CONTRIBUTIONS TO PARTITION IDENTITIES AND SUMS OF POLYGONAL NUMBERS BY USING RAMANUJAN'S THETA FUNCTIONS

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
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Dedicated to my family

(Maa, Deuta, Luku, Neor)

Abstract

In this thesis, we study certain partition identities involving self-conjugate 7- and 9-core partitions, general partition function, and generalized Frobenius partitions. We also study the representations of a number as sums of various polygonal numbers. Using the properties of Ramanujan's general theta functions and representations of product of theta functions as a linear combination of other theta functions, we obtain identities involving self-conjugate 7- and 9- core partitions. Further, the use of elementary properties of Ramanujan's theta functions yields interesting identities involving self-conjugate t-core partitions and double distinct t-core partitions. Using the integer matrix exact covering system, we obtain the q-product representations for the generating functions of generalized Frobenius partitions with 4 and 5 colors and 4-order generalized Frobenius partitions with 4 colors. In the process, we find various new congruences involving generalized Frobenius partitions with 4 colors and 4-order generalized Frobenius partitions with 4 colors. We also deduce identities and recursion relations for general partition functions $p_r(n)$ for different integral values of r wherein we use different properties of the Rogers-Ramanujan continued fraction and Ramanujan's cubic continued fraction. Our investigation on this topic leads to simple proofs of Ramanujan's partition congruences $p(5n + 4) \equiv 0 \pmod 5$ and $p(7n + 5) \equiv 0 \pmod 7$ and yields some identities for Ramanujan's tau function. Finally, we use the dissection of Ramanujan's general theta functions to obtain various identities involving representations of a number as sums of polygonal numbers.

DECLARATION BY THE CANDIDATE

I, Bipul Kumar Sarmah, hereby declare that the subject matter in this thesis entitled, **"Contributions to Partition Identities and Sums of Polygonal Numbers by Using Ramanujan's Theta Functions"**, is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/institute.

This thesis is being submitted to the Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

Place: Tezpur.

Date: 20/12/2011

Signature of the candidate

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CERTIFICATE OF THE SUPERVISOR

This is to certify that the thesis entitled "**Contributions to Partition Identities and Sums of Polygonal Numbers by Using Ramanujan's Theta Functions**", submitted to the School of Science and Technology of Tezpur University in partial fulfillment for the award of the degree of Doctor of Philosophy in Mathematical Sciences is a record of research work carried out by **Mr. Bipul Kumar Sarmah** under my supervision and guidance.

All help received by him from various sources have been duly acknowledged.

No part of this thesis have been submitted elsewhere for award of any other degree.

Place: Tezpur
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(Bipul Kumar Sarmah)

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Place: Tezpur

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Chapter 1

Introduction

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \tag{1.0.1}$$

which is equivalent to Jacobi's classical theta function

$$\vartheta_3(z,q) := \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}, \quad \text{with} \quad |q| < 1.$$

In fact

$$f(a,b) = \vartheta_3(z,q), \quad \text{where} \quad a = qe^{2iz}, \ b = qe^{-2iz}.$$

Various properties of (1.0.1) were explored by Ramanujan to generate numerous results in diverse directions including modular equations, continued fractions, class invariants, partitions etc. Later, several authors contributed and expanded the scope of the use of (1.0.1) in different fields. In this thesis, we shall focus on the use of Ramanujan's general theta function in obtaining partition identities and on representations of an integer as sums of polygonal numbers. Most of our methods are elementary and involve direct application of the properties of Ramanujan's general theta function.

The thesis consists of five chapters including the introductory chapter. In the following few paragraphs we briefly introduce the basic concepts and terminology used in the subsequent chapters.

A *partition* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of a natural number n is a finite sequence of non-increasing positive integer *parts* λ_i such that $n = \sum_{i=1}^{k} \lambda_i$. For example, the partitions of 5 are (5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$, $(1,1,1,1,1)$. If $p(n)$ denotes the number of partitions of n, then the generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where, for $|q| < 1$, $(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n)$.

Ramanujan [51]–[53] found nice congruence properties for $p(n)$ modulo 5, 7, and 11, namely, for any nonnegative integer n,

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.0.2}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{1.0.3}$$

$$p(11n+6) \equiv 0 \pmod{11}.$$

Moreover, Ramanujan offered a more general conjecture which states that if $\delta = 5^a 7^b 11^c$ and λ is an integer such that $24\lambda \equiv 1 \pmod \delta$ then

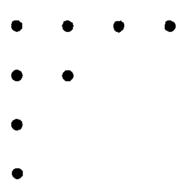
$$p(n\delta + \lambda) \equiv 0 \pmod \delta.$$

Ramanujan also gave a proof of this conjecture for arbitrary a and $b = c = 0$. However, for arbitrary b and $a = c = 0$ the conjecture needs to be corrected as

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta'},$$

where $\delta' = 5^a 7^{b'} 11^c$ with $b' = b$ if $b = 0,\ 1,\ 2$ and $b' = [(b+2)/2]$ if $b > 2$.

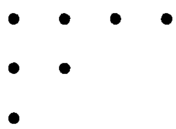
A partition is often represented with the help of a diagram called Ferrers–Young diagram. The Ferrers–Young diagram of the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of n is formed by arranging n nodes in k rows so that the ith row has λ_i nodes. For example, the Ferrers–Young diagram of the partition $\lambda = (4,2,1,1)$ of 8 is



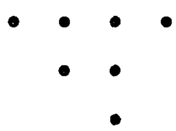
The conjugate of a partition λ, denoted λ', is the partition whose Ferrers–Young diagram is the reflection along the main diagonal of the diagram of λ. Therefore, the conjugate of the partition $(4, 2, 1, 1)$ is the partition $(4, 2, 1, 1)$ itself. A partition λ is *self-conjugate* if $\lambda = \lambda'$. The partition $(4, 2, 1, 1)$ is self-conjugate.

The nodes in the Ferrers–Young diagram of a partition are labeled by row and column coordinates as one would label the entries of a matrix. Let λ'_j denote the number of nodes in column j. The *hook number* $H(i, j)$ of the (i, j) node is defined as the number of nodes directly below and to the right of the node including the node itself. That is, $H(i, j) = \lambda_i + \lambda'_j - j - i + 1$. A *t-core* partition is a partition with no hook number divisible by t. For example the nodes $(1, 1)$, $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 1)$, $(2, 2)$, $(3, 1)$, and $(4, 1)$ in the Ferrers–Young diagram of the partition $\lambda = (4, 2, 1, 1)$ have hook numbers 7, 4, 2, 1, 4, 1, 2 and 1, respectively. Therefore, λ is a 3-core and a 5-core partition, but not a 7-core partition. Obviously, it is a t-core partition for $t \geq 8$.

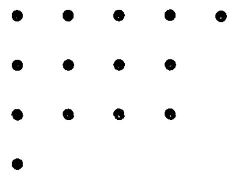
Now, given a partition $\lambda = (\lambda_1, \ \lambda_2, \ \cdots, \ \lambda_k)$ of n with distinct parts, the shifted Ferrers diagram of λ, $S(\lambda)$, is the Ferrers-Young diagram of λ with each row shifted to the right by one node than the previous row. The doubled distinct partition of λ is the partition λ^d of $2n$ obtained by adding λ_i nodes to the $(i-1)$st column of $S(\lambda)$. For example, we consider the partition $(4, 2, 1)$ of 7 whose Ferrers-Young diagram is as follows:



The shifted Ferrers diagram of the above partition is given by the following diagram:



Now adding 4, 2, and 1 nodes respectively to the null, first, and second columns of the above diagram we obtain the Ferrers-Young diagram



which represents the double-distinct partition $(5, 4, 4, 1)$ of 14 corresponding to the partition $(4, 2, 1)$ of 7.

Chapter 2 of this thesis is dedicated to identities involving self-conjugate t- core partitions and double distinct t-core partitions.

Let $asc_t(n)$ and $add_t(n)$ denote, respectively, the number of self-conjugate t-core partitions of n and the number of the double distinct t-core partitions of n. Garvan, Kim and Stanton [25] found the q-product representations of the generating functions for $asc_t(n)$ and $add_t(n)$. Among several results on $asc_t(n)$, Garvan, Kim and Stanton [25] gave bijective proofs of

$$asc_5(2n+1) = asc_5(n), \tag{1.0.4}$$

$$asc_7(n) = 0, \quad \text{if } n+2 = 4^a(8m+1). \tag{1.0.5}$$

Baruah and Berndt [4] proved (1.0.4) and several other results involving 3- and 5-core partitions by using Ramanujan's theta function identities and modular equations. Using product identities for two theta functions [17], we prove that $asc_7(8n+7) = 0$, which is a special case of (1.0.5).

Baldwin, Depweg, Ford, Kunin and Sze [3] proved that if t is an integer with $t = 8$ or $t \geq 10$, then every integer $n > 2$ has a self-conjugate t-core partition. They also gave an infinite sequence of integers that have no self-conjugate 9-core partition. In fact, they proved the following proposition.

If $n = (4^k - 10)/3$ *for some positive* k, *then* n *has no self-conjugate* 9*-core partition.*

In Section 2.4 of our thesis, we show that $asc_9(8n+10) = asc_9(2n)$, and since 2 has no self-conjugate 9-core partition, we see that there is an infinite sequence of positive integers that have no self-conjugate 9-core partitions. We also deduce the above proposition from our result.

In Section 2.5, we establish several results connecting self-conjugate t-core partitions and double distinct t-core partitions.

Chapter 3 is devoted to generating functions and congruences for generalized Frobenius partitions. G. E. Andrews [1] introduced the idea of generalized Frobenius partitions (or simply F-partitions) of n which is a notation of the form

$$\begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix}$$

of non-negative integers a_i's, b_i's with

$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i,$$

where each row is of the same length and each is arranged in non-increasing order. Let $c\phi_k(n)$ represent the number of F-partitions of n with k-colors and strict decrease in each row. Andrews [1] gave the generating function for $c\phi_k(n)$ and obtained the q-product representations of the generating functions for $c\phi_1(n)$, $c\phi_2(n)$, and $c\phi_3(n)$. He further remarked that "after the above results, the expressions quickly become long and messy". In Section 3.2, we present expressions for the generating functions of $c\phi_4(n)$ and $c\phi_5(n)$ similar to those of $c\phi_1(n)$, $c\phi_2(n)$, and $c\phi_3(n)$. Furthermore, Andrews proved the congruences

$$c\phi_2(5n+3) \equiv 0 \pmod 5,$$
$$c\phi_k(n) \equiv 0 \pmod{k^2} \quad \text{if } k \text{ is prime and does not divide } n.$$

Eichhorn and Sellers [18], Lovejoy [40], Ono [47], Paule and Radu [48], and Xiong [59] obtained many congruences and families of congruences involving $c\phi_2(n)$ and $c\phi_3(n)$. In Section 3.3 of this thesis, we present three congruences for $c\phi_4(n)$, all new, which are obtained by using Ramanujan's theta functions.

Again, Kolitsch [35, 37] considered the function $\overline{c\phi}_k(n)$, which denotes the number of F-partitions of n with k colors whose order is k under cyclic permutation of the k colors. The generating function for $\overline{c\phi}_k(n)$ is given by Kolitsch [37] and the q-product representations of the generating functions for $\overline{c\phi}_2(n)$ and $\overline{c\phi}_3(n)$ are known [56, 38]. In Section 3.4 of this thesis, we obtain the generating function for $\overline{c\phi}_4(n)$ in terms of q-products. Further, Kolitsch [35] found for all integers $k \geq 2$, that

$$\overline{c\phi}_k(n) \equiv 0 \pmod{k^2}.$$

More congruences, families of congruences, identities and recurrence relations involving $\overline{c\phi}_k(n)$ have been established by Kolitsch [36, 37, 38], Sellers [55, 56, 57], and Xiong [59]. In particular, Sellers [55, 56] established that

$$\overline{c\phi}_k(kn) \equiv 0 \pmod{k^3} \qquad \text{for} \quad k = 2, 3, 5, 7, \text{ and } 11.$$

It was further remarked in [56, p. 372] that "one question that naturally arises is whether congruences of this form occur for larger primes such as $k = 13$ or 17, or for composite values of k". We give a partial answer to this question by proving three congruences for $\overline{c\phi}_4(n)$ in Section 3.5.

In Chapter 4, we deal with congruences and recurrence relations involving the general partition function and Ramanujan's tau function. For a non-zero integer r, we define the general partition function $p_r(n)$ as the coefficient of q^n in the expansion of $(q;q)_\infty^r$. Therefore,

$$\sum_{n=0}^{\infty} p_r(n)q^n = (q;q)_\infty^r.$$

Note that $p_{-1}(n)$ is the usual partition function $p(n)$. Several authors contributed to congruences and identities satisfied by $p_r(n)$. Newman [44]–[46], Atkin [2], Gandhi [24], Gordon [26], Boylan [16], Kiming and Olsson [34] studied different congruence properties of $p_r(n)$ for certain values of r. Baruah and Ojah [5] used properties of the cubic continued fraction to obtain a few congruences for $p_{-3}(n)$. Recently, Berndt, Gugg and Kim [12] noticed that p. 182 in Ramanujan's lost notebook [54] corresponds to page 5 of an otherwise lost manuscript of Ramanujan in which Ramanujan stated some general congruences for $p_r(n)$. They also proved and discussed further results depending on Ramanujan's ideas. Farkas and Kra [19]–[22] used function theoretic considerations involving Riemann surfaces defined by the action on the upper half plane $\mathbb{H}^2$ of subgroups of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ to obtain several congruences and identities involving $p_r(n)$. In particular, Farkas and Kra [21, Theorem 1] obtained five three term recursion identities and asked whether the list of three term recursions for partition coefficients is complete or not. In Section 4.3 of this thesis, we prove all these five recursion identities and present four more three term recursions for partition coefficients giving an affirmative answer to the question of Farkas and Kra. In the process, we obtain new two term recursion relations for partition coefficients. We also give alternative proofs of several two term recursion relations that appeared in [19]. In the process, we also deduce (1.0.2) and (1.0.3) and several identities for Ramanujan's famous tau function $\tau(n)$ defined by

$$q(q;q)_\infty^{24} = \sum_{n=1}^{\infty} \tau(n) q^n. \tag{1.0.6}$$

In particular, we deduce that

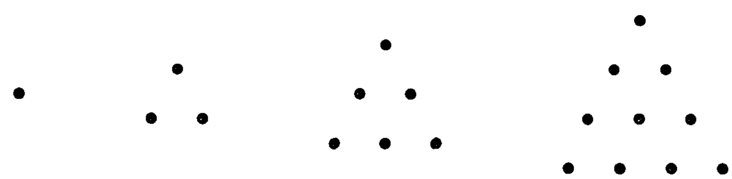
$$\tau(5^2 n) = 4830\tau(5n) - 5^{11}\tau(n),$$

which is a special case for $p = 5$ of Ramanujan's famous conjecture [50]

$$\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}),$$

first proved by Mordell in [43].

The last chapter of this thesis is dedicated to the representations of a number as sums of various polygonal numbers. A polygonal number is a number which can be represented as dots arranged in the shape of a regular polygon. For example 1, 3, 6, 10, $\cdots$ are triangular numbers, because they can be represented as dots arranged in the shape of triangles which is evident from the following diagram.



Jacobi's celebrated two-square theorem [32] states that

If $r\{\square + \square\}(n)$ denotes the number of representations of n as a sum of two squares, then

$$r\{\square + \square\}(n) = 4(d_{1,4}(n) - d_{3,4}(n)), \tag{1.0.7}$$

where $d_{i,j}(n)$ denotes the number of positive divisors of n congruent to i modulo j.

Simple proofs of the above theorem can be found in [15] and [28]. Similar representation theorems involving squares and triangular numbers were found by Dirichlet, Lorenz, Legendre, and Ramanujan [30, Theorems 2, 3, 5, 6]. In [30], Hirschhorn obtained sixteen identities (including those obtained by Legendre and Ramanujan) simply by dissecting the q-series representations of the identities obtained by Jacobi, Dirichlet and Lorenz. Hirschhorn [31] further extended this work and obtained twenty nine more identities involving squares, triangles, pentagons and octagons. Recently, Lam [39] and Melham [41, 42] contributed significantly on this topic. In [42], R. S. Melham presented an informal account of analogues of Jacobi's two-square theorem which are verified using computer algorithms.

In the concluding chapter of this thesis, we obtain twenty five more such identities involving squares, triangular numbers, pentagonal numbers, heptagonal numbers, octagonal numbers, decagonal numbers, hendecagonal numbers, dodecagonal numbers, and octadecagonal numbers. Although most of these identities can be proved without using q-series, our intention is to exhibit the utility of the properties of Ramanujan's theta function in this particular topic.

Chapter 2

Self-conjugate t-core Partitions

2.1 Introduction

In the introductory chapter, we have discussed in detail self conjugate t-core partitions and double distinct t-core partitions and indicated the contributions of Garvan, Kim and Stanton [25], Baruah and Berndt [4], and Baldwin, Depweg, Ford, Kunin, and Sze [3]. Granville and Ono [27] proved that for $t \geq 4$, every natural number n has a t-core partition, and thereby settling a conjecture of Brauer regarding the existence of defect zero characters for finite simple groups. In [25], it was also shown that $asc_t(n)$, the number of self-conjugate t-core partitions of n, is the number of solutions in integers x_i to

$$n = \sum_{i=1}^{t/2} tx_i^2 + (2i-1)x_i, \quad \text{for } t \text{ even},$$

and

$$n = \sum_{i=1}^{(t-1)/2} tx_i^2 + 2ix_i, \quad \text{for } t \text{ odd}.$$

The generating function for $asc_t(n)$ is given as [25, Eqs.(7.1a) and (7.1b)]

$$\sum_{n=0}^{\infty} asc_t(n)q^n = (-q;q^2)_\infty (q^{2t};q^{2t})_\infty^{t/2}, \quad \text{for } t \text{ even} \tag{2.1.1}$$

Note: The contents of this chapter appeared in *International Journal of Number Theory* [7] of World Scientific Publishing Company.

and

$$\sum_{n=0}^{\infty} asc_t(n)q^n = \frac{(-q;q^2)_\infty (q^{2t};q^{2t})_\infty^{(t-1)/2}}{(-q^t;q^{2t})_\infty}, \quad \text{for } t \text{ odd.} \tag{2.1.2}$$

In Section 2.3, we prove that $asc_7(8n+7) = 0$, which is a special case of (1.0.5). In Section 2.4, we show that $asc_9(8n+10) = asc_9(2n)$ and in the process deduce the following proposition proved in [3].

