3, 5 and 15, respectively, over α, and m and m' are the multipliers connecting α, β and γ, δ, respectively.

1.2 Partitions

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a positive integer n is an ordered set of positive integer *parts* λ_i such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. We also often write the partition λ of n as $\lambda_1 + \lambda_2 + \cdots + \lambda_k$. Let $p(n)$ denote the number of partitions of n. For example, $p(5) = 7$, since there are seven partitions of 5, namely,

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Similarly, $(-q;q)_\infty$ is the generating function for the number of partitions of a positive integer into distinct parts.

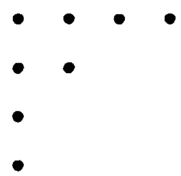
Among many remarkable properties of $p(n)$, we only record Ramanujan's famous partition congruences, namely, for any nonnegative integer n,

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

A partition is often represented with the help of a diagram called a Ferrers–Young diagram. The Ferrers–Young diagram of the partition $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ is formed by arranging n nodes in k rows so that the ith row has λ_i nodes. For example, the Ferrers–Young diagram of the partition $4+2+1+1$ of 8 is



The nodes in the Ferrers–Young diagram of a partition are labeled by row and column coordinates as one would label the entries of a matrix. Let λ'_j denote the number of nodes in column j. The hook number $H(i,j)$ of the (i,j) node is defined as the number of nodes directly below and to the right of the node including the node itself. That is, $H(i,j) = \lambda_i + \lambda'_j - j - i + 1$. A partition λ is said to be a t-core if and only if it has no hook numbers that are multiples of t.

1.3 Overpartitions, overpartition pairs and regular partitions

In [22], S. Corteel and J. Lovejoy defined an *overpartition* of n as a partition of n in which the first occurrence of a number can be overlined. For example, there are 14 overpartitions of 4. These are 4, $\overline{4}$, $3+1$, $\overline{3}+1$, $3+\overline{1}$, $\overline{3}+\overline{1}$, $2+2$, $\overline{2}+2$, $2+1+1$, $\overline{2}+1+1$, $2+\overline{1}+1$, $\overline{2}+\overline{1}+1$, $1+1+1+1$, and $\overline{1}+1+1+1$. Since the overlined parts of an overpartition form a partition into distinct parts and the non-overlined parts of an overpartition form an unrestricted partition, the generating function of $\overline{p}(n)$, the number of overpartitions of n, is

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots .$$

The function $\overline{p}(n)$ has been studied recently by a number of mathematicians including Hirschhorn and Sellers [33, 34], Mahlburg [51] and Kim [40]. Overpartitions have been used in combinatorial proofs of many q-series identities and these partitions arise quite naturally in the study of hypergeometric series (see [22, 23, 24, 49, 52]).

Lovejoy [44] defined an *overpartition pair* of n as a pair of overpartitions (μ, λ) where the sum of all the parts is n. For convenience, it is assumed that there is only one overpartition of zero denoted by $\emptyset$. For example, there are 12 overpartition pairs of 2, namely, $(2,\emptyset)$, $(\overline{2},\emptyset)$, $(1+1,\emptyset)$, $(\overline{1}+1,\emptyset)$, $(1,1)$, $(\overline{1},1)$, $(1,\overline{1})$, $(\overline{1},\overline{1})$, $(\emptyset,2)$, $(\emptyset,\overline{2})$, $(\emptyset,1+1)$, and $(\emptyset,\overline{1}+1)$. The generating function for the number of overpartition pairs, $\overline{pp}(n)$, is given by

$$\sum_{n=0}^{\infty} \overline{pp}(n)q^n = \frac{(-q;q)_\infty^2}{(q;q)_\infty^2} = 1 + 4q + 12q^2 + 32q^3 + 76q^4 + \cdots.$$

Recently, various arithmetic properties of $\overline{pp}(n)$ have been studied by Bringmann and Lovejoy [19], Chen and Lin [21] and Kim [41]. It has become clear that overpartition pairs play an important role in the theory of q-series and partitions. They provide a natural and general setting for the study of q-series identities and q-difference equations [46, 47, 48]. In [35], Hirschhorn and Sellers studied the arithmetic properties of overpartitions having only odd parts. More recently, Lin [43] investigated various arithmetic properties of overpartition pairs into odd parts.

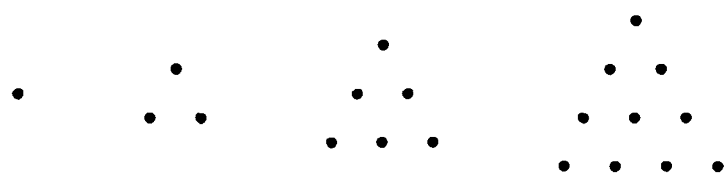
Next, a partition of n is called k-regular if none of its parts are divisible by k. Clearly, the generating function of $b_k(n)$, the number of k-regular partitions of n, is given by

$$\sum_{n=0}^{\infty} b_k(n)q^n = \frac{(q^k;q^k)_\infty}{(q;q)_\infty}$$

The function $b_k(n)$ has been studied by various mathematicians. For example, Hirschhorn and Sellers [38], Furcy and Penniston [28], Webb [61] and Lovejoy and Penniston [50]. Recently, Cui and Gu [26] and Keith [39] found several arithmetic properties and infinite families of congruences for some k-regular partitions.

1.4 Polygonal number

A polygonal number is a number which can be represented as dots arranged in the shape of a regular polygon. For example 1, 3, 6, 10, $\cdots$ are triangular numbers, because they can be represented as dots arranged in the shape of triangles which is evident from the following diagram.



In general, for $k \geq 3$, the n^{th} k-gonal number $F_k(n)$ is given by

$$F_k := F_k(n) = \frac{(k-2)n^2 - (k-4)n}{2}.$$

By allowing the domain for $F_k(n)$ to be the set of all integers, we see that the generating function $G_k(q)$ of $F_k(n)$ is given by

$$G_k(q) = \sum_{n=-\infty}^{\infty} q^{F_k} = \sum_{n=-\infty}^{\infty} q^{((k-2)n^2-(k-4)n)/2}.$$

We take $G_6(q)$ as the generating function for triangular numbers instead of $G_3(q)$ as $G_3(q)$ generates each triangular number twice, while $G_6(q)$ generates them once only. In terms of Ramanujan's general theta function as defined in (1.1.1),

$$G_k(q) = f(q, q^{k-3}). \tag{1.4.1}$$

From (1.4.1), the respective generating functions of squares, triangular numbers, pentagonal numbers, heptagonal numbers, octagonal numbers and dodecagonal numbers are

$$\begin{aligned} G_4(q) &= f(q,q) = \varphi(q), \\ G_6(q) &= f(q,q^3) = \psi(q), \\ G_5(q) &= f(q,q^2), \\ G_7(q) &= f(q,q^4), \\ G_8(q) &= f(q,q^5), \end{aligned}$$

and

$$G_{12}(q) = f(q,q^9).$$

1.5 Work carried out in this thesis

If $r_k(n)$ and $t_k(n)$ denote the number of representations of a positive integer n as a sum of k integer squares and as a sum of k triangular numbers, respectively, then

$$\sum_{n=0}^{\infty} r_k(n)q^n = \left(\sum_{j=-\infty}^{\infty} q^{j^2}\right)^k = \varphi^k(q) \tag{1.5.1}$$

and

$$\sum_{n=0}^{\infty} t_k(n)q^n = \left(\sum_{j=0}^{\infty} q^{j(j+1)/2}\right)^k = \psi^k(q). \tag{1.5.2}$$

Gauss [29] showed that $t_3(n) > 0$, in other words, every number is the sum of three triangular numbers. Andrews [3] provided a proof of this fact via q-series. In [31] and [32], Hirschhorn and Sellers found many arithmetic properties of $r_3(n)$ and $t_3(n)$ by manipulating q-series and theta functions. Their main identities are given in the following two theorems.

Theorem 1.5.1. *We have*

$$\sum_{n=0}^{\infty} r_3(27n+9)q^n = 5\sum_{n=0}^{\infty} r_3(3n+1)q^n, \tag{1.5.3}$$

$$\sum_{n=0}^{\infty} r_3(27n+18)q^n = 3\sum_{n=0}^{\infty} r_3(3n+2)q^n, \tag{1.5.4}$$

and

$$\sum_{n=0}^{\infty} r_3(27n)q^n = 4\sum_{n=0}^{\infty} r_3(3n)q^n - 3\sum_{n=0}^{\infty} r_3(n)q^{3n}. \tag{1.5.5}$$

Theorem 1.5.2. *We have*

$$\sum_{n=0}^{\infty} t_3(27n+3)q^n = 4\sum_{n=0}^{\infty} t_3(3n)q^n - 3\sum_{n=0}^{\infty} t_3(n)q^{3n+1}, \tag{1.5.6}$$

$$\sum_{n=0}^{\infty} t_3(27n+12)q^n = 3\sum_{n=0}^{\infty} t_3(3n+1)q^n, \tag{1.5.7}$$

and

$$\sum_{n=0}^{\infty} t_3(27n+21)q^n = 5\sum_{n=0}^{\infty} t_3(3n+2)q^n. \tag{1.5.8}$$

On p. 310 of his second notebook [53], Ramanujan recorded the following two beautiful theta function identities. If $\varphi(q)$, $\psi(q)$, and $f(-q)$ are as defined in (1.1.3)–(1.1.5), then

$$\frac{\varphi^3(q^{1/3})}{\varphi(q)} = \frac{\varphi^3(q)}{\varphi(q^3)} + 6q^{1/3}\frac{f^3(q^3)}{f(q)} + 12q^{2/3}\frac{f^3(-q^6)}{f(-q^2)} \tag{1.5.9}$$

and

$$\frac{\psi^3(q^{1/3})}{\psi(q)} = \frac{\psi^3(q)}{\psi(q^3)} + 3q^{1/3}\frac{f^3(-q^3)}{f(-q)} + 3q^{2/3}\frac{f^3(-q^6)}{f(-q^2)}. \tag{1.5.10}$$

The first proofs of (1.5.9) and (1.5.10) were given by Berndt [14, p. 185, Entry 33] by using Ramanujan's modular equations and a method of parameterizations. Baruah and Bora [7] gave alternative proofs by using other theta function identities of Ramanujan. Recently, Baruah and Nath [11] deduced (1.5.9) and (1.5.10) while studying the coefficients of $\varphi^3(q)/\varphi(q^3)$ and $\psi^3(q)/\psi(q^3)$.

In Chapter 2, we prove Theorem 1.5.1 and Theorem 1.5.2 by using the above two theta function identities.

Next, Chen and Lin [20] used a Lambert series to prove some infinite families of congruences modulo 5 of the number of two colored partitions (also called bi-partitions) where the odd parts are distinct. Inspired by their work, in Chapter 3 of the thesis, we establish many infinite families of congruences modulo 3 and 7 of coefficients of some theta functions by using Lambert series.

In Chapter 3, we also establish many infinite families of congruences of coefficients for $t_2(n)$, $t_5(n)$ and $t_8(n)$. Some of our results are stated below:

For any $\alpha \in \mathbb{N}$, we have

$$t_2\left(3^{2\alpha}n + \frac{7\times 3^{2\alpha-1}-1}{4}\right) \equiv 0 \pmod 3,$$
$$t_5\left(3^{2\alpha}n + \frac{23\times 3^{2\alpha-1}-5}{8}\right) \equiv 0 \pmod 3,$$
$$t_8(7^{2\alpha}n + 3\times 7^{2\alpha-1} - 1) \equiv 0 \pmod 7,$$
$$t_8(7^{2\alpha}n + 5\times 7^{2\alpha-1} - 1) \equiv 0 \pmod 7,$$

and

$$t_8(7^{2\alpha}n + 6\times 7^{2\alpha-1} - 1) \equiv 0 \pmod 7.$$

We also establish families of congruences modulo 3 for coefficients of each of the functions $\psi^4(q^2)$ and $\dfrac{1}{\psi^4(q)}$.

In [25], S. Cooper found the series expansions for six infinite products of theta functions

$$Z_1(q) = \frac{f^2(-q^3)f^2(-q^5)}{f(-q)f(-q^{15})},$$
$$Z_2(q) = q\frac{f^2(-q)f^2(-q^{15})}{f(-q^3)f(-q^5)},$$
$$Y_1(q) = \frac{f(-q)f(-q^6)f(-q^{10})f(-q^{15})}{f(-q^2)f(-q^{30})},$$
$$Y_2(q) = q\frac{f(-q^2)f(-q^3)f(-q^5)f(-q^{30})}{f(-q)f(-q^{15})},$$
$$Y_3(q) = q\frac{f(-q)f(-q^6)f(-q^{10})f(-q^{15})}{f(-q^3)f(-q^5)},$$

and

$$Y_4(q) = q\frac{f(-q^2)f(-q^3)f(-q^5)f(-q^{30})}{f(-q^6)f(-q^{10})}.$$

He then established new and simple proofs for theorems on the number of representations of a positive integer by the quadratic forms $j^2 + jk + 4k^2$, $2j^2 + jk + 2k^2$, $j^2 + 15k^2$ and $3j^2 + 5k^2$ and by the quadratic polynomials $j(j+1)/2 + 15k(k+1)/2$ and $3j(j+1)/2 + 5k(k+1)/2$. In our thesis, we establish some new arithmetic properties of $Y_2(q)$ and $Y_1(-q)$. We also establish some results on the number of representations of a positive integer by a sum of certain polygonal numbers.

For example, if $D(n)$ is defined by

$$Y_2(q) = \sum_{n=0}^{\infty} D(n)q^n := q\psi(q^3)\psi(q^5) + q^2\psi(q)\psi(q^{15}),$$

then

$$D(3^\alpha n) = D(n),$$

and

$$D(5^\alpha n) = D(n).$$

By using certain dissections, we find, for example, that

$$r\{5G_5 + G_6\}(5n+3) = r\{G_5 + 5G_6\}(n),$$
$$r\{G_5 + 5G_6\}(5n+1) = r\{5G_5 + G_6\}(n),$$
$$r\{5G_5 + G_6\}(n) = r\{5G_5 + G_6\}\left(5^{2\alpha}n + \frac{5^{2\alpha}-1}{3}\right),$$

and

$$r\{G_5 + 5G_6\}(n) = r\{G_5 + 5G_6\}\left(5^{2\alpha}n + \frac{2(5^{2\alpha}-1)}{3}\right),$$

where $r\{G_i + G_j\}(n)$ denotes the number of representations of n as a sum of an i-gonal number and a j-gonal number.

Similarly, if $E(n)$ is defined by

$$Y_1(-q) = \sum_{n=0}^{\infty} E(n)q^n := \varphi(q^3)\varphi(q^5) + \varphi(q)\varphi(q^{15}),$$

then some of our results are

$$E(3^\alpha n) = E(n),$$
$$E(5^\alpha n) = E(n),$$
$$r\{G_4 + 5G_8\}(5n) = r\{5G_4 + G_8\}(n),$$
$$r\{G_4 + 5G_8\}(5n+1) = 2r\{G_8 + B\}(n),$$
$$r\{5G_4 + G_8\}(5n+8) = r\{G_4 + 5G_8\}(n),$$
$$r\{5G_4 + G_8\}(n) = r\{5G_4 + G_8\}\left(5^{2\alpha}n + \frac{5^{2\alpha}-1}{3}\right),$$
$$r\{G_4 + 5G_8\}(n) = r\{G_4 + 5G_8\}\left(5^{2\alpha}n + \frac{5(5^{2\alpha}-1)}{3}\right),$$

and

$$r\{G_4 + 3G_{12}\}(3n+2) = 0.$$

H.M. Farkas and I. Kra [27] observed that theta constant identities or modular equations can be interpreted or can be transcribed into partition identities. Perhaps the most elegant of their three partition theorems is the following result [27, p. 202, Theorem 3].

Theorem 1.5.3. *Let S denote the set consisting of one copy of the positive integers and one additional copy of those positive integers that are multiples of 7. Then for each positive integers k, the number of partitions of $2k$ into even elements of S is equal to the number of partitions of $2k+1$ into odd elements of S.*

Hirschhorn [37] gave a different proof of the above theorem by establishing the equivalent q-series identity. The referee of the paper [37] pointed out that the above theorem is equivalent to a modular equation of degree 7 recorded by Ramanujan. In fact, that modular equation was first proved by Guetzlaff [30] in 1834. Ramanujan also recorded four similar modular equations of degrees 3, 5, 11 and 23. All those five modular equations were established earlier by Russel [55, 56] and Schroter [59]. Berndt [16] established the five partition identities implied by the five modular equations of Schroter, Russel and Ramanujan. S. O. Warnaar [60] found a generalization of the q-series identity associated to Theorem 1.5.3 and gave a combinatorial proof of this more general identity. In fact, Baruah and Berndt [5] showed that Warnaar's identity is equivalent to a theorem of Ramanujan recorded in his second notebook [13, p. 47, corollary]. Baruah and Berndt [5, 6] also showed that certain other modular equations and theta function identities of Ramanujan imply elegant partition identities. Several of these identities are for t-cores.

S. Kim [42] generalized one of the Warnaar's results and provided entirely bijective proofs of the identities for modulo 3 and 7.

Recently, C. Sandon and F. Zanello [57] determined a unified combinatorial framework to look at a large number of colored partition identities, and studied combinatorially the five identities, proved by Berndt [16] corresponding to modular equations of prime degrees 3, 5, 7, 11, and 23 of the Schroter, Russell, and Ramanujan type. In [58], they further found several new and highly non-trivial colored partition identities by using their master bijection, i.e., Theorem 2.1 in [57], and conjectured 29 more identities (in fact, they conjectured 30 identities, but analytic proof of one of the identities was already given by Baruah and Berndt in [5, Theorem 8.1]). Their conjectures are formulated in terms of certain sets of integers satisfying the conditions of their master bijection and the partition identities are stated as corollaries As mentioned by Berndt and Zhou [17], *these conjectures and corollaries are different formulations of the same phenomena, their corollaries are not less general than the corresponding conjectures* Three of their conjectured identities are proved analytically by Berndt and Zhou [17] with the help of Ramanujan's formulas for multipliers In a forthcoming project [18], they proved all the remaining conjectures of Sandon and Zanello [58]

Following Sandon and Zanello [58], for given integers $C \geq 1$, $0 \leq A_i \leq C/2$ and $0 \leq B_i \leq C/2$, let S be the set containing one copy of all positive integers congruent to $\pm A_i$ modulo C for each i, and let T be the set containing one copy of all positive integers congruent to $\pm B_i$ modulo C for each i Let $D_S(N)$ (respectively, $D_T(N)$) be the number of partitions of N into distinct elements of S (respectively, T), where **such partitions require to have an odd number of parts if no A_i (respectively, no B_i) is equal to zero.** Then the theorems and conjectures on colored partitions of Sandon and Zanello in [57] and [58] are identities connecting $D_S(N)$ and $D_T(N)$ For example, Corollary to Conjecture 3 24 in [58] can be stated as follows

Conjecture 1.5.1. *Let S be the set containing one copy of the even positive integers that are not multiples of* 25, *and T be the set containing one copy of the odd positive integers that are not multiples of* 25 *Then, for any $N \geq 4$,*

$$D_S(N) = D_T(N-3)$$

An equivalent form of the above conjecture, which has been proved by Berndt and Zhou [17] by using a 25th degree modular equation of Ramanujan, can be stated as follows

Theorem 1.5.4. *If S and T are as defined in Corollary 1 5 1, then, for any $N \geq 2$, the number of partitions of $2N$ into an odd number of distinct elements of S is equinumerous to the number of partitions of $2N-3$ into distinct elements of S*

As mentioned earlier, Berndt and Zhou [17] also proved two similar conjectures of Sandon and Zanello [58] as well as several new partition identities arising from a certain kind of Ramanujan's modular equations involving multipliers

Now, refer to the bold-faced underlined text in the above, if we do not restrict the parity of number of partitions into distinct elements of S (or, T), then some of the partition identities conjectured by Sandon and Zanello [58] take different forms In Chapter 4, we present new partition identities without restricting the parity of number of partitions into distinct elements of S (or, T) We also prove 17 conjectures of Sandon and Zanello [58] that do not require restrictions on the parity of number of partitions into distinct elements of S (or T) These correspond to corollaries to Conjectures 3 26, 3 32, 3 34–3 38, 3 40, 3 41, 3 43–3 46, and 3 48–3 51 in [58]

In each of our partition identities, the number of partitions of a positive integer n into distinct elements of a particular set A will be denoted by $P_A(n)$ For example, the identity analogous to the previous theorem is

$$P_S(2N) = 2P_T(2N-3) + a(N),$$

where

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{(q,q)_\infty}{(q^{25},q^{25})_\infty}$$

It follows from Euler's famous pentagonal number theorem that $a(N)$ is the difference of the number of partitions of N into an even number of distinct non multiples of 25 and the number of partitions of N into an odd number of distinct non multiples of 25

In the same chapter, we also prove analogous colored partition identities of the remaining 11 conjectural identities [58] by using the theory of Ramanujan's theta functions We also present some new colored partition identities of the same type

It is already mentioned in the previous section that Berndt and Zhou [17, 18] and we found partition identities conjectured by Sandon and Zanello [58] and analogous identities arising from Ramanujan's modular equations involving multipliers In Chapter 5, we find that different partition theoretic interpretations can be obtained from those modular equations involving multipliers Some of the identities are for overpartitions, overpartition pairs and regular partitions

For example, by using the two modular equations [13, p 230, Entry 5(vii)]

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2},$$

$$\frac{9}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2},$$

where β has degree 3 over α and m is the multiplier connecting α and β, we find the following theorem

Theorem 1.5.5. *Let $A(n)$ denote the number of partitions of n into parts congruent to ± 1 modulo 6 having 4 colors, $B(n)$ denote the number of partitions of n into distinct odd parts not multiples of 3 having 4 colors or even parts congruent to ± 4 modulo 12 having 4 colors, and $C(n)$ denote the number of partitions of n into parts congruent to ± 1 modulo 3 having 4 colors Then for $n \geq 1$*

$$A(2n+1) - B(2n-1) = 4C(n)$$

Similarly from the mixed modular equation [13, p 384, Entry 11(ix)]

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} = -\sqrt{\frac{m}{m'}},$$

where β, γ and δ are of degrees 3, 5 and 15, respectively, over α, and m and m' are the multipliers connecting α, β and γ, δ respectively, we find the following result

Theorem 1.5.6. *Let $A(n)$ denote the number of partitions of n into distinct odd parts that are not multiples of 3 and 5 or into even parts that are not multiples of 3 and 5, and $B(n)$ denote the number of overpartitions of n into parts not multiples of 3 and 5 Then, for any $n \geq 2$,*

$$2A(2n+1) = B(n)$$

Several of the results of this kind are discussed in Chapter 5

Chapter 2

Applications of Two Theta Function Identities of Ramanujan

2.1 Introduction

In this chapter, we give natural proofs of Theorem 1.5.1 and Theorem 1.5.2 by using two beautiful identities of Ramanujan, namely (1.5.9) and (1.5.10) mentioned in the introductory chapter. We say "natural" because these two theta function identities of Ramanujan show that their right hand sides can be interpreted as dissections in terms of generating functions of $r_3(3n)$, $r_3(3n+1)$ and $r_3(3n+2)$, and generating functions of $t_3(3n)$, $t_3(3n+1)$ and $t_3(3n+2)$, respectively. In the next section, we present some simple properties of theta functions which will be used in the subsequent sections. In Section 2.3, we prove Theorem 1.5.1 and in Section 2.4, we prove Theorem 1.5.2.

The contents of this chapter appeared in [9].

2.2 Preliminary results

In this section, we state some results which will be used to derive our results related to $r_3(n)$ and $t_3(n)$. After Ramanujan, we also define

$$\chi(q) := (-q;q^2)_\infty,$$

which is the generating function for the number of partitions of a positive integer into distinct odd parts.

Lemma 2.2.1. [13, p. 39, Entries 24(ii)–(iv)] *We have*

$$f^3(-q) = \varphi^2(-q)\psi(q), \tag{2.2.1}$$

$$\chi(q)\chi(-q) = \chi(-q^2), \tag{2.2.2}$$

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \left(\frac{\varphi(q)}{\psi(-q)}\right)^{1/3} = \frac{\varphi(q)}{f(q)} = \frac{f(-q^2)}{\psi(-q)}. \tag{2.2.3}$$

The results in the following lemma can be easily derived by manipulating q-products.

Lemma 2.2.2.

$$\varphi(q) = \frac{f^5(-q^2)}{f^2(-q)f^2(-q^4)}, \quad \psi(q) = \frac{f^2(-q^2)}{f(-q)}, \quad \varphi(-q) = \frac{f^2(-q)}{f(-q^2)},$$
$$\psi(-q) = \frac{f(-q)f(-q^4)}{f(-q^2)}, \quad f(q) = \frac{f^3(-q^2)}{f(-q)f(-q^4)}, \quad \chi(q) = \frac{f^2(-q^2)}{f(-q)f(-q^4)},$$
$$\chi(-q) = \frac{f(-q)}{f(-q^2)}.$$

Lemma 2.2.3. [13, p. 51, Example (v)] *We have*

$$f(q, q^5) = \psi(-q^3)\chi(q). \tag{2.2.4}$$

Lemma 2.2.4. [13, p. 350, Eq. (2.3)] *We have*

$$f(q, q^2) = \frac{\varphi(-q^3)}{\chi(-q)}. \tag{2.2.5}$$

Lemma 2.2.5. [13, p. 49, Corollaries (i) and (ii)] *We have*

$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}) \tag{2.2.6}$$

and

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{2.2.7}$$

Lemma 2.2.6. [7] *We have*

$$1+\frac{\chi^9(-q^3)}{q\chi^3(-q)}=\frac{\psi^4(q)}{q\psi^4(q^3)}. \tag{2.2.8}$$

Lemma 2.2.7. [4, Eq. (53)] *We have*

$$\varphi^4(q)-\varphi^4(q^3)=8q\varphi^2(-q^6)\frac{\chi^2(q)\chi(-q^2)\psi(-q^3)\psi(q^6)}{\chi(-q)}. \tag{2.2.9}$$

Lemma 2.2.8. [2] *If*

$$a(q)=\varphi(-q^3),\ b(q)=\frac{f(-q)f^2(-q^6)}{f(-q^2)f(-q^3)},$$

then

$$a^3(q)-8qb^3(q)=\frac{\varphi^4(-q)}{\varphi(-q^3)}. \tag{2.2.10}$$

Lemma 2.2.9. *If*

$$P(q)=\frac{\varphi(q^3)}{\chi(q)}\quad and \quad Q(q)=\psi(-q^3),$$

then

$$P^3(q)-qQ^3(q)=\frac{\psi^4(-q)}{\psi(-q^3)}. \tag{2.2.11}$$

Proof. Employing (2.2.3), we have

$$\begin{aligned}P^3(q)-qQ^3(q)&=\frac{\varphi^3(q^3)}{\chi^3(q)}-q\psi^3(-q^3)\\&=\frac{\chi^9(q^3)\psi^3(-q^3)}{\chi^3(q)}-q\psi^3(-q^3)\\&=q\psi^3(-q^3)\left(\frac{\chi^9(q^3)}{q\chi^3(q)}-1\right).\end{aligned} \tag{2.2.12}$$

Employing (2.2.8), with q replaced by $-q$, in (2.2.12), we easily arrive at (2.2.11). □

2.3 Proof of Theorem 1.5.1

Replacing q by q^3 in (1.5.9), we find that

$$\begin{aligned}\varphi^3(q) &= \frac{\varphi^4(q^3)}{\varphi(q^9)} + 6q\frac{f^3(q^9)\varphi(q^3)}{f(q^3)} + 12q^2\frac{f^3(-q^{18})\varphi(q^3)}{f(-q^6)} \\ &= A(q^3) + 6qB(q^3) + 12q^2C(q^3), \end{aligned} \tag{2.3.1}$$

where

$$A(q) = \frac{\varphi^4(q)}{\varphi(q^3)}, \quad B(q) = \frac{f^3(q^3)\varphi(q)}{f(q)}, \text{ and } C(q) = \frac{f^3(-q^6)\varphi(q)}{f(-q^2)}.$$

Since

$$\varphi^3(q) = \sum_{n=0}^{\infty} r_3(n)q^n,$$

we readily derive from (2.3.1) that

$$\sum_{n=0}^{\infty} r_3(3n)q^n = A(q) = \frac{\varphi^4(q)}{\varphi(q^3)}, \tag{2.3.2}$$

$$\sum_{n=0}^{\infty} r_3(3n+1)q^n = 6B(q) = 6\frac{f^3(q^3)\varphi(q)}{f(q)} = 6\frac{f^9(-q^6)f^2(-q^2)}{f(-q)f^3(-q^3)f(-q^4)f^3(-q^{12})}, \tag{2.3.3}$$

and

$$\sum_{n=0}^{\infty} r_3(3n+2)q^n = 12C(q) = 12\frac{f^3(-q^6)\varphi(q)}{f(-q^2)} = 12\frac{f^4(-q^2)f^3(-q^6)}{f^2(-q)f^2(-q^4)}. \tag{2.3.4}$$

Using (2.3.1), (2.2.6), and (2.2.4) in (2.3.2), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} r_3(3n)q^n &= \frac{\varphi^4(q)}{\varphi(q^3)} = \frac{\varphi^3(q)}{\varphi(q^3)}\,\varphi(q) \\ &= \left(\frac{\varphi^3(q^3)}{\varphi(q^9)} + 6q\frac{f^3(q^9)}{f(q^3)} + 12q^2\frac{f^3(-q^{18})}{f(-q^6)}\right)\left(\varphi(q^9) + 2qf(q^3,q^{15})\right) \\ &= \left(\frac{\varphi^3(q^3)}{\varphi(q^9)} + 6q\frac{f^3(q^9)}{f(q^3)} + 12q^2\frac{f^3(-q^{18})}{f(-q^6)}\right)\left(\varphi(q^9) + 2q\chi(q^3)\psi(-q^9)\right). \end{aligned} \tag{2.3.5}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} from both sides of (2.3.5), we obtain

$$\sum_{n=0}^{\infty} r_3(9n)q^n = \varphi^3(q) + 24q\chi(q)\psi(-q^3)\frac{f^3(-q^6)}{f(-q^2)}, \tag{2.3.6}$$

$$\sum_{n=0}^{\infty} r_3(9n+3)q^n = 2\chi(q)\psi(-q^3)\frac{\varphi^3(q^3)}{\varphi(q^3)} + 6\varphi(q^3)\frac{f^3(q^3)}{f(q)}, \tag{2.3.7}$$

and

$$\sum_{n=0}^{\infty} r_3(9n+6)q^n = 12\chi(q)\psi(-q^3)\frac{f^3(q^3)}{f(q)} + 12\varphi(q^3)\frac{f^3(-q^6)}{f(-q^2)}, \tag{2.3.8}$$

respectively.

Next, employing (2.2.3), we rewrite (2.3.6) as

$$\sum_{n=0}^{\infty} r_3(9n)q^n = \varphi^3(q) + 24q\frac{f^3(-q^6)\psi(-q^3)}{\psi(-q)}. \tag{2.3.9}$$

Now, replacing q by $-q$ in (2.2.7), and then using (2.2.5), we obtain

$$\begin{aligned}\psi(-q) &= f(-q^3, q^6) - q\psi(-q^9)\\ &= \frac{\varphi(q^9)}{\chi(q^3)} - q\psi(-q^9)\\ &= P(q^3) - qQ(q^3),\end{aligned} \tag{2.3.10}$$

where $P(q)$ and $Q(q)$ are as defined in Lemma 2.2.9. Therefore,

$$\frac{1}{\psi(-q)} = \frac{1}{P(q^3) - qQ(q^3)} = \frac{P^2(q^3) + qP(q^3)Q(q^3) + q^2Q^2(q^3)}{P^3(q^3) - q^3Q^3(q^3)}. \tag{2.3.11}$$

Employing (2.3.1) and (2.3.11) in (2.3.9), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} r_3(9n)q^n &= A(q^3) + 6qB(q^3) + 12q^2C(q^3) + 24qf^3(-q^6)\psi(-q^3)\\ &\quad \times \left(\frac{P^2(q^3) + qP(q^3)Q(q^3) + q^2Q^2(q^3)}{P^3(q^3) - q^3Q^3(q^3)}\right)\\ &= A(q^3) + 24q^3\frac{f^3(-q^6)\psi(-q^3)Q^2(q^3)}{P^3(q^3) - q^3Q^3(q^3)}\\ &\quad + q\left(6B(q^3) + 24\frac{f^3(-q^6)\psi(-q^3)P^2(q^3)}{P^3(q^3) - q^3Q^3(q^3)}\right)\\ &\quad + q^2\left(12C(q^3) + 24\frac{f^3(-q^6)\psi(-q^3)P(q^3)Q(q^3)}{P^3(q^3) - q^3Q^3(q^3)}\right).\end{aligned} \tag{2.3.12}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} from both sides of (2.3.12), we obtain

$$\sum_{n=0}^{\infty} r_3(27n)q^n = A(q) + 24q\frac{f^3(-q^2)\psi(-q)Q^2(q)}{P^3(q)-qQ^3(q)}, \tag{2.3.13}$$

$$\sum_{n=0}^{\infty} r_3(27n+9)q^n = 6B(q) + 24\frac{f^3(-q^2)\psi(-q)P^2(q)}{P^3(q)-qQ^3(q)}, \tag{2.3.14}$$

and

$$\sum_{n=0}^{\infty} r_3(27n+18)q^n = 12C(q) + 24\frac{f^3(-q^2)\psi(-q)P(q)Q(q)}{P^3(q)-qQ^3(q)}, \tag{2.3.15}$$

respectively.

Employing Lemma 2.2.9 and Lemma 2.2.2 in (2.3.14), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} r_3(27n+9)q^n &= 6\frac{f^3(q^3)\varphi(q)}{f(q)} + 24\frac{f^3(-q^2)\psi(-q^3)\varphi^2(q^3)}{\psi^3(-q)\chi^2(q)} \\ &= 6\frac{f^9(-q^6)f^2(-q^2)}{f(-q)f^2(-q^3)f(-q^4)f^3(-q^{12})} \\ &\quad + 24\frac{f^9(-q^6)f^2(-q^2)}{f(-q)f^2(-q^3)f(-q^4)f^3(-q^{12})} \\ &= 30\frac{f^9(-q^6)f^2(-q^2)}{f(-q)f^2(-q^3)f(-q^4)f^3(-q^{12})}.\end{aligned} \tag{2.3.16}$$

Using (2.3.3) in (2.3.16), we readily arrive at (1.5.3).