Proposition 2.1.1. *If $n = (4^k - 10)/3$ for some positive integer k, then n has no self-conjugate 9-core partitions.*

We also deduce two more propositions similar to the above proposition.

Now, the generating function for $add_t(n)$, the number of the double distinct partitions of n that are t-cores, is given by [25, Eq. (8.1a)]

$$\sum_{n=0}^{\infty} add_t(n)q^n = \frac{(-q^2;q^2)_\infty (q^{2t};q^{2t})_\infty^{(t-2)/2}}{(-q^t;q^t)_\infty}, \quad \text{for } t \text{ even} \tag{2.1.3}$$

and

$$\sum_{n=0}^{\infty} add_t(n)q^n = \frac{(-q^2;q^2)_\infty (q^{2t};q^{2t})_\infty^{(t-1)/2}}{(-q^{2t};q^{2t})_\infty}, \quad \text{for } t \text{ odd.} \tag{2.1.4}$$

Observe that $add_t(n) = 0$ if n is odd.

In Section 2.5, we prove certain results involving $asc_t(n)$ and $add_t(n)$.

We will use Ramanujan's theta functions in our proofs in Sections 2.3–2.5. In the next section, we state a few special cases of Ramanujan's theta function and give some preliminary results.

2.2 Ramanujan's theta functions and some preliminary results

We recall that Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

Jacobi's famous triple product identity takes the simple form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.2.1}$$

Three special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{2.2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{2.2.3}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{2.2.4}$$

where the product representations in (2.2.2)–(2.2.4) arise from (2.2.1), and the last equality in (2.2.4) is Euler's famous pentagonal number theorem.

After Ramanujan, we also define

$$\chi(q) = (-q;q^2)_\infty. \tag{2.2.5}$$

Now, the generating functions (2.1.1) and (2.1.2) for $asc_t(n)$ can be rewritten in the form

$$\sum_{n=0}^{\infty} asc_t(n)q^n = \chi(q)f^{t/2}(-q^{2t}), \quad \text{for } t \text{ even}$$

and

$$\sum_{n=0}^{\infty} asc_t(n)q^n = \frac{\chi(q)f^{(t-1)/2}(-q^{2t})}{\chi(q^t)}, \quad \text{for } t \text{ odd.} \tag{2.2.6}$$

In particular,

$$\sum_{n=0}^{\infty} asc_7(n)q^n = \frac{\chi(q)f^3(-q^{14})}{\chi(q^7)}. \tag{2.2.7}$$

and

$$\sum_{n=0}^{\infty} asc_9(n)q^n = \frac{\chi(q)f^4(-q^{18})}{\chi(q^9)}. \tag{2.2.8}$$

In the following two lemmas we state some properties satisfied by $f(a,b)$.

Lemma 2.2.1. [10, p. 46, Entry 30] *We have*

$$f(a,b)+f(-a,-b)=2f(a^3b,ab^3) \tag{2.2.9}$$

and

$$f(a,b)-f(-a,-b)=2af(b/a,a^5b^3). \tag{2.2.10}$$

Lemma 2.2.2. [10, p. 45, Entry 29] *If $ab=cd$, then*

$$f(a,b)f(c,d)+f(-a,-b)f(-c,-d)=2f(ac,bd)f(ad,bc) \tag{2.2.11}$$

and

$$f(a,b)f(c,d)-f(-a,-b)f(-c,-d)=2af(b/c,ac^2d)f(b/d,acd^2). \tag{2.2.12}$$

In [17], Z. Cao developed a general theorem for the product of n theta functions. In our proofs in Sections 2.3 and 2.4, we will also use some of his theorems.

2.3 A Theorem on self-conjugate 7-cores

Theorem 2.3.1. *If $asc_7(n)$ denotes the number of self-conjugate 7-core partitions of n, then*

$$asc_7(8n+7)=0. \tag{2.3.1}$$

Proof. Employing Jacobi's triple product identity (2.2.1) and using the definitions (2.2.4) and (2.2.5), we find that

$$\begin{aligned} f(q,q^{13})f(q^3,q^{11})f(q^5,q^9) &= (-q;q^{14})_\infty(-q^3;q^{14})_\infty(-q^5;q^{14})_\infty(-q^9;q^{14})_\infty \\ &\quad \times (-q^{11};q^{14})_\infty(-q^{13};q^{14})_\infty(q^{14};q^{14})^3_\infty \\ &= \frac{(-q;q^2)_\infty(q^{14};q^{14})^3_\infty}{(-q^7;q^{14})_\infty} \\ &= \frac{\chi(q)f^3(-q^{14})}{\chi(q^7)}. \end{aligned} \tag{2.3.2}$$

From (2.2.7) and (2.3.2), we have

$$\sum_{n=0}^{\infty} asc_7(n)q^n = f(q,q^{13})f(q^3,q^{11})f(q^5,q^9). \tag{2.3.3}$$

Now, from [17, Eq. (3.1)], we recall that, for $q = a_1b_1 = a_2b_2 = a_3b_3$ and $|q| < 1$,

$$\begin{aligned}
&f(a_1,b_1)f(a_2,b_2)f(a_3,b_3)\\
&= f(a_1a_2^2a_3q, b_1b_2^2b_3q)f(a_1b_2a_3, b_1a_2b_3)f(a_1b_3, b_1a_3)\\
&\quad + a_1f(a_1a_2^2a_3q^2, b_1b_2^2b_3)f\left(a_1b_2a_3q, \frac{b_1a_2b_3}{q}\right)f\left(a_1b_3q, \frac{b_1a_3}{q}\right)\\
&\quad + a_2f\left(a_1a_2^2a_3q^3, \frac{b_1b_2^2b_3}{q}\right)f\left(\frac{a_1b_2a_3}{q}, b_1a_2b_3q\right)f(a_1b_3, b_1a_3)\\
&\quad + a_1a_2f\left(a_1a_2^2a_3q^4, \frac{b_1b_2^2b_3}{q^2}\right)f(a_1b_2a_3, b_1a_2b_3)f\left(a_1b_3q, \frac{b_1a_3}{q}\right)\\
&\quad + a_1a_2a_3f\left(a_1a_2^2a_3q^5, \frac{b_1b_2^2b_3}{q^3}\right)f\left(a_1b_2a_3q, \frac{b_1a_2b_3}{q}\right)f(a_1b_3, b_1a_3)\\
&\quad + b_3f(a_1a_2^2a_3, b_1b_2^2b_3q^2)f\left(\frac{a_1b_2a_3}{q}, b_1a_2b_3q\right)f\left(a_1b_3q, \frac{b_1a_3}{q}\right).
\end{aligned} \tag{2.3.4}$$

Replacing q by q^{14} and setting $a_1 = q$, $b_1 = q^{13}$, $a_2 = q^3$, $b_2 = q^{11}$, $a_3 = q^5$ and $b_3 = q^9$ in (2.3.4), we find that

$$\begin{aligned}
&f(q,q^{13})f(q^3,q^{11})f(q^5,q^9)\\
&= f(q^{26},q^{58})f(q^{17},q^{25})f(q^{10},q^{18}) + qf(q^{40},q^{44})f(q^{31},q^{11})f(q^{24},q^4)\\
&\quad + q^3f(q^{54},q^{30})f(q^3,q^{39})f(q^{10},q^{18}) + q^4f(q^{68},q^{16})f(q^{17},q^{25})f(q^{24},q^4)\\
&\quad + q^9f(q^{82},q^2)f(q^{31},q^{11})f(q^{10},q^{18}) + q^9f(q^{12},q^{72})f(q^3,q^{39})f(q^{24},q^4).
\end{aligned} \tag{2.3.5}$$

Replacing q by $-q$ in (2.3.5) and subtracting thc resulting identity from (2.3.5), and then with the aid of (2.3.3), we deduce that

$$\sum_{n=0}^{\infty} asc_7(n)q^n - \sum_{n=0}^{\infty} asc_7(n)(-1)^n q^n$$
$$= f(q^{26},q^{58})f(q^{10},q^{18})[f(q^{17},q^{25}) - f(-q^{17},-q^{25})]$$
$$+ qf(q^{40},q^{44})f(q^{24},q^4)[f(q^{31},q^{11}) + f(-q^{31},-q^{11})]$$
$$+ q^3f(q^{54},q^{30})f(q^{10},q^{18})[f(q^3,q^{39}) + f(-q^3,-q^{39})]$$
$$+ q^4f(q^{68},q^{16})f(q^{24},q^4)[f(q^{17},q^{25}) - f(-q^{17},-q^{25})]$$
$$+ q^9f(q^{82},q^2)f(q^{10},q^{18})[f(q^{31},q^{11}) + f(-q^{31},-q^{11})]$$
$$+ q^9f(q^{12},q^{72})f(q^{24},q^4)[f(q^3,q^{39}) + f(-q^3,-q^{39})]. \tag{2.3.6}$$

Now, from (2.2.9) and (2.2.10), we obtain

$$f(q^{17},q^{25}) - f(-q^{17},-q^{25}) = 2q^{17}f(q^8,q^{160}), \tag{2.3.7}$$

$$f(q^{31},q^{11}) + f(-q^{31},-q^{11}) = 2f(q^{64},q^{104}), \tag{2.3.8}$$

$$f(q^3,q^{39}) + f(-q^3,-q^{39}) = 2f(q^{48},q^{120}). \tag{2.3.9}$$

Employing (2.3.7)–(2.3.9) in (2.3.6), we deduce that

$$\sum_{n=0}^{\infty} asc_7(n)q^n - \sum_{n=0}^{\infty} asc_7(n)(-1)^n q^n = \sum_{n=0}^{\infty} asc_7(2n+1)q^{2n+1}$$
$$= 2q^{17}f(q^{26},q^{58})f(q^8,q^{160})f(q^{10},q^{18}) + 2qf(q^{40},q^{44})f(q^{64},q^{104})f(q^{24},q^4)$$
$$+ 2q^3f(q^{54},q^{30})f(q^{48},q^{120})f(q^{10},q^{18}) + 2q^{21}f(q^{68},q^{16})f(q^8,q^{160})f(q^{24},q^4)$$
$$+ 2q^9f(q^{82},q^2)f(q^{64},q^{104})f(q^{10},q^{18}) + 2q^9f(q^{12},q^{72})f(q^{48},q^{120})f(q^{24},q^4). \tag{2.3.10}$$

Dividing both sides of the second equality in (2.3.10) by q and then replacing q^2 by

q, we arrive at

$$\sum_{n=0}^{\infty} asc_7(2n+1)q^n$$
$$= q^8 f(q^{13}, q^{29}) f(q^4, q^{80}) f(q^5, q^9) + f(q^{20}, q^{22}) f(q^{32}, q^{52}) f(q^{12}, q^2)$$
$$+ q f(q^{27}, q^{15}) f(q^{24}, q^{60}) f(q^5, q^9) + q^{10} f(q^{34}, q^8) f(q^4, q^{80}) f(q^{12}, q^2)$$
$$+ q^4 f(q^{41}, q) f(q^{32}, q^{52}) f(q^5, q^9) + q^4 f(q^6, q^{36}) f(q^{24}, q^{60}) f(q^{12}, q^2). \quad (2.3.11)$$

Replacing q by $-q$ in (2.3.11) and then subtracting the resulting identity from (2.3.11), we obtain

$$\sum_{n=0}^{\infty} asc_7(2n+1)q^n - \sum_{n=0}^{\infty} asc_7(2n+1)(-1)^n q^n$$
$$= q^8 f(q^4, q^{80})[f(q^5, q^9) f(q^{13}, q^{29}) - f(-q^5, -q^9) f(-q^{13}, -q^{29})]$$
$$+ q f(q^{24}, q^{60})[f(q^5, q^9) f(q^{27}, q^{15}) + f(-q^5, -q^9) f(-q^{27}, -q^{15})]$$
$$+ q^4 f(q^{32}, q^{52})[f(q^5, q^9) f(q^{41}, q) - f(-q^5, -q^9) f(-q^{41}, -q)]. \quad (2.3.12)$$

Now we simplify the square bracketed expressions on the right hand side of (2.3.12) by employing the following product identity for two theta functions given in [17, Eq. (2.6)].

If $|ab| < 1$ and $cd = (ab)^{k_1 k_2}$, where k_1 and k_2 are both positive integers, then

$$f(a,b)f(c,d) = \sum_{r=0}^{k_1+k_2-1} a^{\frac{r^2+r}{2}} b^{\frac{r^2-r}{2}} f(a^{\frac{k_1^2+k_1}{2}+k_1 r} b^{\frac{k_1^2-k_1}{2}+k_1 r} d, a^{\frac{k_1^2-k_1}{2}-k_1 r} b^{\frac{k_1^2+k_1}{2}-k_1 r} c)$$
$$\times f(a^{\frac{k_2^2+k_2}{2}+k_2 r} b^{\frac{k_2^2-k_2}{2}+k_2 r} c, a^{\frac{k_2^2-k_2}{2}-k_2 r} b^{\frac{k_2^2+k_2}{2}-k_2 r} d). \quad (2.3.13)$$

Choosing $k_1 = 3$ and $k_2 = 1$ in (2.3.13), we note that, for $(ab)^3 = cd$,

$$f(a,b)f(c,d) = f(a^6 b^3 d, a^3 b^6 c) f(ac, bd) + a f(a^9 b^6 d, b^3 c) f(a^2 bc, a^{-1} d)$$
$$+ a^3 b f(a^{12} b^9 d, a^{-3} c) f(a^3 b^2 c, a^{-2} b^{-1} d)$$
$$+ a^6 b^3 f(a^{15} b^{12} d, a^{-6} b^{-3} c) f(a^4 b^3 c, a^{-3} b^{-2} d). \quad (2.3.14)$$

Setting $a = q^5$, $b = q^9$, $c = q^{13}$, and $d = q^{29}$ in (2.3.14), we find that

$$\begin{aligned}
&f(q^5, q^9)f(q^{13}, q^{29}) \\
&= f(q^{86}, q^{82})f(q^{18}, q^{38}) + q^5 f(q^{128}, q^{40})f(q^{32}, q^{24}) \\
&\quad + q^{24} f(q^{170}, q^{-2})f(q^{46}, q^{10}) + q^{57} f(q^{212}, q^{-44})f(q^{60}, q^{-4}). \\
&= f(q^{86}, q^{82})f(q^{18}, q^{38}) + q^5 f(q^{128}, q^{40})f(q^{32}, q^{24}) \\
&\quad + q^{22} f(q^{166}, q^{2})f(q^{46}, q^{10}) + q^{9} f(q^{124}, q^{44})f(q^{52}, q^{4}),
\end{aligned} \tag{2.3.15}$$

where in the last equality we also used the trivial identity

$$f(a, b) = af\left(a^2 b, a^{-1}\right). \tag{2.3.16}$$

Now, replacing q by $-q$ in (2.3.15) and then subtracting the resulting identity from (2.3.15), we obtain

$$\begin{aligned}
&f(q^5, q^9)f(q^{13}, q^{29}) - f(-q^5, -q^9)f(-q^{13}, -q^{29}) \\
&= 2q^5 f(q^{128}, q^{40})f(q^{32}, q^{24}) + 2q^9 f(q^{124}, q^{44})f(q^{52}, q^4).
\end{aligned} \tag{2.3.17}$$

Similarly, setting $(a, b, c, d) = (q^5, q^9, q^{27}, q^{15})$ and $(a, b, c, d) = (q^5, q^9, q, q^{41})$ in (2.3.14), we can deduce that

$$\begin{aligned}
&f(q^5, q^9)f(q^{27}, q^{15}) + f(-q^5, -q^9)f(-q^{27}, -q^{15}) \\
&= 2f(q^{84}, q^{84})f(q^{20}, q^{36}) + 2q^{24} f(1, q^{168})f(q^{48}, q^8)
\end{aligned} \tag{2.3.18}$$

and

$$\begin{aligned}
&f(q^5, q^9)f(q, q^{41}) - f(-q^5, -q^9)f(-q, -q^{41}) \\
&= 2q^5 f(q^{140}, q^{28})f(q^{20}, q^{26}) + 2qf(q^{112}, q^{56})f(q^{48}, q^8),
\end{aligned} \tag{2.3.19}$$

respectively.

Employing (2.3.17)–(2.3.19) in (2.3.12), we find that

$$\sum_{n=0}^{\infty} asc_7(2n+1)q^n - \sum_{n=0}^{\infty} asc_7(2n+1)(-1)^n q^n$$
$$= 2q^{13}f(q^4,q^{80})f(q^{128},q^{40})f(q^{32},q^{24}) + 2q^{17}f(q^4,q^{80})f(q^{124},q^{44})f(q^{52},q^4)$$
$$+ 2qf(q^{24},q^{60})f(q^{84},q^{84})f(q^{20},q^{36}) + 2q^{25}f(q^{24},q^{60})f(1,q^{168})f(q^{48},q^8)$$
$$+ 2q^9f(q^{32},q^{52})f(q^{140},q^{28})f(q^{20},q^{36}) + 2q^5f(q^{32},q^{52})f(q^{112},q^{56})f(q^{48},q^8).$$

Comparing the coefficients of q^{4n+3} on both sides of the identity above, we readily arrive at (2.3.1) to complete the proof. □

2.4 A Theorem on self-conjugate 9-cores

Theorem 2.4.1. *If $asc_9(n)$ denotes the number of self-conjugate 9-core partitions of n, then*

$$asc_9(8n+10) = asc_9(2n). \tag{2.4.1}$$

Proof. With the help of Jacobi's triple product identity (2.2.1) and the product representations (2.2.4) and (2.2.5), we have

$$f(q,q^{17})f(q^3,q^{15})f(q^5,q^{13})f(q^7,q^{11})$$
$$= (-q;q^{18})_\infty(-q^3;q^{18})_\infty(-q^5;q^{18})_\infty(-q^7;q^{18})_\infty(-q^{11};q^{18})_\infty$$
$$\times (-q^{13};q^{18})_\infty(-q^{15};q^{18})_\infty(-q^{17};q^{18})_\infty(q^{18};q^{18})_\infty^4$$
$$= \frac{(-q;q^2)_\infty(q^{18};q^{18})_\infty^4}{(-q^9;q^{18})_\infty}$$
$$= \frac{\chi(q)f^4(-q^{18})}{\chi(q^9)}. \tag{2.4.2}$$

From (2.4.2) and (2.2.8), we find that

$$\sum_{n=0}^{\infty} asc_9(n)q^n = f(q,q^{17})f(q^3,q^{15})f(q^5,q^{13})f(q^7,q^{11}). \tag{2.4.3}$$

Now, from [17, Corollary 3.14], we recall that, for $q = a_1b_1 = a_2b_2 = a_3b_3 = a_4b_4$ and $|q| < 1$,

$$\begin{aligned}
&\prod_{i=1}^{4} f(a_i, b_i) + \prod_{i=1}^{4} f(-a_i, -b_i) \\
&= 2f(a_1a_2, b_1b_2)f(a_1b_2a_3a_4, b_1a_2b_3b_4)f(a_3b_4, a_4b_3)f(a_1b_2b_3b_4, b_1a_2a_3a_4) \\
&\quad + 2a_1a_4 f\left(a_1a_2q, \frac{b_1b_2}{q}\right) f\left(a_1b_2a_3a_4q^2, \frac{b_1a_2b_3b_4}{q^2}\right) \\
&\quad \times f\left(\frac{a_3b_4}{q}, a_4b_3q\right) f(a_1b_2b_3b_4, b_1a_2a_3a_4) \\
&\quad + 2a_2a_4 f\left(a_1a_2q, \frac{b_1b_2}{q}\right) f(a_1b_2a_3a_4, b_1a_2b_3b_4) \\
&\quad \times f\left(\frac{a_3b_4}{q}, a_4b_3q\right) f\left(\frac{a_1b_2b_3b_4}{q^2}, b_1a_2a_3a_4q^2\right) \\
&\quad + 2a_3a_4 f(a_1a_2, b_1b_2) f\left(a_1b_2a_3a_4q^2, \frac{b_1a_2b_3b_4}{q^2}\right) \\
&\quad \times f(a_3b_4, a_4b_3) f\left(\frac{a_1b_2b_3b_4}{q^2}, b_1a_2a_3a_4q^2\right).
\end{aligned} \tag{2.4.4}$$

Replacing q by q^{18} and then setting $a_1 = q$, $b_1 = q^{17}$, $a_2 = q^3$, $b_2 = q^{15}$, $a_3 = q^5$, $b_3 = q^{13}$ $a_4 = q^7$, and $b_4 = q^{11}$ in (2.4.4), we have

$$\begin{aligned}
&f(q, q^{17})f(q^3, q^{15})f(q^5, q^{13})f(q^7, q^{11}) \\
&\quad + f(-q, -q^{17})f(-q^3, -q^{15})f(-q^5, -q^{13})f(-q^7, -q^{11}) \\
&= 2f(q^4, q^{32})f(q^{28}, q^{44})f(q^{16}, q^{20})f(q^{40}, q^{32}) \\
&\quad + 2q^8 f(q^{22}, q^{14})f(q^{64}, q^8)f(q^{-2}, q^{38})f(q^{40}, q^{32}) \\
&\quad + 2q^{10} f(q^{22}, q^{14})f(q^{28}, q^{44})f(q^{-2}, q^{38})f(q^4, q^{68}) \\
&\quad + 2q^{12} f(q^4, q^{32})f(q^{64}, q^8)f(q^{16}, q^{20})f(q^4, q^{68}) \\
&= 2f(q^4, q^{32})f(q^{28}, q^{44})f(q^{16}, q^{20})f(q^{40}, q^{32}) \\
&\quad + 2q^6 f(q^{22}, q^{14})f(q^{64}, q^8)f(q^2, q^{34})f(q^{40}, q^{32}) \\
&\quad + 2q^8 f(q^{22}, q^{14})f(q^{28}, q^{44})f(q^2, q^{34})f(q^4, q^{68}) \\
&\quad + 2q^{12} f(q^4, q^{32})f(q^{64}, q^8)f(q^{16}, q^{20})f(q^4, q^{68}),
\end{aligned} \tag{2.4.5}$$

where in the last equality we also used (2.3.16).

We rewrite (2.4.5), with the aid of (2.4.3), in the form

$$\sum_{n=0}^{\infty} asc_9(n)q^n + \sum_{n=0}^{\infty} asc_9(n)(-1)^n q^n$$
$$= 2f(q^4,q^{32})f(q^{28},q^{44})f(q^{16},q^{20})f(q^{40},q^{32})$$
$$+ 2q^6 f(q^{22},q^{14})f(q^{64},q^{8})f(q^{2},q^{34})f(q^{40},q^{32})$$
$$+ 2q^8 f(q^{22},q^{14})f(q^{28},q^{44})f(q^{2},q^{34})f(q^{4},q^{68})$$
$$+ 2q^{12} f(q^{4},q^{32})f(q^{64},q^{8})f(q^{16},q^{20})f(q^{4},q^{68}). \qquad (2.4.6)$$

From (2.4.6), we deduce that

$$\sum_{n=0}^{\infty} asc_9(2n)q^n$$
$$= f(q^2,q^{16})f(q^{14},q^{22})f(q^{8},q^{10})f(q^{20},q^{16})$$
$$+ q^3 f(q^{11},q^{7})f(q^{32},q^{4})f(q,q^{17})f(q^{20},q^{16})$$
$$+ q^4 f(q^{11},q^{7})f(q^{14},q^{22})f(q,q^{17})f(q^{2},q^{34})$$
$$+ q^6 f(q^2q^{16})f(q^{32},q^{4})f(q^{8},q^{10})f(q^{2},q^{34}). \qquad (2.4.7)$$

Replacing q by $-q$ in (2.4.7), then subtracting the resulting identity from (2.4.7), we find that

$$\sum_{n=0}^{\infty} asc_9(2n)q^n - \sum_{n=0}^{\infty} asc_9(2n)(-1)^n q^n$$
$$= q^3 f(q^{32},q^{4})f(q^{20},q^{16})[f(q,q^{17})f(q^{11},q^{7}) + f(-q,-q^{17})f(-q^{11},-q^{7})]$$
$$+ q^4 f(q^{14},q^{22})f(q^{2},q^{34})[f(q,q^{17})f(q^{11},q^{7}) - f(-q,-q^{17})f(-q^{11},-q^{7})]. \quad (2.4.8)$$

Now, by (2.2.11) and (2.2.12), we have

$$f(q,q^{17})f(q^{11},q^{7}) + f(-q,-q^{17})f(-q^{11},-q^{7}) = 2f(q^{8},q^{28})f(q^{12},q^{24}), \qquad (2.4.9)$$
$$f(q,q^{17})f(q^{11},q^{7}) - f(-q,-q^{17})f(-q^{11},-q^{7}) = 2qf(q^{10},q^{26})f(q^{6},q^{30}). \qquad (2.4.10)$$

Employing (2.4.9) and (2.4.10) in (2.4.8), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} asc_9(2n)q^n - \sum_{n=0}^{\infty} asc_9(2n)(-1)^n q^n\\&= 2q^3 f(q^{32},q^4)f(q^{20},q^{16})f(q^8,q^{28})f(q^{12},q^{24})\\&\quad + 2q^5 f(q^{14},q^{22})f(q^2,q^{34})f(q^{10},q^{26})f(q^6,q^{30}).\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{n=0}^{\infty} asc_9(4n+2)q^n\\&= qf(q^{16},q^2)f(q^{10},q^8)f(q^4,q^{14})f(q^6,q^{12}) + q^2 f(q^7,q^{11})f(q,q^{17})f(q^5,q^{13})f(q^3,q^{15}),\end{aligned}$$

which can also be written, with the help of (2.4.3), as

$$\sum_{n=0}^{\infty} asc_9(4n+2)q^n = qf(q^{16},q^2)f(q^{10},q^8)f(q^4,q^{14})f(q^6,q^{12}) + q^2\sum_{n=0}^{\infty} asc_9(n)q^n. \tag{2.4.11}$$

Equating the coefficients of q^{2n+2} on both sides of (2.4.11), we readily arrive at (2.4.1) to finish the proof. □

Now we deduce Proposition 2.1.1 from Theorem 2.4.1.

Corollary 2.4.2. *Proposition* 2.1.1 *holds.*

Proof. For a fixed positive integer m, we define a sequence of integers $\langle t_k \rangle$ by the recurrence formula

$$t_1 = 2m, \qquad t_k = 4t_{k-1} + 10, \qquad \text{for} \quad k \geq 2.$$

By mathematical induction, we find that

$$t_k = \frac{4^{k-1}(6m+10) - 10}{3}, \quad k \geq 1.$$

Therefore, from (2.4.1), we obtain

$$asc_9\left(\frac{4^{k-1}(6m+10)-10}{3}\right)=asc_9(2m), \quad k\geq 1. \tag{2.4.12}$$

Setting $m=1$ in (2.4.12), and noting that 2 has no self-conjugate 9-core partition, we arrive at

$$asc_9\left(\frac{4^{k+1}-10}{3}\right)=asc_9(2)=0, \quad k\geq 1,$$

to complete the proof. □

Remark 2.4.1. *The identity* (2.4.12) *also gives results analogous to Proposition* 2.1.1. *For example, setting* $m=2$ *and* $m=4$ *in* (2.4.12), *and noting that* 4 *and* 8 *have exactly one and two self-conjugate* 9*-core partitions, respectively, we readily deduce the following two propositions.*

Proposition 2.4.3. *If* $n=(22\cdot 4^{k-1}-10)/3$ *for some positive integer* k, *then* n *has exactly one self-conjugate* 9*-core partition.*

Proposition 2.4.4. *If* $n=(34\cdot 4^{k-1}-10)/3$ *for some positive integer* k, *then* n *has exactly two self-conjugate* 9*-core partitions.*

2.5 Theorems on double distinct partitions

In this section, we prove a few theorems involving $add_t(n)$ defined in (2.1.3) and (2.1.4).

Theorem 2.5.1. *If* $asc_3(n)$ *denotes the number of* 3*-cores of* n *that are self-conjugates and* $add_3(n)$ *denotes the number of double distinct* 3*-core partitions, then*

$$asc_3(4n)=add_3(n). \tag{2.5.1}$$

Proof. Setting $t = 3$ in (2.2.6), we find that

$$\sum_{n=0}^{\infty} asc_3(n)q^n = \frac{\chi(q)f(-q^6)}{\chi(q^3)}. \tag{2.5.2}$$

Furthermore, applying the Jacobi's triple product identity (2.2.1) and recalling (2.2.4) and (2.2.5), we find that

$$f(q,q^5) = (-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty = \frac{(-q;q^2)_\infty(q^6;q^6)_\infty}{(-q^3;q^6)_\infty} = \frac{\chi(q)f(-q^6)}{\chi(q^3)}. \tag{2.5.3}$$

From (2.5.2) and (2.5.3), we have

$$\sum_{n=0}^{\infty} asc_3(n)q^n = f(q,q^5). \tag{2.5.4}$$

Now, setting $a = q$ and $b = q^5$ in (2.2.9), we find that

$$f(q,q^5) + f(-q,-q^5) = 2f(q^8,q^{16}), \tag{2.5.5}$$

Again, from (2.1.4) and (2.2.1), we obtain

$$\sum_{n=0}^{\infty} add_3(n)q^n = f(q^2,q^4). \tag{2.5.6}$$

Employing (2.5.4) and (2.5.6) in (2.5.5), we find that

$$\sum_{n=0}^{\infty} asc_3(n)q^n + \sum_{n=0}^{\infty}(-1)^n asc_3(n)q^n = 2\sum_{k=-\infty}^{\infty} add_3(n)q^{4n}. \tag{2.5.7}$$

Comparing coefficients of q^{4n} on both sides of (2.5.7), we readily deduce (2.5.1) to complete the proof. ☐

Remark 2.5.1. *Since $add_t(n) = 0$ for n odd [25, Section 8], we deduce from the above theorem that $asc_3(8n+4) = 0$ for all $n \geq 0$.*

Theorem 2.5.2. *If $add_3(n)$ denotes the number of double distinct 3-core partitions of n, then*

$$add_3(2P_k) = 1, \tag{2.5.8}$$

where P_k are the generalized pentagonal numbers $k(3k \pm 1)/2$.

Proof. Comparing the even parts in (2.5.6), we obtain

$$\sum_{n=0}^{\infty} add_3(2n)q^n = f(q, q^2). \tag{2.5.9}$$

Now, from the definition of $f(a, b)$, we see that

$$f(q, q^2) = \sum_{k=-\infty}^{\infty} q^{k(3k-1)/2} \tag{2.5.10}$$

Thus, from (2.5.9) and (2.5.10), we find that

$$\sum_{n=0}^{\infty} add_3(2n)q^n = \sum_{k=-\infty}^{\infty} q^{k(3k-1)/2} \tag{2.5.11}$$

Comparing the coefficients of q^{P_k} on both sides of (2.5.11), we arrive at (2.5.8). □

Theorem 2.5.3. *If $asc_2(n)$ denotes the number of 2-cores of n that are self-conjugates and $add_4(n)$ denotes the number of double distinct 4-core partitions, then*

$$add_4(2n) = asc_2(n) = \begin{cases} 1, & \text{if } n \text{ is a triangular number;} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. From (2.1.3) and (2.2.1) we deduce that

$$\sum_{n=0}^{\infty} add_4(n)q^n = f(q^2, q^6). \tag{2.5.12}$$

Also, from (2.1.1), (2.2.1), and (2.2.3), we obtain

$$\sum_{n=0}^{\infty} asc_2(n)q^n = f(q, q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2}, \tag{2.5.13}$$

and therefore

$$asc_2(n) = \begin{cases} 1, & \text{if } n \text{ is a triangular number;} \\ 0, & \text{otherwise.} \end{cases} \tag{2.5.14}$$

Using (2.5.12) and (2.5.13), we arrive at

$$\sum_{n=0}^{\infty} add_4(n)q^n = \sum_{n=0}^{\infty} asc_2(n)q^{2n}. \tag{2.5.15}$$

Equating the coefficients of q^{2n} in (2.5.15) and then using (2.5.14) we complete the proof. □

Theorem 2.5.4. *If $add_5(n)$ denotes the number of double distinct 5-core partitions and $t_2(n)$ denotes the number of representations of n as a sum of two triangular numbers, then*

$$add_5(10n+2) = t_2(5n+1) - t_2(n). \tag{2.5.16}$$

Proof. From (2.1.4) and (2.2.1), we deduce that

$$\sum_{n=0}^{\infty} add_5(n)q^n = f(q^2, q^8)f(q^4, q^6). \tag{2.5.17}$$

Furthermore, if $\psi(q)$ is as defined in (2.2.3), then $\psi^2(q)$ is the generating function for $t_2(n)$, the number of representations of n as a sum of two triangular numbers, i.e.,

$$\psi^2(q) = \sum_{n=0}^{\infty} t_2(n)q^n. \tag{2.5.18}$$

Now, from [10, p. 262, Entry 10(v)], we recall that

$$\psi^2(q) - q\psi^2(q^5) = f(q, q^4)f(q^2, q^3). \tag{2.5.19}$$

Replacing q by q^2 in (2.5.19), we have

$$f(q^2, q^8)f(q^4, q^6) = \psi^2(q^2) - q^2\psi^2(q^{10}). \tag{2.5.20}$$

Employing (2.5.17) and (2.5.18) in (2.5.20), we find that

$$\sum_{n=0}^{\infty} add_5(n)q^n = \sum_{n=0}^{\infty} t_2(n)q^{2n} - q^2\sum_{n=0}^{\infty} t_2(n)q^{10n}. \tag{2.5.21}$$

We readily arrive at (2.5.16) by equating the coefficients of q^{10n+2} on both sides of (2.5.21). □

Theorem 2.5.5. *If $asc_5(n)$ denotes the number of 5-cores of n that are self-conjugates, and if $add_5(n)$ denotes the number of double distinct 5-core partitions of n, then*

$$asc_5(2n) = add_5(n). \tag{2.5.22}$$

Proof. Setting $a = q$, $b = q^9$, $c = q^3$, and $d = q^7$ in (2.2.11), we deduce that

$$f(q,q^9)f(q^3,q^7) + f(-q,-q^9)f(-q^3,-q^7) = 2f(q^4,q^{16})f(q^8,q^{12}). \tag{2.5.23}$$

With the help of (2.2.6) and (2.2.1), we rewrite (2.5.23) as

$$\sum_{n=0}^{\infty} asc_5(n)q^n + \sum_{n=0}^{\infty}(-1)^n asc_5(n)q^n = 2f(q^4,q^{16})f(q^8,q^{12}). \tag{2.5.24}$$

Employing (2.5.17) in (2.5.24), we arrive at

$$\sum_{n=0}^{\infty} asc_5(n)q^n + \sum_{n=0}^{\infty}(-1)^n asc_5(n)q^n = 2\sum_{n=0}^{\infty} add_5(n)q^{2n}. \tag{2.5.25}$$

Comparing the coefficients of q^{2n} on both sides of (2.5.25), we easily deduce (2.5.22) to finish the proof. □

Remark 2.5.2. *As in Remark 2.5.1, since $add_t(n) = 0$ for n odd, we readily deduce from the above theorem that $asc_5(4n+2) = 0$ for all $n \geq 0$. In [25, Corollary 2(3)], it was shown that $asc_5(m) = 0$ if and only if there exists a prime $q \equiv 3 \bmod 4$ and an odd integer b such that q^b exactly divides $m+1$.*

Theorem 2.5.6. *If $asc_9(n)$ denotes the number of 9-cores of n that are self-conjugates, and if $add_9(n)$ denotes the number of double distinct 9-core partitions of n, then*

$$asc_9(4n+10) - asc_9(n) = add_9(n+1). \tag{2.5.26}$$

Proof. From (2.1.4) and (2.2.1), we find that

$$\sum_{n=0}^{\infty} add_9(n)q^n = f(q^2,q^{16})f(q^4,q^{14})f(q^6,q^{12})f(q^8,q^{10}).$$

Therefore, (2.4.11) can be rewritten as

$$\sum_{n=0}^{\infty} asc_9(4n+2)q^n = q\sum_{n=0}^{\infty} add_9(n)q^n + q^2\sum_{n=0}^{\infty} asc_9(n)q^n. \tag{2.5.27}$$

Equating the coefficients of q^{n+2} on both sides of (2.5.27), we obtain the proffered identity (2.5.26). □

Chapter 3

Generalized Frobenius Partitions

3.1 Introduction

In the introductory chapter, we have defined partitions and their conjugates. G. Frobenius [23] (see also [1]) introduced an idea of representing the conjugate of a partition λ of n once λ was known. This was done by simply removing the dots (say r in numbers) on the main diagonal of the Ferrers-Young diagram of λ and then enumerating the dots above and below the main diagonal by rows and columns respectively to obtain two strictly decreasing finite sequences of non-negative integers $a_1 > a_2 > \cdots > a_r \geq 0$, $b_1 > b_2 > \cdots > b_r \geq 0$. These two sequences are then presented in the Frobenius notation given by

$$\begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix}.$$

Clearly,
$$n = r + \textstyle\sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i.$$

For example, the Ferrers–Young diagram of the partition $\lambda = (6, 4, 3, 1)$ of 14 is

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & & \\ \bullet & & & & & \end{array}$$

Note: The contents of this chapter appeared in Elsevier's *Discrete Mathematics* [6].