Next, employing Lemma 2.2.9, Lemma 2.2.2, and (2.3.4) in (2.3.15), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} r_3(27n+18)q^n &= 12C(q) + 24\frac{f^3(-q^2)\psi(-q)P(q)Q(q)}{P^3(q)-qQ^3(q)} \\ &= 12\frac{f^3(-q^6)\varphi(q)}{f(-q^2)} + 24\frac{f^3(-q^2)\psi(-q)\psi^2(-q^3)\varphi(q^3)}{\chi(q)\psi^4(-q)} \\ &= 36\frac{f^4(-q^2)f^3(-q^6)}{f^2(-q)f^2(-q^4)} \\ &= 3\sum_{n=0}^{\infty} r_3(3n+2)q^n,\end{aligned}$$

to complete the proof of (1.5.4).

Now, from (2.3.13), Lemma 2.2.9, and (2.2.9), we have

$$\begin{aligned}\sum_{n=0}^{\infty} r_3(27n)q^n &= \frac{\varphi^4(q)}{\varphi(q^3)} + 24q\frac{f^3(-q^2)\psi^3(-q^3)}{\psi^3(-q)}\\ &= \frac{\varphi^4(q)}{\varphi(q^3)} + 24q\frac{\chi^2(q)\chi(-q^2)\varphi^2(-q^6)\psi(-q^3)\psi(q^6)}{\varphi(q^3)\chi(-q)}\\ &= \frac{\varphi^4(q)}{\varphi(q^3)} + 3\frac{\varphi^4(q)-\varphi^4(q^3)}{\varphi(q^3)} = 4\frac{\varphi^4(q)}{\varphi(q^3)} - 3\varphi^3(q^3). \qquad (2.3.17)\end{aligned}$$

Employing (2.3.2) and (1.5.1) in the above, we arrive at (1.5.5) to finish the proof.

2.4 Proof of Theorem 1.5.2

Replacing q by q^3 in (1.5.10), we find that

$$\begin{aligned}\psi^3(q) &= \frac{\psi^4(q^3)}{\psi(q^9)} + 3q\frac{f^3(-q^9)\psi(q^3)}{f(-q^3)} + 3q^2\frac{f^3(-q^{18})\psi(q^3)}{f(-q^6)}\\ &= L(q^3) + 3qM(q^3) + 3q^2N(q^3), \qquad (2.4.1)\end{aligned}$$

where

$$L(q) = \frac{\psi^4(q)}{\psi(q^3)}, \quad M(q) = \frac{f^3(-q^3)\psi(q)}{f(-q)}, \quad \text{and } N(q) = \frac{f^3(-q^6)\psi(q)}{f(-q^2)}.$$

Since $\psi^3(q) = \sum_{n=0}^{\infty} t_3(n)q^n$, we deduce from (2.4.1) that

$$\sum_{n=0}^{\infty} t_3(3n)q^n = L(q) = \frac{\psi^4(q)}{\psi(q^3)}, \qquad (2.4.2)$$

$$\sum_{n=0}^{\infty} t_3(3n+1)q^n = 3M(q) = 3\frac{f^3(-q^3)\psi(q)}{f(-q)} = 3\frac{f^3(-q^3)f^2(-q^2)}{f^2(-q)}, \qquad (2.4.3)$$

and

$$\sum_{n=0}^{\infty} t_3(3n+2)q^n = 3N(q) = 3\frac{f^3(-q^6)\psi(q)}{f(-q^2)} = 3\frac{f^3(-q^6)f(-q^2)}{f(-q)}. \qquad (2.4.4)$$

Employing (1.5.10), with q replaced by $-q$, (2.2.7), and (2.2.5) in (2.4.2), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} t_3(3n)q^n &= \left(\frac{\psi^3(q^3)}{\psi(q^9)} + 3q\frac{f^3(-q^9)}{f(-q^3)} + 3q^2\frac{f^3(-q^{18})}{f(-q^6)}\right)\left(f(q^3,q^6) + q\psi(q^9)\right)\\ &= \left(\frac{\psi^3(q^3)}{\psi(q^9)} + 3q\frac{f^3(-q^9)}{f(-q^3)} + 3q^2\frac{f^3(-q^{18})}{f(-q^6)}\right)\left(\frac{\varphi(-q^9)}{\chi(-q^3)} + q\psi(q^9)\right).\end{aligned}$$

(2.4.5)

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} from both sides of (2.4.5), we obtain

$$\sum_{n=0}^{\infty} t_3(9n)q^n = \frac{\psi^3(q)\varphi(-q^3)}{\psi(q^3)\chi(-q)} + 3q\frac{f^3(-q^6)\psi(q^3)}{f(-q^2)}, \tag{2.4.6}$$

$$\sum_{n=0}^{\infty} t_3(9n+3)q^n = \psi^3(q) + 3\frac{\chi(-q^3)f^4(-q^3)}{\varphi(-q)}, \tag{2.4.7}$$

and

$$\sum_{n=0}^{\infty} t_3(9n+6)q^n = 3\left(\frac{f^3(-q^3)}{f(-q)}\psi(q^3) + \frac{f^3(-q^6)}{f(-q^2)}\frac{\varphi(-q^3)}{\chi(-q)}\right) = 6\frac{f^2(-q^6)f^2(-q^3)}{f(-q)}, \tag{2.4.8}$$

respectively, where in the last equality we used the identities in Lemma 2.2.2.

Now, replacing q by $-q$ in (2.2.6), we have

$$\varphi(-q) = a(q^3) - 2qb(q^3),$$

where $a(q)$ and $b(q)$ are as defined in Lemma 2.2.8. Therefore, we have

$$\frac{1}{\varphi(-q)} = \frac{1}{a(q^3) - 2qb(q^3)} = \frac{a^2(q^3) + 2qa(q^3)b(q^3) + 4q^2b^2(q^3)}{a^3(q^3) - 8q^3b(q^3)}. \tag{2.4.9}$$

Employing (2.4.1) and (2.4.9) in (2.4.7), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} t_3(9n+3)q^n &= L(q^3) + 3qM(q^3) + 3q^2N(q^3) + 3\chi(-q^3)f^4(-q^3)\\ &\quad\times \frac{a^2(q^3) + 2qa(q^3)b(q^3) + 4q^2b^2(q^3)}{a^3(q^3) - 8q^3b(q^3))}\\ &= \left(L(q^3) + 3\frac{a^2(q^3)\chi(-q^3)f^4(-q^3)}{a^3(q^3) - 8q^3b(q^3)}\right)\\ &\quad + 3q\left(M(q^3) + 2\frac{a(q^3)b(b^3)\chi(-q^3)f^4(-q^3)}{a^3(q^3) - 8q^3b(q^3)}\right)\\ &\quad + 3q^2\left(N(q^3) + 4\frac{b^2(q^3)\chi(-q^3)f^4(-q^3)}{a^3(q^3) - 8q^3b(q^3)}\right).\end{aligned} \tag{2.4.10}$$

Comparing the terms involving q^{3n} on both sides of the above identity, and then applying Lemma 2.2.8, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} t_3(27n+3)q^n &= L(q) + 3\frac{a^2(q)\chi(-q)f^4(-q)}{a^3(q) - 8q^3(q)})\\ &= \frac{\psi^4(q)}{\psi(q^3)} + 3\frac{\varphi^3(-q^3)\chi(-q)f^4(-q)}{\varphi^4(-q)}.\end{aligned}$$

Employing (2.2.3), the above can be rewritten as

$$\begin{aligned}\sum_{n=0}^{\infty} t_3(27n+3)q^n &= \frac{\psi^4(q)}{\psi(q^3)} + 3\frac{\varphi^3(-q^3)}{\chi^3(-q)} \\ &= \frac{\psi^4(q)}{\psi(q^3)} + 3\frac{\chi^9(-q^3)\psi^3(q^3)}{\chi^3(-q)}. \qquad (2.4.11)\end{aligned}$$

Using (2.2.8) in (2.4.11), we obtain

$$\sum_{n=0}^{\infty} t_3(27n+3)q^n = \frac{\psi^4(q)}{\psi(q^3)} + 3\frac{\psi^4(q) - q\psi^4(q^3)}{\psi(q^3)} = 4\frac{\psi^4(q)}{\psi(q^3)} - 3q\psi^3(q^3), \qquad (2.4.12)$$

to arrive at (1.5.6), with further aids from (2.4.2) and (1.5.2).

Next, comparing the terms involving q^{3n+1} on both sides of (2.4.10), and then applying Lemma 2.2.8 and Lemma 2.2.2, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} t_3(27n+12)q^n &= 3M(q) + 6\frac{a(q)b(q)\chi(-q)f^4(-q)}{a^3(q) - 8qb^3(q)} \\ &= 9\frac{f^3(-q^3)f^2(-q^2)}{f^2(-q)}. \qquad (2.4.13)\end{aligned}$$

The identity (1.5.7) now follows from (2.4.13) and (2.4.3).

Finally, comparing the terms involving q^{3n+2} on both sides of (2.4.10), and then applying Lemma 2.2.8 and Lemma 2.2.2, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} t_3(27n+21)q^n &= 3N(q) + 12\frac{b^2(q)\chi(-q)f^4(-q)}{a^3(q) - 8q^3(q)} \\ &= 15\frac{f^3(-q^6)f(-q^2)}{f(-q)}, \qquad (2.4.14)\end{aligned}$$

and then with the aid of (2.4.4), the identity (1.5.8) follows easily.

Chapter 3

Arithmetic Properties of the Coefficients of Some Theta Functions

3.1 Introduction

Ramanujan recorded many identities involving Lambert series in his notebooks [53] and the lost notebook [54]. In a fragment published with his lost notebook [54, pp. 353–355], Ramanujan provided a list of twenty identities involving Lambert series and products or quotients of theta functions. Several of the identities involving Lambert series have arithmetical interpretations. Some of these are related to the number of representations of an integer as a sum of squares or as a sum of triangular numbers. We state in the introductory chapter that Chen and Lin [20] used a Lambert series to prove some infinite families of congruences modulo 5 of the number of bipartitions where the odd parts are distinct. In this chapter, we discuss several Lambert series identities to establish infinite families of congruences modulo 3 and 7 of the coefficients of some theta functions.

The chapter is organized as follows. In Section 3.2, three infinite families of congruences modulo 7 for $t_8(n)$, the number of representations of n as a sum of eight triangular numbers, are established. In Section 3.3 two infinite families of congruences modulo 3 for $t_2(n)$, the number of representations of n as a sum of two triangular numbers, are obtained. In Sections 3.4–3.6, we discuss some infinite families of congruences modulo 3 for coefficients of three more functions.

In the introductory chapter we mentioned that Cooper [25] found series expansions for the following six infinite products of theta functions as

$$Z_1(q) = \frac{f^2(-q^3)f^2(-q^5)}{f(-q)f(-q^{15})},$$
$$Z_2(q) = q\frac{f^2(-q)f^2(-q^{15})}{f(-q^3)f(-q^5)},$$
$$Y_1(q) = \frac{f(-q)f(-q^6)f(-q^{10})f(-q^{15})}{f(-q^2)f(-q^{30})}, \tag{3.1.1}$$
$$Y_2(q) = q\frac{f(-q^2)f(-q^3)f(-q^5)f(-q^{30})}{f(-q)f(-q^{15})}, \tag{3.1.2}$$
$$Y_3(q) = q\frac{f(-q)f(-q^6)f(-q^{10})f(-q^{15})}{f(-q^3)f(-q^5)},$$

and

$$Y_4(q) = q\frac{f(-q^2)f(-q^3)f(-q^5)f(-q^{30})}{f(-q^6)f(-q^{10})}.$$

From [25, p. 82, Theorem 2.2] we observe that $Y_1(-q) = \varphi(q^3)\varphi(q^5) + \varphi(q)\varphi(q^{15})$ and $Y_2(q) = q\psi(q^3)\psi(q^5) + q^2\psi(q)\psi(q^{15})$. In Section 3.7, we establish some arithmetic properties for $Y_1(-q)$ and $Y_2(q)$ and discuss some results on the number of representations of a positive integer by a sum of certain polygonal numbers.

3.2 Infinite families of congruences modulo 7 for $t_8(n)$

In this section, we establish three infinite families of congruences modulo 7 for $t_8(n)$.

Theorem 3.2.1. *For any $\alpha \in \mathbb{N}$, we have*

$$t_8(49n + 20) \equiv t_8(49n + 34) \equiv t_8(49n + 41) \equiv 0 \pmod 7,$$
$$t_8(7^{2\alpha}n + 3 \times 7^{2\alpha-1} - 1) \equiv 0 \pmod 7,$$
$$t_8(7^{2\alpha}n + 5 \times 7^{2\alpha-1} - 1) \equiv 0 \pmod 7,$$

and

$$t_8(7^{2\alpha}n + 6 \times 7^{2\alpha-1} - 1) \equiv 0 \pmod 7.$$

Before proving the theorem, we present the following lemma.

Lemma 3.2.2. *For any r with $r \not\equiv 0 \pmod 7$, we have*

$$\sum_{n=0}^{\infty} a_r(n)q^n := \sum_{n=0}^{\infty} \frac{q^{14n+r}}{1-q^{14n+r}} = \sum_{n=0}^{\infty} a_r(7n)q^n.$$

Proof of Lemma 3.2.2. Clearly,

$$\sum_{n=0}^{\infty} a_r(n)q^n = \sum_{n=0}^{\infty} \frac{q^{14n+r}}{1-q^{14n+r}} = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} q^{(14n+r)k}. \tag{3.2.1}$$

Now, $(14n+r)k$ is a multiple of 7 if and only if $k \equiv 0 \pmod 7$. It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} a_r(7n)q^{7n} &= \sum_{n=0}^{\infty} \sum_{k\equiv 0 \pmod 7}^{\infty} q^{(14n+r)k} \\ &= \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} q^{(14n+r)7j}.\end{aligned}$$

Replacing q^7 by q and using (3.2.1), we complete the proof. □

Proof of Theorem 3.2.1. From [13, p. 302, Entry 17(i)], we recall the Lambert series for $q\psi(q)\psi(q^7)$ as

$$\begin{aligned}q\psi(q)\psi(q^7) = &\sum_{n=0}^{\infty} \frac{q^{14n+1}}{1-q^{14n+1}} - \sum_{n=0}^{\infty} \frac{q^{14n+3}}{1-q^{14n+3}} - \sum_{n=0}^{\infty} \frac{q^{14n+5}}{1-q^{14n+5}} \\ &+ \sum_{n=0}^{\infty} \frac{q^{14n+9}}{1-q^{14n+9}} + \sum_{n=0}^{\infty} \frac{q^{14n+11}}{1-q^{14n+11}} - \sum_{n=0}^{\infty} \frac{q^{14n+13}}{1-q^{14n+13}}.\end{aligned} \tag{3.2.2}$$

If

$$q\psi(q)\psi(q^7) =: \sum_{n=0}^{\infty} A(n)q^n,$$

then from (3.2.2) and Lemma 3.2.2, we find that

$$q\psi(q)\psi(q^7) = \sum_{n=0}^{\infty} A(7n)q^n.$$

Now, setting $k = 8$ in (1.5.2), we have

$$\psi^8(q) = \sum_{n=0}^{\infty} t_8(n)q^n.$$

Since

$$(1 - q^n)^7 \equiv (1 - q^{7n}) \pmod 7$$

and

$$\psi(q) = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty} = \frac{\prod_{n=1}^{\infty}(1 - q^{2n})^2}{\prod_{n=1}^{\infty}(1 - q^n)}, \tag{3.2.3}$$

we easily deduce that

$$\psi^7(q) \equiv \psi(q^7) \pmod 7.$$

Thus,

$$\begin{aligned} q\sum_{n=0}^{\infty} t_8(n)q^n &= q\psi^8(q) \\ &\equiv q\psi(q)\psi(q^7) \\ &\equiv \sum_{n=0}^{\infty} A(n)q^n \pmod 7. \end{aligned} \tag{3.2.4}$$

Extracting the terms q^{7n} from both sides of the above and then replacing q^7 by q, we obtain

$$\sum_{n=1}^{\infty} t_8(7n-1)q^n \equiv \sum_{n=0}^{\infty} A(7n)q^n = q\psi(q)\psi(q^7) \pmod 7. \tag{3.2.5}$$

From (3.2.4) and (3.2.5),

$$\sum_{n=1}^{\infty} t_8(7n-1)q^n \equiv \sum_{n=0}^{\infty} t_8(n)q^{n+1} \pmod 7.$$

Equating the coefficients of q^{n+1} from both sides, we find that

$$t_8(7n+6) \equiv t_8(n) \pmod 7.$$

Again, setting $a = q$, $b = q^3$ and $n = 7$ in [13, p 48, Entry 31], we have

$$\psi(q) = f(q^{91}, q^{105}) + q^{21}f(q^7, q^{189}) + qf(q^{77} q^{119}) + q^{14}f(q^{21}, q^{175}) + q^3f(q^{63}, q^{210}) + q^7f(q^{35}, q^{161}) + q^6\psi(q^{49}) \tag{3 2 6}$$

Employing (3 2 6) in (3 2 5), and then extracting the terms involving q^{7n+a}, where $a = 3, 5, 6$ and 7, from both sides of the resulting identity, we find that

$$t_8(49n + 20) \equiv t_8(49n + 34) \equiv t_8(49n + 41) \equiv 0 \pmod 7, \tag{3 2 7}$$

and

$$\sum_{n=0}^{\infty} t_8(49n + 48) \equiv \sum_{n=0}^{\infty} t_8(7n + 6) \equiv q\psi(q)\psi(q^7) \tag{3 2 8}$$

From (3 2 4), (3 2 5) and (3 2 8), we arrive at

$$t_8(49n + 48) \equiv t_8(7n + 6) \equiv t_8(n) \pmod 7 \tag{3 2 9}$$

From (3 2 7), (3 2 9) and mathematical induction, we deduce the desired identities of the theorem to complete the proof □

Remark 3.2.3. *Note that* $A(n) = t_{(1\ 7)}(n - 1)$, *where* $t_{(1,7)}(n)$ *is the number of representations of a nonnegative integer* n *as a sum of a triangular number and 7 times of another triangular number*

3.3 Infinite families of congruences modulo 3 for $t_2(n)$

In this section, we establish two infinite families of congruences modulo 3 for $t_2(n)$ For notational convenience, we assume that all congruences from this section of this chapter onward are modulo 3, unless otherwise stated

Theorem 3.3.1. *For any $\alpha \in \mathbb{N}$, we have*

$$t_2(9n+5) \equiv t_2(9n+8) \equiv 0 \pmod 3,$$

$$t_2\left(3^{2\alpha}n + \frac{7 \times 3^{2\alpha-1}-1}{4}\right) \equiv 0 \pmod 3,$$

and

$$t_2\left(3^{2\alpha}n + \frac{11 \times 3^{2\alpha-1}-1}{4}\right) \equiv 0 \pmod 3.$$

We present the following lemma to prove the theorem.

Lemma 3.3.2. *For any r with $r \not\equiv 0 \pmod 3$, we have*

$$\sum_{n=0}^{\infty} b_r(n)q^n := \sum_{n=0}^{\infty} \frac{q^{3n+r}}{1-q^{6n+2r}} = \sum_{n=0}^{\infty} b_r(3n)q^n.$$

Proof of Lemma 3.3.2. Clearly,

$$\sum_{n=0}^{\infty} b_r(n)q^n = \sum_{n=0}^{\infty} \frac{q^{3n+r}}{1-q^{6n+2r}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q^{(3n+r)(2k+1)}. \tag{3.3.1}$$

Since $(3n+r)(2k+1)$ is a multiple of 3 if and only if $k \equiv 1 \pmod 3$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_r(3n)q^{3n} &= \sum_{n=0}^{\infty} \sum_{k\equiv 1 (\mathrm{mod}\ 3)}^{\infty} q^{(3n+r)(2k+1)} \\ &= \sum_{n=0}^{\infty}\sum_{t=0}^{\infty} q^{(3n+r)(6t+3)}.\end{aligned}$$

Replacing q^3 by q and using (3.3.1), we complete the proof of Lemma 3.3.2. □

Proof of Theorem 3.3.1. From [13, p. 226, Entry 4(i)] we consider the Lambert series for $q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3)$ as

$$\begin{aligned}q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3) &= \frac{q}{1-q^2} - \frac{2^2q^2}{1-q^4} + \frac{4^2q^4}{1-q^8} - \frac{5^2q^5}{1-q^{10}} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(3n+1)^2q^{3n+1}}{1-q^{6n+2}} - \sum_{n=0}^{\infty} \frac{(3n+2)^2q^{3n+2}}{1-q^{6n+4}} \\ &\equiv \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1-q^{6n+2}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1-q^{6n+4}} \pmod 3. \end{aligned} \tag{3.3.2}$$

Assuming

$$q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3) =: \sum_{n=0}^{\infty} B(n)q^n,$$

from (3.3.2) and Lemma 3.3.2, we obtain

$$q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3) = \sum_{n=0}^{\infty} B(3n)q^n. \tag{3.3.3}$$

By setting $k = 2$ in (1.5.2), we have

$$\psi^2(q) = \sum_{n=0}^{\infty} t_2(n)q^n$$

Now, since

$$(1 - q^n)^3 \equiv (1 - q^{3n}) \pmod 3$$

and $\psi(q)$ is as defined in (3.2.3), we find that

$$\psi^3(q) \equiv \psi(q^3) \pmod 3. \tag{3.3.4}$$

Thus,

$$\begin{aligned} q\sum_{n=0}^{\infty} t_2(n)q^n &= q\psi^2(q) \\ &= q\frac{\psi^5(q)}{\psi^3(q)} \\ &\equiv \frac{q\psi^5(q)}{\psi(q^3)} \pmod 3 \\ &= \frac{q\psi^5(q)\psi(q^3)}{\psi^2(q^3)} \\ &\equiv \frac{q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3)}{\psi^2(q^3)} \pmod 3 \\ &\equiv \frac{\sum_{n=0}^{\infty} B(n)q^n}{\psi^2(q^3)} \pmod 3. \end{aligned} \tag{3.3.5}$$

Extracting the terms q^{3n} from both sides of the above and then replacing q^3 by q, we arrive at

$$\begin{aligned}\sum_{n=1}^{\infty} t_2(3n-1)q^n &\equiv \frac{\sum_{n=0}^{\infty} B(3n)q^n}{\psi^2(q)} \\ &\equiv \frac{q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3)}{\psi^2(q)} \pmod 3 \\ &\equiv \frac{q\psi^5(q)\psi(q^3)}{\psi^2(q)} \\ &= q\psi^3(q)\psi(q^3) \\ &\equiv q\psi^2(q^3) \pmod 3 \\ &\equiv \sum_{n=0}^{\infty} t_2(n)q^{3n+1}.\end{aligned} \tag{3.3.6}$$

Equating the coefficients of q^{3n+1} from both sides,

$$t_2(9n+2) \equiv t_2(n) \pmod 3. \tag{3.3.7}$$

Now we recall from [13, p. 49, Corollary (ii)] that

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{3.3.8}$$

By the Jacobi triple product identity, (1.1 2), and the definition of $\varphi(q)$ and $\chi(q)$, we have

$$f(q, q^2) = (-q; q^3)_\infty(-q^2; q^3)_\infty(q^3; q^3)_\infty = \frac{\varphi(-q^3)}{\chi(-q)}. \tag{3.3.9}$$

With the help of the above, we rewrite (3.3.8) as

$$\psi(q) = \frac{\varphi(-q^9)}{\chi(-q^3)} + q\psi(q^9). \tag{3.3.10}$$

Employing the above in (3.3.6), and extracting the terms involving q^{3n+1} and q^{3n+2} from both sides of the resulting identity, we find that

$$t_2(9n+5) \equiv t_2(9n+8) \equiv 0 \pmod 3. \tag{3.3.11}$$

From (3.3.7), (3.3.11) and mathematical induction, we derive the desired identities of the theorem and hence complete the proof. □

Remark 3.3.3. *The results discussed in this section can also be derived from the Lambert series [13, p. 223, Entry 3(iii)] for* $q\psi^2(q)\psi^2(q^3)$ *as*

$$q\psi^2(q)\psi^2(q^3) = \frac{q}{1-q^2} + \frac{2q^2}{1-q^4} + \frac{4q^4}{1-q^8} + \frac{5q^5}{1-q^{10}} + \cdots$$

by proceeding in a similar way.

3.4 An infinite family of congruences modulo 3 for $t_5(n)$

In this section, we establish an infinite family of congruences modulo 3 for $t_5(n)$, the number of representations of n as a sum of five triangular numbers.

Theorem 3.4.1. *For any* r *with* $r \not\equiv 0 \pmod 3$*, we have*

$$t_5(9n+8) \equiv 0 \pmod 3$$

and

$$t_5\left(3^{2\alpha}n + \frac{23 \times 3^{2\alpha-1} - 5}{8}\right) \equiv 0 \pmod 3.$$

Proof. From [13, p. 223, Entry 3(iii)], we recall the Lambert series for $q\psi^2(q)\psi^2(q^3)$ as

$$\begin{aligned} q\psi^2(q)\psi^2(q^3) &= \frac{q}{1-q^2} + \frac{2q^2}{1-q^4} + \frac{4q^4}{1-q^8} + \frac{5q^5}{1-q^{10}} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(3n+1)q^{3n+1}}{1-q^{6n+2}} + \sum_{n=0}^{\infty} \frac{(3n+2)q^{3n+2}}{1-q^{6n+4}}. \end{aligned}$$

Thus,

$$q\psi^2(q)\psi^2(q^3) \equiv \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1-q^{6n+2}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1-q^{6n+4}} \pmod 3. \tag{3.4.1}$$

Considering

$$q\psi^2(q)\psi^2(q^3) =: \sum_{n=0}^{\infty} C(n)q^n,$$

from (3.4.1) and Lemma 3.3.2, we arrive at

$$q\psi^2(q)\psi^2(q^3) = \sum_{n=0}^{\infty} C(3n)q^n.$$

Now, setting $k = 5$ in (1.5.2), we have

$$\psi^5(q) = \sum_{n=0}^{\infty} t_5(n)q^n.$$

Employing (3 3.4), we find that

$$\begin{aligned}
q\sum_{n=0}^{\infty} t_5(n)q^n &= q\psi^5(q) \\
&= q\psi^2(q)\psi^3(q) \\
&\equiv q\psi^2(q)\psi(q^3) \\
&\equiv \frac{q\psi^2(q)\psi^2(q^3)}{\psi(q^3)} \\
&\equiv \frac{\sum_{n=0}^{\infty} C(n)q^n}{\psi(q^3)} \pmod 3.
\end{aligned} \tag{3.4.2}$$

Extracting the terms q^{3n} from both sides of the above and then replacing q^3 by q and utilizing (3.3.10), we arrive at

$$\begin{aligned}
\sum_{n=1}^{\infty} t_5(3n-1)q^n &\equiv \frac{\sum_{n=0}^{\infty} C(3n)q^n}{\psi(q)} \\
&= \frac{q\psi^2(q)\psi^2(q^3)}{\psi(q)} \pmod 3 \\
&\equiv q\psi(q)\psi^2(q^3) \\
&\equiv q\psi^2(q^3)\left(\frac{\varphi(-q^9)}{\chi(-q^3)} + q\psi(q^9)\right).
\end{aligned} \tag{3.4.3}$$

Extracting the terms q^{3n+2} from both sides of the above and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} t_5(9n+5)q^n \equiv \psi^2(q)\psi(q^3)$$
$$\equiv \sum_{n=0}^{\infty} t_5(n)q^n.$$

Equating the coefficients of q^n from both sides, we obtain

$$t_5(9n+5) \equiv t_5(n) \pmod 3. \tag{3.4.4}$$

Also, extracting the terms q^{3n+3} from both sides of (3.4.3), we have

$$t_5(9n+8) \equiv 0 \pmod 3. \tag{3.4.5}$$

From (3.4.4), (3.4.5) and mathematical induction, we arrive at the desired identities of the theorem.

□

3.5 An infinite family of congruences modulo 3 for the coefficients of $\psi^4(q^2)$

In this section, we discuss an infinite family of congruences modulo 3 for the coefficients of $\psi^4(q^2)$.

Theorem 3.5.1. *If*

$$\sum_{n=0}^{\infty} L(n)q^n := \psi^4(q^2),$$

then

$$L(9n+5) \equiv 0 \pmod 3$$

and for any $\alpha \in \mathbb{N}$,

$$L\left(3^{2\alpha}n + \frac{19 \times 3^{2\alpha-1} - 17}{8}\right) \equiv 0 \pmod 3.$$

We present the following lemma to prove the theorem.

Lemma 3.5.2. *For any r with $r \equiv \pm 1 \pmod{6}$, we have*

$$\sum_{n=0}^{\infty} d_r(n) q^n := \sum_{n=0}^{\infty} \frac{q^{6n+r}}{1-q^{12n+2r}} = \sum_{n=0}^{\infty} d_r(3n) q^n.$$

Proof. The proof is similar to the proof of Lemma 3.3.2. So details are omitted. □

Proof of Theorem 3.5.1. From [13, p. 223, Entry 3(i)], we recall the Lambert series for $q\psi(q^2)\psi(q^6)$ as

$$\begin{aligned} q\psi(q^2)\psi(q^6) &= \frac{q}{1-q^2} - \frac{q^5}{1-q^{10}} + \frac{q^7}{1-q^{14}} - \frac{q^{11}}{1-q^{22}} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{q^{6n+1}}{1-q^{12n+2}} - \sum_{n=0}^{\infty} \frac{q^{6n+5}}{1-q^{12n+10}}. \end{aligned} \tag{3.5.1}$$

Assuming

$$q\psi(q^2)\psi(q^6) =: \sum_{n=0}^{\infty} A(n) q^n,$$

from (3.5.1) and Lemma 3.5.2, we find that

$$q\psi(q^2)\psi(q^6) = \sum_{n=0}^{\infty} A(3n) q^n.$$

Employing (3.3.4), we have

$$\begin{aligned} q\sum_{n=0}^{\infty} L(n) q^n &= q\psi^4(q^2) \\ &= q\psi(q^2)\psi^3(q^2) \\ &\equiv q\psi(q^2)\psi(q^6) \pmod{3} \\ &\equiv \sum_{n=0}^{\infty} A(n) q^n \pmod{3}. \end{aligned} \tag{3.5.2}$$

Extracting the terms q^{3n} from both sides of the above and then replacing q^3 by q, we find that

$$\sum_{n=1}^{\infty} L(3n-1) q^n \equiv \sum_{n=0}^{\infty} A(3n) q^n = q\psi(q^2)\psi(q^6) \pmod{3}. \tag{3.5.3}$$

From (3.5.2) and (3.5.3), we have

$$\sum_{n=1}^{\infty} L(3n-1)q^n \equiv \sum_{n=0}^{\infty} L(n)q^{n+1} \pmod 3.$$

Equating the coefficients of q^{n+1} from both sides, we obtain

$$L(3n+2) \equiv L(n) \pmod 3. \tag{3.5.4}$$

Again, employing (3.3.10) in (3.5.3) and then extracting the terms involving q^{3n+2} and q^{3n+3}, and replacing q^3 by q, we find that

$$L(9n+5) \equiv 0 \pmod 3, \tag{3.5.5}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} L(9n+8)q^n &\equiv q\psi(q^2)\psi(q^6)\\ &\equiv \sum_{n=0}^{\infty} L(n)q^{n+1} \pmod 3.\end{aligned} \tag{3.5.6}$$

From (3.5.4)–(3.5.6) and mathematical induction, we complete the proof. □

Remark 3.5.3. *Note that $L(n) = 0$ for odd n and $L(n) = t_4(n/2)$ for even n.*

3.6 An infinite family of congruences modulo 3 for the coefficients of $\dfrac{1}{\psi^4(q)}$

In this section, we discuss an infinite family of congruences modulo 3 for the coefficients of $\dfrac{1}{\psi^4(q)}$.

Theorem 3.6.1. *If*

$$\sum_{n=0}^{\infty} P(n)q^n := \frac{1}{\psi^4(q)},$$

then

$$P(9n+8) \equiv 0 \pmod 3,$$

and for any $\alpha \in \mathbb{N}$,

$$P\left(3^{\alpha+1}n + \frac{5 \times 3^{\alpha}+1}{2}\right) \equiv 0 \pmod 3.$$

Proof. We recall the Lambert series for $q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3)$ from (3.3.2) as

$$\begin{aligned} q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3) &= \frac{q}{1-q^2} - \frac{2^2q^2}{1-q^4} + \frac{4^2q^4}{1-q^8} - \frac{5^2q^5}{1-q^{10}} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(3n+1)^2q^{3n+1}}{1-q^{6n+2}} - \sum_{n=0}^{\infty} \frac{(3n+2)^2q^{3n+2}}{1-q^{6n+4}} \\ &\equiv \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1-q^{6n+2}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1-q^{6n+4}} \pmod 3. \end{aligned} \tag{3.6.1}$$

Assuming

$$q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3) =: \sum_{n=0}^{\infty} B(n)q^n,$$

from (3.6.1) and Lemma 3.3.2, we obtain

$$q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3) = \sum_{n=0}^{\infty} B(3n)q^n \tag{3.6.2}$$

Employing (3.3.4), we have

$$\begin{aligned} q\sum_{n=0}^{\infty} P(n)q^n &= \frac{q}{\psi^4(q)} \\ &= \frac{q\psi^5(q)\psi(q^3)}{\psi^9(q)\psi(q^3)} \\ &\equiv \frac{q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3)}{\psi^4(q^3)} \pmod 3 \\ &\equiv \frac{\sum_{n=0}^{\infty} B(n)q^n}{\psi^4(q^3)} \pmod 3. \end{aligned} \tag{3.6.3}$$

Extracting the terms q^{3n} from both sides of the above and then replacing q^3 by q and utilizing (3.3.10), we arrive at

$$\begin{aligned}\sum_{n=1}^{\infty} P(3n-1)q^n &\equiv \frac{\sum_{n=0}^{\infty} B(3n)q^n}{\psi^2(q)} \\ &= \frac{q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3)}{\psi^4(q)} \pmod 3 \\ &\equiv \frac{q\psi^5(q)\psi(q^3)}{\psi^4(q)} \\ &\equiv q\psi(q)\psi(q^3) \\ &\equiv q\psi(q^3)\left(\frac{\varphi(-q^9)}{\chi(-q^3)} + q\psi(q^9)\right). \end{aligned} \tag{3.6.4}$$

Extracting the terms q^{3n+2} from both sides of the above, we have

$$\sum_{n=0}^{\infty} P(9n+5)q^n \equiv \psi(q)\psi(q^3) \equiv \sum_{n=1}^{\infty} P(3n-1)q^{n-1} \pmod 3. \tag{3.6.5}$$

Equating the coefficients of q^n from both sides, we arrive at

$$P(9n+5) \equiv P(3n+2) \pmod 3. \tag{3.6.6}$$

Also, equating the coefficients of q^{3n+3} from (3.6.4), we find that

$$P(9n+8) \equiv 0 \pmod 3. \tag{3.6.7}$$

From (3.6.6), (3.6.7) and mathematical induction, we achieve the desired identities of the theorem to complete the proof. □

3.7 Arithmetic properties of some k-gonal numbers

In the introductory chapter, we define a k-gonal number. Using (1.4.1), we give the respective generating functions of squares, triangular numbers, pentagonal numbers, heptagonal numbers, octagonal numbers and dodecagonal numbers. We define two more theta functions $A(q)$ and $B(q)$ as

$$A(q) := \sum_{n=-\infty}^{\infty} q^{\frac{n(5n-1)}{2}} = f(q^2, q^3)$$

and

$$B(q) := \sum_{n=-\infty}^{\infty} q^{n(5n-2)} = f(q^3, q^7)$$

In this section we discuss arithmetic properties of coefficients of two infinite products of theta functions and some results on the number of representation of a positive integer by sums of polygonal numbers.