Removing the dots on the main diagonal of the above Ferrers-Young diagram, we find that

$$
\begin{array}{cccccc}
\times & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \times & \bullet & \bullet & & \\
\bullet & \bullet & \times & & & \\
\bullet & & & & &
\end{array}
$$

Enumerating the dots above and below the main diagonal by rows and columns respectively, we easily see that λ can be presented in Frobenius notation as

$$\begin{pmatrix} 5 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}.$$

Similarly, the conjugate $\lambda' = (4,3,3,2,1,1)$ of λ in Frobenius notation is

$$\begin{pmatrix} 3 & 1 & 0 \\ 5 & 2 & 0 \end{pmatrix}.$$

Andrews [1] enhanced this idea to introduce generalized Frobenius partitions (or simply F-partitions) of n which is a notation of the form

$$\begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ b_1 & b_2 & . & & . & b_r \end{pmatrix}$$

of non-negative integers a_i's, b_i's with

$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i,$$

where each row is of the same length and each is arranged in non-increasing order. Andrews [1] considered two general classes of F-partitions, in one of which each non-negative integer is allowed to have k-copies(colors) and strict decrease in each row is maintained. Let $c\phi_k(n)$ denotes the number of such F-partitions of n and the generating function for $c\phi_k(n)$ is given by [1]

$$\sum_{n=0}^{\infty} c\phi_k(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k} \sum_{m_1,m_2,\ldots,m_{k-1}=-\infty}^{\infty} q^{Q(m_1,m_2,\ldots,m_{k-1})}, \tag{3.1.1}$$

where

$$Q(m_1, m_2, \cdots, m_{k-1}) = \sum_{j=1}^{k-1} m_j^2 + \sum_{1\le i<j\le k-1} m_i m_j.$$

In particular [1],

$$\begin{aligned}
\sum_{n=0}^{\infty} c\phi_1(n)q^n &= \frac{1}{(q;q)_\infty},\\
\sum_{n=0}^{\infty} c\phi_2(n)q^n &= \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty^4(q^4;q^4)_\infty},\\
\sum_{n=0}^{\infty} c\phi_3(n)q^n &= \frac{(q^{12};q^{12})_\infty(q^6;q^{12})_\infty^3}{(q;q^6)_\infty^5(q^5;q^6)_\infty^5(q^4;q^4)_\infty^2(q^3;q^6)_\infty^7} + 4q\frac{(q^{12};q^{12})_\infty(q^4;q^4)_\infty}{(q^6;q^{12})_\infty(q^2;q^4)_\infty(q;q)_\infty^3}.
\end{aligned} \tag{3.1.2}$$

In Section 3.2, we present expressions for the generating functions of $c\phi_4(n)$ and $c\phi_5(n)$ similar to (3.1.2) above. In Chapter 1, we have discussed the congruences satisfied by $c\phi_2(n)$ and $c\phi_3(n)$. In Section 3.3, we establish the congruences

$$\begin{aligned}
c\phi_4(2n+1) &\equiv 0 \pmod{4^2},\\
c\phi_4(4n+3) &\equiv 0 \pmod{4^4},\\
c\phi_4(4n+2) &\equiv 0 \pmod{4}.
\end{aligned}$$

Next, Kolitsch [35, 37] considered the function $\overline{c\phi}_k(n)$, which denotes the number of F-partitions of n with k colors whose order is k under cyclic permutation of the k colors. For example, the F-partitions enumerated by $\overline{c\phi_2}(2)$ are $\begin{pmatrix} 1_r \\ 0_r \end{pmatrix}$, $\begin{pmatrix} 1_g \\ 0_r \end{pmatrix}$, $\begin{pmatrix} 1_r \\ 0_g \end{pmatrix}$, $\begin{pmatrix} 1_g \\ 0_g \end{pmatrix}$, $\begin{pmatrix} 0_r \\ 1_r \end{pmatrix}$, $\begin{pmatrix} 0_r \\ 1_g \end{pmatrix}$, $\begin{pmatrix} 0_g \\ 1_r \end{pmatrix}$, and $\begin{pmatrix} 0_g \\ 1_g \end{pmatrix}$, where the subscripts represent the two colors viz. red and green of the non negative integers. The generating function for $\overline{c\phi}_k(n)$ is given by [37],

$$\sum_{n=0}^{\infty} \overline{c\phi}_k(n)q^n = \frac{k\sum q^{Q(\mathbf{m})}}{(q;q)_\infty^k}, \tag{3.1.3}$$

where the sum on the right extends over all vectors $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ with $\mathbf{m}\cdot\bar{1} = 1$ and $Q(\mathbf{m}) = \frac{1}{2}\sum_{i=1}^{k}(m_i - m_{i+1})^2$ wherein $\bar{1} = (1,1,1,\cdots,1)$ and $m_{k+1} = m_1$.

In particular (see [56] & [38]),

$$\sum_{n=0}^{\infty} \overline{c\phi}_2(n)q^n = \frac{4q(q^{16};q^{16})_\infty^2}{(q;q)_\infty^2(q^8;q^8)_\infty},$$
$$\sum_{n=0}^{\infty} \overline{c\phi}_3(n)q^n = \frac{9q(q^9;q^9)_\infty^3}{(q;q)_\infty^3(q^3;q^3)_\infty}.$$

In Section 3.4, we obtain the generating function for $\overline{c\phi}_4(n)$ in terms of q-products. In Chapter 1, we have discussed the congruences satisfied by $\overline{c\phi}_k(n)$ for different values of k. In Section 3.5, we prove the following congruences.

$$\overline{c\phi}_4(2n) \equiv 0 \pmod{4^3},$$
$$\overline{c\phi}_4(4n+3) \equiv 0 \pmod{4^4},$$
$$\overline{c\phi}_4(4n) \equiv 0 \pmod{4^4}.$$

We conclude this introduction with a brief discussion of an integer matrix exact covering system as described by Cao [17].

An exact covering system is a partition of the integers into a finite set of arithmetic sequences. An integer matrix exact covering system is a partition of $\mathbb{Z}^n$ into a lattice and a finite number of its translates without overlap.

Let

$$S = \sum_{x_1,x_2,\cdots,x_n=-\infty}^{\infty} f(x_1,x_2,\cdots x_n).$$

We change the variables from x_i to y_i by the transformation $y = Ax$ where A is an integer matrix with $\det A \neq 0$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, and $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$.

By the inverse formula, $x = A^{-1}y = \dfrac{1}{\det A}A^*y$, where A^* is the adjoint of A.
Let

$$\frac{\det A}{d_{n-1}(A)} = s_n(A),$$

where $d_{n-1}(A)$ is the $(n-1)^{th}$ determinantal divisor of A (the k^{th} determinantal divisor of A is the g.c.d. of all $k \times k$ minors of A).
Therefore,

$$x = \frac{1}{s_n(A)} \frac{A^*}{d_{n-1}(A)} y. \tag{3.1.4}$$

Setting $\operatorname{sgn}(s_n(A)) \dfrac{A^*}{d_{n-1}(A)} = B$, $|s_n(A)| = d$, we can rewrite (3.1.4) as

$$x = \frac{1}{d} By.$$

Thus, we have

$$By \equiv 0 \pmod{d}. \tag{3.1.5}$$

If $y \equiv c_r \pmod{d}$ $(r = 0, 1, 2, \cdots, k-1)$ is the solution set of (3.1.5) then we have $x = By + \frac{1}{d}Bc_r$, $(r = 0, 1, 2, \cdots, k-1)$, $\{By + \frac{1}{d}Bc_r\}_{r=0}^{k-1}$ covers $\mathbb{Z}^n$ and there is no overlap between the members thereby giving an integer matrix exact covering system. Corresponding to this integer matrix exact covering system we can write S as a linear combination of k parts.

3.2 Generating functions for $c\phi_4(n)$ and $c\phi_5(n)$

In this section, we find expressions for the generating functions of $c\phi_4(n)$ and $c\phi_5(n)$.

Theorem 3.2.1. *We have*

$$\sum_{n=0}^{\infty} c\phi_4(n) q^n = \frac{\varphi^3(q^2) + 12q\varphi(q^2)\psi^2(q^4)}{(q;q)_\infty^4}.$$

Proof. Setting $k = 4$ in (3.1.1) we have

$$\sum_{n=0}^{\infty} c\phi_4(n) q^n = \frac{1}{(q;q)_\infty^4} \sum_{m_1,m_2,m_3=-\infty}^{\infty} q^{m_1^2+m_2^2+m_3^2+m_1m_2+m_2m_3+m_3m_1}. \tag{3.2.1}$$

Let

$$S=\sum_{m_1,m_2,m_3=-\infty}^{\infty} q^{m_1^2+m_2^2+m_3^2+m_1m_2+m_2m_3+m_3m_1}. \tag{3.2.2}$$

We change the variables from m_1, m_2, m_3 to n_1, n_2, n_3 by an integer matrix exact covering system $\{B\mathbf{n}+\frac{1}{d}Bc_r\}_{r=0}^{k-1}$, where $B=(b_{ij})_{3\times 3}$ is an integer matrix, $\mathbf{n}=\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ and $c_0, c_1, \cdots c_{k-1}$ are the solutions of the congruences $B\mathbf{n}\equiv 0 \pmod{d}$.

We shall further require that the coefficients of n_1n_2, n_2n_3, n_3n_1 in

$$\begin{aligned}&(b_{11}n_1+b_{12}n_2+b_{13}n_3)^2+(b_{21}n_1+b_{22}n_2+b_{23}n_3)^2+(b_{31}n_1+b_{32}n_2+b_{33}n_3)^2\\&+(b_{11}n_1+b_{12}n_2+b_{13}n_3)(b_{21}n_1+b_{22}n_2+b_{23}n_3)\\&+(b_{21}n_1+b_{22}n_2+b_{23}n_3)(b_{31}n_1+b_{32}n_2+b_{33}n_3)\\&+(b_{31}n_1+b_{32}n_2+b_{33}n_3)(b_{11}n_1+b_{12}n_2+b_{13}n_3)\end{aligned}$$

to be zero in order to separate n_i's. Thus, we have the conditions that

$$2b_{11}b_{12}+2b_{21}b_{22}+2b_{31}b_{32}+b_{11}b_{22}+b_{12}b_{21}+b_{21}b_{32}+b_{22}b_{31}+b_{31}b_{12}+b_{32}b_{11}=0, \tag{3.2.3}$$

$$2b_{12}b_{13}+2b_{22}b_{23}+2b_{32}b_{33}+b_{12}b_{23}+b_{13}b_{22}+b_{22}b_{33}+b_{23}b_{32}+b_{32}b_{13}+b_{33}b_{12}=0, \tag{3.2.4}$$

$$2b_{11}b_{13}+2b_{21}b_{23}+2b_{31}b_{33}+b_{11}b_{23}+b_{13}b_{21}+b_{21}b_{33}+b_{23}b_{31}+b_{31}b_{13}+b_{33}b_{11}=0. \tag{3.2.5}$$

The integer matrix exact covering system is obtained by following the general procedure for obtaining series-product identities developed in [17]. Let

$$B'=4\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Clearly, B' satisfies (3.2.3)–(3.2.5) and $\det B' = 4^4 = 16^{3-1}$.

Also $16B'^{-1} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ is an integer matrix.

Now, the system of congruences $B'\mathbf{n} \equiv 0 \pmod{16}$ is equivalent to $B\mathbf{n} \equiv 0 \pmod 4$, where $B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. That is,

$$-n_1 + n_2 + n_3 \equiv 0 \pmod 4,$$
$$n_1 - n_2 + n_3 \equiv 0 \pmod 4,$$
$$n_1 + n_2 - n_3 \equiv 0 \pmod 4.$$

The above system of congruences has four solutions

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \text{ and } \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

modulo 4. Hence we have the integer matrix exact covering system with members

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.2.6}$$

Corresponding to this integer matrix exact covering system, we can write S as a linear combination of four parts as

$$\begin{aligned} S &= \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{\frac{1}{2}\{(2n_3)^2+(2n_1)^2+(2n_2)^2\}} + 3 \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{\frac{1}{2}\{(2n_3+1)^2+(2n_1)^2+(2n_2+1)^2\}} \\ &= \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{2n_1^2+2n_2^2+2n_3^2} + 3 \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{2n_1^2+2n_2^2+2n_3^2+2n_2+2n_3+1} \\ &= \left(\sum_{n_1=-\infty}^{\infty} q^{2n_1^2} \right)^3 + 3q \left(\sum_{n_1=-\infty}^{\infty} q^{2n_1^2} \right) \left(\sum_{n_2=-\infty}^{\infty} q^{2n_2^2+2n_2} \right)^2 \\ &= \varphi^3(q^2) + 12q\varphi(q^2)\psi^2(q^4), \end{aligned} \tag{3.2.7}$$

where we have used (2.2.2) and (2.2.3) to arrive at the last equality. Employing (3.2.7) and (3.2.2) in (3.2.1), we complete the proof. □

In the next theorem, we find an expression for the generating function of $c\phi_5(n)$.

Theorem 3.2.2. *We have*

$$\sum_{n=0}^{\infty} c\phi_5(n)q^n = \frac{1}{(q;q)_\infty^5} \{\varphi(q^{10})\varphi^3(q^2) + 12q\varphi(q^{10})\varphi(q^2)\psi^2(q^4) + 8q\psi(q^5)\psi^3(q) + 12q^3\psi(q^{20})\psi(q^4)\varphi^2(q) + 16q^4\psi(q^{20})\psi^3(q^4)\}.$$

Proof. Setting $k = 5$ in (3.1.1) we have

$$\sum_{n=0}^{\infty} c\phi_5(n)q^n = \frac{1}{(q;q)_\infty^5} \sum_{m_1,m_2,m_3,m_4=-\infty}^{\infty} q^{m_1^2+m_2^2+m_3^2+m_4^2+m_1m_2+m_1m_3+m_1m_4+m_2m_3+m_2m_4+m_3m_4}. \tag{3.2.8}$$

Let

$$T = \sum_{m_1,m_2,m_3,m_4=-\infty}^{\infty} q^{m_1^2+m_2^2+m_3^2+m_4^2+m_1m_2+m_1m_3+m_1m_4+m_2m_3+m_2m_4+m_3m_4}. \quad (3.2.9)$$

We change the variables from m_1, m_2, m_3, m_4 to n_1, n_2, n_3, n_4 by an integer matrix exact covering system $\{B\mathbf{n}+\frac{1}{d}Bc_r\}_{r=0}^{k-1}$, where $B = (b_{ij})_{4\times4}$ is an integer matrix, $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}$ and $c_0, c_1, \cdots c_{k-1}$ are the solutions of the congruences $B\mathbf{n} \equiv 0 \pmod{d}$.