We again recall from [25, p. 82, Theorem 2.2] that

$$Y_1(-q) = \varphi(q^3)\varphi(q^5) + \varphi(q)\varphi(q^{15})$$

and

$$Y_2(q) = q\psi(q^3)\psi(q^5) + q^2\psi(q)\psi(q^{15}),$$

where $Y_1(q)$ and $Y_2(q)$ are as defined in (3.1.1) and (3.1.2), respectively.

Theorem 3.7.1. *Let $D(n)$ be defined by*

$$\sum_{n=0}^{\infty} D(n)q^n := Y_2(q),$$

where $Y_2(q)$ is as defined in (3.1.2)*. Let $r\{G_i + G_j\}(n)$ denote the number of representations of n as a sum of an i-gonal number and a j-gonal number. Then*

$$D(3^\alpha n) = D(n), \tag{3.7.1}$$

$$D(5^\alpha n) = D(n). \tag{3.7.2}$$

$$r\{5G_5 + G_6\}(5n+3) = r\{G_5 + 5G_6\}(n), \tag{3.7.3}$$

$$r\{G_5 + 5G_6\}(5n+1) = r\{5G_5 + G_6\}(n), \tag{3.7.4}$$

$$r\{5G_5 + G_6\}(5n+2) = 0, \tag{3.7.5}$$

$$r\{5G_5 + G_6\}(5n + 4) = 0, \tag{3.7.6}$$

$$r\{G_5 + 5G_6\{(5n + 3) = 0, \tag{3.7.7}$$

$$r\{G_5 + 5G_6\}(5n + 4) = 0, \tag{3.7.8}$$

$$r\{5G_5 + G_6\}(n) = r\{5G_5 + G_6\}\left(5^{2\alpha}n + \frac{5^{2\alpha} - 1}{3}\right), \tag{3.7.9}$$

$$r\{G_5 + 5G_6\}(n) = r\{G_5 + 5G_6\}\left(5^{2\alpha}n + \frac{2(5^{2\alpha} - 1)}{3}\right), \tag{3.7.10}$$

$$r\{5G_5 + G_6\}(5n) = r\{G_5 + A\}(n), \tag{3.7.11}$$

$$r\{5G_5 + G_6\}(5n + 1) = r\{G_5 + G_7\}(n), \tag{3.7.12}$$

$$r\{G_6 + 3A\}(3n + 1) = r\{3G_6 + A\}(n), \tag{3.7.13}$$

$$r\{G_5 + 3A\}(3n + 2) = 0, \tag{3.7.14}$$

$$r\{G_6 + 3A\}(3n) = r\{G_5 + A\}(n), \tag{3.7.15}$$

$$r\{3G_6 + A\}(3n + 2) = r\{G_6 + 3G_7\}(n), \tag{3.7.16}$$

$$r\{3G_6 + A\}(3n + 1) = 0, \tag{3.7.17}$$

$$r\{3G_6 + G_7\}(3n) = r\{G_6 + 3A\}(n), \tag{3.7.18}$$

$$r\{3G_6 + G_7\}(3n + 2) = 0, \tag{3.7.19}$$

$$r\{G_6 + 3G_7\}(3n + 1) = r\{3G_6 + G_7\}(n), \tag{3.7.20}$$

$$r\{G_6 + 3G_7\}(3n) = r\{G_5 + G_6\}(n), \tag{3.7.21}$$

and

$$r\{G_6 + 3G_7\}(3n + 2) = 0. \tag{3.7.22}$$

Proof. We recall the 3- and 5-dissections of $\psi(q)$ from [13, p. 49, Corollary (ii)] as

$$\psi(q) = f(q^3, q^6) + q\psi(q^9) \tag{3.7.23}$$

and

$$\psi(q) = f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3\psi(q^{25}). \tag{3.7.24}$$

Therefore,

$$\sum_{n=0}^{\infty} D(n)q^n = q\psi(q^3)\psi(q^5) + q^2\psi(q)\psi(q^{15})$$

$$= q\psi(q^3)\{f(q^{15}, q^{30}) + q^5\psi(q^{45})\} + q^2\{f(q^3, q^6) + q\psi(q^9)\}\psi(q^{15}). \tag{3.7.25}$$

Extracting the terms q^{3n} from both sides and replacing q^3 by q, we find that

$$\sum_{n=0}^{\infty} D(3n)q^n = \sum_{n=0}^{\infty} D(n)q^n$$

Equating the coefficients of q^n from both sides of the above, we obtain

$$D(n) = D(3n). \tag{3.7.26}$$

Again, by employing (3.7.24), we have

$$\sum_{n=0}^{\infty} D(n)q^n = q\psi(q^3)\psi(q^5) + q^2\psi(q)\psi(q^{15})$$

$$= q\{f(q^{30}, q^{45}) + q^3 f(q^{15}, q^{60}) + q^9\psi(q^{75})\}\psi(q^5)$$

$$+ q^2\{f(q^{10}, q^{15}) + q^3 f(q^5, q^{20}) + q^3\psi(q^{25})\}\psi(q^{15}). \tag{3.7.27}$$

Extracting the terms involving q^{5n} from both sides of the above, replacing q^5 by q and then equating the coefficients of q^n from both sides of the resulting identity, we find that

$$D(n) = D(5n). \tag{3.7.28}$$

From (3.7.26), (3.7.28) and mathematical induction, we readily arrived at (3.7.1) and (3.7.2).

Again, extracting the terms q^{3n+1} and q^{3n+2} from (3.7.25) and replacing q^3 by q, we find that

$$\sum_{n=0}^{\infty} D(3n+1)q^n = \psi(q)f(q^5, q^{10}) = \sum_{n=0}^{\infty} r\{5G_5 + G_6\}(n)q^n \tag{3.7.29}$$

and

$$\sum_{n=0}^{\infty} D(3n+2)q^n = \psi(q^5)f(q,q^2) = \sum_{n=0}^{\infty} r\{G_5+5G_6\}(n)q^n. \tag{3.7.30}$$

Similarly by extracting the terms q^{5n+1}, q^{5n+2}, q^{5n+3} and q^{5n+4} from (3.7.27) and replacing q^5 by q,

$$\sum_{n=0}^{\infty} D(5n+1)q^n = \psi(q)f(q^6,q^9) = \sum_{n=0}^{\infty} r\{G_6+3A\}(n)q^n, \tag{3.7.31}$$

$$\sum_{n=0}^{\infty} D(5n+2)q^n = \psi(q^3)f(q^2,q^3) = \sum_{n=0}^{\infty} r\{3G_6+A\}(n)q^n, \tag{3.7.32}$$

$$\sum_{n=0}^{\infty} D(5n+3)q^n = \psi(q^3)f(q,q^4) = \sum_{n=0}^{\infty} r\{3G_6+G_7\}(n)q^n, \tag{3.7.33}$$

and

$$\sum_{n=0}^{\infty} D(5n+4)q^n = \psi(q)f(q^3,q^{12}) = \sum_{n=0}^{\infty} r\{G_6+3G_7\}(n)q^n. \tag{3.7.34}$$

By employing (3.7.24) in (3.7.29) we easily establish (3.7.3), (3.7.5), (3.7.6), (3.7.11) and (3.7.12). Similarly, using a 5-dissection of $f(q,q^2)$ in (3.7.30), we prove (3.7.4), (3.7.7) and (3.7.8). From (3.7.3) and (3.7.4), we readily conclude (3.7.9) and (3.7.10) by mathematical induction. Employing (3.7.23) in (3.7.31), we arrive at the results (3.7.13)- (3.7.15). Using a 3-dissection of $f(q^2,q^3)$ in (3.7.32), we deduce (3.7.16) and (3.7.17). Similarly, by employing 3-dissections of $f(q,q^4)$ and $\psi(q)$ in (3.7.33) and (3.7.34), respectively, we arrive at the remaining five results to complete the proof of the theorem. ☐

Theorem 3.7.2. *Let $E(n)$ be defined by*

$$\sum_{n=0}^{\infty} E(n)q^n := Y_1(-q),$$

where $Y_1(q)$ is as defined in (3.1.1) *and let $r\{G_i+G_j\}(n)$ is as defined in the previous* *theorem. Then*

$$E(3^\alpha n) = E(n),$$

$$E(5^\alpha n) = E(n),$$

$$r\{G_4 + 5G_8\}(5n) = r\{5G_4 + G_8\}(n),$$

$$r\{G_4 + 5G_8\}(5n+1) = 2r\{G_8 + B\}(n),$$

$$r\{G_4 + 5G_8\}(5n+4) = 2r\{G_8 + G_{12}\}(n),$$

$$r\{G_4 + 5G_8\}(5n+2) = 0,$$

$$r\{G_4 + 5G_8\}(5n+3) = 0,$$

$$r\{5G_4 + G_8\}(5n+8) = r\{G_4 + 5G_8\}(n),$$

$$r\{5G_4 + G_8\}(5n+2) = 0,$$

$$r\{5G_4 + G_8\}(5n+4) = 0,$$

$$r\{5G_4 + G_8\}(n) = r\{5G_4 + G_8\}\left(5^{2\alpha}n + \frac{5^{2\alpha}-1}{3}\right),$$

$$r\{G_4 + 5G_8\}(n) = r\{G_4 + 5G_8\}\left(5^{2\alpha}n + \frac{5(5^{2\alpha}-1)}{3}\right),$$

$$r\{3G_4 + G_{12}\}(3n+1) = r\{G_4 + 3B\}(n),$$

$$r\{3G_4 + G_{12}\}(3n+2) = 0,$$

$$r\{G_4 + 3B\}(3n) = r\{3G_4 + B\}(n),$$

$$r\{G_4 + 3B\}(3n+1) = 2r\{G_8 + B\}(n),$$

$$r\{G_4 + 3B\}(3n+2) = 0,$$

$$r\{3G_4 + B\}(3n+7) = r\{G_4 + 3G_{12}\}(n),$$

$$r\{3G_4 + B\}(3n+2) = 0,$$

$$r\{G_4 + 3G_{12}\}(3n) = r\{3G_4 + G_{12}\}(n),$$

$$r\{G_4 + 3G_{12}\}(3n+1) = 2r\{G_8 + G_{12}\}(n),$$

and

$$r\{G_4 + 3G_{12}\}(3n+2) = 0.$$

Proof. The proof is similar to that of Theorem 3.7.1. So we omit the details. □

Chapter 4

Colored Partition Identities Conjectured by Sandon and Zanello

4.1 Introduction

As mentioned in the introductory chapter, this chapter deals with colored partition identities. In the next six sections, we prove 17 of the conjectures in [58] and find analogous partition identities for the remaining 12 conjectures. It would be clear from our proofs of the partition identities that more such colored partition identities could be found. In the last section of this chapter, we present some new colored partition identities of the same type.

The contents of this chapter appeared in [10].

We now state some theta function identities which will be used in the subsequent sections of this chapter.

Lemma 4.1.1. [13, p. 45, Entry 29] *If $ab = cd$, then*

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc) \tag{4.1.1}$$

and

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(b/c, ac^2d\right) f\left(b/d, acd^2\right). \tag{4.1.2}$$

Lemma 4.1.2. [13, p. 40, Entry 25] *We have*

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4),$$

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8),$$

$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4),$$

and

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2).$$

4.2 Partition identities analogous to Conjectures 3.24, 3.25 and 3.27 of [58]

Conjectures 3.24, 3.25 and 3.27 of [58] have been proved by Berndt and Zhou [18] by employing a certain kind of Ramanujan modular equation involving multipliers. In this section, we present three analogous partition identities without restricting the parity of the number of distinct elements of S (and/or, T). It is worthwhile to mention that the same kind of partition identities may be obtained from other analogous modular equations of Ramanujan involving multipliers.

Theorem 4.2.1. (Analogues to Corollary to Conjecture 3.24 of [58] and to Theorem 3.3 of [17]) *Let S be the set containing one copy of the even positive integers that are not multiples of* 25, *and T be the set containing one copy of the odd positive integers that are not multiples of* 25. *Let $a(N)$ be the difference of the number of partitions of N into an even number of distinct non multiples of* 25 *and the number of partitions of N into an odd number of distinct non multiples of* 25. *Then $P_S(2) = 2 + a(1)$ and for $N > 1$, we have*

$$P_S(2N) = 2P_T(2N-3) + a(N). \tag{4.2.1}$$

Proof. From [13, p. 291, Entry 15(i)], we recall a modular equation of degree 25 as

$$\left(\frac{\beta}{\alpha}\right)^{1/8}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/8}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}-2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}=(mm')^{1/2},$$

where β has degree 25 over α. Transcribing this modular equation with the aid of Lemma 1.1.1, we have

$$q^3\left\{\frac{\psi(q^{25})}{\psi(q)}-\frac{\psi(-q^{25})}{\psi(-q)}\right\}=1+2q^2\frac{f(-q^{50})}{f(-q^2)}-\frac{\varphi(-q^{50})}{\varphi(-q^2)},$$

which can be transformed into the q-product identity

$$\frac{(-q^2;q^2)_\infty}{(-q^{50};q^{50})_\infty}=q^3\left\{\frac{(-q;q^2)_\infty}{(-q^{25};q^{50})_\infty}-\frac{(q;q^2)_\infty}{(q^{25};q^{50})_\infty}\right\}+2q^2+\frac{f(-q^2)}{f(-q^{50})}.$$

Thus,

$$\sum_{n=0}^{\infty}P_S(n)q^n=q^3\left\{\sum_{n=0}^{\infty}P_T(n)q^n-\sum_{n=0}^{\infty}P_T(n)(-q)^n\right\}+2q^2+\frac{f(-q^2)}{f(-q^{50})}.$$

Equating the coefficients of q^{2N} from both sides of the above, and noting that

$$\sum_{n=0}^{\infty}a(n)q^n=\frac{f(-q)}{f(-q^{25})},$$

we easily arrive at (4.2.1). □

Example: $n=5$ in (4.2.1).

Then $P_S(10)$=3; the relevant partitions of 10 are 10, $8+2$ and $6+4$; $P_T(7)$=1; the relevant partition of 7 is 7, and $a(5)=1$.

Corollary 4.2.2. *For $N \geq 0$,*

$$P_S(10N+6)=2P_T(10N+3), \tag{4.2.2}$$

$$P_S(10N+8)=2P_T(10N+5), \tag{4.2.3}$$

and, for $N \geq 1$,

$$P_S(10N+2)=2P_T(10N-1). \tag{4.2.4}$$

Furthermore,

$$a(10) = a(17) = a(20) = a(43) = a(45) = a(67) = a(117) = 0;$$

i.e.,

$$P_S(20) = 2P_T(17), P_S(34) = 2P_T(31), P_S(40) = 2P_T(37), P_S(86) = 2P_T(83),$$

$$P_S(90) = 2P_T(87), P_S(134) = 2P_T(131), P_S(234) = 2P_T(231),$$

and

$$a(25n) > 0, a(25n+5) > 0, a(25n+7) > 0, a(25n+17) > 0, a(25n+22 > 0;$$

$$a(25n+2) < 0, a(25n+10) < 0, a(25n+12) < 0, a(25n+15) < 0, a(25n+20) < 0.$$

Proof. We recall from [13, p. 82] that

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{f(-q)}{f(-q^{25})} = \frac{f(-q^{10}, -q^{15})}{f(-q^5, -q^{20})} - q - q^2 \frac{f(-q^5, -q^{20})}{f(-q^{10}, -q^{15})}.$$

Extracting various terms from both sides of the above, we find that

$$\begin{aligned} &a(1) = -1, \ a(5n+1) = 0, \ \text{for } n \geq 1, \\ &a(5n+3) = 0 = a(5n+4), \ \text{for } n \geq 0, \\ &\sum_{n=0}^{\infty} a(5n)q^n = \frac{f(-q^2, -q^3)}{f(-q, -q^4)}, \end{aligned} \tag{4.2.5}$$

and

$$\sum_{n=0}^{\infty} a(5n+2)q^n = -\frac{f(-q, -q^4)}{f(-q^2, -q^3)}.$$

From (4.2.5) and (4.2.1), we arrive at (4.2.2)–(4.2.4).

Next, let $\gamma(n)$ and $\delta(n)$ be defined by

$$\sum_{n=0}^{\infty} \gamma(n)q^n := \sum_{n=0}^{\infty} a(5n)q^n = \frac{f(-q^2, -q^3)}{f(-q, -q^4)} = \frac{1}{q^{-1/5}R(q)}$$

and

$$\sum_{n=0}^{\infty} \delta(n)q^n := \sum_{n=0}^{\infty} a(5n+2)q^n = -\frac{f(-q,-q^4)}{f(-q^2,-q^3)} = -q^{-1/5}R(q),$$

where $R(q)$ is the famous Rogers-Ramanujan continued fraction, defined by

$$R(q) := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots, \quad |q| < 1. \tag{4.2.6}$$

The coefficients $\gamma(n)$ and $\delta(n)$ have been extensively studied by various authors. We refer to Chapter 4 of [1] for many references. In particular, from [1, pp. 111–113, Corollary 4.2.1 and Corollary 4.2.2], we have

$$\gamma(2) = \gamma(4) = \gamma(9) = 0, \delta(3) = \delta(8) = \delta(13) = \delta(23) = 0,$$

$$\gamma(5n) > 0,\ \gamma(5n+1) > 0,\ \gamma(5n+2) < 0,\ \gamma(5n+3) < 0,\ \gamma(5n+4) < 0,$$

$$\delta(5n) < 0,\ \delta(5n+2) < 0,\ \delta(5n+1) > 0,\ \delta(5n+3) > 0,\ \delta(5n+4) > 0.$$

Therefore,

$$a(10) = a(20) = a(17) = a(42) = a(45) = a(67) = a(117) = 0,$$

$$a(25n) > 0,\ a(25n+5) > 0,\ a(25n+7) > 0,\ a(25n+17) > 0,\ a(25n+22) > 0,$$

$$a(25n+2) < 0,\ a(25n+10) < 0,\ a(25n+12) < 0,\ a(25n+15) < 0,$$

$$a(25n+20) < 0,$$

which completes the proof. □

Theorem 4.2.3. (Analogues to Corollary to Conjecture 3.25 of [58] and to Theorem 2.7 of [17]) *Let S be the set containing 2 copies of the even positive integers that are not multiples of 13, and T be the set containing 2 copies of the odd positive integers that are not multiples of 13. Let $a(N)$ be the difference of the number of 2-colored partitions of N into an even number of distinct non multiples of 13 and the number* *of 2-colored partitions of N into an odd number of distinct non multiples of* 13. *Then* $P_S(2) = 4 + a(1)$ *and for any* $N > 1$, *we have*

$$P_S(2N) = 2P_T(2N-3) + a(N). \tag{4.2.7}$$

Proof. If β has degree 13 over α and m is the multiplier connecting α and β, then from [13, p. 376, Entry 8(iii)], we have

$$\left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} - 4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6} = m,$$

which can be transcribed, with the help of the identities in Lemma 1.1.1, into

$$\frac{\varphi^2(-q^{26})}{\varphi^2(-q^2)} = q^3\frac{\psi(q^{26})}{\psi(q^2)}\left\{\frac{\varphi(-q^{13})}{\varphi(-q)} - \frac{\varphi(q^{13})}{\varphi(q)}\right\} + 4q^2\frac{f^4(q^{13})\varphi^2(q)}{f^4(q)\varphi^2(q^{13})} + 1.$$

The above can be further transformed into

$$\frac{(-q^2;q^2)^2_\infty}{(-q^{26};q^{26})^2_\infty} = q^3\left\{\frac{(-q;q^2)^2_\infty}{(-q^{13};q^{26})^2_\infty} - \frac{(q;q^2)^2_\infty}{(q^{13};q^{26})^2_\infty}\right\} + 4q^2 + \frac{f^2(-q^2)}{f^2(-q^{26})}.$$

Thus,

$$\sum_{n=0}^{\infty} P_S(n)q^n = q^3\left\{\sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\} + 4q^2 + \frac{f^2(-q^2)}{f^2(-q^{26})}.$$

Now, the generating function of $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{f^2(-q)}{f^2(-q^{13})}.$$

Equating the coefficients of q^{2N} from both sides of the above, we easily arrive at the desired identity to complete the proof. □

Example: $n = 5$ *in* (4.2.7).

Then $P_S(10) = 14$ as there are two copies of each of the types 10, $6+2+2$ and $4+4+2$ and 4 additional copies of each of the forms $8+2$ and $6+4$; $P_T(7) = 6$, the relevant partitions of 7 are 2 copies each of the types 7, $5+1+1$ and $3+3+1$, and $a(5) = 2$.

Theorem 4.2.4. (Analogues to Corollary to Conjecture 3.27 of [58] and to Theorem 2.5 of [17]) *Let S be the set containing 4 copies of the even positive integers that are not multiples of 7, and T the set containing 4 copies of the odd positive integers that are not multiples of 7. Furthermore, let $a(N)$ be the difference of the number of 4-colored partitions of N into an even number of distinct non multiples of 7 and the number of 4-colored partitions of N into an odd number of distinct non multiples of 7. Then*

$$P_S(2) = 8 + a(1) \quad \textit{and} \quad \textit{for } N > 1, \quad P_S(2N) = 2P_T(2N-3) + a(N). \tag{4.2.8}$$

Proof. If β has degree 7 over α and m is the multiplier connecting α and β, then, from [13, p. 314, Entry 19(v)]

$$\left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} - 8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3} = m^2,$$

which can be transcribed, with the help of Lemma 1.1.1, into

$$\frac{\varphi^4(-q^{14})}{\varphi^4(-q^2)} = 1 + q^3\left\{\frac{\psi^4(-q^7)}{\psi^4(-q)} - \frac{\psi^4(q^7)}{\psi^4(q)}\right\} + 8q^2\frac{f^4(-q^{14})}{f^4(-q^2)}.$$

Transforming the theta functions into q-products, with the aid of (1.1.3)–(1.1.5), we find that

$$\frac{(-q^2;q^2)^4_\infty}{(-q^{14};q^{14})^4_\infty} = q^3\left\{\frac{(-q;q^2)^4_\infty}{(-q^7;q^{14})^4_\infty} - \frac{(q;q^2)^4_\infty}{(q^7;q^{14})^4_\infty}\right\} + 8q^2 + \frac{f^4(-q^2)}{f^4(-q^{14})},$$

which can be written as

$$\sum_{n=0}^{\infty} P_S(n)q^n = q^3\left\{\sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\} + 8q^2 + \frac{f^4(-q^2)}{f^4(-q^{14})}. \tag{4.2.9}$$

Now, the generating function of $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{f^4(-q)}{f^4(-q^7)}.$$

Thus, equating the coefficients of q^{2N} from both sides of (4.2.9), we easily arrived at (4.2.8) to complete the proof. □

Example $n = 3$ *in* (4 2 8)

Then $P_S(6) = 24$, the relevant partitions of 6 are 4 copies each of the types 6 and $2+2+2$ and 16 additional copies of the form $4+2$, $P_T(3) = 8$, the relevant partitions of 3 are 4 copies each of the types 3 and $1+1+1$, and $a(3) = 8$

4.3 Conjectures 3.51 and 3.26 of [58]

Theorem 4.3.1. (Corollary to Conjecture 3 51 of [58]) *Let S be the set containing one copy of the even positive integers, 2 copies of the odd positive integers, one more copy of the positive multiples of 14, and 2 more copies of the odd positive multiples of 7, let T be the set containing 2 copies of the even positive integers, one copy of the odd positive integers, 2 more copies of the positive multiples of 14, and one more copy of the odd positive multiples of 7 Then, for any $N \geq 1$,*

$$D_S(N) = 2D_T(N-1),$$

or equivalently,

$$P_S(N) = 2P_T(N-1)$$

Proof We recall from Berndt s book [13, p 304] that

$$\varphi(q)\varphi(q^7) - \varphi(-q^2)\varphi(-q^{14}) = 2q\psi(q)\psi(q^7) \qquad (4\ 3\ 1)$$

Transforming this into q-products with the aid of (1 1 3)–(1 1 5) and canceling $(q^2, q^2)_\infty(q^{14}, q^{14})_\infty$ from both sides we find that

$$(-q, q^2)_\infty^2(-q^7, q^{14})_\infty^2 - (q^2, q^4)_\infty(q^{14}, q^{28})_\infty = \frac{2q}{(q, q^2)_\infty(q^7, q^{14})_\infty}$$

Multiplying both sides by $(-q^2, q^2)_\infty(-q^{14}, q^{14})_\infty$ and then using Euler's identity $(-q, q)_\infty = (q, q^2)_\infty^{-1}$, we obtain

$$\begin{aligned}&(-q, q^2)_\infty^2(-q^7, q^{14})_\infty^2(-q^2, q^2)_\infty(-q^{14}, q^{14})_\infty \\ &\quad = 2q(-q, q^2)_\infty(-q^7, q^{14})_\infty(-q^2\ q^2)_\infty^2(-q^{14}, q^{14})_\infty^2 + 1,\end{aligned}$$

which is equivalent to

$$\sum_{n=0}^{\infty} D_S(n)q^n = 2q\sum_{n=0}^{\infty} D_T(n)q^n + 1.$$

Equating the coefficients of q^N, we arrive at the desired result. □

Example: $n = 5$.

Then $P_S(5) = 8$, the relevant partitions of 5 are 2 copies each of the types 5, $4+1$, $3+2$ and $3+1+1$; $P_T(4) = 4$, the relevant partitions of 4 are 2 copies of 4 and 1 copy each of the types $3+1$ and $2+2$.

Conjecture 4.3.1. (Corollary to Conjecture 3.26 of [58]) *Let S be the set containing 3 copies of the odd positive integers and 3 more copies of the odd positive multiples of 7, and T the set containing 3 copies of the even positive integers and 3 more copies of the positive multiples of 14. Then, for any $N \geq 3$,*

$$D_S(N) = 4D_T(N-3).$$

We prove the following equivalent version of the conjecture.

Theorem 4.3.2. *Let S and T be as defined in Conjecture 4.3.1. Then, for any $N \geq 1$,*

$$D_S(2N+1) = 4D_T(2N-2), \tag{4.3.2}$$

or equivalently,

$$P_S(2N+1) = 4P_T(2N-2). \tag{4.3.3}$$

Proof. Cubing (4.3.1), we find that

$$\varphi^3(q)\varphi^3(q^7) - \varphi^3(-q^2)\varphi^3(-q^{14}) = 8q^3\psi^3(q)\psi^3(q^7)$$
$$+ 6q\psi(q)\psi(q^7)\varphi(q)\varphi(q^7)\varphi(-q^2)\varphi(-q^{14}),$$

which can be transformed, with the aid of (1.1.3)–(1.1.5), into

$$\{(-q;q^2)_\infty^6(-q^7;q^{14})_\infty^6 - (q^2;q^4)_\infty^3(q^{14};q^{28})_\infty^3\}(q^2;q^2)_\infty^3(q^{14};q^{14})_\infty^3$$
$$= 8q^3\frac{(q^2;q^2)_\infty^3(q^{14};q^{14})_\infty^3}{(q;q^2)_\infty^3(q^7;q^{14})_\infty^3} + 6q(q^2;q^2)_\infty^3(q^{14};q^{14})_\infty^3(-q;q^2)_\infty^3(-q^7;q^{14})_\infty^3.$$

Dividing both sides of the above identity by $(q^2;q^2)_\infty^3(q^{14};q^{14})_\infty^3(-q;q^2)_\infty^3(-q^7;q^{14})_\infty^3$ and then using the trivial identity $(q^2;q^4)_\infty = (q;q^2)_\infty(-q;q^2)_\infty$, we arrive at

$$(-q;q^2)_\infty^3(-q^7;q^{14})_\infty^3 - (q;q^2)_\infty^3(q^7;q^{14})_\infty^3 = 8q^3\frac{1}{(q^2;q^4)_\infty^3(q^{14};q^{28})_\infty^3} + 6q,$$

which by Euler's identity $(-q;q)_\infty = (q,q^2)_\infty^{-1}$ reduces to

$$(-q;q^2)_\infty^3(-q^7;q^{14})_\infty^3 - (q;q^2)_\infty^3(q^7;q^{14})_\infty^3 = 8q^3(-q^2;q^2)_\infty^3(-q^{14};q^{14})_\infty^3 + 6q.$$

Thus,

$$\sum_{n=0}^{\infty} D_S(n)q^n - \sum_{n=0}^{\infty} D_S(n)(-q)^n = 8q^3\sum_{n=0}^{\infty} D_T(n)q^n + 6q$$

or

$$\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n = 8q^3\sum_{n=0}^{\infty} P_T(n)q^n + 6q.$$

Comparing the coefficients of q^{2N+1} from both sides of the above identities, we readily arrive at (4.3.2) and (4.3.3) to complete the proof. □

Example: $n = 3$ in (4.3.3).

Then $P_S(7) = 24$, the relevant partitions of 7 are 6 copies of the type 7 and 9 copies each of the types $5+1+1$ and $3+3+1$; $P_T(4) = 6$, the relevant partitions of 4 are 3 copies each of the types 4 and $2+2$.

Remark 4.3.3. *The above two theorems have also been proved by Berndt and Zhou [18] by using Ramanujan's modular equations.*

4.4 Partition identities in Conjectures 3.38, 3.28, 3.30, and 3.42 of [58]

Conjecture 4.4.1. (Corollary to Conjecture 3.38 of [58]) *Let S be the set containing 3 copies of the odd positive integers that are not multiples of 3, one copy of the odd positive multiples of 3 that are not multiples of 9, and 4 copies of the odd positive multiples of 9; let T be the set containing 3 copies of the even positive integers that are not multiples of 3, one copy of the positive multiples of 6 that are not multiples of 18, and 4 copies of the positive multiples of 18. Then, for any $N \geq 3$,*

$$D_S(N) = 2D_T(N-3).$$

We prove the following equivalent theorem.

Theorem 4.4.1. *If S and T are as defined in Conjecture 4.4.1, then $D_S(1) = 3$ and for $N > 1$,*

$$D_S(2N+1) = 2D_T(2N-2) \tag{4.4.1}$$

or equivalently,

$$P_S(2N+1) = 2P_T(2N-2) \tag{4.4.2}$$

Proof. First we recall from (3.3.10) that

$$\psi(q) = \frac{\varphi(-q^9)}{\chi(-q^3)} + q\psi(q^9).$$

Replacing q by $-q$ and then employing the trivial identity $\chi^3(q) = \varphi(q)/\psi(-q)$, we have

$$\frac{\chi^3(q^9)}{\chi(q^3)} = q + \frac{\psi(-q)}{\psi(-q^9)} \tag{4.4.3}$$

Again, from [14, p. 202, Entry 50(i)], we recall that

$$\frac{\chi^3(q)}{\chi(q^3)} = 1 + 3q\frac{\psi(-q^9)}{\psi(-q)}. \tag{4.4.4}$$

Multiplying the previous two identities, we have

$$\frac{\chi^3(q)\chi^3(q^9)}{\chi^2(q^3)} = 4q + 3q^2\frac{\psi(-q^9)}{\psi(-q)} + \frac{\psi(-q)}{\psi(-q^9)}. \tag{4.4.5}$$

Next, by [13, p. 358, Entries 4(i) and (ii)],

$$\frac{\varphi(-q^{18})}{\varphi(-q^2)} + q\left(\frac{\psi(q^9)}{\psi(q)} - \frac{\psi(-q^9)}{\psi(-q)}\right) = 1$$

and

$$\frac{\varphi(-q^2)}{\varphi(-q^{18})} + \frac{1}{q}\left(\frac{\psi(q)}{\psi(q^9)} - \frac{\psi(-q)}{\psi(-q^9)}\right) = 3.$$

Replacing q by $-q$ in (4.4.5) and then subtracting from (4.4.5) and using the above two identities, we obtain

$$\begin{aligned}\frac{\chi^3(q)\chi^3(q^9)}{\chi^2(q^3)} - \frac{\chi^3(-q)\chi^3(-q^9)}{\chi^2(-q^3)} &= 2q + 3q\frac{\varphi(-q^{18})}{\varphi(-q^2)} + q\frac{\varphi(-q^2)}{\varphi(-q^{18})}, \\ &= 2q + \frac{q}{\varphi(-q^2)\varphi(-q^{18})}\left\{\varphi^2(-q^2) + 3\varphi^2(-q^{18})\right\}.\end{aligned} \tag{4.4.6}$$

Now, by [13, p. 49, Corollary (i)],

$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}).$$

Noting by (1.1.2), that

$$f(q, q^5) = (-q; q^6)_\infty(-q^5; q^6)_\infty(q^6; q^6)_\infty = \psi(-q^3)\chi(q),$$

we rewrite the previous identity as

$$\varphi(q) = \varphi(q^9) + 2q\psi(-q^9)\chi(q^3), \tag{4.4.7}$$

that is,

$$\varphi(q) - \varphi(q^9) = 2q\psi(-q^9)\chi(q^3).$$

Again, by [7, Eq. (3.37)],

$$3\varphi(q^9) - \varphi(q) = 2\psi(-q)\chi(q^3).$$

Multiplying the above two identities and then replacing q by $-q^2$, we find that

$$\varphi^2(-q^2) + 3\varphi^2(-q^{18}) = 4\varphi(-q^2)\varphi(-q^{18}) + 4q^2\psi(q^2)\psi(q^{18})\chi^2(-q^6). \tag{4.4.8}$$

Employing (4.4.8) in (4.4.6), we obatin

$$\begin{aligned}\frac{\chi^3(q)\chi^3(q^9)}{\chi^2(q^3)} - \frac{\chi^3(-q)\chi^3(-q^9)}{\chi^2(-q^3)} &= 6q + 4q^3\frac{\psi(q^2)\psi(q^{18})}{\varphi(-q^2)\varphi(-q^{18})}\chi^2(-q^6), \\ &= 6q + 4q^3\frac{\chi^2(-q^6)}{\chi^3(-q^2)\chi^3(-q^{18})}. \end{aligned} \tag{4.4.9}$$

The above identity can be rewritten as

$$\sum_{n=0}^{\infty} D_S(n)q^n - \sum_{n=0}^{\infty} D_S(n)(-q)^n = 6q + 4q^3\sum_{n=0}^{\infty} D_T(n)q^n$$

or

$$\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n = 6q + 4q^3\sum_{n=0}^{\infty} P_T(n)q^n.$$

Equating the coefficients of q^{2N+1} from both sides of the above identities, we readily arrive at (4.4.1) and (4.4.2) to finish the proof. □

Example: $n = 3$ *in* (4.4.2).

Then $P_S(7) = 12$, the relevant partitions of 7 are 3 copies of 7 and 9 copies of the type $5 + 1 + 1$; $P_T(4) = 6$, the relevant partitions of 4 are 3 copies each of the types 4 and $2 + 2$.

Remark 4.4.2. *The identity* (4.4.1) *has also been established by Berndt and Zhou [18] by using Ramanujan's modular equations.*

Conjecture 4.4.2. (Corollary to Conjecture 3 28 of [58]) *Let S be the set containing 3 copies of the even positive integers that are not multiples of 9, and T be the set containing 3 copies of the odd positive integers that are not multiples of 9 Then, for any $N \geq 4$,*

$$D_S(N) = D_T(N-3)$$

An equivalent form of the above conjecture has been proved by Berndt and Zhou [18] Here we find the following analogous result

Theorem 4.4.3. *Let S and T be as defined in Conjecture 4 4 2 and let $a(N)$ be the difference of the number of 3-colored partitions of N into an even number of distinct non multiples of 9 and the number of 3-colored partitions of N into an odd number of distinct non multiples of 9 Then, $P_S(2) = 3 + a(1)$ and for $N > 1$, we have*

$$P_S(2) = 3 + a(1) \text{ and } P_S(2N) = 2P_T(2N-3) + a(N), \text{ for } N > 1 \tag{4 4 10}$$

Proof Multiplying (3 3 10) by $\varphi(q^9)$ and (4 4 7) by $q\psi(q^9)$ and then subtracting the second from the first, we have

$$\psi(q)\varphi(q^9) - q\varphi(q)\psi(q^9) = \frac{\varphi^2(-q^{18}) - 2q^2\psi(q^{18})\varphi(-q^{18})\chi(-q^6)}{\chi(-q^3)},$$

where we have also used the trivial identities $\chi(q)\chi(-q) = \chi(-q^2)$, $\varphi(q)\varphi(-q) = \varphi^2(-q^2)$ and $\psi(q)\psi(-q) = \psi(q^2)\varphi(-q^2)$ Replacing q by $-q^2$ in (4 4 7) and then using it in the above identity, we obtain

$$\psi(q)\varphi(q^9) - q\varphi(q)\psi(q^9) = \frac{\varphi(-q^2)\varphi(-q^{18})}{\chi(-q^3)} \tag{4 4 11}$$

Cubing, we have

$$\begin{aligned}
&\psi^3(q)\varphi^3(q^9) - q^3\varphi^3(q)\psi^3(q^9) \\
&\quad = \frac{\varphi^3(-q^2)\varphi^3(-q^{18})}{\chi^3(-q^3)} + 3q\psi(q)\varphi(q^9)\varphi(q)\psi(q^9)\frac{\varphi(-q^2)\varphi(-q^{18})}{\chi(-q^3)}
\end{aligned}$$

Dividing both sides of the above by $\psi^3(q)\psi^3(q^9)$ and using $\dfrac{\varphi(q)}{\psi(q)} = \chi(q)\chi(-q^2)$ and then simplifying further, we find that

$$\frac{\chi^3(-q^{18})}{\chi^3(-q^2)} = q^3\frac{\chi^3(q)}{\chi^3(q^9)} + \frac{\chi^3(-q)\chi^6(-q^9)}{\chi^3(-q^3)} + 3q\frac{\chi^3(-q^9)}{\chi(-q^3)}. \tag{4.4.12}$$

Replacing q by $-q$ in (4.4.12) and then adding the resulting identity with (4.4.12), we have

$$2\frac{\chi^3(-q^{18})}{\chi^3(-q^2)} = q^3\left\{\frac{\chi^3(q)}{\chi^3(q^9)} - \frac{\chi^3(-q)}{\chi^3(-q^9)}\right\} + \frac{\chi^3(-q)\chi^6(-q^9)}{\chi^3(-q^3)} + \frac{\chi^3(q)\chi^6(q^9)}{\chi^3(q^3)}$$
$$+ 3q\left\{\frac{\chi^3(-q^9)}{\chi(-q^3)} - \frac{\chi^3(q^9)}{\chi(q^3)}\right\}. \tag{4.4.13}$$

Now, Ramanujan's third degree modular equation

$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^3}{\alpha}\right)^{1/8} = 1$$

can be transformed into (see [5, Theorem 4.1])

$$\frac{f^3(-q^3)}{f(-q)} - \frac{f^3(q^3)}{f(q)} = 2q\frac{f^3(-q^{12})}{f(-q^4)}.$$

Multiplying both sides of the above by $\dfrac{f(-q^2)}{f^3(-q^6)}$ and noting that $f(-q) = \chi(-q)f(-q^2)$, we find that

$$\frac{\chi^3(-q^3)}{\chi(-q)} - \frac{\chi^3(q^3)}{\chi(q)} = 2q\frac{\chi(-q^2)}{\chi^3(-q^6)}. \tag{4.4.14}$$

Employing the above, with q replaced by q^3, in (4.4.13), we obtain

$$2\frac{\chi^3(-q^{18})}{\chi^3(-q^2)} = q^3\left\{\frac{\chi^3(q)}{\chi^3(q^9)} - \frac{\chi^3(-q)}{\chi^3(-q^9)}\right\} + \frac{\chi^3(-q)\chi^6(-q^9)}{\chi^3(-q^3)}$$
$$+ \frac{\chi^3(q)\chi^6(q^9)}{\chi^3(q^3)} + 6q^4\frac{\chi(-q^6)}{\chi^3(-q^{18})},$$

which can be recast as

$$2L = q^3R + A + 6q^4\frac{\chi(-q^6)}{\chi^3(-q^{18})}, \tag{4.4.15}$$

where

$$L = \frac{\chi^3(-q^{18})}{\chi^3(-q^2)} = \sum_{n=0}^{\infty} P_S(n)q^n,$$

$$R = \left\{\frac{\chi^3(q)}{\chi^3(q^9)} - \frac{\chi^3(-q)}{\chi^3(-q^9)}\right\} = \sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n,$$

and

$$A = \frac{\chi^3(-q)\chi^6(-q^9)}{\chi^3(-q^3)} + \frac{\chi^3(q)\chi^6(q^9)}{\chi^3(q^3)}.$$

From (4.4.9) and (4.4.14), we have

$$\frac{\chi^3(q)\chi^3(q^9)}{\chi^2(q^3)} - \frac{\chi^3(-q)\chi^3(-q^9)}{\chi^2(-q^3)} = 6q + 4q^3\frac{\chi^2(-q^6)}{\chi^3(-q^2)\chi^3(-q^{18})} \tag{4.4.16}$$

and

$$\frac{\chi^3(q^9)}{\chi(q^3)} - \frac{\chi^3(-q^9)}{\chi(-q^3)} = -2q^3\frac{\chi(-q^6)}{\chi^3(-q^{18})}. \tag{4.4.17}$$

Multiplying (4.4.16) and (4.4.17), we find that

$$2L = A + 12q^4\frac{\chi(-q^6)}{\chi^3(-q^{18})} + 8q^6\frac{\chi^3(-q^6)}{\chi^3(-q^2)\chi^6(-q^{18})}. \tag{4.4.18}$$

Multiplying (4.4.15) by 2 and then subtracting (4.4.18), we obtain

$$2L = 2q^3R + A - 8q^6\frac{\chi^3(-q^6)}{\chi^3(-q^2)\chi^6(-q^{18})}. \tag{4.4.19}$$

We want a simplified expression for $A - 8q^6\dfrac{\chi^3(-q^6)}{\chi^3(-q^2)\chi^6(-q^{18})}$. We do this by employing some results involving Ramanujan's cubic continued fraction [13, p. 345], $G(q)$, defined by

$$G(q) := \frac{q^{1/3}\chi(-q)}{\chi^3(-q^3)} = \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \cdots, \quad |q| < 1. \tag{4.4.20}$$

We note from [1, pp. 95–96, Theorem 3.3.1] that

$$G(q)G(-q) + G(q^2) = 0, \tag{4.4.21}$$

$$G(q) + G(-q) + 2G^2(-q)G^2(q) = 0, \tag{4.4.22}$$

and

$$G^2(q) + 2G^2(q^2)G(q) - G(q^2) = 0. \tag{4.4.23}$$

Now, by (4.4.3) and (4.4.4),

$$\frac{\psi(-q)}{\psi(-q^9)} = -q + \frac{\chi^3(q^9)}{\chi(q^3)}$$

and

$$3q\frac{\psi(-q^9)}{\psi(-q)} = -1 + \frac{\chi^3(q)}{\chi(q^3)}.$$

Multiplying the above two identities, we obtain

$$\frac{\chi^3(q)}{\chi(q^3)} = \frac{1-2G(-q^3)}{1+G(-q^3)}.$$

Thus,

$$\begin{aligned} A - 8q^6\frac{\chi^3(-q^6)}{\chi^3(-q^2)\chi^6(-q^{18})} &= \frac{q^2}{G^2(q^3)}\left(\frac{1-2G(q^3)}{1+G(q^3)}\right) + \frac{q^2}{G^2(-q^3)}\left(\frac{1-2G(-q^3)}{1+G(-q^3)}\right) \\ &\quad - 8q^2G^2(q^6)\left(\frac{1+G(q^6)}{1-2G(q^6)}\right) \\ &= 2q^2\left(\frac{1}{G(q^6)} + 3 + 4G^2(q^6)\right), \end{aligned} \tag{4.4.24}$$

where (4.4.21)–(4.4.23) have also been utilized to arrive at the last expression.

But, by [13, p. 95, Entry 1(iv)],

$$4G^2(q) - 3 + \frac{1}{G(q)} = \frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)}.$$

Employing the above, with q replaced by q^6, in (4.4.24),

$$A - 8q^6\frac{\chi^3(-q^6)}{\chi^3(-q^2)\chi^6(-q^{18})} = 12q^2 + 2\frac{f^3(-q^2)}{f^3(-q^{18})},$$

and hence, (4.4.19) reduces to

$$L = q^3R + 6q^2 + \frac{f^3(-q^2)}{f^3(-q^{18})}.$$

In terms of $P_S(n)$ and $P_T(n)$, the above can be recast as

$$\sum_{n=0}^{\infty} P_S(n)q^n = q^3\left\{\sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\} + 6q^2 + \frac{f^3(-q^2)}{f^3(-q^{18})}.$$

Equating the coefficients of q^{2N} from both sides of the above, and also noting that Note that

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{f^3(-q)}{f^3(-q^9)},$$

we arrive at the desired result (4.4.10).

□

Example: $n = 3$ in (4.4.10).

Then $P_S(6) = 13$, the relevant partitions of 6 are 3 copies of the form 6, 9 copies of the form 4+2 and one copy of the form 2+2+2; $P_T(3) = 4$, the relevant partitions of 3 are 3 copies of the form 3 and one copy of the form $1 + 1 + 1$ and $a(3) = 5$.

Example: $n = 5$.

Then $P_S(10) = 42$, the relevant partitions of 10 are 3 copies each of 10 and $4 + 2 + 2 + 2$ and 9 copies each of the type $8 + 2$, $6 + 4$, $6 + 2 + 2$ and $4 + 4 + 2$; $P_T(7) = 21$, the relevant partitions of 7 are 3 copies each of 7 and 9 copies each of the type $5 + 1 + 1$ and $3 + 3 + 1$ and $a(5) = 0$.

Theorem 4.4.4. (Corollary to Conjecture 3.30 of [58]) *Let S be the set containing one copy of the odd positive integers that are not multiples of 9 and 2 copies of the even positive integers that are not multiples of 9, and T be the set containing 2 copies of the odd positive integers that are not multiples of 9 and one copy of the even positive integers that are not multiples of 9. Then, for any $N \geq 2$,*

$$D_S(N) = D_T(N-1).$$

Berndt and Zhou [18] proved the above theorem. Here we present an analogous theorem involving $P_S(n)$ and $P_T(n)$.

Theorem 4.4.5. *If S and T are as defined in Conjecture 4.4.4, then*

$$P_S(3N+1) = P_T(3N), \tag{4.4.25}$$

$$P_S(3N+2) = P_T(3N+1), \tag{4.4.26}$$

$$P_S(3N) = P_T(3N-1) + a(N), \tag{4.4.27}$$

where

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{\chi^3(-q^3)}{\chi(-q)} = \frac{1}{C(q)},$$

with $C(q) = q^{-1/3}G(q)$, *where* $G(q)$ *is Ramanujan's cubic continued fraction as defined by* (4.4.20).

Furthermore, $a(n)$ *is nonzero except if* $n = 5$ *and* 8.

Proof. Transforming the theta functions in (4.4.11) into q-products by using (1.1.3)–(1.1.5), we find that

$$\frac{(-q;q^2)_\infty(-q^2;q^2)_\infty^2}{(-q^9;q^{18})_\infty(-q^{18};q^{18})_\infty^2} = q\frac{(-q;q^2)_\infty^2(-q^2;q^2)_\infty}{(-q^9;q^{18})_\infty^2(-q^{18};q^{18})_\infty} + \frac{\chi^3(-q^9)}{\chi(-q^3)}.$$

In terms of $P_S(n)$ and $P_T(n)$, the above can be written as

$$\sum_{n=0}^{\infty} P_S(n)q^n = q\sum_{n=0}^{\infty} P_T(n)q^n + \frac{\chi^3(-q^9)}{\chi(-q^3)}.$$

Equating the coefficients of q^{3N+1}, q^{3N+2} and q^{3N}, from both sides of the above, we readily arrive at (4.4.25)–(4.4.27), respectively.

Now, Hirschhorn and Roselin [36, Theorem 1.5] proved that $a_{6n} > 0$, $a_{6n+1} > 0$, $a_{6n+2} > 0$, $a_{6n+3} < 0$, $a_{6n+4} < 0$, $a_{6n+5} < 0$ except $a_5 = a_8 = 0$. Thus we finish the proof. □

Example: $n = 2$ *in* (4.4.25).

Then $P_S(7) = 12$, the relevant partitions of 7 are 7, $3+2+2$, 2 copies each of the types $6+1$, $5+2$ and $4+3$ and 4 additional copies of the form $4+2+1$; $P_T(6) = 12$, the relevant partitions of 6 are 6, $4+2$, $3+3$, $4+2+1$ and 4 copies each of the types $5+1$ and $3+2+1$.

Example $n = 2$ *in* (4 4 26)

Then $P_S(8) = 16$, the relevant partitions of 8 are $7+1$, $5+3$, $4+4$, $3+2+2+1$, 2 copies each of the types 8, $5+2+1$, $4+3+1$ and $4+2+2$ and 4 additional copies of the form $6+2$, $P_T(7) = 16$, the relevant partitions of 7 are 2 copies each of the types 7, $6+1$, $5+2$, $5+1+1$, $4+3$, $4+2+1$, $3+3+1$ and $3+2+1+1$

Example $n = 2$ *in* (4 4 27)

Then $P_S(6) = 9$, the relevant partitions of 6 are $5+1$, 2 copies each of the types 6 and $3+2+1$ and 4 additional copies of the form $4+2$, $P_T(5) = 8$, the relevant partitions of 5 are two copies each of the types 5, $4+1$, $3+2$ and $3+1+1$ and $a(2) = 1$

Theorem 4.4.6. (Analogue to Corollary to Conjecture 3 42 of [58]) *Let S be the set containing 2 copies of the positive integers that are not multiples of 4, and 2 more copies of the positive multiples of 3 that are not multiples of 4, let T be the set containing 2 copies of the positive integers that are not congruent to 2 modulo 4, and 2 more copies of the positive multiples of 3 that are not congruent to 2 modulo 4 Then $P_S(1) = 2$ and for $N \geq 1$,*

$$P_S(2N+1) = 4P_T(2N-1)$$

Proof We recall from [13, p 232] that

$$\varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3) = 4q\psi(q^2)\psi(q^6),$$

which can be transformed into

$$(-q,q^2)^2_\infty(-q^3,q^6)^2_\infty - (q,q^2)^2_\infty(q^3,q^6)^2_\infty = \frac{4q}{(q^2,q^4)^2_\infty(q^6,q^{12})^2_\infty} \qquad (4\,4\,28)$$

Replacing q by q^2 in the above, multiplying both sides of the resulting identity by $(-q,q^2)^2_\infty(-q^3,q^6)^2_\infty$ and then using Euler's identity and the trivial identity

$(-q;q^2)_\infty(q;q^2)_\infty=(q^2;q^4)_\infty$, we find that

$$\frac{(-q;q)_\infty^2(-q^3;q^3)_\infty^2}{(-q^4;q^4)_\infty^2(-q^{12};q^{12})_\infty^2}-(-q;q^2)_\infty^2(-q^3;q^6)_\infty^2(q^2;q^4)_\infty^2(q^6;q^{12})_\infty^2$$
$$=\frac{4q^2(-q;q)_\infty^2(-q^3;q^3)_\infty^2}{(-q^2;q^4)_\infty^2(-q^6;q^{12})_\infty^2}.$$

We rewrite the above as

$$\sum_{n=0}^{\infty}P_S(n)q^n-(-q;q^2)_\infty^2(-q^3;q^6)_\infty^2(q^2;q^4)_\infty^2(q^6;q^{12})_\infty^2=4q^2\sum_{n=0}^{\infty}P_T(n)q^n. \quad (4.4.29)$$

Replacing q by $-q$ in (4.4.29) and then subtracting the resulting identity from (4.4.29),

$$\sum_{n=0}^{\infty}P_S(n)q^n-\sum_{n=0}^{\infty}P_S(n)(-q)^n \quad (4.4.30)$$
$$-(q^2;q^4)_\infty^2(q^6;q^{12})_\infty^2\left\{(-q;q^2)_\infty^2(-q^3;q^6)_\infty^2-(-q;q^2)_\infty^2(-q^3;q^6)_\infty^2\right\}$$
$$=4q^2\left\{\sum_{n=0}^{\infty}P_T(n)q^n-\sum_{n=0}^{\infty}P_T(n)(-q)^n\right\}. \quad (4.4.31)$$

Employing (4.4.28) in (4.4.30),

$$\sum_{n=0}^{\infty}P_S(n)q^n-\sum_{n=0}^{\infty}P_S(n)(-q)^n=4q+4q^2\left\{\sum_{n=0}^{\infty}P_T(n)q^n-\sum_{n=0}^{\infty}P_T(n)(-q)^n\right\}.$$

Equating the coefficients of q^{2N+1} from both sides, we complete the proof. □

Example: $n=2$.

Then $P_S(5)=16$, the relevant partitions of 5 are 2 copies each of the types 5 and $2+2+1$, 4 copies of the type $3+1+1$ and 8 additional copies of the form $3+2$; $P_T(3)=4$, the relevant partitions of 3 are 4 copies of 3.

4.5 Conjectures 3.32, 3.33, 3.31, 3.35 – 3.37 and 3.52 of [58]

Theorem 4.5.1. (Corollary to Conjecture 3.32 of [58]) *Let S be the set containing 2 copies of the positive integers that are either odd or multiples of* 8, *and* 7 *copies of* *the positive integer that are congruent to* 2 *modulo* 4*; let* T *be the set containing* 4 *copies of the positive integers that are either odd or multiples of* 8*, and* 2 *copies of the positive integers that are congruent to* 2 *modulo* 4*. Then, for any* $N \geq 1$,

$$D_S(N) = 2D_T(N-1)$$

or equivalently,

$$P_S(N) = 2P_T(N-1).$$

Proof. By Lemma 4.1.2,

$$\frac{\varphi(q)-\varphi(-q)}{\varphi(q)+\varphi(-q)} = 2q\frac{\psi(q^8)}{\varphi(q^4)}.$$

Thus,

$$\frac{\varphi^2(q)-\varphi(q)\varphi(-q)}{\varphi(q)+\varphi(-q)} = 2q\frac{\varphi(q)\psi(q^8)}{\varphi(q^4)}.$$

Adding $\varphi(-q)$ to both sides of the above and then using Lemma 4.1.2 again,

$$\frac{\varphi^2(q^2)}{\varphi(q^4)} = 2q\frac{\varphi(q)\psi(q^8)}{\varphi(q^4)} + \varphi(-q).$$

Dividing both sides by $\varphi(-q)$,

$$\frac{\varphi^2(q^2)}{\varphi(-q)\varphi(q^4)} = 2q\frac{\varphi(q)\psi(q^8)}{\varphi(-q)\varphi(q^4)} + 1,$$

which can be transformed into

$$(-q;q^2)^2_\infty(-q^8;q^8)^2_\infty(-q^2;q^4)^7_\infty = 2q(-q;q^2)^4_\infty(-q^8;q^8)^4_\infty(-q^2;q^4)^2_\infty + 1, \quad (4.5.1)$$

where we also applied Euler's identity $(-q;q)_\infty = (q;q^2)^{-1}_\infty$ and the trivial identity $(q;q^2)_\infty(-q;q^2)_\infty = (q^2;q^4)_\infty$.

Since the above is equivalent to

$$\sum_{n=0}^{\infty} D_S(n)q^n = 2q\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q\sum_{n=0}^{\infty} P_T(n)q^n + 1,$$

equating the coefficients of q^N from both sides, we readily arrive at the desired result. □

Example: $n = 5$.

Then $P_S(5) = 58$, the relevant partitions of 5 are 2 copies of the type 5, 14 copies of the type $3 + 2$ and 42 additional copies of the type $2 + 2 + 1$; $P_T(4) = 29$, the relevant partitions of 4 are $2 + 2$, 12 copies of the type $2 + 1 + 1$ and 16 additional copies of the type $3 + 1$.

Theorem 4.5.2. (Corollary to Conjecture 3.33 of [58]) *Let S be the set containing 4 copies of the positive integers that are either odd or congruent to 4 modulo 8, and 2 copies of the positive integers that are congruent to 2 modulo 4; let T be the set containing 2 copies of the positive integers that are either odd or multiples of 8, and 7 copies of the positive integers that are congruent to 2 modulo 4. Then, for any $N \geq 2$,*

$$D_S(N) = D_T(N-1).$$

The above theorem has been proved by Berndt and Zhou [18]. In the following theorem we prove an analogous result.

Theorem 4.5.3. *If S and T are as defined in Theorem 4.5.2, then $P_S(1) = 4$, $P_T(0) = 1$, and for $N \geq 1$,*

$$P_S(2N+1) = 2P_T(2N) \tag{4.5.2}$$

and

$$P_S(2N) = 2P_T(2N-1) + a(N), \tag{4.5.3}$$

where

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{\chi^8(-q^2)}{\chi^4(-q)}.$$

Proof. It is easy to see, or by Lemma 4.1.2,

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{4.5.4}$$

Multiplying both sides by $\varphi^2(q^2)$ and again using the identities of Lemma 4.1.2,

$$\varphi(q)\left(2\varphi^2(q^4) - \varphi^2(-q^2)\right) = 2q\psi(q^8)\varphi^2(q^2) + \varphi^2(q^2)\left(\varphi(-q) + 2q\psi(q^8)\right),$$

that is,

$$\begin{aligned}\varphi(q)\varphi^2(q^4) &= 2q\psi(q^8)\varphi^2(q^2) + \frac{1}{2}\varphi(-q)\varphi^2(q^2) + \frac{1}{2}\varphi(-q)\varphi^2(q)\\ &= 2q\psi(q^8)\varphi^2(q^2) + \frac{1}{2}\varphi(-q)\varphi^2(q^2) + \frac{1}{2}\varphi(-q)\left\{\varphi^2(q^2) + 4q\psi^2(q^4)\right\}\\ &= 2q\psi(q^8)\varphi^2(q^2) + \varphi(-q)\varphi^2(q^2) + 2q\varphi(-q)\psi^2(q^4).\end{aligned}$$

Dividing both sides by $\varphi(-q)\psi^2(q^4)$ and simplifying by using $\varphi(q)\psi(q^2) = \psi^2(q)$,

$$\frac{\varphi(q)\varphi(q^4)}{\varphi(-q)\psi(q^8)} = 2q\frac{\varphi^2(q^2)}{\varphi(-q)\varphi(q^4)} + 2q + \frac{\varphi^2(q^2)}{\psi^2(q^4)}.$$

Expressing in q-products,

$$(-q;q^2)^4_\infty(-q^4:q^8)^4_\infty(-q^2;q^4)^2_\infty = 2q(-q;q^2)^2_\infty(-q^8:q^8)^2_\infty(-q^2;q^4)^7_\infty + 2q + \frac{\chi^8(-q^4)}{\chi^4(-q^2)}, \tag{4.5.5}$$

which is equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q\sum_{n=0}^{\infty} P_T(n)q^n + 2q + \frac{\chi^8(-q^4)}{\chi^4(-q^2)}.$$

Equating the coefficients of q^{2N+1} and q^{2N} on both sides, we easily arrive at (4.5.2) and (4.5.3), respectively. □

Example $n = 2$ *in* (4.5.2)

Then $P_S(5) = 64$, the relevant partitions of 5 are 4 copies each of the types 5 and $2+2+1$, 8 copies each of the types $3+2$ and $2+1+1+1$, 16 copies of the form $4+1$ and 24 additional copies of the form $3+1+1$; $P_T(4) = 32$, the relevant partitions of 4 are 4 copies of the form $3+1$, 7 copies of the type $2+1+1$ and 21 additional copies of the form $2+2$.

Example: $n = 2$ *in* (4.5.3).

Then $P_S(4) = 34$, the relevant partitions of 4 are $2+2$, $1+1+1+1$, 4 copies of the type 4, 12 copies of the type of $2+1+1$ and 16 additional copies of the form $3+1$; $P_T(3) = 16$, the relevant partitions of 3 are 2 copies of 3 and 14 additional copies of the form $2+1$ and $a(2) = 2$

Theorem 4.5.4. (Corollary to Conjecture 3.31 of [58]) *Let S be the set containing 4 copies of the positive integers that are either odd or congruent to 4 modulo 8, and 2 copies of the positive integers that are congruent to 2 modulo 4, let T be the set containing 4 copies of the positive integers that are either odd or multiples of 8, and 2 copies of the positive integers that are congruent to 2 modulo 4. Then, for any $N \geq 2$,*

$$D_S(N) = 2D_T(N-2)$$

A proof of the above theorem can be found in Berndt and Zhou [18] We find the following result.

Theorem 4.5.5. *If S and T are as defined in Theorem 4.5.4, then $P_S(1) = 4$ and for $N \geq 1$,*

$$P_S(2N+1) = 4P_T(2N-1), \tag{4 5.6}$$

$$P_S(2N) = 4P_T(2N-2) + a(N), \tag{4.5.7}$$

where

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{\chi^8(-q^2)}{\chi^4(-q)}$$

Proof. From (4.5.1) and (4.5.5), we have

$$(-q;q^2)_\infty^2(-q^8;q^8)_\infty^2(-q^2;q^4)_\infty^7 = 2q(-q;q^2)_\infty^4(-q^8;q^8)_\infty^4(-q^2;q^4)_\infty^2 + 1$$

and

$$(-q;q^2)_\infty^4(-q^4;q^8)_\infty^4(-q^2;q^4)_\infty^2 = 2q(-q;q^2)_\infty^2(-q^8;q^8)_\infty^2(-q^2;q^4)_\infty^7 + 2q + \frac{\chi^8(-q^4)}{\chi^4(-q^2)}.$$

Thus,

$$(-q;q^2)_\infty^4(-q^4;q^8)_\infty^4(-q^2;q^4)_\infty^2 = 4q^2(-q;q^2)_\infty^4(-q^4;q^8)_\infty^4(-q^2;q^4)_\infty^2 + 4q + \frac{\chi^8(-q^4)}{\chi^4(-q^2)},$$

which is equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n = 4q^2\sum_{n=0}^{\infty} P_T(n)q^n + 4q + \frac{\chi^8(-q^4)}{\chi^4(-q^2)}.$$

Equating the coefficients of q^{2N+1} and q^{2N} from both sides, we readily deduce (4.5.6) and (4.5.7). □

Example: $n = 2$ *in* (4.5.6).

Then $P_S(5) = 64$, the relevant partitions of 5 are 4 copies each of the types 5 and $2+2+1$, 8 copies each of the types of $3+2$ and $2+1+1+1$, 16 copies of the form $4+1$ and 24 additional copies of the form $3+1+1$; $P_T(3) = 16$, the relevant partitions of 3 are 4 copies of each of the types 3 and $1+1+1$, 8 additional copies of the form $2+1$.

Example: $n = 2$ *in* (4.5.7).

Then $P_S(4) = 34$, the relevant partitions of 4 are $2+2$, $1+1+1+1$, 4 copies of the type 4, 12 additional copies of the form $2+1+1$ and 16 more copies of the type of $3+1$; $P_T(2) = 8$, the relevant partitions of 2 are 2 copies of 2 and 6 additional copies of the type $1+1$ and $a(2) = 2$.

Theorem 4.5.6. (Corollary to Conjecture 3.35 of [58]) *Let S be the set containing 2 copies of the odd positive integers, 3 copies of the positive integers that are congruent* *to 2 modulo 4, 6 copies of the positive integers that are congruent to 4 modulo 8, and 4 copies of the positive multiples of 8; let T be the set containing 2 copies of the odd positive integers, 3 copies of the positive integers that are congruent to 2 modulo 4, 4 copies of the positive integers that are congruent to 4 modulo 8, and 6 copies of the positive multiples of 8. Then, for any $N \geq 1$,*

$$D_S(N) = 2D_T(N-1)$$

or equivalently,

$$P_S(N) = 2P_T(N-1).$$

Proof. Replacing q by $-q$ in (4.5.4) and then dividing both side by $\varphi(-q)$,

$$\frac{\varphi(q^4)}{\varphi(-q)} = 2q\frac{\psi(q^8)}{\varphi(-q)} + 1,$$

which can be transformed into

$$\begin{aligned}(-q;q^2)_\infty^2(-q^2;q^4)_\infty^3(-q^4;q^8)_\infty^6(-q^8;q^8)_\infty^4 \\ = 2q(-q;q^2)_\infty^2(-q^2;q^4)_\infty^3(-q^4;q^8)_\infty^4(-q^8;q^8)_\infty^6 + 1.\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} D_S(n)q^n = 2q\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q\sum_{n=0}^{\infty} P_T(n)q^n + 1.$$

Equating the coefficients of q^N from both sides, we arrive at the desired result. □

Example: $n = 5$.

Then $P_S(5) = 28$, the relevant partitions of 5 are 2 copies each of the types 5 and $3+1+1$, 6 copies each of the types $3+2$ and $2+2+1$ and 12 additional copies of the form $4+1$; $P_T(4) = 14$, the relevant partitions of 4 are 3 copies of each of the types $2+2$ and $2+1+1$ and 4 additional copies of the forms 4 and $3+1$.

Theorem 4.5.7. (Corollary to Conjecture 3.36 of [58]) *Let S be the set containing 2 copies of the positive integers and 2 more copies of the odd positive integers; let T be the set containing 2 copies of the odd positive integers, 3 copies of the positive integers that are congruent to 2 modulo 4, 4 copies of the positive integers that are congruent to 4 modulo 8, and 6 copies of the positive multiples of 8. Then, for any $N \geq 1$,*

$$D_S(N) = 4D_T(N-1)$$

or equivalently,

$$P_S(N) = 4P_T(N-1).$$

Proof. From Lemma 4.1.2,

$$\varphi(q) = 4q\psi(q^8) + \varphi(-q).$$

Dividing both sides by $\varphi(-q)$ and then transforming into q-products, we find that

$$(-q;q^2)^2_\infty(-q;q)^2_\infty = 4q(-q;q^2)^2_\infty(-q^2;q^4)^3_\infty(-q^4;q^8)^4_\infty(-q^8;q^8)^6_\infty + 1.$$

Since the above is equivalent to

$$\sum_{n=0}^{\infty} D_S(n)q^n = 4q\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

or, equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n = 4q\sum_{n=0}^{\infty} P_T(n)q^n + 1,$$

we complete the proof by comparing the coefficients of q^N from both sides. □

Example: $n = 5$.

Then $P_S(5) = 56$, the relevant partitions of 5 are 4 copies each of the types 5 and $2+2+1$, 8 copies each of the types $4+1$, $3+2$ and $2+1+1+1$ and 24 additional copies of the form $3+1+1$; $P_T(4)=14$, the relevant partitions of 4 are 3 copies of each of the types $2+2$ and $2+1+1$ and 4 additional copies of the forms 4 and $3+1$.