We also require the following additional conditions in order to separate n_i's.

$$2b_{11}b_{12} + 2b_{21}b_{22} + 2b_{31}b_{32} + 2b_{41}b_{42} + b_{11}b_{22} + b_{12}b_{21} + b_{11}b_{32} + b_{12}b_{31}$$
$$+ b_{11}b_{42} + b_{12}b_{41} + b_{21}b_{32} + b_{22}b_{31} + b_{21}b_{42} + b_{22}b_{41} + b_{31}b_{42} + b_{32}b_{41} = 0,$$
$$2b_{11}b_{13} + 2b_{21}b_{23} + 2b_{31}b_{33} + 2b_{41}b_{43} + b_{11}b_{23} + b_{13}b_{21} + b_{11}b_{33} + b_{13}b_{31}$$
$$+ b_{11}b_{43} + b_{13}b_{41} + b_{21}b_{33} + b_{23}b_{31} + b_{21}b_{43} + b_{23}b_{41} + b_{31}b_{43} + b_{33}b_{41} = 0,$$
$$2b_{11}b_{14} + 2b_{21}b_{24} + 2b_{31}b_{34} + 2b_{41}b_{44} + b_{11}b_{24} + b_{14}b_{21} + b_{11}b_{34} + b_{14}b_{31}$$
$$+ b_{11}b_{44} + b_{14}b_{41} + b_{21}b_{34} + b_{24}b_{31} + b_{21}b_{44} + b_{24}b_{41} + b_{31}b_{44} + b_{34}b_{41} = 0,$$
$$2b_{12}b_{13} + 2b_{22}b_{23} + 2b_{32}b_{33} + 2b_{42}b_{43} + b_{12}b_{23} + b_{13}b_{22} + b_{12}b_{33} + b_{13}b_{32}$$
$$+ b_{12}b_{43} + b_{13}b_{42} + b_{22}b_{33} + b_{23}b_{32} + b_{22}b_{43} + b_{23}b_{42} + b_{32}b_{43} + b_{33}b_{42} = 0,$$
$$2b_{12}b_{14} + 2b_{22}b_{24} + 2b_{32}b_{34} + 2b_{42}b_{44} + b_{12}b_{24} + b_{14}b_{22} + b_{12}b_{34} + b_{14}b_{32}$$
$$+ b_{12}b_{44} + b_{14}b_{42} + b_{22}b_{34} + b_{24}b_{32} + b_{22}b_{44} + b_{24}b_{42} + b_{32}b_{44} + b_{34}b_{42} = 0,$$
$$2b_{13}b_{14} + 2b_{23}b_{24} + 2b_{33}b_{34} + 2b_{43}b_{44} + b_{13}b_{24} + b_{14}b_{23} + b_{13}b_{34} + b_{14}b_{33}$$
$$+ b_{13}b_{44} + b_{14}b_{43} + b_{23}b_{34} + b_{24}b_{33} + b_{23}b_{44} + b_{24}b_{43} + b_{33}b_{44} + b_{34}b_{43} = 0.$$

Proceeding as in the previous proof, our task reduces to obtaining the solutions modulo 16 of the system of congruences $B\mathbf{n} \equiv 0 \pmod{16}$, where

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

That is, we require to solve the simultaneous congruences

$$n_1 + n_2 + n_3 + n_4 \equiv 0 \pmod{16},$$

$$n_1 + n_2 - n_3 - n_4 \equiv 0 \pmod{16},$$

$$n_1 - n_2 - n_3 + n_4 \equiv 0 \pmod{16},$$

$$n_1 - n_2 + n_3 - n_4 \equiv 0 \pmod{16}.$$

The above system of congruences has sixteen solutions

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 8 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 8 \\ 0 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 8 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 12 \\ 12 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 12 \\ 4 \\ 12 \end{pmatrix},$$

$$\begin{pmatrix} 4 \\ 4 \\ 12 \\ 12 \end{pmatrix}, \begin{pmatrix} 8 \\ 8 \\ 8 \\ 8 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \\ 0 \\ 8 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 8 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 12 \\ 12 \\ 12 \\ 12 \end{pmatrix}, \begin{pmatrix} 12 \\ 4 \\ 4 \\ 12 \end{pmatrix},$$

$$\begin{pmatrix} 12 \\ 4 \\ 12 \\ 4 \end{pmatrix}, \text{ and } \begin{pmatrix} 12 \\ 12 \\ 4 \\ 4 \end{pmatrix} \text{ modulo 16.}$$

Hence we have the integer matrix exact covering system with members

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Corresponding to this integer matrix exact covering system, we can write T as a linear combination of sixteen parts as

$$
\begin{aligned}
T = &\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2} \\
&+3\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+2n_2+2n_3+1} \\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+5n_1+n_2+n_3+n_4+1} \\
&+3\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+5n_1+3n_2+3n_3+n_4+3} \\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+10n_1+2n_2+2n_3+2n_4+4} \\
&+3\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+10n_1+2n_4+3} \\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+15n_1+3n_2+3n_3+3n_4+9} \\
&+3\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+15n_1+n_2+n_3+3n_4+7} \\
= &\ \varphi(q^{10})\varphi^3(q^2) + 12q\varphi(q^{10})\varphi(q^2)\psi^2(q^4) + 8q\psi(q^5)\psi^3(q) \\
&+ 12q^3\psi(q^{20})\psi(q^4)\varphi^2(q) + 16q^4\psi(q^{20})\psi^3(q^4), \qquad (3.2.10)
\end{aligned}
$$

where we have used (2.2.2), (2.2.3), and (2.3.16) to arrive at the last equality. Employing (3.2.10) and (3.2.9) in (3.2.8), we complete the proof.

□

3.3 Congruences involving $c\phi_4(n)$

In this section, we find some congruences involving $c\phi_4(n)$ arising from the expression of the generating function for $c\phi_4(n)$ obtained in the previous section.

Theorem 3.3.1. *We have*

$$c\phi_4(2n+1) \equiv 0 \pmod{4^2}, \tag{3.3.1}$$

$$c\phi_4(4n+3) \equiv 0 \pmod{4^4}, \tag{3.3.2}$$

$$c\phi_4(4n+2) \equiv 0 \pmod{4}. \tag{3.3.3}$$

Proof. From Theorem 3.2.1, we note that

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_4(n)q^n &= \frac{\varphi^3(q^2)}{(q;q)_\infty^4} + 12q\frac{\varphi(q^2)\psi^2(q^4)}{(q;q)_\infty^4} \\ &= \frac{\varphi^3(q^2)}{(q^2;q^2)_\infty^4(q;q^2)_\infty^4} + 12q\frac{\varphi(q^2)\psi^2(q^4)}{(q^2;q^2)_\infty^4(q;q^2)_\infty^4}.\end{aligned} \tag{3.3.4}$$

Replacing q by $-q$ in (3.3.4) and then subtracting the resulting identity from (3.3.4), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} c\phi_4(n)q^n - \sum_{n=0}^{\infty} c\phi_4(n)(-1)^n q^n \\ &= \frac{\varphi^3(q^2)}{(q^2;q^2)_\infty^4}\left\{\frac{1}{(q;q^2)_\infty^4} - \frac{1}{(-q;q^2)_\infty^4}\right\} + 12q\frac{\varphi(q^2)\psi^2(q^4)}{(q^2;q^2)_\infty^4}\left\{\frac{1}{(q;q^2)_\infty^4} + \frac{1}{(-q;q^2)_\infty^4}\right\} \\ &= \frac{\varphi^3(q^2)}{(q^2;q^2)_\infty^4(q^2;q^4)_\infty^4}\left\{(-q;q^2)_\infty^4 - (q;q^2)_\infty^4\right\} \\ &\quad + 12q\frac{\varphi(q^2)\psi^2(q^4)}{(q^2;q^2)_\infty^4(q^2;q^4)_\infty^4}\left\{(-q;q^2)_\infty^4 + (q;q^2)_\infty^4\right\}.\end{aligned} \tag{3.3.5}$$

From (2.2.2) and (2.2.3), we find that

$$\varphi(q) = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}, \tag{3.3.6}$$

and

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{3.3.7}$$

Again, from Entry 25 [10, p. 40], we have

$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4), \tag{3.3.8}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \tag{3.3.9}$$

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2). \tag{3.3.10}$$

Using (2.2.2) on the left hand sides of (3.3.8)–(3.3.10), we obtain

$$(-q;q^2)_\infty^4-(q;q^2)_\infty^4=8q\frac{\psi^2(q^4)}{(q^2;q^2)_\infty^2}, \tag{3.3.11}$$

$$(-q;q^2)_\infty^4+(q;q^2)_\infty^4=2\frac{\varphi^2(q^2)}{(q^2;q^2)_\infty^2}, \tag{3.3.12}$$

$$(-q;q^2)_\infty^8-(q;q^2)_\infty^8=16q\frac{\psi^4(q^2)}{(q^2;q^2)_\infty^4}. \tag{3.3.13}$$

Employing (3.3.11) and (3.3.12) in (3.3.5), we deduce that

$$\sum_{n=0}^{\infty}c\phi_4(n)q^n-\sum_{n=0}^{\infty}c\phi_4(n)(-1)^nq^n=32q\frac{\varphi^3(q^2)\psi^2(q^4)(q^4;q^4)_\infty^4}{(q^2;q^2)_\infty^{10}}. \tag{3.3.14}$$

Comparing the terms involving q^{2n+1} on both sides of (3.3.14), and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty}c\phi_4(2n+1)q^n=16\frac{\varphi^3(q)\psi^2(q^2)(q^2;q^2)_\infty^4}{(q;q)_\infty^{10}}, \tag{3.3.15}$$

from which we readily deduce the congruence (3.3.1).

Now, employing (2.2.2) and (2.2.3) in (3.3.15) and then simplifying, we have

$$\sum_{n=0}^{\infty}c\phi_4(2n+1)q^n=16\,\frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty^2(q;q^2)_\infty^{16}}. \tag{3.3.16}$$

Replacing q by $-q$ in (3.3.16) and then subtracting the resulting identity from (3.3.16), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty}c\phi_4(2n+1)q^n-\sum_{n=0}^{\infty}c\phi_4(2n+1)(-1)^nq^n\\
&=16\,\frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty^2}\left\{\frac{1}{(q;q^2)_\infty^{16}}-\frac{1}{(-q;q^2)_\infty^{16}}\right\}\\
&=16\,\frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty^2(q^2;q^4)_\infty^{16}}\left\{(-q;q^2)_\infty^{16}-(q;q^2)_\infty^{16}\right\}\\
&=16\,\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{15}}\left\{(-q;q^2)_\infty^8-(q;q^2)_\infty^8\right\}\left\{\left\{(-q;q^2)_\infty^4-(q;q^2)_\infty^4\right\}^2+2(q^2;q^4)_\infty^4\right\}.
\end{aligned} \tag{3.3.17}$$

Using (3.3.11), (3.3.13) and (2.2.3) in (3.3.17), we deduce that

$$\sum_{n=0}^{\infty} c\phi_4(2n+1)q^n - \sum_{n=0}^{\infty} c\phi_4(2n+1)(-1)^n q^n$$
$$= 256q\frac{(q^4;q^4)_\infty^{22}}{(q^2;q^2)_\infty^{27}}\left\{64q^2\psi^4(q^4) + 2(q^2;q^2)_\infty^4(q^2;q^4)_\infty^4\right\}. \tag{3.3.18}$$

Equating the terms involving q^{2n+1} on both sides of (3.3.18), and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty} c\phi_4(4n+3)q^n = 256\,\frac{(q^2;q^2)_\infty^{22}}{(q;q)_\infty^{27}}\left\{32q\psi^4(q^2) + (q;q)_\infty^4(q;q^2)_\infty^4\right\}. \tag{3.3.19}$$

Now (3.3.2) easily follows from (3.3.19).

Again, replacing q by $-q$ in (3.3.4) and then adding the resulting identity with (3.3.4), we obtain

$$\sum_{n=0}^{\infty} c\phi_4(n)q^n + \sum_{n=0}^{\infty} c\phi_4(n)(-1)^n q^n$$
$$= \frac{\varphi^3(q^2)}{(q^2;q^2)_\infty^4}\left\{\frac{1}{(q;q^2)_\infty^4} + \frac{1}{(-q;q^2)_\infty^4}\right\} + 12q\frac{\varphi(q^2)\psi^2(q^4)}{(q^2;q^2)_\infty^4}\left\{\frac{1}{(q;q^2)_\infty^4} - \frac{1}{(-q;q^2)_\infty^4}\right\}$$
$$= \frac{\varphi^3(q^2)}{(q^2;q^2)_\infty^4(q^2;q^4)_\infty^4}\left\{(-q;q^2)_\infty^4 + (q;q^2)_\infty^4\right\}$$
$$+ 12q\frac{\varphi(q^2)\psi^2(q^4)}{(q^2;q^2)_\infty^4(q^2;q^4)_\infty^4}\left\{(-q;q^2)_\infty^4 - (q;q^2)_\infty^4\right\}$$
$$= 2\,\frac{\varphi^5(q^2)(q^4;q^4)_\infty^4}{(q^2;q^2)_\infty^{10}} + 96q^2\frac{\varphi(q^2)\psi^4(q^4)(q^4;q^4)_\infty^4}{(q^2;q^2)_\infty^{10}}, \tag{3.3.20}$$

where we have used (3.3.11) and (3.3.12) in the last equality.

Comparing the terms involving q^{2n} on both sides of (3.3.20), and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty} c\phi_4(2n)q^n = \frac{\varphi^5(q)(q^2;q^2)_\infty^4}{(q;q)_\infty^{10}} + 48q\frac{\varphi(q)\psi^4(q^2)(q^2;q^2)_\infty^4}{(q;q)_\infty^{10}}. \tag{3.3.21}$$

Employing (2.2.2) and (2.2.3) in (3.3.21), we deduce that

$$\sum_{n=0}^{\infty} c\phi_4(2n)q^n = \frac{(q^2;q^2)_\infty^{29}}{(q;q)_\infty^{20}(q^4;q^4)_\infty^{10}} + 48q\frac{(q^2;q^2)_\infty^5(q^4;q^4)_\infty^6}{(q;q)_\infty^{12}}$$
$$= \frac{(q^2;q^2)_\infty^9}{(q;q^2)_\infty^{20}(q^4;q^4)_\infty^{10}} + 48q\frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^7(q;q^2)_\infty^{12}}. \tag{3.3.22}$$

Replacing q by $-q$ in (3.3.22) and then subtracting the resulting identity from (3.3.22), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} c\phi_4(2n)q^n - \sum_{n=0}^{\infty} c\phi_4(2n)(-1)^n q^n \\
&= \frac{(q^2;q^2)_\infty^9}{(q^4;q^4)_\infty^{10}}\left\{\frac{1}{(q;q^2)_\infty^{20}} - \frac{1}{(-q;q^2)_\infty^{20}}\right\} + 48q\frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^7}\left\{\frac{1}{(q;q^2)_\infty^{12}} + \frac{1}{(-q;q^2)_\infty^{12}}\right\} \\
&= \frac{(q^2;q^2)_\infty^9}{(q^4;q^4)_\infty^{10}(q^2;q^4)_\infty^{20}}\left\{(-q;q^2)_\infty^{20} - (q;q^2)_\infty^{20}\right\} \\
&\quad + 48q\frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^7(q^2;q^4)_\infty^{12}}\left\{(-q;q^2)_\infty^{12} + (q;q^2)_\infty^{12}\right\} \\
&= \frac{(q^4;q^4)_\infty^{10}}{(q^2;q^2)_\infty^{11}}\left\{A^5 - B^5\right\} + 48q\frac{(q^4;q^4)_\infty^{18}}{(q^2;q^2)_\infty^{19}}\left\{A^3 + B^3\right\},
\end{aligned} \tag{3.3.23}$$

where $A = (-q;q^2)_\infty^4$ and $B = (q;q^2)_\infty^4$.

Now, by the binomial theorem, we have

$$(A-B)^5 = A^5 - B^5 - 5AB(A^3 - B^3) + 10A^2B^2(A-B).$$

Therefore,

$$\begin{aligned}
A^5 - B^5 &= (A-B)^5 + 5AB(A^3 - B^3) - 10A^2B^2(A-B) \\
&= (A-B)\left\{(A-B)^4 + 5AB(A^2 + AB + B^2) - 10A^2B^2\right\} \\
&= (A-B)\left\{(A-B)^4 + 5AB(A-B)^2 + 5A^2B^2\right\}.
\end{aligned} \tag{3.3.24}$$

Also,

$$A^3 + B^3 = (A+B)\left\{(A+B)^2 - 3AB\right\}. \tag{3.3.25}$$

Employing (3.3.11) and (3.3.12) in (3.3.24) and (3.3.25), we find that

$$A^5 - B^5 = 8q\frac{\psi^2(q^4)}{(q^2;q^2)_\infty^2}\left\{4096q^4\frac{\psi^8(q^4)}{(q^2;q^2)_\infty^8} + 320q^2\frac{\psi^4(q^4)}{(q^4;q^4)_\infty^4} + 5(q^2;q^4)_\infty^8\right\}, \tag{3.3.26}$$

$$A^3 + B^3 = 2\frac{\varphi^2(q^2)}{(q^2;q^2)_\infty^2}\left\{4\frac{\varphi^4(q^2)}{(q^2;q^2)_\infty^4} - 3(q^2;q^4)_\infty^4\right\}. \tag{3.3.27}$$

Using (3.3.26) and (3.3.27) in (3.3.23), we deduce that

$$\sum_{n=0}^{\infty} c\phi_4(2n)q^n - \sum_{n=0}^{\infty} c\phi_4(2n)(-1)^n q^n$$
$$= 8q\frac{(q^4;q^4)_\infty^{10}\psi^2(q^4)}{(q^2;q^2)_\infty^{13}}\left\{4096q^4\frac{\psi^8(q^4)}{(q^2;q^2)_\infty^8} + 320q^2\frac{\psi^4(q^4)}{(q^4;q^4)_\infty^4} + 5(q^2;q^4)_\infty^8\right\}$$
$$+ 96q\frac{(q^4;q^4)_\infty^{18}\varphi^2(q^2)}{(q^2;q^2)_\infty^{21}}\left\{4\frac{\varphi^4(q^2)}{(q^2;q^2)_\infty^4} - 3(q^2;q^4)_\infty^4\right\}. \tag{3.3.28}$$

Equating the coefficients of the terms involving q^{2n+1} in (3.3.28), we readily arrive at (3.3.3) to finish the proof. □

3.4 Generating function for $\overline{c\phi_4}(n)$

In this section, we find an expression for the generating function of $\overline{c\phi_4}(n)$.

Theorem 3.4.1. *We have*

$$\sum_{n=0}^{\infty} \overline{c\phi_4}(n)q^n = 16q\frac{\psi^2(q^2)\psi(q^4)}{(q;q)_\infty^4}. \tag{3.4.1}$$

Proof. Setting $k = 4$ in (3.1.3) and utilizing the subsequent conditions, we have

$$\sum_{n=0}^{\infty} \overline{c\phi_4}(n)q^n$$
$$= \frac{4}{(q;q)_\infty^4}\sum_{m_1,m_2,m_3=-\infty}^{\infty} q^{3m_1^2+2m_2^2+3m_3^2+2m_1m_2+2m_2m_3+4m_3m_1-3m_1-2m_2-3m_3+1}. \tag{3.4.2}$$

Let

$$S' = \sum_{m_1,m_2,m_3=-\infty}^{\infty} q^{3m_1^2+2m_2^2+3m_3^2+2m_1m_2+2m_2m_3+4m_3m_1-3m_1-2m_2-3m_3+1}. \tag{3.4.3}$$

We change the variables from m_1, m_2, m_3 to n_1, n_2, n_3 by the integer matrix exact covering system $\{B\mathbf{n} + \frac{1}{d}Bc_r\}_{r=0}^{k-1}$, where $B = (b_{ij})_{3\times 3}$ is an integer matrix, $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ and c_0, c_1, $\cdots$, c_{k-1} are the solutions of the congruences $B\mathbf{n} \equiv 0 \pmod{d}$.