Theorem 4.5.8. (Corollary to Conjecture 3.37 of [58]) *Let S be the set containing 2 copies of the odd positive integers, 3 copies of the positive integers that are congruent to 2 modulo 4, 6 copies of the positive integers that are congruent to 4 modulo 8, and 4 copies of the positive multiples of 8; let T be the set containing 2 copies of the positive integers and 2 more copies of the odd positive integers. Then, for any $N \geq 1$,*

$$D_S(N) = \frac{1}{2} D_T(N)$$

or equivalently,

$$P_S(N) = \frac{1}{2} P_T(N).$$

Proof. It is easy to see, or by Lemma 4.1.2,

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4),$$

which can be transformed into

$$\begin{aligned}(-q;q^2)_\infty^2 + (q;q^2)_\infty^2 &= 2\frac{(-q^4;q^8)_\infty^2 (q^8;q^8)_\infty}{(q^2;q^2)_\infty} \\ &= 2\frac{(-q^4;q^8)_\infty^2}{(q^2;q^4)_\infty (q^4;q^8)_\infty}.\end{aligned}$$

Dividing both sides by $2(q;q^2)_\infty^2$, and then employing Euler's theorem $(q;q^2)_\infty =$

$(-q;q)_\infty^{-1}$,

$$\begin{aligned}\frac{1}{2}(-q;q)_\infty^2(-q,q^2)_\infty^2+\frac{1}{2}&=\frac{(-q^4;q^8)_\infty^2}{(q;q^2)_\infty^2(q^2;q^4)_\infty(q^4;q^8)_\infty}\\&=\frac{(-q;q^2)_\infty^2(-q^4;q^8)_\infty^2}{(q^2;q^4)_\infty^3(q^4;q^8)_\infty}\\&=\frac{(-q,q^2)_\infty^2(-q^2;q^4)_\infty^3(-q^4;q^8)_\infty^2}{(q^4,q^8)_\infty^4}\\&=\frac{(-q;q^2)_\infty^2(-q^2;q^4)_\infty^3(-q^4;q^8)_\infty^6}{(q^8,q^{16})_\infty^4}\\&=(-q^4;q^8)_\infty^6(-q;q^2)_\infty^2(-q^2;q^4)_\infty^3(-q^8;q^8)_\infty^4,\end{aligned}$$

which can be rewritten in either of the forms

$$\frac{1}{2}\sum_{n=0}^{\infty}D_T(n)q^n+\frac{1}{2}=\sum_{n=0}^{\infty}D_S(n)q^n$$

and

$$\frac{1}{2}\sum_{n=0}^{\infty}P_T(n)q^n+\frac{1}{2}=\sum_{n=0}^{\infty}P_S(n)q^n$$

Comparing the coefficients of q^N from both sides, we finish the proof □

Example: $n = 5$.

Then $P_S(5) = 28$, the relevant partitions of 5 are 2 copies each of the types 5 and $3+1+1$, 6 copies each of the types $3+2$ and $2+2+1$ and 12 additional copies of the form $4+1$, $P_T(5) = 56$, the relevant partitions of 5 are 4 copies of each of the types 5 and $2+2+1$, 8 copies each of the types $4+1$, $3+2$ and $2+1+1+1$ and 24 additional copies of the form $3+1+1$.

Theorem 4.5.9. (Corollary to Conjecture 3 52 of [58]) *Let S be the set containing 2 copies of the odd positive integers, one copy of the even positive integers that are not multiples of 16, and one more copy of the positive odd multiples of 8; let T be the set containing 2 copies of the odd positive integers, one copy of the even positive integers that are not odd multiples of 8, and one more copy of the positive multiples* *of* 16. *Then, for any* $N \geq 2$,

$$D_S(N) = D_T(N-2).$$

Theorem 4.5.9 has been proved by Berndt and Zhou [18]. We present the following result involving $P_S(n)$ and $P_T(n)$.

Theorem 4.5.10. *If S and T are as defined in Theorem 4.5.9, then $P_S(1) = 2$, and for $N \geq 1$,*

$$P_S(4N+1) = 2P_T(4N-1), \tag{4.5.8}$$

$$P_S(4N+2) = 2P_T(4N), \tag{4.5.9}$$

$$P_S(4N+3) = 2P_T(4N+1), \tag{4.5.10}$$

$$P_S(4N) = 2P_T(4N-2) + a(N), \tag{4.5.11}$$

where

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{\chi^4(-q^2)}{\chi^2(-q)}.$$

Proof. With repeated applications of the identities in Lemma 4.1.2, we have

$$\begin{aligned}
\varphi(q)\left(\varphi^2(q^8) - 2q^2\psi^2(q^8)\right) &= \varphi(q)\left(\left(\frac{\varphi(q^2)+\varphi(-q^2)}{2}\right)^2 - 2q^2\left(\frac{\varphi^2(q^2)-\varphi^2(-q^2)}{8}\right)\right) \\
&= \frac{1}{2}\varphi(q)\left(\varphi^2(-q^2) + \varphi^2(-q^4)\right) \\
&= \frac{1}{2}\varphi(q)\varphi(-q^2)\left(\varphi(-q^2) + \varphi(q^2)\right) \\
&= \varphi(q)\varphi(-q^2)\varphi(q^8) \\
&= \varphi(-q^2)\varphi(q^8)\left(2q\psi(q^8) + \varphi(q^4)\right) \\
&= 2q\varphi(-q^2)\varphi(q^8)\psi(q^8) + \varphi(-q^2)\varphi(q^8)\varphi(q^4).
\end{aligned}$$

Hence,

$$\varphi(q)\varphi^2(q^8) = 2q^2\varphi(q)\psi^2(q^8) + 2q\varphi(-q^2)\varphi(q^8)\psi(q^8) + \varphi(-q^2)\varphi(q^8)\varphi(q^4).$$

Dividing both sides by $\varphi(-q^2)\varphi(q^8)\psi(q^8)$,

$$\frac{\varphi(q)\varphi(q^8)}{\varphi(-q^2)\psi(q^8)} = 2q^2\frac{\varphi(q)\psi(q^8)}{\varphi(-q^2)\varphi(q^8)} + 2q + \frac{\varphi(q^4)}{\psi(q^8)},$$

which can be transformed into

$$\frac{(-q;q^2)^2_\infty(-q^2;q^2)_\infty(-q^8;q^{16})_\infty}{(-q^{16};q^{16})_\infty} = 2q^2\frac{(-q;q^2)^2_\infty(-q^2;q^2)_\infty(-q^{16};q^{16})_\infty}{(-q^8;q^{16})_\infty} + 2q + \frac{\chi^4(-q^8)}{\chi^2(-q^4)}.$$

Thus,

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q^2\sum_{n=0}^{\infty} P_T(n)q^n + 2q + \frac{\chi^4(-q^8)}{\chi^2(-q^4)}.$$

Equating the coefficients of q^{4N+1}, q^{4N+2}, q^{4N+3}, and q^{4N} from both sides of the above, we finish the proof. □

Example: $n = 1$ *in* (4.5.8).

Then $P_S(5) = 8$, the relevant partitions of 5 are 2 copies of each of the types 5, $4+1$, $3+2$ and $3+1+1$; $P_T(3) = 4$, the relevant partitions of 3 are 2 copies of each of the types 3 and $2+1$.

Example: $n = 1$ *in* (4.5.9).

Then $P_S(6) = 12$, the relevant partitions of 6 are 6, $4+2$, $4+1+1$, $3+3$ and 4 copies of each of the types $5+1$ and $3+2+1$; $P_T(4) = 6$, the relevant partitions of 4 are 4, $2+1+1$, 4 copies of the type $3+1$.

Example: $n = 1$ *in* (4.5.10).

Then $P_S(7) = 16$, the relevant partitions of 7 are 2 copies of each of the types 7, $6+1$, $5+2$, $5+1+1$, $4+3$, $4+2+1$, $3+3+1$ and $3+2+1+1$; $P_T(5) = 8$, the relevant partitions of 5 are 2 copies of each of the types 5, $4+1$, $3+2$ and $3+1+1$.

Example: $n = 2$ *in* (4.5.11).

Then $P_S(8) = 23$, the relevant partitions of 8 are $6+2$, $6+1+1$, $4+2+1+1$, $3+3+2$, $3+3+1+1$, 2 copies of the type 8 and 4 copies of each of the types $7+1$, $5+3$, $5+2+1$ and $4+3+1$; $P_T(6) = 12$, the relevant partitions of 6 are 6, $4+2$, $4+1+1$, $3+3$, 4 copies of each of the types $5+1$ and $3+2+1$ and $a(2) = -1$.

4.6 Conjectures 3.39 and 3.40 of [58]

Theorem 4.6.1. (Corollary to Conjecture 3.39 of [58]) *Let S be the set containing 2 copies of the positive integers that are not multiples of* 10, *one more copy of the odd positive integers, and one more copy of the odd positive multiples of* 5; *let T be the set containing* 2 *copies of the positive integers that are not odd multiples of* 5, *one more copy of the even positive integers, and one more copy of the positive multiples of* 10. *Then, for any $N \geq 2$,*

$$D_S(N) = 2D_T(N-2).$$

Berndt and Zhou [18] have proved Theorem 4.6.1. Here we give an analogous result.

Theorem 4.6.2. *If S and T are as defined in Theorem 4.6.1, then $P_S(1) = 2 + b(1)$ and for $N > 1$*

$$P_S(N) = 4P_T(N-2) + b(N),$$

where

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{(q^5;q^{10})_\infty^5}{(q;q^2)_\infty} = \frac{\chi^5(-q^5)}{\chi(-q)}.$$

Proof. Recall from [5, p. 1039, equation (7.16)] that

$$\begin{aligned}(-q;q^2)_\infty(-q^5;q^{10})_\infty^3 - (q;q^2)_\infty(q^5;q^{10})_\infty^3 &= 4q^2(-q^2;q^2)_\infty(-q^{10};q^{10})_\infty^3 \\ &\quad + 2q\frac{(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2}.\end{aligned}$$

Employing Euler's identity $(-q;q)_\infty = (q;q^2)_\infty^{-1}$, the above can be written as

$$\begin{aligned}&\frac{(-q;q)_\infty^2}{(-q^{10};q^{10})_\infty^2}(-q^5;q^{10})_\infty(-q;q^2)_\infty - \frac{(q^5;q^{10})_\infty^5}{(q;q^2)_\infty} \\ &= 4q^2\frac{(-q;q)_\infty^2}{(-q^5;q^{10})_\infty^2}(-q^2;q^2)_\infty(-q^{10};q^{10})_\infty + 2q.\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} P_S(n)q^n = 4q^2 \sum_{n=0}^{\infty} P_T(n)q^n + 2q + \frac{(q^5;q^{10})_\infty^5}{(q;q^2)_\infty}.$$

Equating the coefficients of q^N from both sides, we finish the proof. □

Example: $n = 5$.

Then $P_S(5) = 30$, the relevant partitions of 5 are 2 copies of the type $2+1+1+1$, 3 copies of the type $2+2+1$, 4 copies of the form 5, 6 copies of each of the forms $4+1$ and $3+2$, 9 additional copies of the form $3+1+1$; $P_T(3) = 8$, the relevant partitions of 3 are 2 copies of the type 3 and 6 copies of the type $2+1$ and $b(5) = -2$.

Theorem 4.6.3. (Corollary to Conjecture 3.40 of [58]) *Let S be the set containing 3 copies of the even positive integers, one copy of the odd positive integers, 3 more copies of the odd positive multiples of 5, and one more copy of the positive multiples of 10; let T be the set containing 3 copies of the odd positive integers, one copy of the even positive integers, one more copy of the odd positive multiples of 5, and 3 more copies of the positive multiples of 10. Then, for any $N \geq 1$,*

$$D_S(N) = D_T(N-1)$$

or equivalently,

$$P_S(N) = P_T(N-1).$$

Proof. From [1, p. 28, Entries 1.7.1 (i), (iv)], we have

$$\varphi(q) + \varphi(q^5) = 2q^{4/5} f(q,q^9) R^{-1}(q^4)$$

and

$$\psi(q^2) - q\psi(q^{10}) = q^{-1/5} f(q^4,q^6) R(q),$$

where $R(q)$ is the Rogers-Ramanujan continued fraction as defined in (4.2.6). Multiplying the above identities and simplifying by using the trivial identity $\varphi(q)\psi(q^2) = \psi^2(q)$, we find that

$$\varphi(q^5)\psi(q^2) - q\varphi(q)\psi(q^{10}) + \psi^2(q) - q\psi^2(q^5) = 2q^{3/5}f(q,q^9)f(q^4,q^6)\frac{R(q)}{R(q^4)}. \quad (4.6.1)$$

Since, by [13, p. 262, Entry 10(v)],

$$\psi^2(q) - q\psi^2(q^5) = f(q,q^4)f(q^2,q^2),$$

identity (4.6.1) reduces to

$$\varphi(q^5)\psi(q^2) - q\varphi(q)\psi(q^{10}) + f(q,q^4)f(q^2,q^3) = 2q^{3/5}f(q,q^9)f(q^4,q^6)\frac{R(q)}{R(q^4)}. \quad (4.6.2)$$

Now, setting $a = q$, $b = q^4$, $c = q^2$, and $d = q^3$ in (4.1.1),

$$f(q,q^4)f(q^2,q^3) + f(-q,-q^4)f(-q^2,-q^3) = 2f(q^3,q^7)f(q^4,q^6). \quad (4.6.3)$$

But, by Jacobi's triple product identity, (1.1.2),

$$\begin{aligned} f(-q,-q^4)f(-q^2,-q^3) &= (q;q^5)_\infty(q^2;q^5)_\infty(q^3;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)^2_\infty \\ &= (q;q)_\infty(q^5;q^5)_\infty, \end{aligned}$$

and hence, from (4.6.3), we have

$$f(q,q^4)f(q^2,q^3) = 2f(q^3,q^7)f(q^4,q^6) - (q;q)_\infty(q^5;q^5)_\infty. \quad (4.6.4)$$

Employing (4.6.4) in (4.6.2),

$$\begin{aligned} \varphi(q^5)\psi(q^2) - q\varphi(q)\psi(q^{10}) = &\left\{2q^{3/5}f(q,q^9)f(q^4,q^6)\frac{R(q)}{R(q^4)} - 2f(q^3,q^7)f(q^4,q^6)\right\} \\ &+ (q;q)_\infty(q^5;q^5)_\infty. \end{aligned} \quad (4.6.5)$$

Now, $R(q)$ has the product representation

$$R(q) = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty},$$

and therefore,

$$\frac{R(q)}{R(q^4)} = q^{-3/5}\frac{(q,q^6,q^9,q^{11},q^{14},q^{19};q^{20})_\infty}{(q^2,q^3,q^7,q^{13},q^{17},q^{18};q^{20})_\infty}. \tag{4.6.6}$$

On the other hand, by employing Jacobi's triple product identity, (1.1.2), and changing the base to q^{20}, we have

$$\frac{f(q^3,q^7)}{f(q,q^9)} = \frac{(q,q^6,q^9,q^{11},q^{14},q^{19};q^{20})_\infty}{(q^2,q^3,q^7,q^{13},q^{17},q^{18};q^{20})_\infty}. \tag{4.6.7}$$

From (4.6.6) and (4.6.7),

$$q^{3/5}\frac{R(q)}{R(q^4)}f(q,q^9)f(q^4,q^6) - f(q^3,q^7)f(q^4,q^6) = 0,$$

and hence, from (4.6.5),

$$\varphi(q^5)\psi(q^2) - q\varphi(q)\psi(q^{10}) = (q;q)_\infty(q^5;q^5)_\infty.$$

The above is equivalent to

$$(-q^5;q^{10})^2_\infty(q^{10};q^{10})_\infty\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty} - q(-q;q^2)^2_\infty(q^2;q^2)_\infty\frac{(q^{20};q^{20})_\infty}{(q^{10};q^{20})_\infty} = (q;q)_\infty(q^5;q^5)_\infty.$$

Dividing both sides by $(q;q)_\infty(q^5;q^5)_\infty$ and then employing $(q;q)_\infty = (q;q^2)_\infty(q^2;q^2)_\infty$ and Euler's identity, we find that

$$\begin{aligned}(-q^2;q^2)^3_\infty(-q;q^2)_\infty(-q^5;q^{10})^3_\infty(-q^{10};q^{10})_\infty\\ = q(-q;q^2)^3_\infty(-q^2;q^2)_\infty(-q^5;q^{10})_\infty(-q^{10};q^{10})^3_\infty + 1.\end{aligned}$$

Since the above can put either of the forms

$$\sum_{n=0}^{\infty} D_S(n)q^n = q\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

and

$$\sum_{n=0}^{\infty} P_S(n)q^n = q\sum_{n=0}^{\infty} P_T(n)q^n + 1,$$

we complete the proof by equating the coefficients of q^N from both sides of the above two identities. □

Example $n = 5$

Then $P_S(5) = 13$, the relevant partitions of 5 are 3 copies of each of the types $4+1$, $3+2$ and $2+2+1$ and 4 additional copies of the type 5, $P_T(4) = 13$, the relevant partitions of 4 are 4, 3 copies of the type $2+1+1$ and 9 additional copies of the type $3+1$

4.7 Conjectures 3.34, 3.29, 3.41, 3.43 – 3.50 of [58]

Theorem 4.7.1. (Corollary to Conjecture 3 34 of [58]) *Let S be the set containing one copy of the positive integers congruent to ± 1 modulo 6, 5 copies of the positive integers congruent to ± 2 modulo 6, and 6 copies of the positive multiples of 3, let T be the set containing 5 copies of the positive integers congruent to ± 1 modulo 6, one copy of the positive integers congruent to ± 2 modulo 6, and 6 copies of the positive multiples of 3 Then, for any $N \geq 1$,*

$$D_S(N) = D_T(N-1)$$

or equivalently,

$$P_S(N) = P_T(N-1)$$

Proof Adding (4 1 1) and (4 1 2), we find that

$$f(a,b)f(c,d) = af(b/c, ac^2d)f(b/d, acd^2) + f(ac, bd)f(ad, bc) \tag{4 7 1}$$

Setting $a = q$, $b = q^5$, $c = q^3$ and $d = q^3$ in the above, we have

$$f(q, q^5)\varphi(q^3) = qf^2(q^2, q^{10}) + f^2(q^4, q^8) \tag{4 7 2}$$

Replacing q by $-q$ in the above, we have

$$f(-q, -q^5)\varphi(-q^3) = -qf^2(q^2, q^{10}) + f^2(q^4, q^8) \tag{4 7 3}$$

Multiplying the previous two identities, we find that

$$\begin{aligned} f^4(q^4,q^8) - q^2 f^4(q^2,q^{10}) &= f(q,q^5)f(-q,-q^5)\varphi(q^3)\varphi(-q^3) \\ &= f(q,q^5)f(-q,-q^5)\varphi^2(-q^6). \end{aligned} \tag{4.7.4}$$

Now, setting $a = q$, $b = q^5$, $c = -q$ and $d = -q^5$ in (4.7.1) and noting that $f(-1,u) = 0$, we find that

$$f(q,q^5)f(-q,-q^5) = \varphi(-q^6)f(-q^2,-q^{10}). \tag{4.7.5}$$

Using the above in (4.7.4), we have

$$f^4(q^4,q^8) = q^2 f^4(q^2,q^{10}) + \varphi^3(-q^6)f(-q^2,-q^{10}). \tag{4.7.6}$$

Replacing q^2 by q in (4.7.6) and then noting, by (1.1.2), that

$$f(-q,-q^5) = (q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty = \psi(q^3)\chi(-q),$$

we have

$$f^4(q^2,q^4) = qf^4(q,q^5) + \varphi^3(-q^3)\psi(q^3)\chi(-q).$$

Dividing both sides by the last expression, employing (1.1.2), and then simplifying, we deduce that

$$(-q^{\pm 1};q^6)_\infty(-q^{\pm 2};q^6)^5_\infty(-q^3;q^3)^6_\infty = q(-q^{\pm 1};q^6)^5_\infty(-q^{\pm 2};q^6)_\infty(-q^3;q^3)^6_\infty + 1, \tag{4.7.7}$$

where, here and the sequel,

$$(-q^{\pm r};q^s)_\infty := (-q^r;q^s)_\infty(q^{s-r};q^s)_\infty.$$

Since (4.7.7) can be written

$$\sum_{n=0}^{\infty} D_S(n)q^n = q\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(n)q^n = q\sum_{n=0}^{\infty} P_T(n)q^n + 1,$$

we complete the proof by equating the coefficients of q^N from both sides. □

Example: $n = 5$.

Then $P_S(5) = 46$, the relevant partitions of 5 are 5, 5 copies of the type $4+1$, 10 copies of the type $2+2+1$ and 30 additional copies of the type $3+2$; $P_T(4) = 46$, the relevant partitions of 4 are 4, 5 copies of the type $1+1+1+1$, 10 copies of the type $2+1+1$ and 30 additional copies of the type $3+1$

Theorem 4.7.2. (Analogue to Corollary to Conjecture 3 29 of [58]) *Let S be the set containing 4 copies of the positive integers that are either congruent to ±1 modulo 6 or to ±4 modulo 12, and T be the set containing 4 copies of the positive integers that are either congruent to ±1 modulo 6 or to ±2 modulo 12 Then, $P_S(1) = 4$ and for $N \geq 1$,*

$$P_S(2N+1) = P_T(2N-1) \tag{4.7.8}$$

Furthermore, let U be the set containing one copy of the even positive integers and one more copy of the even positive multiples of 3, V be the set containing two copies of the odd positive integers and two more copies of the odd positive multiples of 3, W be the set containing two copies of the even positive integers and two more copies of the even positive multiples of 3. If $S' = S \cup U$ and $T' = T \cup U$, then

$$P_{S'}(2N) = P_{T'}(2N-2) + P_V(N) + 4P_W(N-1) \tag{4.7.9}$$

Proof. Dividing both sides of (4.7.6) by $f^4(-q^{12})$ and then transforming into q-products, we find that

$$(-q^{\pm4}, q^{12})_\infty^4 = q^2(-q^{\pm2}; q^{12})_\infty^4 + (q^2; q^4)_\infty(q^6, q^{12})_\infty^5$$

Multiplying both sides of the above by $(-q^{\pm 1};q^6)_\infty^4$ and then simplifying by using Euler's identity, we have

$$(-q^{\pm 1};q^6)_\infty^4(-q^{\pm 4};q^{12})_\infty^4 = q^2(-q^{\pm 1};q^6)_\infty^4(-q^{\pm 2};q^{12})_\infty^4 \\ + (-q;q^2)_\infty^4(q^2;q^4)_\infty(q^3;q^6)_\infty^4(q^6;q^{12})_\infty,$$

which can put in the form

$$\sum_{n=0}^{\infty} P_S(n)q^n = q^2\sum_{n=0}^{\infty} P_T(n)q^n + (q^2;q^4)_\infty(q^6;q^{12})_\infty(-q;q^2)_\infty^4(q^3;q^6)_\infty^4. \qquad (4.7.10)$$

Replacing q by $-q$ in (4.7.10) and then subtracting the identity from (4.7.10), we have

$$\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n = q^2\left\{\sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\} \\ + (q^2;q^4)_\infty(q^6;q^{12})_\infty \times \left\{(-q;q^2)_\infty^4(q^3;q^6)_\infty^4 - (q;q^2)_\infty^4(-q^3;q^6)_\infty^4\right\}. \qquad (4.7.11)$$

Now, we note from [62, p. 84, Corollary 3.3] that

$$\varphi(q)\varphi(-q^3) = \varphi(-q^4)\varphi(-q^{12}) + 2q\psi(-q^2)\psi(-q^6).$$

Squaring, we get

$$\varphi^2(q)\varphi^2(-q^3) = \varphi^2(-q^4)\varphi^2(-q^{12}) + 4q^2\psi^2(-q^2)\psi^2(-q^6) \\ + 4q\varphi(-q^4)\varphi(-q^{12})\psi(-q^2)\psi(-q^6). \qquad (4.7.12)$$

Replacing q by $-q$ in (4.7.12) and then subtracting the resulting identity from (4.7.12), we find that

$$\varphi^2(q)\varphi^2(-q^3) - \varphi^2(-q)\varphi^2(q^3) = 8q\psi(-q^2)\varphi(-q^4)\psi(-q^6)\varphi(-q^{12}),$$

which can be transformed into

$$(-q;q^2)_\infty^4(q^3;q^6)_\infty^4 - (q;q^2)_\infty^4(-q^3;q^6)_\infty^4 = \frac{8q}{(q^2;q^4)_\infty(q^6;q^{12})_\infty}. \qquad (4.7.13)$$

Employing (4.7.13) in (4.7.11),

$$\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n = q^2\left\{\sum_{n=0}^{\infty} P_T(n)q^n - \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\} + 8q.$$

from which, by equating the coefficients of q^{2N+1} from both sides, we arrive at (4.7.8).

Now we prove (4.7.9).

Replacing q by $-q$ in (4.7.10) and then adding the resulting identity with (4.7.10),

$$\begin{aligned}
&\sum_{n=0}^{\infty} P_S(n)q^n + \sum_{n=0}^{\infty} P_S(n)(-q)^n \\
&\quad = q^2\left\{\sum_{n=0}^{\infty} P_T(n)q^n + \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\} + (q^2;q^4)_\infty(q^6;q^{12})_\infty \\
&\quad \times \left\{(-q;q^2)_\infty^4(q^3;q^6)_\infty^4 + (q;q^2)_\infty^4(-q^3;q^6)_\infty^4\right\}, \\
&\quad = q^2\left\{\sum_{n=0}^{\infty} P_T(n)q^n + \sum_{n=0}^{\infty} P_T(n)(-q)^n\right\} \\
&\quad + \frac{(-q;q^2)_\infty^4(q^3;q^6)_\infty^4 + (q;q^2)_\infty^4(-q^3;q^6)_\infty^4}{(-q^2;q^2)_\infty(-q^6;q^6)_\infty},
\end{aligned}$$

that is,

$$\begin{aligned}
\sum_{n=0}^{\infty} P_{S'}(n)q^n + \sum_{n=0}^{\infty} P_{S'}(n)(-q)^n &= q^2\left\{\sum_{n=0}^{\infty} P_{T'}(n)q^n + \sum_{n=0}^{\infty} P_{T'}(n)(-q)^n\right\} \\
&+ (-q;q^2)_\infty^4(q^3;q^6)_\infty^4 + (q;q^2)_\infty^4(-q^3;q^6)_\infty^4.
\end{aligned} \tag{4.7.14}$$

Again, replacing q by $-q$ in (4.7.12) and then adding the resulting identity with (4.7.12), we have

$$\varphi^2(q)\varphi^2(-q^3) + \varphi^2(-q)\varphi^2(q^3) = 2\varphi^2(-q^4)\varphi^2(-q^{12}) + 8q^2\psi^2(-q^2)\psi^2(-q^6),$$

which is equivalent to

$$\begin{aligned}
(-q;q^2)_\infty^4(q^3;q^6)_\infty^4 + (q;q^2)_\infty^4(-q^3;q^6)_\infty^4 &= 2(-q^2;q^4)_\infty^2(-q^6;q^{12})_\infty^2 \\
&+ 8q^2(-q^4;q^4)_\infty^2(-q^{12};q^{12})_\infty^2.
\end{aligned}$$

Employing the above identity in (4.7.14), we have

$$\sum_{n=0}^{\infty} P_{S'}(n)q^n + \sum_{n=0}^{\infty} P_{S'}(n)(-q)^n = q^2\left\{\sum_{n=0}^{\infty} P_{T'}(n)q^n + \sum_{n=0}^{\infty} P_{T'}(n)(-q)^n\right\}$$
$$+ 2(-q^2;q^4)_\infty^2(-q^6;q^{12})_\infty^2 + 8q^2(-q^4;q^4)_\infty^2(-q^{12};q^{12})_\infty^2.$$

Equating the coefficients of q^{2N} from both sides of the above and noting that

$$\sum_{n=0}^{\infty} P_V(n)q^n = (-q;q^2)_\infty^2(-q^3;q^6)_\infty^2$$

and

$$\sum_{n=0}^{\infty} P_W(n)q^n = (-q^2;q^2)_\infty^2(-q^6;q^6)_\infty^2,$$

we readily arrive at (4.7.9) to finish the proof □

Example: $n = 3$ *in* (4.7.8).

Then $P_S(7) = 44$, the relevant partitions of 7 are 4 copies of the type 7, 16 copies of the type $4+1+1+1$, 24 additional copies of the type $5+1+1$; $P_T(5) = 44$, the relevant partitions of 5 are 4 copies of the type 5, 16 copies of the type $2+1+1+1$ and 24 additional copies of the type $2+2+1$.

Corollary 4.7.3. *If S' and T' are defined in Theorem 4.7.2, then*

$$P_{S'}(4N+2) = P_{T'}(4N) + 3P_V(2N+1)$$

and

$$P_{S'}(4N) = P_{T'}(4N-2) + P_V(2N).$$

Proof. It is known from Berndt's paper [16, Theorem 3.1] that $P_V(2N+1) = 2P_W(2N)$. Therefore, from (4.7.9),

$$P_{S'}(4N+2) = P_{T'}(4N) + P_V(2N+1) + 4P_W(2N) = P_{T'}(4N) + 3P_V(2N+1)$$

and

$$\begin{aligned} P_{S'}(4N) &= P_{T'}(4N-2) + P_V(2N) + 4P_W(2N-1) \\ &= P_{T'}(4N-2) + P_V(2N+1), \end{aligned}$$

since $P_W(2N-1) = 0$ as W contains only even elements. □

Theorem 4.7.4. (Corollary to Conjecture 3.41 of [58]) *Let S be the set containing 2 copies of the even positive integers, 2 more copies of the positive integers congruent to ± 2 modulo 12, and 4 copies of the odd multiples of 3; let T be the set containing 2 copies of the even positive integers, 4 more copies of the positive multiples of 12, one copy of the odd positive integers, and one more copy of the odd multiples of 3. Then, for any $N \geq 2$,*

$$D_S(N) = 4D_T(N-2)$$

or equivalently,

$$P_S(N) = 4P_T(N-2).$$

Proof. Setting $a = q$, $b = q^5$, $c = -q^3$, and $d = -q^3$ in (4.1.2), we find that

$$-f(-q,-q^5)\varphi(q^3) = -f(q,q^5)\varphi(-q^3) + 2qf^2(-q^2,-q^{10}). \tag{4.7.15}$$

Multiplying both sides of (4.7.15) by $\varphi(-q^3)$ and then adding $\varphi^2(q^3)f(q,q^5)$ to both sides, we have

$$\begin{aligned} \varphi(q^3)\left\{\varphi(q^3)f(q,q^5) - \varphi(-q^3)f(-q,-q^5)\right\} &= f(q,q^5)\left\{\varphi^2(q^3) - \varphi^2(-q^3)\right\} \\ &\quad + 2q\varphi(-q^3)f^2(-q^2,-q^{10}). \end{aligned} \tag{4.7.16}$$

Again applying (4.7.15) and the third identity of Lemma 4.1.2, with q replaced by q^3, in (4.7.16), we find that

$$\varphi(q^3)f^2(q^2,q^{10}) = 4q^2f(q,q^5)\psi^2(q^{12}) + \varphi(-q^3)f^2(-q^2,-q^{10}).$$

Dividing both sides of the above by $\varphi(-q^3)f^2(-q^2,-q^{10})$ and then transforming to q-products, we obtain

$$(-q^2;q^2)^2_\infty(-q^{\pm 2};q^{12})^2_\infty(-q^3;q^6)^4_\infty = 4q^2(-q^2;q^2)^2_\infty(-q^{12};q^{12})^4_\infty(-q;q^2)_\infty(-q^3;q^6)_\infty + 1,$$

which is clearly

$$\sum_{n=0}^{\infty} D_S(n)q^n = 4q^2\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(n)q^n = 4q^2\sum_{n=0}^{\infty} P_T(n)q^n + 1.$$

Equating the coefficients of q^N from both sides of the above, we find the desired result. □

Example: $n = 6$.

Then $P_S(6) = 20$, the relevant partitions of 6 are 2 copies of the type 6, 4 copies of the type $2+2+2$, 6 copies of the type $3+3$ and 8 additional copies of the type $4+2$; $P_T(4) = 5$, the relevant partitions of 4 are $2+2$, 2 copies of each of the types 4 and $3+1$.

Theorem 4.7.5. (Corollary to Conjecture 3.44 of [58]) *Let S be the set containing one copy of the positive integers that are not odd multiples of* 6, *one more copy of the positive multiples of* 3 *that are not odd multiples of* 6, 2 *more copies of the positive integers that are congruent to* ± 2 *modulo* 12, *and* 3 *more copies of the positive integers that are congruent to* ± 4 *modulo* 12; *let T be the set containing* 2 *copies of the positive integers that are not congruent to* 6 *or* ± 4 *modulo* 12, *one copy of the positive integers that are congruent to* ± 4 *modulo* 12, *and one more copy of the positive integers that are congruent to* ± 1 *modulo* 6. *Then, for any $N \geq 1$,*

$$D_S(N) = D_T(N-1)$$

or equivalently,

$$P_S(N) = P_T(N-1).$$

Proof. Setting, in turn, $a = c = q$, $b = d = q^5$ and $a = c = q$, $b = d = q^2$, in Lemma 4.1.1, we find that

$$f^2(q,q^5) + f^2(-q,-q^5) = 2f(q^2,q^{10})\varphi(q^6), \tag{4.7.17}$$

$$f^2(q,q^5) - f^2(-q,-q^5) = 4qf(q^4,q^8)\psi(q^{12}), \tag{4.7.18}$$

$$f^2(q,q^2) + f^2(-q,-q^2) = 2f(q^2,q^4)\varphi(q^3), \tag{4.7.19}$$

and

$$f^2(q,q^2) - f^2(-q,-q^2) = 4qf(q,q^5)\psi(q^6). \tag{4.7.20}$$

Multiplying (4.7.18) by $qf(q^2,q^{10})$ and then using (4.7.20) with q replaced by q^2, we have

$$\begin{aligned} &qf^2(q,q^5)f(q^2,q^{10}) \\ &\quad= 4q^2f(q^4,q^8)f^2(q^2,q^{10})\psi(q^{12}) + qf(q^2,q^{10})f^2(-q,-q^5) \\ &\quad= f(q^4,q^8)\left(f^2(q^2,q^4) - f^2(-q^2,-q^4)\right) + qf(q^2,q^{10})f^2(-q,-q^5) \\ &\quad= f^2(q^2,q^4)f(q^4,q^8) - \left(f^2(-q^2,-q^4)f(q^4,q^8) - qf(q^2,q^{10})f^2(-q,-q^5)\right). \end{aligned} \tag{4.7.21}$$

Again, by (4.7.17)-(4.7.20),

$$\begin{aligned} &f^2(-q^2,-q^4)f(q^4,q^8) - qf(q^2,q^{10})f^2(-q,-q^5) \\ &\quad= f^2(q^4,q^8)\varphi(q^6) - 2q^2f(q^2,q^{10})f(q^4,q^8)\psi(q^{12}) - qf^2(q^2,q^{10})\varphi(q^6) \\ &\qquad+ 2q^2f(q^2,q^{10})f(q^4,q^8)\psi(q^{12}) \\ &\quad= \varphi(q^6)\left(f^2(q^4,q^8) - qf^2(q^2,q^{10})\right) \\ &\quad= \varphi(q^6)\varphi(-q^3)f(-q,-q^5), \end{aligned} \tag{4.7.22}$$

where (4.7.3) is used to get the last equality.

Employing (4.7.22) in (4.7.21), we get

$$f^2(q^2,q^4)f(q^4,q^8) = qf^2(q,q^5)f(q^2,q^{10}) + \varphi(-q^3)\varphi(q^6)f(-q,-q^5).$$

Multiplying the equation by $f(q^4,q^8)$ and then transforming the terms into q-products, and then simplifying further, we find that

$$\begin{aligned}&\frac{(-q;q)_\infty}{(-q^6;q^{12})_\infty}\frac{(-q^3;q^3)_\infty}{(-q^6;q^{12})_\infty}(-q^{\pm 2};q^{12})^2_\infty(-q^{\pm 4};q^{12})^3_\infty \\ &\quad= \frac{(-q;q)^2_\infty}{(-q^6;q^{12})^2_\infty(-q^{\pm 4};q^{12})^2_\infty}(-q^{\pm 4};q^{12})_\infty(-q^{\pm 1};q^6)_\infty + 1. \end{aligned} \tag{4.7.23}$$

Thus,

$$\sum_{n=0}^{\infty} D_S(n)q^n = q\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(n)q^n = q\sum_{n=0}^{\infty} P_T(n)q^n + 1.$$

Equating the coefficients of q^N from both sides, we complete the proof. □

Example: $n = 7$.

Then $P_S(7) = 32$, the relevant partitions of 7 are 7, $3+3+1$, $2+2+2+1$, 3 copies of the type $5+2$, 6 copies of the type $3+2+2$, 8 copies of the type $4+3$ and 12 additional copies of the type $4+2+1$; $P_T(6) = 32$, the relevant partitions of 6 are $3+3$, 2 copies of each of the types $4+2$ and $3+1+1+1$, 3 copies of each of the types $4+1+1$ and $2+2+1+1$, 9 copies of the type $5+1$ and 12 additional copies of the type $3+2+1$.

Theorem 4.7.6. (Analogue to Corollary to Conjecture 3.45 of [58]) *Let S be the set containing 2 copies of the positive integers that are not congruent to 0 or ± 2 modulo 12, one copy of the positive integers that are congruent to ± 2 modulo 12,* *and one more copy of the positive integers that are congruent to* ± 1 *modulo* 6*; let* T *be the set containing one copy of the positive integers that are not odd multiples of* 6*, one more copy of the positive multiples of* 3 *that are not odd multiples of* 6*,* 2 *more copies of the positive integers that are congruent to* ± 2 *modulo* 12*, and* 3 *more copies of the positive integers that are congruent to* ± 4 *modulo* 12*. Then, for any* $N \geq 1$,

$$P_S(2N) = 2P_T(2N-1) + a(2N) \tag{4.7.24}$$

and

$$P_S(2N+1) = 2P_T(2N), \tag{4.7.25}$$

where

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{\chi^3(-q^3)\chi^3(-q^6)}{\chi(-q)\chi(-q^2)}. \tag{4.7.26}$$

Proof. From (4.7.17)–(4.7.20) and Lemma 4.1.2, we have

$$\begin{aligned}
&\varphi(q^6)f^2(q,q^5) - 2q\psi(q^{12})f^2(q^2,q^4) \\
&= \left(f(q^2,q^{10})\varphi^2(q^6) + 2qf(q^4,q^6)\varphi(q^6)\psi(q^{12})\right) \\
&\quad - 2q\left(f(q^4,q^6)\varphi(q^6)\psi(q^{12}) + 2q^2 f(q^2,q^{10})\psi^2(q^{12})\right) \\
&= f(q^2,q^{10})\left(\varphi^2(q^6) - 4q^3\psi^2(q^{12})\right) = f(q^2,q^{10})\varphi^2(-q^3).
\end{aligned}$$

Transforming the above into q-products, multiplying both sides by $(-q;q)_\infty(-q^{\pm 4};q^{12})_\infty/(-q^6;q^{12})^2_\infty$ and then simplifying further, we deduce that

$$\begin{aligned}
\frac{(-q;q)^2_\infty(-q^{\pm 2};q^{12})_\infty(-q^{\pm 1};q^6)_\infty}{(-q^{12};q^{12})^2_\infty(-q^{\pm 2};q^{12})^2_\infty} &= 2q\frac{(-q;q)_\infty(-q^3;q^3)_\infty(-q^{\pm 2};q^{12})^2_\infty(-q^{\pm 4};q^{12})^3_\infty}{(-q^6;q^{12})^2_\infty} \\
&\quad + \frac{\chi^3(-q^3)\chi^3(-q^6)}{\chi(-q)\chi(-q^2)},
\end{aligned} \tag{4.7.27}$$

which also states that

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q\sum_{n=0}^{\infty} P_T(n)q^n + \sum_{n=0}^{\infty} a(n)q^n, \tag{4.7.28}$$

where $a(n)$ is as defined in (4.7.26). Equating the coefficients of q^{2N} and q^{2N+1} from both sides of (4.7.28), we deduce (4.7.24) and

$$P_S(2N+1) = 2P_T(2N) + a(2N+1), \tag{4.7.29}$$

respectively.

But, by (4.4.14), we have

$$\sum_{n=0}^{\infty} a(n)q^n - \sum_{n=0}^{\infty} a(n)(-q)^n = 2q.$$

Equating the coefficients of q^{2N+1} from both sides of the above, we have $a(1) = 1$ and for $n \geq 1$, $a(2n+1) = 0$, and therefore, (4.7.29) reduces to (4.7.25). $\square$

Example: $n = 2$ *in* (4.7.25).

Then $P_S(5) = 18$, the relevant partitions of 5 are $2+1+1+1$, 2 copies of the type $3+2$, 3 copies of the type 5, 6 copies of each of the types $4+1$ and $3+1+1$; $P_T(4) = 9$, the relevant partitions of 4 are 2 copies of the type $3+1$, 3 copies of the type $2+2$ and 4 additional copies of the type 4.

Theorem 4.7.7. (Analogue to Corollary to Conjecture 3.43 of [58]) *Let S be the set containing 2 copies of the positive integers that are not congruent to 0 or ± 2 modulo 12, one copy of the positive integers that are congruent to ± 2 modulo 12, and one more copy of the positive integers that are congruent to ± 1 modulo 6; let T be the set containing 2 copies of the positive integers that are not congruent to 6 or ± 4 modulo 12, one copy of the positive integers that are congruent to ± 4 modulo 12, and one more copy of the positive integers that are congruent to ± 1 modulo 6. Then, for any $N \geq 1$,*

$$P_S(2N) = 2P_T(2N-2) + a(2N) \tag{4.7.30}$$

and

$$P_S(2N+1) = 2P_T(2N-1), \tag{4.7.31}$$

where $a(n)$ is as defined in (4.7.26).

Proof. Define

$$A := \frac{(-q;q)_\infty}{(-q^6;q^{12})_\infty}\frac{(-q^3;q^3)_\infty}{(-q^6;q^{12})_\infty}(-q^{\pm 2};q^{12})^2_\infty(-q^{\pm 4};q^{12})^3_\infty,$$
$$B := \frac{(-q;q)^2_\infty}{(-q^6;q^{12})^2_\infty(-q^{\pm 4};q^{12})^2_\infty}(-q^{\pm 4};q^{12})_\infty(-q^{\pm 1};q^6)_\infty,$$

and

$$C := \frac{(-q;q)^2_\infty(-q^{\pm 2};q^{12})_\infty(-q^{\pm 1};q^6)_\infty}{(-q^{12};q^{12})^2_\infty(-q^{\pm 2};q^{12})^2_\infty}.$$

From (4.7.23) and (4.7.27), we have

$$A = qB + 1$$

and

$$C = 2qA + \sum_{n=0}^{\infty} a(n)q^n,$$

where $a(n)$ is defined by (4.7.26). It is easily seen from the above that

$$C = 2q^2B + 2q + \sum_{n=0}^{\infty} a(n)q^n,$$

which is equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q^2\sum_{n=0}^{\infty} P_T(n)q^n + \sum_{n=0}^{\infty} a(n)q^n + 2q,$$

where S and T are as given in the statement of the theorem. Equating the coefficients of q^{2N} and q^{2N+1}, respectively, from both sides of the above, and also noting that $a(2N+1) = 0$, we readily arrive at (4.7.30) and (4.7.31) to complete the proof. □

Example: $n = 2$ *in* (4.7.31).

Then $P_S(5) = 18$, the relevant partitions of 5 are $2+1+1+1$, 2 copies of the type $3+2$, 3 copies of the type 5, 6 copies of each of the types $4+1$ and $3+1+1$; $P_T(3) = 9$, the relevant partitions of 3 are $1+1+1$, 2 copies of the type 3 and 6 copies of the type $2+1$.

Theorem 4.7.8. (Corollary to Conjecture 3.46 of [58]) *Let S be the set containing 2 copies of the positive integers that are not odd multiples of 3, one more copy of the positive integers that are congruent to ± 2 modulo 12, and 2 more copies of the positive odd multiples of 6; let T be the set containing 2 copies of the positive integers that are not odd multiples of 3, one more copy of the positive integers that are congruent to ± 4 modulo 12, and 2 more copies of the positive multiples of 12. Then, for any $N \geq 1$,*

$$D_S(N) = 2D_T(N-1)$$

or equivalently,

$$P_S(N) = 2P_T(N-1).$$

Proof. Replacing q by q^2 in (4.7.2), we have

$$\begin{aligned} f(q^2, q^{10})\varphi(q^6) &= q^2 f^2(q^4, q^{20}) + f^2(q^8, q^{16}), \\ &= \left(f(q^8, q^{16}) - qf(q^4, q^{20})\right)^2 + 2qf(q^4, q^{20})f(q^8, q^{16}). \end{aligned} \tag{4.7.32}$$

But, from [13, p. 46, Entries 30(ii) and 30(iii)],

$$f(-q, -q^5) = f(q^8, q^{16}) - qf(q^4, q^{20}). \tag{4.7.33}$$

Employing (4.7.33) in (4.7.32), we get

$$f(q^2, q^{10})\varphi(q^6) = 2qf(q^4, q^{20})f(q^8, q^{16}) + f^2(-q, -q^5).$$

Transforming into q-products and then simplifying, we find that

$$\frac{(-q;q)_\infty^2}{(-q^3;q^6)_\infty^2}(-q^{\pm 2};q^{12})_\infty(-q^6;q^{12})_\infty^2 = 2q\frac{(-q;q)_\infty^2}{(-q^3;q^6)_\infty^2}(-q^{\pm 4};q^{12})_\infty(-q^{12};q^{12})_\infty^2 + 1,$$

that is

$$\sum_{n=0}^{\infty} D_S(n)q^n = 2q\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2q\sum_{n=0}^{\infty} P_T(n)q^n + 1$$

The proffered partition identity of the theorem follows immediately. □

Example: $n = 6$.

Then $P_S(6) = 20$, the relevant partitions of 6 are $2+2+2$, 2 copies of the type $4+1+1$, 3 copies of the type $2+2+1+1$, 4 copies of each of the types 6 and $5+1$ and 6 additional copies of the type $4+2$; $P_T(5) = 10$, the relevant partitions of 5 are 2 copies of each of the types 5 and $2+2+1$ and 6 additional copies of the type $4+1$.

Theorem 4.7.9. (Analogue to Corollary to Conjecture 3 47 of [58]) *Let S be the set containing 2 copies of the positive integers that are not congruent to 2 modulo 4, one more copy of the positive integers that are congruent to ±1 modulo 6, and one more copy of the positive integers that are congruent to ±4 modulo 12; let T be the set containing 2 copies of the positive integers that are not multiples of 4, one more copy of the positive integers that are congruent to ±1 modulo 6, and one more copy of the positive integers that are congruent to ±2 modulo 12. Then, for any $N > 2$*

$$P_T(N) = P_S(N) + 3U(N-2), \tag{4.7.34}$$

where $U(N)$ is defined by

$$\sum_{n=0}^{\infty} U(n)q^n := \frac{(-q;q^2)^2_\infty(-q^{\pm1};q^6)_\infty(-q^4;q^4)_\infty(-q^{12};q^{12})_\infty}{(q^8;q^{24})^2_\infty(q^{16};q^{24})^2_\infty} \tag{4 7 35}$$

Proof. Baruah and Nath [12, Equation 3.17] proved that

$$\varphi(q)f(q,q^5) = \psi(q^2)f(q^2,q^4) + 3q\frac{f^3(-q^{12})}{f(-q^4)}$$

Replacing q by q^2 and then multiplying both sides by $f(q,q^5)$, we find

$$\varphi(q^2)f(q^2,q^{10})f(q,q^5) = f(q,q^5)\psi(q^4)f(q^4,q^8) + 3q^2f(q.q^5)\frac{f^3(-q^{24})}{f(-q^8)}$$

Transforming into q-products and simplifying, we get

$$\begin{aligned}&(-q^{\pm1};q^6)_\infty(-q^{\pm2};q^{12})_\infty(-q^2;q^4)_\infty^2\\&=(-q^{\pm1};q^6)_\infty(-q^{\pm4};q^{12})_\infty\frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty(q^4;q^4)_\infty}+3q^2\frac{(q^{24};q^{24})_\infty^3(q^{\pm1};q^6)_\infty}{(q^4;q^4)_\infty(q^{12};q^{12})_\infty(q^8;q^8)_\infty}.\end{aligned}$$

Multiplying both sides of the above by $(-q;q^2)_\infty^2$ and simplifying by using Euler's identity, we find that

$$\begin{aligned}&(-q^{\pm1};q^6)_\infty(-q^{\pm2};q^{12})_\infty\frac{(-q;q)_\infty^2}{(-q^4;q^4)_\infty^2}\\&=(-q^{\pm1};q^6)_\infty(-q^{\pm4};q^{12})_\infty\frac{(-q;q)_\infty^2}{(-q^2;q^4)_\infty^2}+3q^2\frac{(-q;q^2)_\infty^2(q^{24};q^{24})_\infty^3(q^{\pm1};q^6)_\infty}{(q^4;q^4)_\infty(q^{12};q^{12})_\infty(q^8;q^8)_\infty}\\&=(-q^{\pm1};q^6)_\infty(-q^{\pm4};q^{12})_\infty\frac{(-q;q)_\infty^2}{(-q^2;q^4)_\infty^2}\\&\quad+3q^2\frac{(-q;q^2)_\infty^2(-q^{\pm1};q^6)_\infty(-q^4;q^4)_\infty(-q^{12};q^{12})_\infty}{(q^8;q^{24})_\infty^2(q^{16};q^{24})_\infty^2},\end{aligned}$$

which is equivalent to

$$\sum_{n=0}^{\infty}P_T(n)q^n+3q^2\sum_{n=0}^{\infty}U(n)q^n=\sum_{n=0}^{\infty}P_S(n)q^n. \tag{4.7.36}$$

Equating the coefficients of q^N from both sides, we easily arrive at the desired identity. □

Example: $n=5$.

Then $P_S(5)=18$, the relevant partitions of 5 are 3 copies of the type 5, 6 copies of the types $3+1+1$ and 9 additional copies of the type $4+1$; $P_T(5)=27$, the relevant partitions of 5 are 3 copies of each of the types 5 and $2+1+1+1$, 6 copies of each of the types $3+2$ and $3+1+1$, 9 additional copies of the type $2+2+1$ and $U(3)=3$, the relevant partitions of 3 are $1+1+1$ and 2 copies of the form 3.

Theorem 4.7.10. (Corollary to Conjecture 3.49 of [58]) *Let S be the set containing 2 copies of the positive multiples of 6, 2 copies of the positive integers that are congruent to ±1 modulo 6, one copy of the positive integers that are congruent to ±2 modulo 6, and 4 copies of the odd positive multiples of 3; let T be the set containing*

4 copies of the positive multiples of 6*, one copy of the positive integers that are congruent to* ± 1 *modulo* 6*, one copy of the positive integers that are congruent to* ± 2 *modulo* 6*,* 2 *more copies of the positive integers that are congruent to* ± 2 *modulo* 12*, and* 2 *copies of the odd positive multiples of* 3*. Then, for any* $N \geq 1$,

$$D_S(N) = 2D_T(N-1)$$

or equivalently,

$$P_S(N) = 2P_T(N-1).$$

Proof. Setting $a = q$, $b = q^5$, $c = q^3$, and $d = q^3$ in (4.1.2),

$$f(q, q^5)\varphi(q^3) = 2qf^2(q^2, q^{10}) + f(-q, -q^5)\varphi(-q^3),$$

which can be rewritten, with the aid of the Jacobi triple product identity, (1.1.2), as

$$\begin{aligned}(-q^{\pm 1}; q^6)_\infty(-q^3; q^6)^2_\infty &= 2q(-q^6; q^6)^2_\infty(-q^{\pm 2}; q^{12})^2_\infty + (q^{\pm 1}; q^6)_\infty(q^3; q^6)^2_\infty \\ &= 2q(-q^6; q^6)^2_\infty(-q^{\pm 2}; q^{12})^2_\infty + (q; q^2)_\infty(q^3; q^6)_\infty \\ &= 2q(-q^6; q^6)^2_\infty(-q^{\pm 2}; q^{12})^2_\infty + \frac{1}{(-q; q)_\infty(-q^3; q^3)_\infty},\end{aligned}$$

where Euler's identity is used in the last equality. The above can be put in the form

$$\begin{aligned}&(-q^6; q^6)^2_\infty(-q^{\pm 1}; q^6)^2_\infty(-q^{\pm 2}; q^6)_\infty(-q^3; q^6)^4_\infty \\ &= 2q(-q^6; q^6)^4_\infty(-q^{\pm 1}; q^6)_\infty(-q^{\pm 2}; q^6)_\infty(-q^{\pm 2}; q^{12})^2_\infty(-q^3; q^6)^2_\infty + 1, \qquad (4.7.37)\end{aligned}$$

which is

$$\sum_{n=0}^{\infty} D_S(N)q^n = 2q\sum_{n=0}^{\infty} D_T(N)q^n + 1,$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(N)q^n = 2q\sum_{n=0}^{\infty} P_T(N)q^n + 1.$$

Equating the coefficients of q^N from both sides of the above two identities, we complete the proof. □

Example: $n = 7$.

Then $P_S(7) = 32$, the relevant partitions of 7 are 2 copies of each of the types 7, $5+2$, $5+1+1$ and $4+2+1$, 4 copies each of the forms $6+1$, $4+3$ and $3+2+1+1$ and 12 additional copies of the form $3+3+1$; $P_T(6) = 16$, the relevant partitions of 6 are $5+1$, $3+3$, $2+2+2$, 3 copies of the type $4+2$, 4 copies of the type 6 and 6 additional copies of the type $3+2+1$.

Theorem 4.7.11. (Corollary to Conjecture 3.50 of [58]) *Let S be the set containing 4 copies of the positive multiples of 6, one copy of the positive integers that are congruent to ± 1 modulo 6, one copy of the positive integers that are congruent to ± 2 modulo 6, 2 more copies of the positive integers that are congruent to ± 4 modulo 12, and 2 copies of the odd positive multiples of 3; let T be the set containing 2 copies of the positive multiples of 6, 2 copies of the positive integers that are congruent to ± 1 modulo 6, one copy of the positive integers that are congruent to ± 2 modulo 6, and 4 copies of the odd positive multiples of 3. Then, for any $N \geq 1$,*

$$D_S(N) = \frac{1}{2}\, D_T(N)$$

or equivalently,

$$P_S(N) = \frac{1}{2}\, P_T(N).$$

Proof. Setting $a = q$, $b = q^5$, $c = q^3$, and $d = q^3$ in (4.1.1),

$$2f^2(q^4, q^8) = f(q, q^5)\varphi(q^3) + f(-q, -q^5)\varphi(-q^3), \tag{4.7.38}$$

which can be rewritten, with the help of the Jacobi triple product identity and Euler's identity, as

$$2(-q^6;q^6)_\infty^2(-q^{\pm 4};q^{12})_\infty^2 = (-q^{\pm 1};q^6)_\infty(-q^3;q^6)_\infty^2 + \frac{1}{(-q;q)_\infty(-q^3;q^3)_\infty}.$$

After simplification, the above gives,

$$(-q^6;q^6)_\infty^4(-q^{\pm1};q^6)_\infty(-q^{\pm2};q^6)_\infty(-q^{\pm4};q^{12})_\infty^2(-q^3;q^6)_\infty^2$$
$$= \frac{1}{2}(-q^{\pm1};q^6)_\infty^2(-q^3;q^6)_\infty^4(-q^{\pm2};q^6)_\infty(-q^6;q^6)_\infty^2 + \frac{1}{2}, \tag{4.7.39}$$

which is equivalent to

$$\sum_{n=0}^{\infty} D_S(n)q^n = \frac{1}{2}\sum_{n=0}^{\infty} D_T(n)q^n + \frac{1}{2}$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(n)q^n = \frac{1}{2}\sum_{n=0}^{\infty} P_T(n)q^n + \frac{1}{2}.$$

We complete the proof by equating the coefficients of q^N from both sides of the above two identities. □

Example: $n = 6$.

Then $P_S(6) = 11$, the relevant partitions of 6 are $5+1$, $3+3$, 2 copies of the types $3+2+1$, 3 copies of the type $4+2$ and 4 additional copies of the type 6; $P_T(6) = 22$, the relevant partitions of 6 are $4+2$, $4+1+1$, 2 copies of the type 6, 4 copies of the type $5+1$, 6 copies of the form $3+3$ and 8 additional copies of the form $3+2+1$.

Theorem 4.7.12. (Corollary to Conjecture 3.48 of [58]) *Let S be the set containing 4 copies of the positive multiples of 6, one copy of the positive integers that are congruent to ±1 modulo 6, one copy of the positive integers that are congruent to ±2 modulo 6, 2 more copies of the positive integers that are congruent to ±4 modulo 12, and 2 copies of the odd positive multiples of 3; let T be the set containing 4 copies of the positive multiples of 6, one copy of the positive integers that are congruent to ±1 modulo 6, one copy of the positive integers that are congruent to ±4 modulo 12, 2 copies of the odd positive multiples of 3, and 3 copies of the positive integers that* *are congruent to* ± 2 *modulo* 12. *Then, for any* $N \geq 1$,

$$D_S(N) = D_T(N-1)$$

or equivalently,

$$P_S(N) = P_T(N-1).$$

Proof. From (4.7.39) and (4.7.37), we find that

$$\begin{aligned}&(-q^6;q^6)_\infty^4(-q^{\pm1};q^6)_\infty(-q^{\pm2};q^6)_\infty(-q^{\pm4};q^{12})_\infty^2(-q^3;q^6)_\infty^2\\&= q(-q^6;q^6)_\infty^4(-q^{\pm1};q^6)_\infty(-q^{\pm2};q^6)_\infty(-q^{\pm2};q^{12})_\infty^2(-q^3;q^6)_\infty^2+1.\end{aligned}$$

With the help of the trivial identity $(-q^{\pm2};q^6)_\infty = (-q^{\pm4};q^{12})_\infty(-q^{\pm2};q^{12})_\infty$, the above reduces to

$$\begin{aligned}&(-q^6;q^6)_\infty^4(-q^{\pm1};q^6)_\infty(-q^{\pm2};q^6)_\infty(-q^{\pm4};q^{12})_\infty^2(-q^3;q^6)_\infty^2\\&= q(-q^6;q^6)_\infty^4(-q^{\pm1};q^6)_\infty(-q^{\pm4};q^{12})_\infty(-q^{\pm2};q^{12})_\infty^3(-q^3;q^6)_\infty^2+1,\end{aligned}$$

which can be rewritten as

$$\sum_{n=0}^{\infty} D_S(n)q^n = q\sum_{n=0}^{\infty} D_T(n)q^n + 1$$

or equivalently,

$$\sum_{n=0}^{\infty} P_S(n)q^n = q\sum_{n=0}^{\infty} P_T(n)q^n + 1.$$

Equating the coefficients of q^N, we complete the proof. □

Example: $n = 7$.

Then $P_S(7) = 16$, the relevant partitions of 7 are 7, $5+2$, $3+3+1$, 3 copies of the type $4+2+1$, 4 copies of the type $6+1$ and 6 additional copies of the type $4+3$; $P_T(6) = 16$, the relevant partitions of 6 are $5+1$, $3+3$, $2+2+2$, 3 copies of the type $4+2$, 4 copies of the type 6, and 6 additional copies of the type $3+2+1$.

To conclude this section, for completeness, we state the following theorem which is Corollary to Conjecture 3.53 of [58], and an analytic proof of this theorem has already been given by Baruah and Berndt [5, Theorem 8.1].

Theorem 4.7.13. (Corollary to Conjecture 3.53 of [58]) *Let S be the set containing one copy of the odd positive integers, one more copy of the odd positive multiples of* 3, *one more of the odd positive multiples of* 5, *and one more of the odd positive multiples of* 15; *let T be the set containing one copy of the even positive integers, one more copy of the positive multiples of* 6, *one more of the positive multiples of* 10, *and one more of the positive multiples of* 30. *Then, for any $N \geq 3$,*

$$D_S(N) = 2D_T(N-3)$$

or equivalently,

$$P_S(N) = 2P_T(N-3).$$

4.8 Some more colored partition identities.

In this section, we present some more colored partition identities which are analogous to the partition identities discussed in the previous sections.

Theorem 4.8.1. *Let S be the set containing* 2 *copies of the even positive integers,* 4 *more copies of the odd positive multiples of* 3, *and* 2 *copies of the positive multiples of* 4 *that are not multiples of* 12; *let T be the set containing one copy of the odd positive integers, one more copy of the odd positive multiples of* 3, *two copies of the even positive integers, and* 4 *more copies of the odd multiples of* 6. *Then, for any $N \geq 1$,*

$$P_S(N) = P_T(N).$$

Proof. Multiplying (4.7.15) by $\varphi(q^3)$ and then adding $\varphi^2(q^3)f(q,q^5)$ to both sides, we have

$$\begin{aligned}\varphi(q^3)\left\{\varphi(q^3)f(q,q^5)+\varphi(-q^3)f(-q,-q^5)\right\} =& f(q,q^5)\left\{\varphi^2(q^3)+\varphi^2(-q^3)\right\}\\ &-2q\varphi(-q^3)f^2(-q^2,-q^{10}),\end{aligned}$$

which can be rewritten, with the aid of (4.7.38) and Lemma 4.1.2, as

$$\varphi(q^3)f^2(q^4,q^8)=\varphi^2(q^6)f(q,q^5)-q\varphi(-q^3)f^2(-q^2,-q^{10}).$$

Transforming the above into q-products and then simplifying, we obtain

$$(-q^2;q^2)^2_\infty(-q^3;q^6)^4_\infty\frac{(-q^4;q^4)^2_\infty}{(-q^{12};q^{12})^2_\infty}=(-q;q^2)_\infty(-q^3;q^6)_\infty(-q^2;q^2)^2_\infty(-q^6;q^{12})^4_\infty-q,$$

which is equivalent to

$$\sum_{n=0}^{\infty}P_S(n)q^n=\sum_{n=0}^{\infty}P_T(n)q^n-q.$$

We complete the proof by equating the coefficients of q^N from both sides of the above. □

Example: $n=6$.

Then $P_S(6)=16$, the relevant partitions of 6 are 2 copies of the type 6, 6 copies of the type $3+3$ and 8 additional copies of the type $4+2$; $P_T(6)=16$, the relevant partitions of 6 are $5+1$, $3+3$, 4 copies of each of the types $4+2$ and $3+2+1$ and 6 additional copies of the type 6.

Theorem 4.8.2. *Let S be the set containing 4 copies of the positive integers that are congruent to ±1 modulo 6 and 2 copies of the even positive integers that are not multiples of 6; let T be the set containing 2 copies of the even positive integers, 2 copies of the positive multiples of 12, two more copies of the positive integers that are congruent to ±1 modulo 6 and one more copy of the positive integers that are congruent to ±4 modulo 12. Then, for any $N\geq1$,*

$$P_S(N)=4P_T(N-1).$$

Proof. With the help of (4.7.5), we can rewrite (4.7.18) as

$$f^4(q,q^5) = 4qf(q^4,q^8)f^2(q,q^5)\psi(q^{12}) + \varphi^2(-q^6)f^2(-q^2,-q^{10}).$$

Transcribing the above into q-products, we find that

$$(-q^{\pm 1};q^6)_\infty^4\frac{(-q^2;q^2)_\infty^2}{(-q^6;q^6)_\infty^2} = 4q(-q^2;q^2)_\infty^2(-q^{12};q^{12})_\infty^2(-q^{\pm 1};q^6)_\infty^2(-q^{\pm 4};q^{12})_\infty + 1,$$

which is equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n = 4q\sum_{n=0}^{\infty} P_T(n)q^n + 1.$$

Now the proffered partition identity is apparent. □

Example: $n = 7$.

Then $P_S(7) = 64$, the relevant partitions of 7 are 4 copies of each of the types 7 and $2+2+1+1+1$, 8 copies of each of the types $5+2$ and $4+1+1+1$, 16 copies of the type $4+2+1$ and 24 additional copies of the type $5+1+1$; $P_T(6) = 16$, the relevant partitions of 6 are $2+2+1+1$, 2 copies of the type 6, 3 copies of the type $4+1+1$, 4 copies of the type $5+1$ and 6 additional copies of the type $4+2$.

The next two theorems easily follow from (4.7.38) and (4.7.15), respectively. We omit the proofs.

Theorem 4.8.3. *Let S be the set containing one copy of the odd positive integers, one copy of the positive odd multiples of 3, one copy each of the positive integers and the positive multiples of 3; let T be the set containing one copy of the positive integers, 2 copies of positive integers that are odd multiples of 6, one copy of the positive multiples of 3 and 2 more copies of the positive multiples of 4. Then, for any $N \geq 1$,*

$$P_S(N) = 2P_T(N).$$

Theorem 4.8.4. *Let S be the set containing one copy of the odd positive integers, one more copy of the positive odd multiples of of 3, one copy each of the positive* *integers and the positive multiples of of* 3, *let* T *be the set containing one copy of the positive integers,* 2 *copies of the positive integers that are odd multiples of* 2, *one copy of the positive multiples of* 3 *and* 2 *more copies of the positive multiples of* 12 *Then, for any* $N \geq 1$,

$$P_S(N) = 2P_T(N-1)$$

Theorem 4.8.5. *Let* S *be the set containing* 3 *copies each of the positive integers, the odd positive integers, the positive multiples of* 3 *and the odd positive multiples of* 3; *let* T *be the set containing* 6 *copies each of the odd positive multiples of* 2 *and the positive multiples of* 12 *and* 3 *copies each of the positive integers and the positive multiples of* 3, *and let* U *be the set containing* 2 *copies each of the positive integers, the positive multiples of* 3, *the odd positive multiples of* 2 *and the positive multiples of* 12, *one copy each of the odd positive integers and the odd positive multiples of* 3 *Then, for any* $N \geq 1$,

$$P_S(N) = 8P_T(N-3) + 6P_U(N-1)$$

Proof. We recall from [14, p. 198, Entry 45] that

$$\psi(q)\psi(q^3) - \psi(-q)\psi(-q^3) = 2q\varphi(q^2)\psi(q^{12})$$

Cubing and then dividing both sides by $\psi^3(-q)\psi^3(-q^3)$, we obtain

$$\frac{\psi^3(q)\psi^3(q^3)}{\psi^3(-q)\psi^3(-q^3)} - 1 = 8q^3\frac{\varphi^3(q^2)\psi^3(q^{12})}{\psi^3(-q)\psi^3(-q^3)} + 6q\frac{\varphi(q^2)\psi(q^{12})\psi(q)\psi(q^3)}{\psi^2(-q)\psi^2(-q^3)},$$

which can be easily transformed, with the aid of Euler's identity, into

$$\begin{aligned}
&(-q;q^2)^3_\infty(-q;q)^3_\infty(-q^3;q^6)^3_\infty(-q^3;q^3)^3_\infty \\
&= 8q^3(-q^2;q^4)^6_\infty(-q^{12},q^{12})^6_\infty(-q,q)^3_\infty(-q^3;q^3)^3_\infty \\
&\quad + 6q(-q^2;q^4)^2_\infty(-q;q)^2_\infty(-q^3,q^3)^2_\infty(-q^{12};q^{12})^2_\infty(-q;q^2)_\infty(-q^3,q^6)_\infty + 1
\end{aligned}$$

Since the above is equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n = 8q^3\sum_{n=0}^{\infty} P_T(n)q^n + 6q\sum_{n=0}^{\infty} P_U(n)q^n + 1,$$

we complete the proof by equating the coefficients of q^N from both sides. □

Example: $n = 4$.

Then $P_S(4) = 138$, the relevant partitions of 4 are 3 copies of each of the forms 4 and $2+2$, 15 copies of the form $1+1+1+1$, 45 copies of the form $2+1+1$ and 72 additional copies of the form $3+1$; $P_T(1) = 3$, the relevant partitions of 1 are 3 copies of the form 1 and $P_U(3) = 19$, the relevant partitions of 3 are $1+1+1$, 6 copies of the form 3 and 12 additional copies of the form $2+1$.

Theorem 4.8.6. *Let S be the set containing 6 copies of the odd positive integers that are not multiple of 5 and one copy of the even positive integers; T be the set containing 4 copies of the even positive integers and 9 copies of the positive multiples of 10 and let U be the set containing 5 copies of the positive multiples of 10. Then, for any $N \geq 1$,*

$$P_S(2N+1) = 32P_T(2N-2) + 6P_U(2N).$$

Proof. From [13, p. 278], we recall that

$$\varphi(q)\varphi(-q^5) - \varphi(-q)\varphi(q^5) = 4qf(-q^4)f(-q^{20}).$$

Cubing, we have

$$\begin{aligned}\varphi^3(q)\varphi^3(-q^5) - \varphi^3(-q)\varphi^3(q^5) &= 64q^3f^3(-q^4)f^3(-q^{20}) \\ &\quad + 12q\varphi^2(-q^2)\varphi^2(-q^{10})f(-q^4)f(-q^{20}).\end{aligned}$$

Transforming into q-products, we obtain

$$\frac{(-q;q^2)^6_\infty(q^5;q^{10})^6_\infty}{(q^2;q^4)_\infty(q^{10};q^{20})_\infty} - \frac{(q;q^2)^6_\infty(-q^5;q^{10})^6_\infty}{(q^2;q^4)_\infty(q^{10};q^{20})_\infty} = 64q^3(-q^2;q^2)^4_\infty(-q^{10};q^{10})^4_\infty + 12q,$$

which can further be reduced to

$$\frac{(-q;q^2)_\infty^6(-q^2;q^2)_\infty}{(-q^5;q^{10})_\infty^6} - \frac{(q;q^2)_\infty^6(-q^2;q^2)_\infty}{(q^5;q^{10})_\infty^6} = 64q^3(-q^2;q^2)_\infty^4(-q^{10};q^{10})_\infty^9 + 12q(-q^{10};q^{10})_\infty^5.$$

Thus,

$$\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n = 64q^3\sum_{n=0}^{\infty} P_T(n)q^n + 12q\sum_{n=0}^{\infty} P_U(n)q^n.$$

Equating the coefficients of q^{2N+1} from both sides, we easily arrive at the desired partition identity. □

Example: $n = 2$.