We further require that the coefficients of n_1n_2, n_2n_3, n_3n_1 in

$$3(b_{11}n_1 + b_{12}n_2 + b_{13}n_3)^2 + 2(b_{21}n_1 + b_{22}n_2 + b_{23}n_3)^2 + 3(b_{31}n_1 + b_{32}n_2 + b_{33}n_3)^2$$
$$+ 2(b_{11}n_1 + b_{12}n_2 + b_{13}n_3)(b_{21}n_1 + b_{22}n_2 + b_{23}n_3)$$
$$+ 2(b_{21}n_1 + b_{22}n_2 + b_{23}n_3)(b_{31}n_1 + b_{32}n_2 + b_{33}n_3)$$
$$+ 4(b_{31}n_1 + b_{32}n_2 + b_{33}n_3)(b_{11}n_1 + b_{12}n_2 + b_{13}n_3)$$

to be zero in order to separate n_i's. Thus we have the conditions that

$$3b_{11}b_{12} + 2b_{21}b_{22} + 3b_{31}b_{32} + b_{11}b_{22} + b_{12}b_{21} + b_{21}b_{32} + b_{22}b_{31} + 2b_{31}b_{12} + 2b_{32}b_{11} = 0, \tag{3.4.4}$$

$$3b_{12}b_{13} + 2b_{22}b_{23} + 3b_{32}b_{33} + b_{12}b_{23} + b_{13}b_{22} + b_{22}b_{33} + b_{23}b_{32} + 2b_{32}b_{13} + 2b_{33}b_{12} = 0, \tag{3.4.5}$$

$$3b_{11}b_{13} + 2b_{21}b_{23} + 3b_{31}b_{33} + b_{11}b_{23} + b_{13}b_{21} + b_{21}b_{33} + b_{23}b_{31} + 2b_{31}b_{13} + 2b_{33}b_{11} = 0. \tag{3.4.6}$$

Now, we consider the same integer matrix exact covering system (3.2.6) as in Theorem 3.2.1, because the matrix $B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ also satisfies the conditions (3.4.4)–(3.4.6). Corresponding to the integer matrix exact covering system (3.2.6) we can write S' as a linear combination of 4 parts as follows

$$S' = \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{4n_1^2+8n_2^2+4n_3^2-2n_1-4n_2-2n_3+1} + \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{4n_1^2+8n_2^2+4n_3^2-2n_1+4n_2+2n_3+1}$$
$$+ \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{4n_1^2+8n_2^2+4n_3^2+2n_1-4n_2+2n_3+1} + \sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{4n_1^2+8n_2^2+4n_3^2+2n_1+4n_2-2n_3+1}.$$

It is easy to see that each part of the above sum is equal to $q\psi^2(q^2)\psi(q^4)$. Therefore, we have

$$S' = 4q\psi^2(q^2)\psi(q^4). \tag{3.4.7}$$

From (3.4.7), (3.4.3), and (3.4.2), we readily arrive at (3.4.1). □

3.5 Congruences involving $\overline{c\phi}_4(n)$

In this section, we find some interesting congruences involving $\overline{c\phi}_4(n)$.

Theorem 3.5.1. *We have*

$$\overline{c\phi}_4(2n) \equiv 0 \pmod{4^3}, \tag{3.5.1}$$

$$\overline{c\phi}_4(4n+3) \equiv 0 \pmod{4^4}, \tag{3.5.2}$$

$$\overline{c\phi}_4(4n) \equiv 0 \pmod{4^4}. \tag{3.5.3}$$

Proof. From Theorem 3.4.1, we have

$$\sum_{n=0}^{\infty} \overline{c\phi}_4(n)q^n = 16q\frac{\psi^2(q^2)\psi(q^4)}{(q;q)_\infty^4} = 16q\frac{\psi^2(q^2)\psi(q^4)}{(q^2;q^2)_\infty^4(q;q^2)_\infty^4}. \tag{3.5.4}$$

Replacing q by $-q$ in (3.5.4) and then adding the resulting identity with (3.5.4), we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{c\phi}_4(n)q^n + \sum_{n=0}^{\infty} \overline{c\phi}_4(n)(-1)^n q^n \\
&= 16q\frac{\psi^2(q^2)\psi(q^4)}{(q^2;q^2)_\infty^4}\left\{\frac{1}{(q;q^2)_\infty^4} - \frac{1}{(-q;q^2)_\infty^4}\right\} \\
&= 16q\frac{\psi^2(q^2)\psi(q^4)}{(q^2;q^2)_\infty^4(q^2;q^4)_\infty^4}\left\{(-q;q^2)_\infty^4 - (q;q^2)_\infty^4\right\}.
\end{aligned} \tag{3.5.5}$$

Employing (3.3.11) in (3.5.5), we obtain

$$\sum_{n=0}^{\infty} \overline{c\phi}_4(n)q^n + \sum_{n=0}^{\infty} \overline{c\phi}_4(n)(-1)^n q^n = 128q^2\frac{\psi^2(q^2)\psi^3(q^4)}{(q^2;q^2)_\infty^6(q^2;q^4)_\infty^4}. \tag{3.5.6}$$

Extracting those terms on both sides of (3.5.6) that involve q^{2n} only, and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty} \overline{c\phi}_4(2n)q^n = 64q\frac{\psi^2(q)\psi^3(q^2)}{(q;q)_\infty^6(q;q^2)_\infty^4}. \tag{3.5.7}$$

Now congruence (3.5.1) readily follows from (3.5.7).

Next, with the help of (2.2.3), we can rewrite (3.5.7) as

$$\sum_{n=0}^{\infty}\overline{c\phi}_4(2n)q^n = 64q\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty^6}{(q;q)_\infty^8(q;q^2)_\infty^4} = 64q\frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^7(q;q^2)_\infty^{12}}. \tag{3.5.8}$$

Replacing q by $-q$ in (3.5.8) and then adding the resulting identity with (3.5.8), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{c\phi}_4(2n)q^n + \sum_{n=0}^{\infty}\overline{c\phi}_4(2n)(-1)^nq^n \\
&= 64q\frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^7}\left\{\frac{1}{(q;q^2)_\infty^{12}} - \frac{1}{(-q;q^2)_\infty^{12}}\right\} \\
&= 64q\frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^7(q^2;q^4)_\infty^{12}}\left\{(-q;q^2)_\infty^{12} - (q;q^2)_\infty^{12}\right\} \\
&= 64q\frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^7(q^2;q^4)_\infty^{12}}\left\{(-q;q^2)_\infty^{4} - (q;q^2)_\infty^{4}\right\} \\
&\quad\times\left\{(-q;q^2)_\infty^8 + (q^2;q^4)_\infty^4 + (q;q^2)_\infty^8\right\} \\
&= 64q\frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^7(q^2;q^4)_\infty^{12}}\left\{(-q;q^2)_\infty^{4} - (q;q^2)_\infty^{4}\right\} \\
&\quad\times\left\{\{(-q;q^2)_\infty^4 - (q;q^2)_\infty^4\}^2 + 3(q^2;q^4)_\infty^4\right\}.
\end{aligned} \tag{3.5.9}$$

Employing (3.3.11) and (2.2.3) in (3.5.9), we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{c\phi}_4(2n)q^n + \sum_{n=0}^{\infty}\overline{c\phi}_4(2n)(-1)^nq^n \\
&= 512q^2\frac{(q^4;q^4)_\infty^4(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^9(q^2;q^4)_\infty^{12}}\left\{64q^2\frac{(q^8;q^8)_\infty^8}{(q^2;q^2)_\infty^4(q^4;q^4)_\infty^4} + 3(q^2;q^4)_\infty^4\right\}.
\end{aligned} \tag{3.5.10}$$

Equating the terms involving q^{2n} on both sides of (3.5.10), and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty}\overline{c\phi}_4(4n)q^n = 256q\frac{(q^2;q^2)_\infty^4(q^4;q^4)_\infty^4}{(q;q)_\infty^9(q;q^2)_\infty^{12}}\left\{64q\frac{(q^4;q^4)_\infty^8}{(q;q)_\infty^4(q^2;q^2)_\infty^4} + 3(q;q^2)_\infty^4\right\},$$

from which we easily arrive at (3.5.3).

Finally, replacing q by $-q$ in (3.5.4) and then subtracting the resulting identity from (3.5.4), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{c\phi}_4(n)q^n-\sum_{n=0}^{\infty}\overline{c\phi}_4(n)(-1)^nq^n\\
&=16q\frac{\psi^2(q^2)\psi(q^4)}{(q^2;q^2)_\infty^4}\left\{\frac{1}{(q;q^2)_\infty^4}+\frac{1}{(-q;q^2)_\infty^4}\right\}\\
&=16q\frac{\psi^2(q^2)\psi(q^4)}{(q^2;q^2)_\infty^4(q^2;q^4)_\infty^4}\left\{(q;q^2)_\infty^4+(-q;q^2)_\infty^4\right\}. \qquad (3.5.11)
\end{aligned}$$

Using (3.3.12), (2.2.2) and (2.2.3), we deduce that

$$\sum_{n=0}^{\infty}\overline{c\phi}_4(n)q^n-\sum_{n=0}^{\infty}\overline{c\phi}_4(n)(-1)^nq^n=32q\frac{(q^4;q^4)_\infty}{(q^8;q^8)_\infty^2(q^2;q^4)_\infty^{16}}. \qquad (3.5.12)$$

Extracting from both sides of (3.5.12) those terms involving only q^{2n+1}, and then dividing both sides by q and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty}\overline{c\phi}_4(2n+1)q^n=16\frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty^2(q;q^2)_\infty^{16}}. \qquad (3.5.13)$$

Replacing q by $-q$ in (3.5.13) and then subtracting the resulting identity from (3.5.13), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{c\phi}_4(2n+1)q^n-\sum_{n=0}^{\infty}\overline{c\phi}_4(2n+1)(-1)^nq^n\\
&=16\frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty^2}\left\{\frac{1}{(q;q^2)_\infty^{16}}-\frac{1}{(-q;q^2)_\infty^{16}}\right\}\\
&=16\frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty^2(q^2;q^4)_\infty^{16}}\left\{(-q;q^2)_\infty^{16}-(q;q^2)_\infty^{16}\right\}\\
&=16\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{15}}\left\{(-q;q^2)_\infty^{8}-(q;q^2)_\infty^{8}\right\}\left\{\left\{(-q;q^2)_\infty^4+(q;q^2)_\infty^4\right\}^2-2(q^2;q^4)_\infty^4\right\}.
\end{aligned}$$

(3.5.14)

Employing (3.3.12) and (3.3.13) in (3.5.14), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{c\phi}_4(2n+1)q^n-\sum_{n=0}^{\infty}\overline{c\phi}_4(2n+1)(-1)^nq^n\\
&=256q\frac{(q^4;q^4)_\infty^{22}}{(q^2;q^2)_\infty^{27}}\left\{4\varphi^4(q^2)-2(q^2;q^2)_\infty^4(q^2;q^4)_\infty^4\right\}. \qquad (3.5.15)
\end{aligned}$$

Equating the coefficients of q^{2n+1} on both sides of (3.5.15), we readily arrive at (3.5.2) to complete the proof. □

Chapter 4

General Partition Function and Tau Function

4.1 Introduction

For a nonzero integer r and a nonnegative integer n, the general partition function $p_r(n)$ is defined as the coefficient of q^n in the expansion of $(q;q)_\infty^r$. Thus,

$$\sum_{n=0}^{\infty} p_r(n)q^n = (q;q)_\infty^r.$$

In particular, $p_{-1}(n)$ is the usual partition function $p(n)$, i.e., the number of unrestricted partitions of n, for which Ramanujan [51]–[53] found nice congruence properties modulo powers of 5, 7, and 11, as discussed in Chapter 1. Ramanathan [49] considered the generalization of these congruences modulo powers of 5 and 7 for all r, but these results were found to be incorrect by Atkin [2]. Newman [44]–[46] studied the function $p_r(n)$ and obtained several interesting congruences and identities involving $p_r(n)$. Using the theory of elliptic modular functions, Newman established the following theorem [44].

Theorem 4.1.1. *Suppose that r is one of the numbers* 2, 4, 6, 8, 10, 14, 26. *Let l be a prime greater than* 3 *such that $r(l+1) \equiv 0 (mod \quad 24)$. Let $\triangle = r(l^2-1)/24$ and define $p_r(\alpha)$ as zero if α is not a nonnegative integer. Then*

$$p_r(nl+\triangle) = (-l)^{\frac{r}{2}-1} p_r\left(\frac{n}{l}\right). \tag{4.1.1}$$

Atkin [2], Gandhi [24], Gordon [26], Boylan [16], Kiming and Olsson [34] also studied various congruence properties of $p_r(n)$ for certain negative values of r. Recent works of Baruah and Ojah [5] and Berndt, Gugg and Kim [12] have been discussed in the introductory chapter. Farkas and Kra [19]–[22] obtained several congruences and identities involving $p_r(n)$. In particular, Farkas and Kra [21, Theorem 1] obtained five three term recursion identities and remarked whether the list of three term recursions for partition coefficients is complete or not. In Section 4.3, we provide alternative proofs of these recursion relations and present four more such three term recursions for partition coefficients. In the process, we obtain new two term recursion relations for partition coefficients. We also give alternative proofs of several two term recursion relations that appeared in [19]. In the process, we also deduce Ramanujan's partition congruences (1.0.2) and (1.0.3) and several identities for Ramanujan's tau function $\tau(n)$.

Our approach involves 2-, 4-, 5-, and 7- dissections of $(q;q)_\infty^r$ for some particular values of r, where a t-dissection of a power series $P(q)$ in q is a representation of the type

$$P(q) = \sum_{k=0}^{t-1} q^k P_k(q^t).$$

We also require elementary properties of the Rogers-Ramanujan continued fraction $R(q)$ and Ramanujan's cubic continued fraction $G(q)$ defined by

$$R(q) := \frac{q^{1/5}}{1\,+}\,\frac{q}{1\,+}\,\frac{q^2}{1\,+}\,\cdots, \qquad |q| < 1$$

and

$$G(q) := \frac{q^{1/3}}{1\,+}\,\frac{q+q^2}{1\,+}\,\frac{q^2+q^4}{1\,+}\,\cdots, \qquad |q| < 1.$$

Two interesting identities satisfied by $F(q) := q^{-1/5}R(q)$ are

$$F^{-1}(q^5) - q - q^2F(q^5) = \frac{(q;q)_\infty}{(q^{25};q^{25})_\infty}, \tag{4.1.2}$$

$$F^{-5}(q) - 11q - q^2F^5(q) = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}. \tag{4.1.3}$$

These two identities were found by Watson [58] in Ramanujan's notebooks. Using (4.1.2) and (4.1.3), Hirschhorn [29] proved that

$$\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^6}{(q^5;q^5)_\infty^6}(F^4(q^5) + qF^3(q^5) + 2q^2F^2(q^5) + 3q^3F(q^5)$$
$$+ 5q^4 - 3q^5F^{-1}(q^5) + 2q^6F^{-2}(q^5) - q^7F^{-3}(q^5) + q^8F^{-4}(q^5)). \qquad (4.1.4)$$

This identity is also in Ramanujan's unpublished menuscripts on $p(n)$ and $\tau(n)$ [54, 13]. Clearly, (4.1.2) and (4.1.4) give 5-dissections of $(q;q)_\infty$ and $\frac{1}{(q;q)_\infty}$, respectively.

Also, if $W(q) = q^{-1/3}G(q)$ then from [10, p. 345, Entry 1(iv)], we find that

$$(q;q)_\infty^3 = (q^9;q^9)_\infty^3 \left(4q^3W^2(q^3) - 3q + W^{-1}(q^3)\right), \qquad (4.1.5)$$

$$(q;q)_\infty^{12} = (q^3;q^3)_\infty^{12} \left(\left(4qW^2(q) + W^{-1}(q)\right)^3 - 27q\right). \qquad (4.1.6)$$

Replacing q by $-q$ in (2.2.2), we find that

$$\varphi(-q) = (q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}. \qquad (4.1.7)$$

By [10, p. 40, Entry 25], we have

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \qquad (4.1.8)$$

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4), \qquad (4.1.9)$$

$$16q\psi^4(q^2) = \varphi^4(q) - \varphi^4(-q), \qquad (4.1.10)$$

$$\psi^2(q) = \varphi(q)\psi(q^2). \qquad (4.1.11)$$

Next, by [10, p. 303, Entry 17(v)], we have

$$(q;q)_\infty = (q^{49};q^{49})_\infty \left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \qquad (4.1.12)$$

where $A(q)$, $B(q)$ and $C(q)$ are defined by

$$A(q) := \frac{f(-q^3,-q^4)}{f(-q^2)}, \quad B(q) := \frac{f(-q^2,-q^5)}{f(-q^2)}, \quad \text{and } C(q) := \frac{f(-q,-q^6)}{f(-q^2)}.$$

Clearly, (4.1.12) gives a 7-dissection of $(q;q)_\infty$.

Similarly, setting

$$I(q) := f(-q,-q^{12}), \quad J(q) := f(-q^2,-q^{11}), \quad K(q) := f(-q^3,-q^{10}),$$

$$L(q) := f(-q^4,-q^9), \quad M(q) := f(-q^5,-q^8), \quad \text{and } N(q) := f(-q^6,-q^7)$$

in [10, p. 373, Entry 8(i), Eq. (8.1)], we find that

$$(q;q)_\infty = (q^{169};q^{169})_\infty \left(\frac{L(q^{13})}{J(q^{13})} - q\frac{N(q^{13})}{K(q^{13})} - q^2\frac{J(q^{13})}{I(q^{13})} + q^5\frac{M(q^{13})}{L(q^{13})} \right.$$
$$\left. + q^7 - q^{12}\frac{K(q^{13})}{M(q^{13})} + q^{22}\frac{I(q^{13})}{N(q^{13})} \right), \tag{4.1.13}$$

which is a 13-dissection of $(q;q)_\infty$.

We will also need the following algebraic identities in our subsequent sections:

$$X^5 + Y^5 = (X+Y)((X+Y)^4 - 5XY(X+Y)^2 + 5X^2Y^2), \tag{4.1.14}$$

$$X^3 + Y^3 = (X+Y)((X+Y)^2 - 3XY), \tag{4.1.15}$$

$$X^7 + Y^7 = (X+Y)((X+Y)^6 - 7XY(X+Y)^4 + 14X^2Y^2(X+Y)^2 - 7X^3Y^3), \tag{4.1.16}$$

$$X^4 - Y^4 = (X-Y)(X+Y)((X+Y)^2 - 2XY). \tag{4.1.17}$$

The rest of the chapter is organized in the following way. In Section 4.2, we give alternative proofs of thirteen two-term identities that appeared in [19] and also find a simple new identity (4.2.10). Furthermore, we also deduce simple proofs of (1.0.2) and (1.0.3). In Section 4.3, we prove nine three-term identities of which (4.3.2), (4.3.3), (4.3.8), and (4.3.9) are new identities. The other five identities appeared in [19, Theorem 1]. In the process, we also deduce a known identity for Ramanujan's tau function $\tau(n)$. In Section 4.4, we find a general three-term identity involving $p_{24}(n)$, which eventually yields a general identity involving Ramanujan's tau function $\tau(n)$. Finally, in Section 4.5, we prove a few congruences for $p_r(n)$ for some negative values of r.