Then $P_S(5) = 128$, the relevant partitions of 5 are 6 copies of each of the types $4+1$, $3+2$, $1+1+1+1+1+1$, 20 copies of the type $2+1+1+1$ and 90 additional copies of the type $3+1+1$; $P_T(2) = 4$, the relevant partitions of 2 are 4 copies of the type 2 and $P_U(4) = 0$.

Theorem 4.8.7. *Let S be the set containing one copy of the positive integers that are congruent to ± 1 modulo 6 and 2 copies of the positive integers that are odd multiples of 3; T be the set containing 2 copies of the positive integers that are multiples of 6 and 2 more copies of the positive integers that are congruent to ± 2 modulo 12 and let U be the set containing 2 copies of the positive multiples of 4 and 2 more copies of the odd multiple of 6. Then, for any $N \geq 1$,*

$$P_S(N) = P_T(N-1) + P_U(N).$$

Proof. We can easily transform (4.7.2) into

$$(-q^{\pm 1};q^6)_\infty(-q^3;q^6)_\infty^2 = q(-q^6;q^6)_\infty^2(-q^{\pm 2};q^{12})_\infty^2 + (-q^6;q^{12})_\infty^2(-q^4;q^4)_\infty^2,$$

which also states that

$$\sum_{n=0}^{\infty} P_S(n)q^n = q\sum_{n=0}^{\infty} P_T(n)q^n + \sum_{n=0}^{\infty} P_U(n)q^n.$$

Equating the coefficients of q^N from both sides, we complete the proof. □

Example: $n = 9$.

Then $P_S(9) = 4$, the relevant partitions of 9 are 2 copies of each of the types 9 and $5 + 3 + 1$; $P_T(8) = 4$, the relevant partitions of 8 are 4 copies of the type $6 + 2$ and $P_U(9) = 0$.

Theorem 4.8.8. *Let S be the set containing one copy of the odd positive integers that are not multiples of 9 and 2 copies of the even positive integers that are not multiples of 18; let T be the set containing 2 copies of the even integers and one more copy of the even positive integers that are not multiples of 6. Then, for any $N \geq 1$,*

$$P_S(2N + 1) = P_T(2N).$$

Proof. We recall from [8, Eq. (8.12)] that

$$\psi(q)\psi(-q^9) - \psi(-q)\psi(q^9) = 2q\psi(q^{18})\psi(q^2)\chi(-q^6).$$

Dividing both sides by $\varphi(-q^2)\psi(q^{18})$, transforming into q-products, and then simplifying by using Euler's identity, we deduce that

$$\frac{(-q;q^2)_\infty(-q^2;q^2)^2_\infty}{(-q^9;q^{18})_\infty(-q^{18};q^{18})^2_\infty} - \frac{(q;q^2)_\infty(-q^2;q^2)^2_\infty}{(q^9;q^{18})_\infty(-q^{18};q^{18})^2_\infty} = 2q\,\frac{(-q^2;q^2)_\infty}{(-q^6;q^6)_\infty}(-q^2;q^2)^2_\infty.$$

Thus,

$$\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n = 2q\sum_{n=0}^{\infty} P_T(n)q^n.$$

Equating the coefficients of q^{2N+1} from both sides, we finish the proof. □

Example: $n = 3$.

Then $P_S(7) = 12$, the relevant partitions of 7 are 7, $3 + 2 + 2$, 2 copies of each of the types $6 + 1$, $5 + 2$ and $4 + 3$ and 4 additional copies of the type $4 + 2 + 1$; $P_T(6) = 12$, the relevant partitions of 6 are $2 + 2 + 2$, 2 copies of the type 6 and 9 additional copies of the type $4 + 2$.

Theorem 4.8.9. *Let S be the set containing one copy of the odd positive integers that are not multiples of* 9 *and let T be the set containing one copy of the even positive integers that are not multiple of* 6 *and* 2 *more copies of the positive integers that are multiples of* 18. *Then, for any $N \geq 1$,*

$$P_S(2N+1) = P_T(2N). \tag{4.8.1}$$

Proof. From [13, p. 358, Entry 4(i)], we have

$$\frac{\varphi(-q^{18})}{\varphi(-q^2)} + q\left(\frac{\psi(q^9)}{\psi(q)} - \frac{\psi(-q^9)}{\psi(-q)}\right) = 1. \tag{4.8.2}$$

Now, replacing q by $-q^2$ in (4.4.7), we have

$$\varphi(-q^2) = \varphi(-q^{18}) - 2q^2\psi(q^{18})\chi(-q^6).$$

Employing the above in (4.8.2), we find that

$$\frac{\psi(-q^9)}{\psi(-q)} - \frac{\psi(q^9)}{\psi(q)} = 2q\frac{\psi(q^{18})\chi(-q^6)}{\varphi(-q^2)},$$

which can be transformed into

$$\frac{(-q;q^2)_\infty}{(-q^9;q^{18})_\infty} - \frac{(q;q^2)_\infty}{(q^9;q^{18})_\infty} = 2q\frac{(-q^2;q^2)_\infty}{(-q^6;q^6)_\infty}(-q^{18};q^{18})^2_\infty.$$

Since the above is equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n = 2q\sum_{n=0}^{\infty} P_T(n)q^n,$$

we complete the proof by equating the coefficients of q^{2N+1} from both sides. □

Example: $n = 7$.

Then $P_S(15) = 3$, the relevant partitions of 15 are 15, $11+3+1$ and $7+5+3$; $P_T(14) = 3$, the relevant partitions of 14 are 14, $10+4$ and $8+4+2$.

Theorem 4.8.10. *Let S be the set containing* 2 *copies of the odd positive integers that are not multiples of* 5 *and let T be the set containing one copy of the even* *positive integers and* 3 *more copies of the positive integers that are multiples of* 10. *Then, for any* $N \geq 1$,

$$P_S(2N+1) = 2P_T(2N).$$

Proof. We recall from From [13, p. 276] that

$$\frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)} + q\left(\frac{\psi^2(q^5)}{\psi^2(q)} - \frac{\psi^2(-q^5)}{\psi^2(-q)}\right) = 1. \tag{4.8.3}$$

But, from Entries 9(vii) and 10(iv) of [13, p. 258 and p. 262],

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q,q^9)f(q^3,q^7) = 4q\chi(q)f(-q^5)f(-q^{20}).$$

Replacing q by q^2 in the above and employing it in (4.8.3), we deduce

$$\frac{\psi^2(-q^5)}{\psi^2(-q)} - \frac{\psi^2(q^5)}{\psi^2(q)} = 4q\frac{\chi(-q^2)f(q^{10})f(-q^{40})}{\varphi^2(-q^2)},$$

which can be transformed into

$$\frac{(-q;q^2)^2_\infty}{(-q^5;q^{10})^2_\infty} - \frac{(q;q^2)^2_\infty}{(q^5;q^{10})^2_\infty} = 4q(-q^2;q^2)_\infty(-q^{10};q^{10})^3_\infty.$$

The above is equivalent to

$$\sum_{n=0}^{\infty} P_S(n)q^n - \sum_{n=0}^{\infty} P_S(n)(-q)^n = 4q\sum_{n=0}^{\infty} P_T(n)q^n,$$

and equating the coefficients of q^{2N+1} from both sides, we finish the proof. ☐

Example: $n = 5$.

Then $P_S(11) = 12$, the relevant partitions of 11 are 2 copies of each of the types 11 and $9+1+1$ and 8 additional copies of the type $7+3+1$; $P_T(10) = 6$, the relevant partitions of 10 are $6+4$, $8+2$ and 4 additional copies of the type 10.

Theorem 4.8.11. *Let* S *be the set containing one copy of the positive integers that are not multiples of* 3, *one more copy of the odd positive integers that are not multiples of* 3, 2 *more copies of the positive integers and* 2 *more copies of the odd* *positive integers; let T be the set containing one copy of the positive integers that are not multiples of* 3*, one more copy of the even positive integers that are not multiples of* 6*,* 2 *more copies of the positive integers and* 2 *more copies of even positive integers. Then, for any $N \geq 1$,*

$$P_S(N) = 2P_T(N)$$

Proof. From [13, p. 359], we have

$$\frac{\varphi(-q)\psi(q^3)}{\psi(q)} + \frac{\varphi(q)\psi(-q^3)}{\psi(-q)} = 2\frac{\psi(q^2)\varphi(-q^6)}{\varphi(-q^2)}$$

Dividing both sides by $\dfrac{\varphi(-q)\psi(q^3)}{\psi(q)}$ and then transforming into q-products, we have

$$\frac{(-q;q)_\infty(-q;q^2)_\infty}{(-q^3;q^3)_\infty(-q^3;q^6)_\infty}(-q;q^2)^2_\infty(-q;q)^2_\infty = 2\frac{(-q,q)_\infty(-q^2;q^2)_\infty}{(-q^3,q^3)_\infty(-q^6;q^6)_\infty}(-q;q)^2_\infty(-q^2;q^2)^2_\infty - 1.$$

which is clearly

$$\sum_{n=0}^{\infty} P_S(n)q^n = 2\sum_{n=0}^{\infty} P_T(n)q^n - 1$$

Equating the coefficients of q^N from both sides, we readily arrive at the desired identity. □

Example: $n = 5$.

Then $P_S(5) = 180$, the relevant partitions of 5 are 6 copies of each of the types 5 and $1+1+1+1+1+1$, 12 copies of the type $3+2$, 18 copies of each of the types $4+1$ and $2+2+1$ and 60 additional copies of each of the types $3+1+1$ and $2+1+1+1$; $P_T(5) = 90$, the relevant partitions of 5 are 3 copies of the type 5, 6 copies of each of the types $3+1+1$ and $2+1+1+1$, 12 copies of the type $3+2$, 18 copies of the type $4+1$ and 45 additional copies of the type $2+2+1$.

Theorem 4.8.12. *Let S be the set containing two copies each of the positive integers, the odd positive integers, positive multiples of* 3 *and odd positive multiples of*

3 and T be the set containing two copies each of the positive integers, even positive integers, positive multiples of 3 *and positive multiples of* 6. *Then, for any* $N \geq 1$,

$$P_S(N) = 4P_T(N-1).$$

Proof. We note from [14, p. 198, Entry 45] that

$$\psi(q)\psi(q^3) - \psi(-q)\psi(-q^3) = 2q\varphi(q^2)\psi(q^{12})$$

and

$$\psi(q)\psi(q^3) + \psi(-q)\psi(-q^3) = 2\varphi(q^6)\psi(q^4).$$

Multiplying together, we have

$$\psi^2(q)\psi^2(q^3) - \psi^2(-q)\psi^2(-q^3) = 4q\varphi(q^2)\psi(q^{12})\varphi(q^6)\psi(q^4).$$

Dividing both sides by $\psi^2(-q)\psi^2(-q^3)$ and simplifying by Euler's identity, we obtain

$$(-q;q^2)^2_\infty(-q;q)^2_\infty(-q^3;q^3)^2_\infty(-q^3;q^6)^2_\infty = 4q(-q;q)^2_\infty(-q^2;q^2)^2_\infty(-q^3;q^3)^2_\infty(-q^6;q^6)^2_\infty + 1,$$

which is

$$\sum_{n=0}^{\infty} P_S(n)q^n = 4q\sum_{n=0}^{\infty} P_T(n)q^n + 1.$$

Equating the coefficients of q^N from both sides, we finish the proof. □

Example: $n = 5$.

Then $P_S(5) = 88$, the relevant partitions of 5 are 4 copies of each of the types 5 and $2+2+1$, 8 copies of each of the types $4+1$ and $2+1+1+1$, 16 copies of the type $3+2$ and 48 additional copies of the type $3+1+1$; $P_T(4) = 22$, the relevant partitions of 4 are 4 copies of each of the types 4 and $2+1+1$, 6 copies of the type $2+2$ and 8 additional copies of the type $3+1$.

Chapter 5

Partition Identities Arising from Ramanujan's Modular Equations Involving Multipliers

5.1 Introduction

In continuation to the results in the previous chapter, in this chapter, we establish several more new theorems on partition identities arising from Ramanujan's modular equations with multipliers. In Section 5.2, we establish 7 partition identities arising from modular equations of degrees 3, 5, 7, 9, 13 and 25 involving multipliers. In Section 5.3, we establish 6 more partition identities arising from mixed modular equations of composite degrees 3, 5, 15; 3, 7, 21; 3, 11, 33; 5, 7, 35 and 3, 13, 39. Some of the identities are for overpartitions, overpartition pairs and regular partitions defined in the introductory chapter.

5.2 Partition identities arising from modular equations of degrees 3, 5, 7, 9, 13 and 25.

Theorem 5.2.1. *Let $A(n)$ denote the number of partitions of n into parts congruent to ± 1 modulo 6 having 4 colors, $B(n)$ denote the number of partitions of n into distinct odd parts that are not multiples of 3 each having 4 colors or even parts* *congruent to* ± 4 *modulo* 12 *having* 4 *colors and* $C(n)$ *denote the number of partitions of* n *into parts congruent to* ± 1 *modulo* 3 *having* 4 *colors. Then, for any* $n \geq 1$,

$$A(2n+1) - B(2n-1) = 4C(n). \tag{5.2.1}$$

Proof. We recall the following two modular equations from [13, p. 230, Entry 5(vii)] which are reciprocal to each other. If β has degree 3 over α and m is the multiplier connecting α and β, then

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}$$

and

$$\frac{9}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}.$$

With the aid of Lemma 1.1.1, the above can be transcribed into

$$\frac{\varphi^4(-q^6)}{\varphi^4(-q^2)} + q\left(\frac{\psi^4(q^3)}{\psi^4(q)} - \frac{\psi^4(-q^3)}{\psi^4(-q)}\right) = 1$$

and

$$\frac{\varphi^4(-q^2)}{\varphi^4(-q^6)} + \frac{1}{q}\left(\frac{\psi^4(q)}{\psi^4(q^3)} - \frac{\psi^4(-q)}{\psi^4(-q^3)}\right) = 9,$$

which can further be reduced to the q-product identities

$$q\left\{\frac{(-q;q^2)^4_\infty}{(-q^3;q^6)^4_\infty} - \frac{(q;q^2)^4_\infty}{(q^3;q^6)^4_\infty}\right\} = \frac{(-q^2;q^2)^4_\infty}{(-q^6;q^6)^4_\infty} - \frac{(q^2;q^2)^4_\infty}{(q^6;q^6)^4_\infty} \tag{5.2.2}$$

and

$$\frac{(q^3;q^6)^4_\infty}{(q;q^2)^4_\infty} - \frac{(-q^3;q^6)^4_\infty}{(-q;q^2)^4_\infty} = q\left\{9\frac{(q^6;q^6)^4_\infty}{(q^2;q^2)^4_\infty} - \frac{(-q^6;q^6)^4_\infty}{(-q^2;q^2)^4_\infty}\right\}. \tag{5.2.3}$$

Multiplying (5.2.2) by $q\dfrac{(q^{12};q^{12})^4_\infty}{(q^4;q^4)^4_\infty}$ and then subtracting from (5.2.3), we obtain

$$\frac{(q^3;q^6)^4_\infty}{(q;q^2)^4_\infty} - \frac{(-q^3;q^6)^4_\infty}{(-q;q^2)^4_\infty} - q^2\left\{\frac{(-q;q^2)^4_\infty(q^{12};q^{12})^4_\infty}{(-q^3;q^6)^4_\infty(q^4;q^4)^4_\infty} - \frac{(q;q^2)^4_\infty(q^{12};q^{12})^4_\infty}{(q^3;q^6)^4_\infty(q^4;q^4)^4_\infty}\right\}$$
$$= 8q\frac{(q^6;q^6)^4_\infty}{(q^2;q^2)^4_\infty}.$$

Thus,

$$\left\{\sum_{n=0}^{\infty} A(n)q^n - \sum_{n=0}^{\infty} A(n)(-q)^n\right\} - q^2\left\{\sum_{n=0}^{\infty} B(n)q^n - \sum_{n=0}^{\infty} B(n)(-q)^n\right\}$$
$$= 8q\sum_{n=0}^{\infty} C(n)q^{2n}.$$

Equating the coefficients of q^{2n+1}, we arrive at (5.2.1). □

Example: $n = 1$.

Then $A(3) = 20$ as there are 20 copies of the type $1+1+1$, $B(1) = C(1) = 4$ as there are 4 copies of 1.

Example: $n = 2$.

Then $A(5) = 60$ as there are 4 copies of 5 and 56 additional copies of the type $1+1+1+1+1$, $B(3) = 4$ as there are 4 copies of the type $1+1+1$, and $C(2) = 14$ as there are 4 copies of 2 and 10 additional copies of the type $1+1$.

Theorem 5.2.2. *Let $A(n)$ denote the number of partitions of n into odd parts not multiples of 5 with 2 colors, $B(n)$ denote the number of partitions of n into distinct odd parts not multiples of 5 having 2 colors or even parts not multiple of 5 having 2 colors and $C(n)$ denote the number of partitions of n into parts not multiples of 5 with 2 colors. Then, for any $n \geq 1$,*

$$A(2n+1) - B(2n-1) = 2C(n). \tag{5.2.4}$$

Proof. If β has degree 5 over α and m is the multiplier connecting α and β, then from [13, p. 281, Entry 13(xii)], we have

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}$$

and

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}.$$

The above two modular equations are reciprocal to each other. With the help of the formulas in Lemma 1.1.1, we transcribe the equations into

$$\frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)}+q\left(\frac{\psi^2(q^5)}{\psi^2(q)}-\frac{\psi^2(-q^5)}{\psi^2(-q)}\right)=1$$

and

$$\frac{\varphi^2(-q^2)}{\varphi^2(-q^{10})}+\frac{1}{q}\left(\frac{\psi^2(q)}{\psi^2(q^5)}-\frac{\psi^2(-q)}{\psi^2(-q^5)}\right)=5,$$

which can be transformed into the q-product identities

$$q\left\{\frac{(-q;q^2)^2_\infty}{(-q^5;q^{10})^2_\infty}-\frac{(q;q^2)^2_\infty}{(q^5;q^{10})^2_\infty}\right\}=\frac{(-q^2;q^2)^2_\infty}{(-q^{10};q^{10})^2_\infty}-\frac{(q^2;q^2)^2_\infty}{(q^{10};q^{10})^2_\infty} \tag{5.2.5}$$

and

$$\frac{(q^5;q^{10})^2_\infty}{(q;q^2)^2_\infty}-\frac{(-q^5;q^{10})^2_\infty}{(-q;q^2)^2_\infty}=q\left\{5\frac{(q^{10};q^{10})^2_\infty}{(q^2;q^2)^2_\infty}-\frac{(-q^{10};q^{10})^2_\infty}{(-q^2;q^2)^2_\infty}\right\}, \tag{5.2.6}$$

respectively. Multiplying (5.2.5) by $q\dfrac{(q^{20};q^{20})^2_\infty}{(q^4;q^4)^2_\infty}$ and then subtracting from (5.2.6), we find that

$$\begin{aligned}\frac{(q^5;q^{10})^2_\infty}{(q;q^2)^2_\infty}-\frac{(-q^5;q^{10})^2_\infty}{(-q;q^2)^2_\infty}-q^2\left\{\frac{(-q;q^2)^2_\infty(q^{20};q^{20})^2_\infty}{(-q^5;q^{10})^2_\infty(q^4;q^4)^2_\infty}-\frac{(q;q^2)^2_\infty(q^{20};q^{20})^2_\infty}{(q^5;q^{10})^2_\infty(q^4;q^4)^2_\infty}\right\}\\ =4q\frac{(q^{10};q^{10})^2_\infty}{(q^2;q^2)^2_\infty},\end{aligned}$$

which is equivalent to

$$\begin{aligned}\left\{\sum_{n=0}^{\infty}A(n)q^n-\sum_{n=0}^{\infty}A(n)(-q)^n\right\}-q^2\left\{\sum_{n=0}^{\infty}B(n)q^n-\sum_{n=0}^{\infty}B(n)(-q)^n\right\}\\ =4q\sum_{n=0}^{\infty}C(n)q^{2n}.\end{aligned}$$

Equating the coefficients of q^{2n+1} from both sides, we deduce (5.2.4) to finish the proof. □

Example: $n=2$.

Then $A(5)=12$ as there are 6 copies of the type $3+1+1$ and 6 additional copies of the type $1+1+1+1+1$, $B(3)=2$ as there are 2 copies of 3, and $C(2)=5$ as there are 2 copies of 2 and 3 additional copies of the type $1+1+1$.

Theorem 5.2.3. *Let $A(n)$ denote the number of partitions of n into odd parts not multiples of 7 having 4 colors, $B(n)$ denote the number of partitions of n into distinct odd parts not multiples of 7 having 4 colors or even parts not multiples of 7 having 4 colors, $C(n)$ denote the number of partitions of n into parts not multiples of 7 having 4 colors. Then, for any $n \geq 2$,*

$$A(4n+1) - B(4n-5) = 24C(2n-1) + 4C(n-1) \tag{5.2.7}$$

and, for any $n \geq 1$,

$$A(4n+3) - B(4n-3) = 24C(2n). \tag{5.2.8}$$

Proof. We recall the following pair of modular equations, reciprocal to each other, from [13, p. 314, Entry 19(v)]. If β has degree 7 over α and m is the multiplier connecting α and β, then

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} - 8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3}$$

and

$$\frac{49}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} - 8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3}.$$

These can be easily transcribed, with the help of the formulas in Lemma 1.1.1, into

$$\frac{\varphi^4(-q^{14})}{\varphi^4(-q^2)} + q^3\left(\frac{\psi^4(q^7)}{\psi^4(q)} - \frac{\psi^4(-q^7)}{\psi^4(-q)}\right) - 8q^2\frac{f^4(-q^{14})}{f^4(-q^2)} = 1$$

and

$$q^3\frac{\varphi^4(-q^2)}{\varphi^4(-q^{14})} + \left(\frac{\psi^4(q)}{\psi^4(q^7)} - \frac{\psi^4(-q)}{\psi^4(-q^7)}\right) - 8q\frac{f^4(-q^2)}{f^4(-q^{14})} = 49q^3.$$

Transforming each of the theta functions in the above into q-products, we find that

$$q^3\left\{\frac{(-q;q^2)_\infty^4}{(-q^7;q^{14})_\infty^4} - \frac{(q;q^2)_\infty^4}{(q^7;q^{14})_\infty^4}\right\} = \frac{(-q^2;q^2)_\infty^4}{(-q^{14};q^{14})_\infty^4} - \frac{(q^2;q^2)_\infty^4}{(q^{14};q^{14})_\infty^4} - 8q^2 \tag{5.2.9}$$

and

$$\frac{(q^7;q^{14})_\infty^4}{(q;q^2)_\infty^4} - \frac{(-q^7;q^{14})_\infty^4}{(-q;q^2)_\infty^4} = q^3\left\{49\frac{(q^{14};q^{14})_\infty^4}{(q^2;q^2)_\infty^4} - \frac{(-q^{14};q^{14})_\infty^4}{(-q^2;q^2)_\infty^4}\right\} + 8q. \tag{5.2.10}$$

Multiplying (5.2.9) by $q^3\dfrac{(q^{28};q^{28})_\infty^4}{(q^4;q^4)_\infty^4}$ and then subtracting from (5.2.10),

$$\begin{aligned}\frac{(q^7;q^{14})_\infty^4}{(q;q^2)_\infty^4} - \frac{(-q^7;q^{14})_\infty^4}{(-q;q^2)_\infty^4} - q^6\left\{\frac{(-q;q^2)_\infty^4(q^{28};q^{28})_\infty^4}{(-q^7;q^{14})_\infty^4(q^4;q^4)_\infty^4} - \frac{(q;q^2)_\infty^4(q^{28};q^{28})_\infty^4}{(q^7;q^{14})_\infty^4(q^4;q^4)_\infty^4}\right\}\\ = 48q^3\frac{(q^{14};q^{14})_\infty^4}{(q^2;q^2)_\infty^4} + 8q + 8q^5\frac{(q^{28};q^{28})_\infty^4}{(q^4;q^4)_\infty^4}.\end{aligned}$$

Thus,

$$\begin{aligned}&\left\{\sum_{n=0}^{\infty}A(n)q^n - \sum_{n=0}^{\infty}A(n)(-q)^n\right\} - q^6\left\{\sum_{n=0}^{\infty}B(n)q^n - \sum_{n=0}^{\infty}B(n)(-q)^n\right\}\\ &= 48q^3\sum_{n=0}^{\infty}C(n)q^{2n} + 8q + 8q^5\sum_{n=0}^{\infty}C(n)q^{4n}.\end{aligned}$$

Equating the coefficients, in turn, of q^{4n+1} and q^{4n+3} from both sides, we easily arrive at (5.2.7) and (5.2.8), respectively, to finish the proof. □

Example: $n = 2$ in (5.2.7).

Then $A(9) = 984$ as there are 4 relevant partitions of 9, 64 relevant partitions of the type $5 + 3 + 1$, 140 relevant partitions of the type $5 + 1 + 1 + 1 + 1$, 20 relevant partitions of the type $3 + 3 + 3$, 200 relevant partitions of the type $3 + 3 + 1 + 1 + 1$, 336 relevant partitions of the type $3+1+1+1+1+1+1$ and 220 relevant partitions of the type $1+1+1+1+1+1+1+1+1$; $B(3) = 4$ as there are 4 relevant partitions of 3; $C(3) = 40$ as there are 4 relevant partitions of 3 of the type 3, 16 relevant partitions of the type $2 + 1$ and 20 additional relevant partitions of 3 of the type $1 + 1 + 1$; and $C(1) = 4$ as there are 4 relevant partitions of 1.

Example: $n = 1$ in (5.2.8).

Then $A(7) = 340$ as there are 40 relevant partitions each of the types $5 + 1 + 1$ and $3 + 3 + 1$, 120 relevant partitions of the type $3 + 1 + 1 + 1 + 1$, 120 relevant partitions of the type $1+1+1+1+1+1+1$; $B(1)=4$ as there are 4 relevant partitions of 1; and $C(2)=14$ as there are 4 relevant partitions of 2 of the type 2 and 10 additional relevant partitions of 2 of the type $1+1$.

Theorem 5.2.4. *Let $A(n)$ denote the number of partitions of n into parts not multiples of 9; $B(n)$ denote the number of partitions of n into distinct parts not multiples of 9 or multiples of 4 that are not multiples of 9 and $b_9(n)$ denote the number of 9-regular partitions of n. Then, for any $n \geq 1$,*

$$A(2n+1) - B(2n-1) = b_9(n). \tag{5.2.11}$$

Proof. If β has degree 9 over α and m is the multiplier connecting α and β, then from [13, p. 352, Entry 3(x) and (xi)], we note that

$$\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} = \sqrt{m}$$

and

$$\left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} = \frac{3}{\sqrt{m}},$$

which are reciprocal to each other. With the help of Lemma 1.1.1, the above can be transcribed into

$$\frac{\varphi(-q^{18})}{\varphi(-q^2)} + q\left(\frac{\psi(q^9)}{\psi(q)} - \frac{\psi(-q^9)}{\psi(-q)}\right) = 1$$

and

$$\frac{\varphi(-q^2)}{\varphi(-q^{18})} + \frac{1}{q}\left(\frac{\psi(q)}{\psi(q^9)} - \frac{\psi(-q)}{\psi(-q^9)}\right) = 3,$$

which can further be transformed into the q-product identities

$$q\left\{\frac{(-q;q^2)_\infty}{(-q^9;q^{18})_\infty} - \frac{(q;q^2)_\infty}{(q^9;q^{18})_\infty}\right\} = \frac{(-q^2;q^2)_\infty}{(-q^{18};q^{18})_\infty} - \frac{(q^2;q^2)_\infty}{(q^{18};q^{18})_\infty} \tag{5.2.12}$$

and

$$\frac{(q^9;q^{18})_\infty}{(q;q^2)_\infty}-\frac{(-q^9;q^{18})_\infty}{(-q;q^2)_\infty}=q\left\{3\frac{(q^{18};q^{18})_\infty}{(q^2;q^2)_\infty}-\frac{(-q^{18};q^{18})_\infty}{(-q^2;q^2)_\infty}\right\}. \qquad (5.2.13)$$

Multiplying (5.2.12) by $q\dfrac{(q^{36};q^{36})_\infty}{(q^4;q^4)_\infty}$ and then subtracting from (5.2.13),

$$\frac{(q^9;q^{18})_\infty}{(q;q^2)_\infty}-\frac{(-q^9;q^{18})_\infty}{(-q;q^2)_\infty}-q^2\left\{\frac{(-q;q^2)_\infty(q^{36};q^{36})_\infty}{(-q^9;q^{18})_\infty(q^4;q^4)_\infty}-\frac{(q;q^2)_\infty(q^{36};q^{36})_\infty}{(q^9;q^{18})_\infty(q^4;q^4)_\infty}\right\}$$
$$=2q\frac{(q^{18};q^{18})_\infty}{(q^2;q^2)_\infty},$$

which can be written as

$$\left\{\sum_{n=0}^{\infty}A(n)q^n-\sum_{n=0}^{\infty}A(n)(-q)^n\right\}-q^2\left\{\sum_{n=0}^{\infty}B(n)q^n-\sum_{n=0}^{\infty}B(n)(-q)^n\right\}$$
$$=2q\sum_{n=0}^{\infty}b_9(n)q^{2n}.$$

Equating the coefficients of q^{2n+1} from both sides of the above, we deduce (5.2.11).

□

Example: $n=3$.

Then $A(7)=5$, the relevant partitions of 7 are 7, 5+1+1, 3+3+1, 3+1+1+1+1 and $1+1+1+1+1+1+1$, $B(5)=2$, the relevant partitions of 5 are 5 and $4+1$, and $b_9(3)=3$, the 9-regular partitions of 3 are 3, $2+1$ and $1+1+1$.

Now we present another partition-theoretic identity arising form (5.2.12).

Theorem 5.2.5. *Let $q_9(n)$ denote the number of partitions of n into distinct odd parts that are not multiples of 9 or into even parts that are not multiples of 9 and let $\overline{p}_9$ denote the number of overpartitions of n that are not multiples of 9. Then, for any $n\geq 1$,*

$$2q_9(2n-1)=\overline{p}_9(n) \qquad (5.2.14)$$

Proof. Changing the base of each of the q-products into q^{18}, we can rewrite (5.2.12) in the form

$$q\left\{(-q^{1,3,5,7,11,13,15,17};q^{18})_\infty-(q^{1,3,5,7,11,13,15,17};q^{18})_\infty\right\}=(-q^{2,4,\ \ ,16};q^{18})_\infty$$
$$-(q^{2,4,\ \ ,16},q^{18})_\infty.$$

Dividing both sides of the above by $(q^{2,4,\ \ ,16};q^{18})_\infty$,

$$q\left\{\frac{(-q^{1,3,5,7,11,13,15,17};q^{18})_\infty}{(q^{2,4,\ \ ,16};q^{18})_\infty}-\frac{(q^{1,3,5,7,11,13,15,17};q^{18})_\infty}{(q^{2,4,\ \ ,16};q^{18})_\infty}\right\}=\frac{(-q^{2,4,\ \ ,16};q^{18})_\infty}{(q^{2,4,\ \ ,16};q^{18})_\infty}-1,$$

which is equivalent to

$$q\left\{\sum_{n=0}^{\infty}q_9(n)q^n-\sum_{n=0}^{\infty}q_9(n)(-q)^n\right\}=\sum_{n=0}^{\infty}\overline{p}_9(n)q^{2n}-1$$

Equating the coefficients of q^{2n} from both sides, we readily arrive at (5.2.14) to complete the proof. □

Example. $n=5$.

Then $q_9(9)=12$ and $\overline{p}_9(5)=24$. The relevant partitions of 9 are $8+1$, $7+2$, $6+3$, $6+2+1$, $5+4$, $5+3+1$, $5+2+2$, $4+4+1$, $4+3+2$, $2+2+2+2+1$, $4+2+2+1$, and $3+2+2+2$, and the relevant overpartitions of 5 are 5, $\overline{5}$, $4+1$, $\overline{4}+1$, $\overline{4}+\overline{1}$, $4+\overline{1}$, $3+2$, $\overline{3}+2$, $3+\overline{2}$, $\overline{3}+\overline{2}$, $3+1+1$, $\overline{3}+1+1$, $3+\overline{1}+1$, $\overline{3}+\overline{1}+1$, $2+2+1$, $\overline{2}+2+1$, $2+2+\overline{1}$, $\overline{2}+2+\overline{1}$, $2+1+1+1$, $\overline{2}+1+1+1$, $\overline{2}+\overline{1}+1+1$, $2+\overline{1}+1+1$, $1+1+1+1+1$, and $\overline{1}+1+1+1+1$.