4.2 Two term recursion relations

In this section, we prove some two term recursion relations by using the dissections and theta function identities given in the previous section. We also deduce Ramanujan's partition congruences (1.0.2) and (1.0.3).

Theorem 4.2.1. For a positive integer k, we have

$$p_3\left(3^{2k}n + \frac{3^{2k}-1}{8}\right) = (-3)^k p_3(n), \tag{4.2.1}$$

$$p_6\left(3^{2k}n + \frac{3^{2k}-1}{4}\right) = 3^{2k} p_6(n), \tag{4.2.2}$$

$$p_1\left(5^{2k}n + \frac{5^{2k}-1}{24}\right) = (-1)^k p_1(n), \tag{4.2.3}$$

$$p_2\left(5^{2k}n + \frac{5^{2k}-1}{12}\right) = (-1)^k p_2(n), \tag{4.2.4}$$

$$p_3\left(5^{2k}n + \frac{5^{2k}-1}{8}\right) = 5^k p_3(n), \tag{4.2.5}$$

$$p_4\left(5^{2k}n + \frac{5^{2k}-1}{6}\right) = (-5)^k p_4(n), \tag{4.2.6}$$

$$p_8\left(5^{2k}n + \frac{5^{2k}-1}{3}\right) = (-5)^{3k} p_8(n), \tag{4.2.7}$$

$$p_{14}\left(5^{2k}n + \frac{7(5^{2k}-1)}{12}\right) = (-1)^k 5^{6k} p_{14}(n), \tag{4.2.8}$$

$$p_8\left(2^{2k}n + \frac{2^{2k}-1}{3}\right) = (-8)^k p_8(n), \tag{4.2.9}$$

$$p_8\left(4n+3\right) = 0, \tag{4.2.10}$$

$$p_1\left(7^{2k}n + \frac{7^{2k}-1}{24}\right) = (-1)^k p_1(n), \tag{4.2.11}$$

$$p_2\left(7^{2k}n + \frac{7^{2k}-1}{12}\right) = p_2(n), \tag{4.2.12}$$

$$p_3\left(7^{2k}n + \frac{7^{2k}-1}{8}\right) = (-7)^k p_3(n), \tag{4.2.13}$$

$$p_6\left(7^{2k}n + \frac{7^{2k}-1}{4}\right) = 7^{2k} p_6(n). \tag{4.2.14}$$

We note that, the identities (4.2.6), (4.2.7), and (4.2.9) are the special cases of (4.1.1) with (r, q) being chosen as $(4, 5)$, $(8, 5)$, and $(8, 2)$, respectively. The identities (4.2.1)–(4.2.9), (4.2.11)–(4.2.14) are proved in [19]. Our method of proofs are different from the previous authors. The identity (4.2.10) appears to be new.

Proof. From (4.1.5), we have

$$\sum_{n=0}^{\infty} p_3(n)q^n = (q;q)_\infty^3$$

$$= (q^9;q^9)_\infty^3 \left(4q^3W^2(q^3) - 3q + W^{-1}(q^3)\right). \tag{4.2.15}$$

Extracting the terms in (4.2.15) that involve q^{3n+1}, then dividing by q and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} p_3(3n+1)q^n = -3(q^3;q^3)_\infty^3 = -3\sum_{n=0}^{\infty} p_3(n)q^{3n}. \tag{4.2.16}$$

Equating the coefficients of q^{3n} on both sides of (4.2.16), we deduce that

$$p_3(3^2n+1) = -3p_3(n),$$

which can be iterated to arrive at (4.2.1).

From (4.1.5), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_6(n)q^n \\
&= (q;q)_\infty^6 \\
&= (q^9;q^9)_\infty^6 \left(4q^3W^2(q^3) - 3q + W^{-1}(q^3)\right)^2 \\
&= (q^9;q^9)_\infty^6 \Big(9q^2 + W^{-2}(q^3) - 6qW^{-1}(q^3) + 8q^3W(q^3) \\
&\quad - 24q^4W^2(q^3) + 16q^6W^4(q^3)\Big).
\end{aligned} \tag{4.2.17}$$

Collecting the terms in (4.2.17) that involve q^{3n+2}, then dividing by q^2 and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} p_6(3n+2)q^n = 3^2(q^3;q^3)_\infty^6 = 3^2\sum_{n=0}^{\infty} p_6(n)q^{3n}. \tag{4.2.18}$$

Comparing the coefficients of q^{3n} on both sides of (4.2.18), we deduce that

$$p_6(3^2n+2) = 3^2p_6(n),$$

which can be iterated to arrive at (4.2.2).

The proofs of (4.2.3)–(4.2.8) are similar. Here we prove only (4.2.6) in details and give a brief outline of the proofs of (4.2.3)–(4.2.5), (4.2.7), and (4.2.8).

By (4.1.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_4(n)q^n &= (q;q)_\infty^4 \\ &= (q^{25};q^{25})_\infty^4 (F^{-1}(q^5) - q - q^2F(q^5))^4 \\ &= (q^{25};q^{25})_\infty^4 (F^{-4}(q^5) - 4qF^{-3}(q^5) + 2q^2F^{-2}(q^5) + 8q^3F^{-1}(q^5) \\ &\quad - 5q^4 - 8q^5F(q^5) + 2q^6F^2(q^5) + 4q^7F^3(q^5) + q^8F^4(q^5)). \end{aligned} \tag{4.2.19}$$

Extracting the terms involving q^{5n+4} in (4.2.19), then dividing the resulting identity by q^4 and replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} p_4(5n+4)q^n = -5(q^5;q^5)_\infty^4 = -5\sum_{n=0}^{\infty} p_4(n)q^{5n}. \tag{4.2.20}$$

Equating coefficients of different powers of q modulo 5 in (4.2.20), we obtain

$$p_4(5^2n+4) = -5p_4(n) \tag{4.2.21}$$

and

$$p_4(5^2n+9) = p_4(5^2n+14) = p_4(5^2n+19) = p_4(5^2n+24) = 0. \tag{4.2.22}$$

Iterating (4.2.21), we easily arrive at (4.2.6) by mathematical induction.

Now, from (4.1.2), we find that

$$\sum_{n=0}^{\infty} p_1(n)q^n = (q;q)_\infty = (q^{25};q^{25})_\infty (F^{-1}(q^5) - q - q^2F(q^5)), \tag{4.2.23}$$

$$\begin{aligned}\sum_{n=0}^{\infty} p_2(n)q^n = (q;q)_\infty^2 = (q^{25};q^{25})_\infty^2 (F^{-2}(q^5) - 2qF^{-1}(q^5) - q^2 \\ + 2q^3F(q^5) + q^4F^2(q^5)),\end{aligned} \tag{4.2.24}$$

$$\begin{aligned}\sum_{n=0}^{\infty} p_3(n)q^n = (q;q)_\infty^3 = (q^{25};q^{25})_\infty^3 (F^{-3}(q^5) - 3qF^{-2}(q^5) + 5q^3 \\ - 3q^5F^2(q^5) - q^6F^3(q^5)),\end{aligned} \tag{4.2.25}$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} p_8(n)q^n \\
&= (q;q)_\infty^8 \\
&= (q^{25};q^{25})_\infty^8\big(F^{-8}(q^5) - 8qF^{-7}(q^5) + 20q^2F^{-6}(q^5) - 70q^4F^{-4}(q^5) + 56q^5F^{-3}(q^5) \\
&\quad + 112q^6F^{-2}(q^5) - 120q^7F^{-1}(q^5) - 125q^8 + 120q^9F(q^5) + 112q^{10}F^2(q^5) \\
&\quad - 56q^{11}F^3(q^5) - 70q^{12}F^4(q^5) + 20q^{14}F^6(q^5) + 8q^{15}F^7(q^5) + q^{16}F^8(q^5)\big), \quad (4.2.26)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} p_{14}(n)q^n \\
&= (q;q)_\infty^{14} \\
&= (q^{25};q^{25})_\infty^{14}(F^{-14}(q^5) - 14qF^{-13}(q^5) + 77q^2F^{-12}(q^5) - 182q^3F^{-11}(q^5) \\
&\quad + 910q^5F^{-9}(q^5) - 1365q^6F^{-8}(q^5) - 1430q^7F^{-7}(q^5) + 5005q^8F^{-6}(q^5) \\
&\quad - 10010q^{10}F^{-4}(q^5) + 3640q^{11}F^{-3}(q^5) + 14105q^{12}F^{-2}(q^5) - 6930q^{13}F^{-1}(q^5) \\
&\quad - 15625q^{14} + 6930q^{15}F(q^5) + 14105q^{16}F^2(q^5) - 3640q^{17}F^3(q^5) - 10010q^{18}F^4(q^5) \\
&\quad + 5005q^{20}F^6(q^5) + 1430q^{21}F^7(q^5) - 1365q^{22}F^8(q^5) - 910q^{23}F^9(q^5) \\
&\quad + 182q^{25}F^{11}(q^5) + 77q^{26}F^{12}(q^5) + 14q^{27}F^{13}(q^5) + q^{28}F^{14}(q^5)). \quad (4.2.27)
\end{aligned}
$$

Extracting the terms involving q^{5n+1}, q^{5n+2}, q^{5n+3}, q^{5n+3}, and q^{5n+4} in (4.2.23), (4.2.24), (4.2.25), (4.2.26), and (4.2.27) respectively and proceeding as in the proof of (4.2.6) we obtain (4.2.3), (4.2.4), (4.2.5), (4.2.7), and (4.2.8).

Next to prove (4.2.9) and (4.2.10), we employ (4.1.7) and (4.1.8) to obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_8(n)q^n &= (q;q)_\infty^8 \\
&= (q^4;q^4)_\infty^2\varphi^2(-q^2)\varphi^4(-q) \\
&= (q^4;q^4)_\infty^2(\varphi(q^8) - 2q^2\psi(q^{16}))^2(\varphi(q^4) - 2q\psi(q^8))^4. \quad (4.2.28)
\end{aligned}
$$

Collecting the terms involving q^{4n+1} in (4.2.28), then dividing the resulting identity by q and replacing q^4 by q, we find that

$$\sum_{n=0}^{\infty} p_8(4n+1)q^n$$
$$= -8(q;q)_\infty^2(\varphi^2(q^2)\varphi^3(q)\psi(q^2) + 4q\psi^2(q^4)\varphi^3(q)\psi(q^2) - 16q\varphi(q^2)\psi(q^4)\varphi(q)\psi^3(q^2))$$
$$= -8(q;q)_\infty^2(\varphi^3(q)\psi(q^2)(\varphi^2(q^2) + 4q\psi^2(q^4)) - 16q\varphi(q^2)\psi(q^4)\varphi(q)\psi^3(q^2)). \quad (4.2.29)$$

With the aid of (4.1.9) and (4.1.11), we rewrite (4.2.29) as

$$\sum_{n=0}^{\infty} p_8(4n+1)q^n = -8(q;q)_\infty^2\psi^2(q)(\varphi^4(q) - 16q\psi^4(q^2)). \quad (4.2.30)$$

Using (4.1.10), (4.1.7), and (2.2.3) in (4.2.30), we obtain

$$\sum_{n=0}^{\infty} p_8(4n+1)q^n = -8(q;q)_\infty^8 = -8\sum_{n=0}^{\infty} p_8(n)q^n. \quad (4.2.31)$$

Iterating (4.2.31), we arrive at (4.2.9) by mathematical induction.

Again, extracting the terms in (4.2.28) in which the powers of q are congruent to 3 modulo 4, we find that

$$\sum_{n=0}^{\infty} p_8(4n+3)q^n$$
$$= 32(q;q)_\infty^2(\varphi(q^2)\psi(q^4)\varphi^3(q)\psi(q^2) - \varphi^2(q^2)\varphi(q)\psi^3(q^2) - 4q\psi^2(q^4)\varphi(q)\psi^3(q^2))$$
$$= 32(q;q)_\infty^2(\varphi(q^2)\psi(q^4)\varphi^3(q)\psi(q^2) - \varphi(q)\psi^3(q^2)(\varphi^2(q^2) + 4q\psi^2(q^4))). \quad (4.2.32)$$

Invoking (4.1.9) and (4.1.11), we obtain

$$\varphi(q)\psi^3(q^2)(\varphi^2(q^2) + 4q\psi^2(q^4)) = \varphi^3(q)\psi^3(q^2) = \varphi^3(q)\varphi(q^2)\psi(q^4)\psi(q^2). \quad (4.2.33)$$

Using (4.2.33) in (4.2.32), we conclude that

$$\sum_{n=0}^{\infty} p_8(4n+3)q^n = 0,$$

which immediately yields (4.2.10).

The proofs of (4.2.11)–(4.2.14) are similar. First we prove (4.2.14). Employing (4.1.12), we have

$$\sum_{n=0}^{\infty} p_6(n)q^n = (q;q)_\infty^6 = (q^{49};q^{49})_\infty^6 \left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right)^6. \quad (4.2.34)$$

Expanding the left hand side of (4.2.34) and extracting the terms in which the powers of q are congruent to 5 modulo 7, we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} p_6(7n+5)q^n \\ &= (q^7;q^7)_\infty^6 \Bigg(211q + 6\left(\frac{B^5(q)}{A(q)C^4(q)} - \frac{A^5(q)}{C(q)B^4(q)} - q^3\frac{C^5(q)}{B(q)A^4(q)}\right) \\ &\quad - 60\left(\frac{A(q)B^2(q)}{C^3(q)} + q\frac{C(q)A^2(q)}{B^3(q)} - q^2\frac{B(q)C^2(q)}{A^3(q)}\right) \\ &\quad + 60\left(\frac{A^3(q)}{B(q)C^2(q)} - q\frac{B^3(q)}{C(q)A^2(q)} - q^2\frac{C^3(q)}{A(q)B^2(q)}\right)\Bigg). \quad (4.2.35)\end{aligned}$$

Now, writing [11, p. 174, Entry 31] and [14, Eq. 3.15] in terms of $A(q)$, $B(q)$, and $C(q)$, we obtain

$$\frac{A(q)B^2(q)}{C^3(q)} + q\frac{C(q)A^2(q)}{B^3(q)} - q^2\frac{B(q)C^2(q)}{A^3(q)} = 8q + \frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4}, \quad (4.2.36)$$

$$\frac{A^3(q)}{B(q)C^2(q)} - q\frac{B^3(q)}{C(q)A^2(q)} - q^2\frac{C^3(q)}{A(q)B^2(q)} = 5q + \frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4}, \quad (4.2.37)$$

$$\frac{B^5(q)}{A(q)C^4(q)} - \frac{A^5(q)}{C(q)B^4(q)} - q^3\frac{C^5(q)}{B(q)A^4(q)} = 3q. \quad (4.2.38)$$

Using (4.2.36), (4.2.37), and (4.2.38) in (4.2.35) and simplifying, we deduce that

$$\sum_{n=0}^{\infty} p_6(7n+5)q^n = 49q(q^7;q^7)_\infty^6 = 49\sum_{n=0}^{\infty} p_6(n)q^{7n+1}. \quad (4.2.39)$$

Equating coefficients of different powers of q modulo 7 in (4.2.39), we obtain

$$p_6(7^2n+12) = 7^2 p_6(n) \quad (4.2.40)$$

and

$$p_6(7^2n+5) = p_6(7^2n+19) = p_6(7^2n+26) = p_6(7^2n+33) = p_6(7^2n+40)$$
$$= p_6(7^2n+47) = 0.$$

Iterating (4.2.40), we arrive at (4.2.14) by mathematical induction.

Also by (4.1.12), we have

$$\sum_{n=0}^{\infty} p_1(n)q^n = (q;q)_\infty = (q^{49};q^{49})_\infty \left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{4.2.41}$$

$$\begin{aligned}\sum_{n=0}^{\infty} p_2(n)q^n &= (q;q)_\infty^2 \\ &= (q^{49};q^{49})_\infty^2 \left(\frac{B^2(q^7)}{C^2(q^7)} - 2q\frac{A(q^7)}{C(q^7)} + q^2\frac{A^2(q^7)}{B^2(q^7)} - 2q^2\frac{B(q^7)}{C(q^7)} + 2q^3\frac{A(q^7)}{B(q^7)}\right.\\ &\quad \left. + q^4 + 2q^5\frac{B(q^7)}{A(q^7)} - 2q^6\frac{C(q^7)}{B(q^7)} - 2q^7\frac{C(q^7)}{A(q^7)} + q^{10}\frac{C^2(q^7)}{A^2(q^7)}\right),\end{aligned} \tag{4.2.42}$$

$$\begin{aligned}\sum_{n=0}^{\infty} p_3(n)q^n &= (q;q)_\infty^3 \\ &= (q^{49};q^{49})_\infty^3 \left(\frac{B^3(q^7)}{C^3(q^7)} - 3q\frac{A(q^7)B(q^7)}{C^2(q^7)} - 3q^2\frac{B^2(q^7)}{C^2(q^7)} + 3q^2\frac{A^2(q^7)}{B(q^7)C(q^7)}\right.\\ &\quad - q^3\frac{A^3(q^7)}{B^3(q^7)} + 6q^3\frac{A(q^7)}{C(q^7)} - 3q^4\frac{A^2(q^7)}{B^2(q^7)} + 3q^4\frac{B(q^7)}{C(q^7)} - 3q^5\frac{A(q^7)}{B(q^7)} \\ &\quad + 3q^5\frac{B^2(q^7)}{A(q^7)C(q^7)} - 7q^6 - 6q^7\frac{B(q^7)}{A(q^7)} + 3q^7\frac{A(q^7)C(q^7)}{B^2(q^7)} + 6q^8\frac{C(q^7)}{B(q^7)} \\ &\quad + 3q^9\frac{C(q^7)}{A(q^7)} + 3q^{10}\frac{B(q^7)C(q^7)}{A^2(q^7)} - 3q^{11}\frac{C^2(q^7)}{A(q^7)B(q^7)} - 3q^{12}\frac{C^2(q^7)}{A^2(q^7)} \\ &\quad \left. + q^{15}\frac{C^3(q^7)}{A^3(q^7)}\right).\end{aligned} \tag{4.2.43}$$

Extracting the terms involving q^{7n+2}, q^{7n+4}, and q^{7n+6} in (4.2.41), (4.2.42), and (4.2.43) respectively and proceeding as in the proof of (4.2.6), we arrive at (4.2.11), (4.2.12), and (4.2.13). □

Corollary 4.2.2. *Ramanujan's congruences*

$$p(5n+4) \equiv 0 \pmod 5, \tag{4.2.44}$$

$$p(7n+5) \equiv 0 \pmod 7 \tag{4.2.45}$$

hold.