Theorem 5.2.6. *Let $A(n)$ denote the number of partitions of n into odd parts not multiples of 13 having 2 colors; $B(n)$ denote the number of partitions of n into distinct odd parts not multiples of 13 having 2 colors or into parts that are multiples of 4 but are not multiples of 13 having 2 colors, $C(n)$ denote the number of partitions of n into parts not multiples of 13 having 2 colors. Then, for any $n\geq 2$,*

$$A(4n+1)-B(4n-5)=6C(2n-1)+2C(n-1) \qquad (5.2.15)$$

and, for any $n \geq 1$,

$$A(4n+3) - B(4n-3) = 6C(2n). \tag{5.2.16}$$

Proof. If β has degree 13 over α and m is the multiplier connecting α and β, then we recall from [13, p. 376, Entry 8(iii), (iv)] that

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} - 4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}$$

and

$$\frac{13}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} - 4\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6},$$

which are reciprocal to each other. Transcribing the above modular equations, with the aid of Lemma 1.1.1, we have

$$q^3\frac{\psi(q^{26})}{\psi(q^2)}\left\{\frac{\varphi(q^{13})}{\varphi(q)} - \frac{\varphi(-q^{13})}{\varphi(-q)}\right\} = 1 + 4q^2\frac{f^4(q^{13})\varphi^2(q)}{f^4(q)\varphi^2(q^{13})} - \frac{\varphi(-q^{13})\varphi(q^{13})}{\varphi(-q)\varphi(q)}$$

and

$$\frac{\psi(q^2)}{\psi(q^{26})}\left\{\frac{\varphi(q)}{\varphi(q^{13})} - \frac{\varphi(-q)}{\varphi(-q^{13})}\right\} = 13q^3 + 4q\frac{f^4(q)\varphi^2(q^{13})}{f^4(q^{13})\varphi^2(q)} - q^3\frac{\varphi(-q)\varphi(q)}{\varphi(-q^{13})\varphi(q^{13})},$$

which can be reduced into

$$q^3\left\{\frac{(-q;q^2)^2_\infty}{(-q^{13};q^{26})^4_\infty} - \frac{(q;q^2)^2_\infty}{(q^{13};q^{26})^4_\infty}\right\} = \frac{(-q^2;q^2)^2_\infty}{(-q^{26};q^{26})^2_\infty} - \frac{(q^2;q^2)^2_\infty}{(q^{26};q^{26})^2_\infty} - 4q^2 \tag{5.2.17}$$

and

$$\frac{(q^{13};q^{26})^2_\infty}{(q;q^2)^2_\infty} - \frac{(-q^{13};q^{26})^2_\infty}{(-q;q^2)^2_\infty} = q^3\left\{13\frac{(q^{26};q^{26})^2_\infty}{(q^2;q^2)^2_\infty} - \frac{(-q^{26};q^{26})^2_\infty}{(-q^2;q^2)^2_\infty}\right\} + 4q. \tag{5.2.18}$$

Multiplying (5.2.17) by $q^3\dfrac{(q^{52};q^{52})^2_\infty}{(q^4;q^4)^2_\infty}$ and then subtracting from (5.2.18),

$$\frac{(q^{13};q^{26})^2_\infty}{(q;q^2)^2_\infty} - \frac{(-q^{13};q^{26})^2_\infty}{(-q;q^2)^2_\infty} - q^6\left\{\frac{(-q;q^2)^2_\infty(q^{52};q^{52})^2_\infty}{(-q^{13};q^{26})_\infty(q^4;q^4)^2_\infty} - \frac{(q;q^2)^2_\infty(q^{52};q^{52})^2_\infty}{(q^{13};q^{26})_\infty(q^4;q^4)^2_\infty}\right\}$$
$$= 12q^3\frac{(q^{26};q^{26})^2_\infty}{(q^2;q^2)^2_\infty} + 4q + 4q^5\frac{(q^{52};q^{52})^2_\infty}{(q^4;q^4)^2_\infty},$$

which is equivalent to

$$\left\{\sum_{n=0}^{\infty} A(n)q^n - \sum_{n=0}^{\infty} A(n)(-q)^n\right\} - q^6\left\{\sum_{n=0}^{\infty} B(n)q^n - \sum_{n=0}^{\infty} B(n)(-q)^n\right\}$$
$$= 12q^3\sum_{n=0}^{\infty} C(n)q^{2n} + 4q + 4q^5\sum_{n=0}^{\infty} C(n)q^{4n}.$$

Equating the coefficients, in turn, of q^{4n+1} and q^{4n+3} from both sides, we readily arrive at (5.2.15) and (5.2.16), respectively, to accomplish the proof. □

Example: $n = 2$ *in* (5.2.15).

Then $A(9) = 66$, the relevant partitions of 9 are 2 copies 9, 6 copies of the form $7+1+1$, 8 copies of the form $5+3+1$, 10 copies of the form $5+1+1+1+1$, 4 copies of the form $3+3+3$, 12 copies of the form $3+3+1+1+1$, 14 copies of the form $3+1+1+1+1+1+1$ and 10 copies of the form $1+1+1+1+1+1+1+1+1$, $B(3) = 2$, the relevant partitions of 3 are 2 copies of 3, $C(3) = 10$, the relevant partitions of 3 are 2 copies of the form 3, 4 copies each of the forms $2+1$ and $1+1+1$, and $C(1) = 2$, the relevant partitions of 1 are 2 copies of 1.

Example: $n = 1$ *in* (5.2.16).

Then $A(7) = 32$, the relevant partitions of 7 are 2 copies of 7, 6 copies of the form $5+1+1$, 6 copies of the form $3+3+1$, 10 copies of the form $3+1+1+1+1$, 8 copies of the form $1+1+1+1+1+1+1$, $B(1) = 2$, the relevant partitions of 1 are 2 copies of 1, and $C(2) = 5$, the relevant partitions of 2 are 2 copies of 2 and 3 copies of the form $1+1$.

Theorem 5.2.7. *Let $A(n)$ denote the number of partitions of n into odd parts not multiples of 25, $B(n)$ denote the number of partitions of n into distinct odd parts not multiples of 25 or into multiples of 4 but not multiples of 25, $b_{25}(n)$ denote the 25-regular partitions of n. Then, for any $n \geq 2$,*

$$A(4n+1) - B(4n-5) = 2b_{25}(2n-1) + b_{25}(n-1) \tag{5.2.19}$$

and, for any $n \geq 1$,

$$A(4n+3) - B(4n-3) = 2b_{25}(2n). \tag{5.2.20}$$

Proof. If γ is of degree 25 over α and m is the multiplier connecting α and γ, then from [13, p. 291, Entry 15(i), (ii)], we recall that

$$\sqrt{m} = \left(\frac{\gamma}{\alpha}\right)^{1/8} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8} - \left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/2}$$

and

$$\frac{5}{\sqrt{m}} = \left(\frac{\alpha}{\gamma}\right)^{1/8} + \left(\frac{1-\alpha}{1-\gamma}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/4} - 4\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/2}.$$

Proceeding as in the previous theorem, we find that

$$q^3\left\{\frac{(-q;q^2)_\infty}{(-q^{25};q^{50})_\infty} - \frac{(q;q^2)_\infty}{(q^{25};q^{50})_\infty}\right\} = \frac{(-q^2;q^2)_\infty}{(-q^{50};q^{50})_\infty} - \frac{(q^2;q^2)_\infty}{(q^{50};q^{50})_\infty} - 2q^2 \tag{5.2.21}$$

and

$$\frac{(q^{25};q^{50})_\infty}{(q;q^2)_\infty} - \frac{(-q^{25};q^{50})_\infty}{(-q;q^2)_\infty} = q^3\left\{5\frac{(q^{50};q^{50})_\infty}{(q^2;q^2)_\infty} - \frac{(-q^{50};q^{50})_\infty}{(-q^2;q^2)_\infty}\right\} + 2q. \tag{5.2.22}$$

Multiplying (5.2.21) by $q^3\dfrac{(q^{100};q^{100})_\infty}{(q^4;q^4)_\infty}$ and then subtracting from (5.2.22),

$$\frac{(q^{25};q^{50})_\infty}{(q;q^2)_\infty} - \frac{(-q^{25};q^{50})_\infty}{(-q;q^2)_\infty} - q^6\left\{\frac{(-q;q^2)_\infty(q^{100};q^{100})_\infty}{(-q^{25};q^{50})_\infty(q^4;q^4)_\infty} - \frac{(q;q^2)_\infty(q^{100};q^{100})_\infty}{(q^{25};q^{50})_\infty(q^4;q^4)_\infty}\right\}$$
$$= 4q^3\frac{(q^{50};q^{50})_\infty}{(q^2;q^2)_\infty} + 2q + 2q^5\frac{(q^{100};q^{100})_\infty}{(q^4;q^4)_\infty},$$

which is clearly equivalent to

$$\left\{\sum_{n=0}^{\infty} A(n)q^n - \sum_{n=0}^{\infty} A(n)(-q)^n\right\} - q^6\left\{\sum_{n=0}^{\infty} B(n)q^n - \sum_{n=0}^{\infty} B(n)(-q)^n\right\}$$
$$= 4q^3\sum_{n=0}^{\infty} b_{25}(n)q^{2n} + 2q + 2q^5\sum_{n=0}^{\infty} b_{25}(n)q^{4n}.$$

Equating the coefficients, in turn, of q^{4n+1} and q^{4n+3} from both sides of the above, we deduce (5.2.19) and (5.2.20), respectively. □

Example: $n = 2$ *in* (5.2.19).

Then $A(9) = 8$, the relevant partitions of 9 are 9, 7+1+1, 5+3+1, 5+1+1+1+1, 3+3+3, 3+3+1+1+1, 3+1+1+1+1+1+1 and 1+1 = 1+1+1+1+1+1+1, $B(3) = 1$, the relevant partition of 3 is 3, $b_{25}(3) = 3$, the relevant partitions are 3, $2+1$ and $1+1+1$, and $b_{25}(1) = 1$, the relevant partition is 1.

Example: $n = 2$ *in* (5.2.20).

Then $A(11) = 11$, the relevant partitions of 11 are 11, $9+1+1$, $7+3+1$, $7+1+1+1+1$, $5+5+1$, $5+3+3$, $5+3+1+1+1$, $5+1+1+1+1+1+1$, $3+3+3+1+1$, $3+3+1+1+1+1+1$, $3+1+1+1+1+1+1+1+1$, $1+1+1+1+1+1+1+1+1+1+1$, $B(5) = 1$, the relevant partitions of 5 are 5 and $4+1$, and $b_{25}(4) = 5$, the relevant partitions are 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$.

5.3 Partition identities arising from mixed modular equations

Theorem 5.3.1. *Let $A(n)$ denote the number of partitions of n into distinct odd parts that are not multiples of* 3 *and* 5 *or into even parts that are not multiples of* 3 *and* 5 *and $B(n)$ denote the number of overpartitions of n into parts not multiples of* 3 *and* 5. *Then, for any $n \geq 2$,*

$$2A(2n+1) = B(n). \tag{5.3.1}$$

Proof. From [13, p. 384, Entry 11(ix)], we have

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} = -\sqrt{\frac{m}{m'}},$$

where β, γ and δ are of degrees 3, 5 and 15, respectively, over α, and m and m' are the multipliers connecting α, β and γ, δ, respectively. Transcribing the above modular equation into q-products, we find that

$$(-q^{\pm1,\pm7,\pm11,\pm13};q^{30})_\infty-(q^{\pm1,\pm7,\pm11,\pm13};q^{30})_\infty=q\Big\{(-q^{\pm2,\pm4,\pm8,\pm14};q^{30})_\infty + (q^{\pm2,\pm4,\pm8,\pm14};q^{30})_\infty\Big\},$$

which can be rewritten as

$$\frac{(-q^{\pm1,\pm7,\pm11,\pm13};q^{30})_\infty}{(q^{\pm2,\pm4,\pm8,\pm14};q^{30})_\infty}-\frac{(q^{\pm1,\pm7,\pm11,\pm13};q^{30})_\infty}{(q^{\pm2,\pm4,\pm8,\pm14};q^{30})_\infty}=q\frac{(-q^{\pm2,\pm4,\pm8,\pm14};q^{30})_\infty}{(q^{\pm2,\pm4,\pm8,\pm14};q^{30})_\infty}+q.$$

Using the definitions of $A(n)$ and $B(n)$ in the above, we have

$$\sum_{n=0}^{\infty}A(n)q^n-\sum_{n=0}^{\infty}A(n)(-q)^n=q\sum_{n=0}^{\infty}B(n)q^{2n}+q.$$

Equating the coefficients of q^{2n+1} from both sides of the above, we arrive at (5.3.1) to finish the proof. □

Example: $n=4$.

Then $A(9)=5$, the relevant partitions of 9 are $8+1$, $7+2$, $4+4+1$, $4+2+2+1$ and $2+2+2+2+1$, $B(4)=10$, the relevant overpartitions of 4 are 4, $\overline{4}$, $2+2$, $\overline{2}+2$, $2+1+1$, $\overline{2}+1+1$, $2+\overline{1}+1$, $\overline{2}+\overline{1}+1$, $1+1+1+1$ and $\overline{1}+1+1+1$.

Theorem 5.3.2. *Let $A(n)$ denote the number of 2-colored partitions of n into distinct odd parts that are not multiples of 3 and additional two colors of the parts that are ±5 modulo 30 or into even parts that are not multiples of 6 having 2 colors with additional 2 colors of the parts that are ±10 modulo 60. Let $B(n)$ denote the number of partitions of n into parts that are not multiples of 3 having 2 colors each with additional 2 colors of the parts that are ±5 modulo 15 and $C(n)$ denote the number of overpartition pairs that are not multiples of 3 with additional 2 copies of the parts that are ±5 modulo 15. Then, for any $n\geq 2$,*

$$2A(2n-3)+4B(n-1)=C(n). \tag{5.3.2}$$

Proof. From [13, p. 384, Entry 11(x)], we have

$$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4}-\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}$$
$$-4\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/6}=mm',$$

where, as in the proof of the previous theorem, β, γ and δ are of degrees 3, 5 and 15, respectively, over α, and m and m' are the multipliers connecting α, β and γ, δ, respectively. We can transcribe the above modular equation into

$$q^3\left\{(q^{\pm1,\pm5,\pm5,\pm7,\pm11,\pm13};q^{30})_\infty^2-(-q^{\pm1,\pm5,\pm5,\pm7,\pm11,\pm13};q^{30})_\infty^2\right\}$$
$$=4q^2+\left\{(q^{\pm2,\pm4,\pm8,\pm10,\pm10,\pm14};q^{30})_\infty^2-(-q^{\pm2,\pm4,\pm8,\pm10,\pm10,\pm14};q^{30})_\infty^2\right\}.$$

Dividing both sides of the above by $(q^{\pm2,\pm4,\pm8,\pm10,\pm10,\pm14};q^{30})_\infty^2$,

$$q^3\left\{\frac{(q^{\pm1,\pm5,\pm5,\pm7,\pm11,\pm13};q^{30})_\infty^2}{(q^{\pm2,\pm4,\pm8,\pm10,\pm10,\pm14};q^{30})_\infty^2}-\frac{(-q^{\pm1,\pm5,\pm5,\pm7,\pm11,\pm13};q^{30})_\infty^2}{(q^{\pm2,\pm4,\pm8,\pm10,\pm10,\pm14};q^{30})_\infty^2}\right\}$$
$$=4q^2\frac{1}{(q^{\pm2,\pm4,\pm8,\pm10,\pm10,\pm14};q^{30})_\infty^2}-\frac{(-q^{\pm2,\pm4,\pm8,\pm10,\pm10,\pm14};q^{30})_\infty^2}{(q^{\pm2,\pm4,\pm8,\pm10,\pm10,\pm14};q^{30})_\infty^2}+1,$$

which can be put in the form

$$-q^3\left\{\sum_{n=0}^{\infty}A(n)q^n-\sum_{n=0}^{\infty}A(n)(-q)^n\right\}=4q^2\sum_{n=0}^{\infty}B(n)q^{2n}-\sum_{n=0}^{\infty}C(n)q^{2n}+1.$$

Equating the coefficients of q^{2n} from both sides, we deduce (5.3.3) to finish the proof. $\square$

Example: $n = 3$.

Then $A(3) = 4$, the relevant partitions of 3 are 4 copies of the form $2 + 1$, $B(2) = 5$, the relevant partitions of 2 are 2 copies of 2 and 3 copies of the form $1+1$, and $C(3) = 28$, the relevant overpartition pairs of 3 are $(2+1,\emptyset)$, $(\bar{2}+1,\emptyset)$, $(2+\bar{1},\emptyset)$, $(\bar{2}+\bar{1},\emptyset)$, $(\emptyset,2+1)$, $(\emptyset,\bar{2}+1)$, $(\emptyset,2+\bar{1})$, $(\emptyset,\bar{2}+\bar{1})$, $(2,1)$, $(\bar{2},1)$, $(2,\bar{1})$, $(\bar{2},\bar{1})$, $(1,2)$, $(1,\bar{2})$, $(\bar{1},2)$, $(\bar{1},\bar{2})$, $(1+1,1)$, $(\bar{1}+1,1)$, $(1+1,\bar{1})$, $(\bar{1}+1,\bar{1})$, $(1,1+1)$, $(1,\bar{1}+1)$, $(\bar{1},1+1)$, $(\bar{1},\bar{1}+1)$, $(1+1+1,\emptyset)$, $(\bar{1}+1+1,\emptyset)$, $(\emptyset,1+1+1)$, and $(\emptyset,\bar{1}+1+1)$.

Theorem 5.3.3. *Let $A(n)$ denote the number of 2-colored partitions of n into distinct odd parts without multiples of 3 and 7 or into even parts that are not multiples of 6 and 14 having 2 colors. Let $B(n)$ denote the number of partitions of n into parts that are not multiples of 3 and 7 having 2 colors and $C(n)$ denote the number of overpartition pairs of n that are not multiples of 3 and 7. Then, for any $n \geq 2$,*

$$2A(2n+1) = 4B(n) + C(n-1). \tag{5.3.3}$$

Proof. Let β, γ and δ are of degrees 3, 7 and 21, respectively, over α, and m and m' be the multipliers connecting α, β and γ, δ, respectively. From [13, p. 401, Entry 13(i)], we have

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/4} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/4} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4} + 4\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/6} = \frac{m}{m'}.$$

Transcribing into q-products,

$$\frac{(q;q^2)_\infty^2(q^{21};q^{42})_\infty^2}{(q^3;q^6)_\infty^2(q^7;q^{14})_\infty^2} - \frac{(-q;q^2)_\infty^2(-q^{21};q^{42})_\infty^2}{(-q^3;q^6)_\infty^2(-q^7;q^{14})_\infty^2} + 4q = q^3\left\{\frac{(q^2;q^2)_\infty^2(q^{42};q^{42})_\infty^2}{(q^6;q^6)_\infty^2(q^{14};q^{14})_\infty^2} - \frac{(-q^2;q^2)_\infty^2(-q^{42};q^{42})_\infty^2}{(-q^6;q^6)_\infty^2(-q^{14};q^{14})_\infty^2}\right\},$$

which can be rewritten as

$$\left\{(-q^{\pm1,\pm5,\pm11,\pm13,\pm17,\pm19};q^{42})_\infty^2 - (q^{\pm1,\pm5,\pm11,\pm13,\pm17,\pm19};q^{42})_\infty^2\right\} = 4q + q^3\left\{(-q^{\pm2,\pm4,\pm8,\pm10,\pm16,\pm20};q^{42})_\infty^2 - (q^{\pm2,\pm4,\pm8,\pm10,\pm16,\pm20};q^{42})_\infty^2\right\}.$$

Dividing both sides by $(q^{\pm2,\pm4,\pm8,\pm10,\pm16,\pm20};q^{42})_\infty^2$,

$$\left\{\frac{(-q^{\pm1,\pm5,\pm11,\pm13,\pm17,\pm19};q^{42})_\infty^2}{(q^{\pm2,\pm4,\pm8,\pm10,\pm16,\pm20};q^{42})_\infty^2} - \frac{(q^{\pm1,\pm5,\pm11,\pm13,\pm17,\pm19};q^{42})_\infty^2}{(q^{\pm2,\pm4,\pm8,\pm10,\pm16,\pm20};q^{42})_\infty^2}\right\} = \frac{4q}{(q^{\pm2,\pm4,\pm8,\pm10,\pm16,\pm20};q^{42})_\infty^2} + q^3\frac{(-q^{\pm2,\pm4,\pm8,\pm10,\pm16,\pm20};q^{42})_\infty^2}{(q^{\pm2,\pm4,\pm8,\pm10,\pm16,\pm20};q^{42})_\infty^2} - q^3.$$

Thus,

$$\left\{\sum_{n=0}^{\infty} A(n)q^n - \sum_{n=0}^{\infty} A(n)(-q)^n\right\} = 4q\sum_{n=0}^{\infty} B(n)q^{2n} + q^3\sum_{n=0}^{\infty} C(n)q^{2n} - q^3.$$

Equating the coefficients of q^{2n+1} from both sides of the above, we easily deduce (5.3.3). □

Example: $n = 2$.

Then $A(5) = 12$, the relevant partitions of 5 are 2 copies of 5, 4 copies of the type $4+1$ and 6 copies of the type $2+2+1$, $B(2) = 5$, the relevant partitions of 2 are 2 copies of the type 2, and 3 copies of the type $1+1$, and $C(1) = 4$, the relevant overpartition pairs of 1 are $(1, \emptyset)$, $(\bar{1}, \emptyset)$, $(\emptyset, 1)$ $(\emptyset, \bar{1})$.

Theorem 5.3.4. *Let $A(n)$ denote the number of partitions of n into distinct odd parts that are not multiples of 3 and multiples of 11 have two colors or into even parts that are not multiples of 6 and the multiples of 22 have two colors. Let $B(n)$ denote the number of partitions of n into parts that are not multiples of 3 and multiples of 11 have two colors and $C(n)$ denote the overpartitions of n that are not multiples of 3 and multiples of 11 have two colors. Then, for any $n \geq 2$,*

$$2A(2n-3) + 2B(n-1) = C(n). \tag{5.3.4}$$

Proof. From [13, p. 408, Entry 14(i)], we note that

$$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8} + \left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8} - \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8} - 2\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/12} = \sqrt{mm'}, \tag{5.3.5}$$

where β, γ and δ are of degrees 3, 11 and 33, respectively, over α, and m is the multiplier connecting α and β and m' is the multiplier connecting γ and δ. Transcribing the above modular equation,

$$q^3\left\{\frac{\psi(q^3)\psi(q^{33})}{\psi(q)\psi(q^{11})} - \frac{\psi(-q^3)\psi(-q^{33})}{\psi(-q)\psi(-q^{11})}\right\} + \frac{\varphi(-q^6)\varphi(-q^{66})}{\varphi(-q^2)\varphi(-q^{22})} - 2q^2\frac{f(-q^6)f(-q^{66})}{f(-q^2)f(-q^{22})} = 1,$$

which can further be transformed into the q-product identity

$$q^3\left\{\frac{(q;q^2)_\infty(q^{11};q^{22})_\infty}{(q^3;q^6)_\infty(q^{33};q^{66})_\infty}-\frac{(-q;q^2)_\infty(-q^{11};q^{22})_\infty}{(-q^3;q^6)_\infty(-q^{33};q^{66})_\infty}\right\}$$
$$=2q^2+\frac{(q^2;q^2)_\infty(q^{22};q^{22})_\infty}{(q^6;q^6)_\infty(q^{66};q^{66})_\infty}-\frac{(-q^2;q^2)_\infty(-q^{22};q^{22})_\infty}{(-q^6;q^6)_\infty(-q^{66};q^{66})_\infty}. \tag{5.3.6}$$

Thus,

$$q^3\{(-q^{\pm1,\pm5,\pm7,\pm11,\pm11,\pm13,\pm17,\pm19,\pm23,\pm25,\pm29,\pm31};q^{66})_\infty$$
$$-(q^{\pm1,\pm7,\pm11,\pm11,\pm13,\pm17,\pm19,\pm23,\pm25,\pm29,\pm31};q^{66})_\infty\}+2q^2$$
$$=(-q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm22,\pm26,\pm28,\pm32};q^{66})_\infty$$
$$-(q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm22,\pm26,\pm28,\pm32};q^{66})_\infty.$$

Dividing both sides by $(q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm22,\pm26,\pm28,\pm32};q^{66})_\infty$,

$$q^3\left\{\frac{(-q^{\pm1,\pm5,\pm7,\pm11,\pm11,\pm13,\pm17,\pm19,\pm23,\pm25,\pm29,\pm31};q^{66})_\infty}{(q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm22,\pm26,\pm28,\pm32};q^{66})_\infty}\right.$$
$$\left.-\frac{(q^{\pm1,\pm5,\pm7,\pm11,\pm11,\pm13,\pm17,\pm19,\pm23,\pm25,\pm29,\pm31};q^{66})_\infty}{(q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm22,\pm26,\pm28,\pm32};q^{66})_\infty}\right\}$$
$$+\frac{2q^2}{(q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm22,\pm26,\pm28,\pm32};q^{66})_\infty}$$
$$=\frac{(-q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm22,\pm26,\pm28,\pm32};q^{66})_\infty}{(q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm22,\pm26,\pm28,\pm32};q^{66})_\infty}-1.$$

The above can be put in the form

$$q^3\left\{\sum_{n=0}^{\infty}A(n)q^n-\sum_{n=0}^{\infty}A(n)(-q)^n\right\}+2q^2\sum_{n=0}^{\infty}B(n)q^{2n}=\sum_{n=0}^{\infty}C(n)q^{2n}-1,$$

from which we readily arrived at the proffered partition identity (5.3.4). □

Example: $n=4$.

Then $A(5)=3$, the relevant partitions of 5 are 5, $4+1$ and $2+2+1$, $B(3)=2$, the relevant partitions of 3 are $2+1$ and $1+1+1$, and $C(4)=10$, the relevant overpartitions of 4 are 4, $\overline{4}$, $2+2$, $\overline{2}+2$, $2+1+1$, $\overline{2}+1+1$, $2+\overline{1}+1$, $\overline{2}+\overline{1}+1$, $1+1+1+1$ and $\overline{1}+1+1+1$.

Theorem 5.3.5. *Let $A(n)$ denote the number of partitions of n into distinct odd parts that are not multiples of* 5 *and* 7 *or into even parts not multiples of* 5 *and* 7*; $B(n)$ denote the number of overpartitions of n into parts not multiples of* 5 *and* 7 *and $C(n)$ denote the number of partitions of n into even parts not multiples of* 5 *and* 7*. Then, for any $n \geq 1$,*

$$2A(2n+1) = B(n-1) + 2C(n). \tag{5.3.7}$$

Proof. Let β, γ and δ are of degrees 5, 7 and 35, respectively, over α, m be the multiplier connecting α and β, and m' be the multiplier connecting γ and δ. Then, from [13, p. 423, Entry 18(vii)], we note that

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} + 2\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12} = -\sqrt{\frac{m}{m'}}.$$

Transcribing the modular equation into q-products, we obtain

$$\begin{aligned}
&(-q^{\pm1,\pm3,\pm9,\pm11,\pm13,\pm17,\pm19,\pm23,\pm27,\pm29,\pm31,\pm33};q^{70})_\infty \\
&\quad - (q^{\pm1,\pm3,\pm9,\pm11,\pm13,\pm17,\pm19,\pm23,\pm27,\pm29,\pm31,\pm33};q^{70})_\infty \\
&= q^3\big\{(-q^{\pm2,\pm4,\pm6,\pm8,\pm12,\pm16,\pm18,\pm22,\pm24,\pm26,\pm32,\pm34};q^{70})_\infty \\
&\quad + (q^{\pm2,\pm4,\pm6,\pm8,\pm12,\pm16,\pm18,\pm22,\pm24,\pm26,\pm32,\pm34};q^{70})_\infty\big\} + 2q.
\end{aligned}$$

Dividing both sides by $(q^{\pm2,\pm4,\pm6,\pm8,\pm12,\pm16,\pm18,\pm22,\pm24,\pm26,\pm32,\pm34};q^{70})_\infty$,

$$\begin{aligned}
&\frac{(-q^{\pm1,\pm3,\pm9,\pm11,\pm13,\pm17,\pm19,\pm23,\pm27,\pm29,\pm31,\pm33};q^{70})_\infty}{(q^{\pm2,\pm4,\pm6,\pm8,\pm12,\pm16,\pm18,\pm22,\pm24,\pm26,\pm32,\pm34};q^{70})_\infty} \\
&\quad - \frac{(q^{\pm1,\pm3,\pm9,\pm11,\pm13,\pm17,\pm19,\pm23,\pm27,\pm29,\pm31,\pm33};q^{70})_\infty}{(q^{\pm2,\pm4,\pm6,\pm8,\pm12,\pm16,\pm18,\pm22,\pm24,\pm26,\pm32,\pm34};q^{70})_\infty} \\
&= q^3 + q^3\frac{(-q^{\pm2,\pm4,\pm6,\pm8,\pm12,\pm16,\pm18,\pm22,\pm24,\pm26,\pm32,\pm34};q^{70})_\infty}{(q^{\pm2,\pm4,\pm6,\pm8,\pm12,\pm16,\pm18,\pm22,\pm24,\pm26,\pm32,\pm34};q^{70})_\infty} \\
&\quad + \frac{2q}{(q^{\pm2,\pm4,\pm6,\pm8,\pm12,\pm16,\pm18,\pm22,\pm24,\pm26,\pm32,\pm34};q^{70})_\infty},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}\sum_{n=0}^{\infty} A(n)q^n - \sum_{n=0}^{\infty} A(n)(-q)^n \\ = q^3 + q^3\frac{(-q^{\pm 2,\pm 4,\pm 6,\pm 8,\pm 12,\pm 16,\pm 18,\pm 22,\pm 24,\pm 26,\pm 32,\pm 34};q^{70})_\infty}{(q^{\pm 2,\pm 4,\pm 6,\pm 8,\pm 12,\pm 16,\pm 18,\pm 22,\pm 24,\pm 26,\pm 32,\pm 34};q^{70})_\infty} \\ + \frac{2q}{(q^{\pm 2,\pm 4,\pm 6,\pm 8,\pm 12,\pm 16,\pm 18,\pm 22,\pm 24,\pm 26,\pm 32,\pm 34};q^{70})_\infty}.\end{aligned}$$

Extracting the terms involving q^{2n+1}, dividing by q, and then replacing q^2 by q, we find that

$$\begin{aligned}2\sum_{n=0}^{\infty} A(2n+1)q^n &= q + q\frac{(-q^{\pm 1,\pm 2,\pm 3,\pm 4,\pm 6,\pm 8,\pm 9,\pm 11,\pm 12,\pm 13,\pm 16,\pm 17};q^{35})_\infty}{(q^{\pm 1,\pm 2,\pm 3,\pm 4,\pm 6,\pm 8,\pm 9,\pm 11,\pm 12,\pm 13,\pm 16,\pm 17};q^{35})_\infty} \\ &\quad + \frac{2}{(q^{\pm 1,\pm 2,\pm 3,\pm 4,\pm 6,\pm 8,\pm 9,\pm 11,\pm 12,\pm 13,\pm 16,\pm 17};q^{35})_\infty} \\ &= q + q\sum_{n=0}^{\infty} B(n)q^n + 2\sum_{n=0}^{\infty} C(n)q^n.\end{aligned}$$

Equating the coefficients of q^n, we readily deduce (5.3.7). □

Example: $n = 4$.

Then $A(9) = 9$, the relevant partitions of 9 are 9, $8+1$, $6+3$, $6+2+1$, $4+4+1$, $4+3+2$, $4+2+2+1$, $3+2+2+2$ and $2+2+2+2+1$, $B(3) = 8$, the relevant overpartitions of 3 are 3, $\overline{3}$, $2+1$, $\overline{2}+1$, $2+\overline{12}+\overline{1}$, $1+1+1$ and $\overline{1}+1+1$, and $C(4) = 5$, the relevant partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$.

Corollary 5.3.6. *If $B(n)$ denote the number of overpartitions of n into parts not multiples of 5 and 7, then for any $n \geq 1$, $B(n) \equiv 0 \pmod 2$.*

Proof. It follows easily from (5.3.7). □

Theorem 5.3.7. *Let $A(n)$ denote the number of partitions of n into distinct odd parts not multiples of 3 and 13 or into even parts not multiples of 3 and 13; $B(n)$ denote the number of overpartitions of n into parts not multiples of 3 and 13 and*

$C(n)$ denote the number of partitions of n into parts not multiples of 3 and 13. Then, for any $n \geq 2$,

$$2A(2n+1) = B(n-1) + 2C(n). \tag{5.3.8}$$

Proof. If β, γ and δ are of degrees 3, 13 and 39, respectively, over α, and m and m' are the multipliers connecting α, β and γ, δ, respectively, then from [13, p. 426, Entry 19(iv, 2nd part)], we note that

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} + 2\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12} = \sqrt{\frac{m}{m'}},$$

As in the previous theorems, we can transform the above mixed modular equation into

$$\begin{aligned}&(-q^{\pm1,\pm5,\pm7,\pm11,\pm17,\pm19,\pm23,\pm25,\pm29,\pm31,\pm35,\pm37};q^{78})_\infty\\ &\quad-(q^{\pm1,\pm5,\pm7,\pm11,\pm17,\pm19,\pm23,\pm25,\pm29,\pm31,\pm35,\pm37};q^{78})_\infty\\ &=q^3\{(-q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm28,\pm32,\pm34,\pm38};q^{78})_\infty\\ &\quad-(q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm28,\pm32,\pm34,\pm38};q^{78})_\infty\}+2q.\end{aligned}$$

Dividing both sides by $(q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm28,\pm32,\pm34,\pm38};q^{78})_\infty$, the above idenity reduces to

$$\begin{aligned}&\frac{(-q^{\pm1,\pm5,\pm7,\pm11,\pm17,\pm19,\pm23,\pm25,\pm29,\pm31,\pm35,\pm37};q^{78})_\infty}{(q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm28,\pm32,\pm34,\pm38};q^{78})_\infty}\\ &\quad-\frac{(q^{\pm1,\pm5,\pm7,\pm11,\pm17,\pm19,\pm23,\pm25,\pm29,\pm31,\pm35,\pm37};q^{78})_\infty}{(q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm28,\pm32,\pm34,\pm38};q^{78})_\infty}\\ &=q^3\frac{(-q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm28,\pm32,\pm34,\pm38};q^{78})_\infty}{(q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm28,\pm32,\pm34,\pm38};q^{78})_\infty}\\ &\quad+\frac{2q}{(q^{\pm2,\pm4,\pm8,\pm10,\pm14,\pm16,\pm20,\pm22,\pm28,\pm32,\pm34,\pm38};q^{78})_\infty}-q^3.\end{aligned}$$

Equivalently,

$$\sum_{n=0}^{\infty} A(n)q^n - \sum_{n=0}^{\infty} A(n)(-q)^n = q^3\sum_{n=0}^{\infty} B(n)q^{2n} + 2q\sum_{n=0}^{\infty} C(n)q^{2n} - q^3.$$

Equating the coefficients of q^n, we easily arrive at the proffered partition identity. □

Example: $n = 2$.

Then $A(5) = 3$, the relevant partitions of 5 are 5, $4 + 1$, $2 + 2 + 1$, $B(1) = 2$, the relevant overpartitions of 1 are 1 and $\overline{1}$, and $C(2) = 2$, the relevant partitions of 2 are 2 and $1 + 1$.

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