Proof. By the binomial theorem, we have

$$(q;q)_\infty^5 \equiv (q^5;q^5)_\infty \pmod 5 \text{ and } (q;q)_\infty^7 \equiv (q^7;q^7)_\infty \pmod 7.$$

Therefore,

$$\begin{aligned}(q;q)_\infty^4 = \frac{(q;q)_\infty^5}{(q;q)_\infty} &\equiv \frac{(q^5;q^5)_\infty}{(q;q)_\infty} \pmod{5} \\ &\equiv (q^5;q^5)_\infty \sum_{n=0}^{\infty} p(n)q^n \pmod{5}. \end{aligned} \tag{4.2.46}$$

Now, from (4.2.19), we observe that the coefficients of q^{5n+4} in $(q;q)_\infty^4$ are divisible by 5. Thus, comparing the terms involving q^{5n+4} in (4.2.46), we readily arrive at (4.2.44).

Again, we have

$$(q;q)_\infty^6 = \frac{(q;q)_\infty^7}{(q;q)_\infty} \equiv \frac{(q^7;q^7)_\infty}{(q;q)_\infty} \pmod{7}. \tag{4.2.47}$$

Employing (4.1.12) in (4.2.47), we find that

$$(q^{49};q^{49})_\infty^6 \left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right)^6 \equiv (q^7;q^7)_\infty \sum_{n=0}^{\infty} p(n)q^n \pmod{7}. \tag{4.2.48}$$

While proving (4.2.14) above, we have already noticed that the terms involving q^{7n+5} in the expansion of the left hand side of (4.2.48) are multiples of 7. Thus, Ramanujan's congruence (4.2.45) readily follows from (4.2.48). □

Remark 4.2.1. *The following identities which are obvious from the pentagonal number theorem, can also be derived while proving* (4.2.3) *and* (4.2.11).

$$p_1(5^2n+6) = p_1(5^2n+11) = p_1(5^2n+16) = p_1(5^2n+21) = 0, \tag{4.2.49}$$

$$\begin{aligned}p_1(7^2n+9) &= p_1(7^2n+16) = p_1(7^2n+23) = p_1(7^2n+30) \\ &= p_1(7^2n+37) = p_1(7^2n+44) = 0. \end{aligned} \tag{4.2.50}$$

Similarly, the identities analogous to (4.2.22) *that can be derived while proving* (4.2.5) *and* (4.2.13) *are*

$$p_3(5^2n+8) = p_3(5^2n+13) = p_3(5^2n+18) = p_3(5^2n+23) = 0, \tag{4.2.51}$$

$$\begin{aligned} p_3(7^2n+13) &= p_3(7^2n+20) = p_3(7^2n+27) = p_3(7^2n+34) \\ &= p_3(7^2n+41) = p_3(7^2n+48) = 0. \end{aligned} \tag{4.2.52}$$

Now, if we recall [10, p. 39, Entry 24(ii)], Jacobi's famous identity

$$(q;q)_\infty^3 = \sum_{\ell=0}^{\infty}(-1)^\ell(2\ell+1)q^{\ell(\ell+1)/2}, \tag{4.2.53}$$

then from (4.2.51)-(4.2.53)*, we conclude that no numbers of the form* 5^2n+j *with* $j = 8,\ 13,\ 18,\ 23$ *or* 7^2n+k *with* $k = 13,\ 20,\ 27,\ 34,\ 41,\ 48$ *are triangular numbers.*

Furthermore, from (4.2.53) *and the definition of* $p_3(n)$*, we have* $p_3\left(\dfrac{m(m+1)}{2}\right) = (-1)^m(2m+1)$ *for each positive integer* m*. Therefore, we can easily deduce from* (4.2.5) *that*

$$p_3\left(\frac{5^{2k}(2m+1)^2-1}{8}\right) = 5^k(-1)^m(2m+1).$$

4.3 Three term recursion relations

In this section, we prove nine three term identities of which (4.3.2), (4.3.3), (4.3.8), and (4.3.9) are new. The other five identities were proved by Farkas and Kra [19, Theorem 1]. The process also yields several two term identities which we record in Remark 4.3.1 and Remark 4.3.2. We also deduce several results on Ramanujan's tau function $\tau(n)$ defined in (1.0.6).

Theorem 4.3.1. We have

$$p_6(5^2n+6) = -6p_6(5n+1) - 5^2p_6(n), \tag{4.3.1}$$

$$p_{12}(5^2n+12) = 54p_{12}(5n+2) - 5^5p_{12}(n), \tag{4.3.2}$$

$$p_{18}(5^2n+18) = -510p_{18}(5n+3) - 5^8p_{18}(n), \tag{4.3.3}$$

$$p_{24}(5^2n+24) = 4830p_{24}(5n+4) - 5^{11}p_{24}(n), \tag{4.3.4}$$

$$p_4(7^2n+8) = -4p_4(7n+1) - 7p_4(n), \tag{4.3.5}$$

$$p_2(13^2n+14) = -2p_2(13n+1) - p_2(n), \tag{4.3.6}$$

$$p_{12}(3^2n+4) = -12p_{12}(3n+1) - 3^5p_{12}(n), \tag{4.3.7}$$

$$p_9(3^4n+30) = -12p_9(3^2n+3) - 3^7p_9(n), \tag{4.3.8}$$

$$p_{15}(3^4n+50) = 1836p_{15}(3^2n+5) - 3^{13}p_{15}(n). \tag{4.3.9}$$

Proof. The proofs of (4.3.1)–(4.3.4) are similar. Therefore, we prove (4.3.4) in details and give a brief outline of the proofs of (4.3.1)–(4.3.3).

From (4.1.2), we have

$$\sum_{n=0}^{\infty} p_{24}(n)q^n = (q;q)_\infty^{24} = (q^{25};q^{25})_\infty^{24}(F^{-1}(q^5) - q - q^2F(q^5))^{24}. \tag{4.3.10}$$

Extracting the terms involving q^{5n+4} in (4.3.10), then dividing the resulting identity by q^4 and replacing q^5 by q, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{24}(5n+4)q^n \\
&= (q^5;q^5)_\infty^{24}(4830F^{-20}(q) - 212520qF^{-15}(q) + 3487260q^2F^{-10}(q) \\
&\quad - 25077360q^3F^{-5}(q) + 14903725q^4 + 25077360q^5F^5(q) + 3487260q^6F^{10}(q) \\
&\quad + 212520q^7F^{15}(q) + 4830q^8F^{20}(q)) \\
&= (q^5;q^5)_\infty^{24}(4830(F^{-20}(q) + q^8F^{20}(q)) - 212520q(F^{-15}(q) - q^6F^{15}(q)) \\
&\quad + 3487260q^2(F^{-10}(q) + q^4F^{10}(q)) - 25077360q^3\big(F^{-5}(q) - q^2F^5(q)\big) \\
&\quad + 14903725q^4\big)
\end{aligned}$$

$$= (q^5;q^5)_\infty^{24}\Big(4830\big(((F^{-5}(q) - q^2F^5(q))^2 + 2q^2)^2 - 2q^4\big)$$
$$- 212520q\big((F^{-5}(q) - q^2F^5(q))^3 + 3q^2(F^{-5}(q) - q^2F^5(q))\big)$$
$$+ 3487260q^2\big((F^{-5}(q) - q^2F^5(q))^2 + 2q^2\big) - 25077360q^3(F^{-5}(q)$$
$$- q^2F^5(q)) + 14903725q^4\Big). \tag{4.3.11}$$

Employing (4.1.3) in (4.3.11), we obtain

$$\sum_{n=0}^{\infty} p_{24}(5n+4)q^n = (q^5;q^5)_\infty^{24}\left(4830\left(\left(\left(11q + \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}\right)^2 + 2q^2\right)^2 - 2q^4\right)\right.$$
$$- 212520q\left(\left(11q + \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}\right)^3 + 3q^2\left(11q + \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}\right)\right)$$
$$+ 3487260q^2\left(\left(11q + \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}\right)^2 + 2q^2\right)$$
$$\left. - 25077360q^3\left(11q + \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}\right) + 14903725q^4\right)$$
$$= 4830(q:q)_\infty^{24} - 48828125q^4(q^5;q^5)_\infty^{24}$$
$$= 4830\sum_{n=0}^{\infty} p_{24}(n)q^n - 48828125\sum_{n=0}^{\infty} p_{24}(n)q^{5n+4}. \tag{4.3.12}$$

Equating coefficients of q^{5n+4} in (4.3.12), we arrive at (4.3.4).

Now, from (4.1.2), we find that

$$\sum_{n=0}^{\infty} p_6(n)q^n = (q;q)_\infty^6$$
$$= (q^{25};q^{25})_\infty^6\Big(F^{-6}(q^5) - 6qF^{-5}(q^5) + 9q^2F^{-4}(q^5) + 10q^3F^{-3}(q^5)$$
$$- 30q^4F^{-2}(q^5) - 6q^5F^{-1}(q^5) + 41q^6 + 6q^7F(q^5) - 30q^8F^2(q^5)$$
$$- 10q^9F^3(q^5) + 9q^{10}F^4(q^5) + 6q^{11}F^5(q^5) + q^{12}F^6(q^5)\Big), \tag{4.3.13}$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} p_{12}(n)q^n \\
&= (q;q)_\infty^{12} \\
&= (q^{25};q^{25})_\infty^{12}\Big(F^{-12}(q^5) - 12qF^{-11}(q^5) + 54q^2F^{-10}(q^5) - 88q^3F^{-9}(q^5) \\
&\quad - 99q^4F^{-8}(q^5) + 528q^5F^{-7}(q^5) - 286q^6F^{-6}(q^5) - 1188q^7F^{-5}(q^5) \\
&\quad + 1386q^8F^{-4}(q^5) + 1628q^9F^{-3}(q^5) - 2706q^{10}F^{-2}(q^5) - 1728q^{11}F^{-1}(q^5) \\
&\quad + 3301q^{12} + 1728q^{13}F(q^5) - 2706q^{14}F^2(q^5) - 1628q^{15}F^3(q^5) \\
&\quad + 1386q^{16}F^4(q^5) + 1188q^{17}F^5(q^5) - 286q^{18}F^6(q^5) - 528q^{19}F^7(q^5) \\
&\quad - 99q^{20}F^8(q^5) + 88q^{21}F^9(q^5) + 54q^{22}F^{10}(q^5) + 12q^{23}F^{11}(q^5) + q^{24}F^{12}(q^5)\Big),
\end{aligned} \tag{4.3.14}
$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} p_{18}(n)q^n \\
&= (q;q)_\infty^{18} \\
&= (q^{25};q^{25})_\infty^{18}\Big(F^{-18}(q^5) - 18qF^{-17}(q^5) + 135q^2F^{-16}(q^5) - 510q^3F^{-15}(q^5) \\
&\quad + 765q^4F^{-14}(q^5) + 1224q^5F^{-13}(q^5) - 6732q^6F^{-12}(q^5) + 6120q^7F^{-11}(q^5) \\
&\quad - 38420q^9F^{-9}(q^5) - 12546q^{10}F^{-8}(q^5) + 103428q^{11}F^{-7}(q^5) \\
&\quad - 33150q^{12}F^{-6}(q^5) - 183600q^{13}F^{-5}(q^5) + 122400q^{14}F^{-4}(q^5) \\
&\quad + 248880q^{15}F^{-3}(q^5) - 214965q^{16}F^{-2}(q^5) - 282150q^{17}F^{-1}(q^5) + 254525q^{18} \\
&\quad + 282150q^{19}F(q^5) - 214965q^{20}F^2(q^5) - 248880q^{21}F^3(q^5) + 122400q^{22}F^4(q^5) \\
&\quad + 183600q^{23}F^5(q^5) - 33150q^{24}F^6(q^5) - 103428q^{25}F^7(q^5) - 12546q^{26}F^8(q^5) \\
&\quad + 38420q^{27}F^9(q^5) + 16830q^{28}F^{10}(q^5) - 6120q^{29}F^{11}(q^5) - 6732q^{30}F^{12}(q^5) \\
&\quad - 1224q^{31}F^{13}(q^5) + 765q^{32}F^{14}(q^5) + 510q^{33}F^{15}(q^5) + 135q^{34}F^{16}(q^5) \\
&\quad + 18q^{35}F^{17}(q^5) + q^{36}F^{18}(q^5)\Big).
\end{aligned} \tag{4.3.15}
$$

Extracting the terms involving q^{5n+1}, q^{5n+2}, and q^{5n+3} in (4.3.13), (4.3.14), and

(4.3.15) respectively and proceeding similarly as in the proof of (4.3.4) we arrive at (4.3.1), (4.3.2), and (4.3.3).

Next, from (4.1.12), we have

$$\sum_{n=0}^{\infty} p_4(n)q^n = (q;q)_\infty^4 = (q^{49};q^{49})_\infty^4 \left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right)^4. \quad (4.3.16)$$

Expanding the right hand side of (4.3.16) and extracting the terms involving q^{7n+1}, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_4(7n+1)q^n \\
&= (q^7;q^7)_\infty^4 \left(-4\frac{A(q)B^2(q)}{C^3(q)} + 25q - 4q\frac{C(q)A^2(q)}{B^3(q)} + 4q^2\frac{B(q)C^2(q)}{A^3(q)}\right) \\
&= (q^7;q^7)_\infty^4 \left(25q - 4\left(\frac{A(q)B^2(q)}{C^3(q)} + q\frac{C(q)A^2(q)}{B^3(q)} - q^2\frac{B(q)C^2(q)}{A^3(q)}\right)\right). \quad (4.3.17)
\end{aligned}$$

Using (4.2.36) in (4.3.17), we deduce that

$$\begin{aligned}
\sum_{n=0}^{\infty} p_4(7n+1)q^n &= -4(q;q)_\infty^4 - 7q(q^7;q^7)_\infty^4 \\
&= -4\sum_{n=0}^{\infty} p_4(n)q^n - 7\sum_{n=0}^{\infty} p_4(n)q^{7n+1}. \quad (4.3.18)
\end{aligned}$$

Equating coefficients of q^{7n+1} in (4.3.18), we arrive at (4.3.5).

Similarly using (4.1.13) and [10, p. 373, Entry 8(i), Eq. 8.2], we can prove (4.3.6).

Also by (4.1.5), we have

$$\sum_{n=0}^{\infty} p_{12}(n)q^n = (q;q)_\infty^{12} = (q^9;q^9)_\infty^{12} \left(4q^3W^2(q^3) - 3q + W^{-1}(q^3)\right)^4. \quad (4.3.19)$$

Expanding the right right hand side of (4.3.19) and collecting the terms involving q^{3n+1}, we find that

$$\sum_{n=0}^{\infty} p_{12}(3n+1)q^n = (q^3;q^3)_\infty^{12} \left(-63q - 12\left(W^{-3}(q) + 48q^2W^3(q) + 64q^3W^6(q)\right)\right).$$

(4.3.20)

Using (4.1.6) in (4.3.20), we obtain

$$\sum_{n=0}^{\infty} p_{12}(3n+1)q^n = -12(q;q)_\infty^{12} - 243q(q^3;q^3)_\infty^{12}$$
$$= -12\sum_{n=0}^{\infty} p_{12}(n)q^n - 243\sum_{n=0}^{\infty} p_{12}(n)q^{3n+1},$$

from which we easily deduce (4.3.7).

Proceeding similarly as in the proof of (4.3.7), we find that

$$\sum_{n=0}^{\infty} p_9(3n)q^n = \frac{(q;q)_\infty^{12}}{(q^3;q^3)_\infty^3}. \tag{4.3.21}$$

With the aid of (4.1.5), we rewrite (4.3.21) as

$$\sum_{n=0}^{\infty} p_9(3n)q^n = \frac{(q^9;q^9)_\infty^{12}}{(q^3;q^3)_\infty^3}\left(4q^3W^2(q^3) - 3q + W^{-1}(q^3)\right)^4. \tag{4.3.22}$$

Extracting the terms in (4.3.22) that involve q^{3n+1}, then dividing the resulting identity by q and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} p_9(3^2n+3)q^n = \frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^3}\left(-63q - 12\left(W^{-3}(q) + 48q^2W^3(q) + 64q^3W^6(q)\right)\right). \tag{4.3.23}$$

Using (4.1.6) in (4.3.23) and simplifying, we find that

$$\sum_{n=0}^{\infty} p_9(3^2n+3)q^n + 12\sum_{n=0}^{\infty} p_9(n)q^n = -3^5q\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^3}. \tag{4.3.24}$$

We recall [5, Eq. (3.9)] which states that

$$\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^3}$$
$$= (q^9;q^9)_\infty^9\Big(W^{-2}(q^3) + 3qW^{-1}(q^3) + 9q^2 + 8q^3W^3(q^3) + 12q^4W^2(q^3)$$
$$+ 16q^6W^4(q^3)\Big). \tag{4.3.25}$$

Employing (4.3.25) in (4.3.24), we find that

$$\sum_{n=0}^{\infty} p_9(3^2n+3)q^n + 12\sum_{n=0}^{\infty} p_9(n)q^n$$
$$= -3^5q(q^9;q^9)_\infty^9\Big(W^{-2}(q^3) + 3qW^{-1}(q^3) + 9q^2 + 8q^3W^3(q^3) + 12q^4W^2(q^3)$$
$$+ 16q^6W^4(q^3)\Big). \tag{4.3.26}$$

Extracting the terms involving q^{3n} in (4.3.26) and then replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} p_9(3^3n+3)q^n + 12\sum_{n=0}^{\infty} p_9(3n)q^n = -3^7 q(q^3;q^3)_\infty^9 = -3^7\sum_{n=0}^{\infty} p_9(n)q^{3n+1}. \tag{4.3.27}$$

Comparing the coefficients of q^{3n+1} in (4.3.27), we readily arrive at (4.3.8).

Following the steps as in the proof of (4.3.7), we deduce that

$$\sum_{n=0}^{\infty} p_{15}(3n+2)q^n = 3^5 q(q^3;q^3)_\infty^{15} + 90(q^3;q^3)_\infty^3 (q;q)_\infty^{12}. \tag{4.3.28}$$

With the help of (4.1.5), we write (4.3.28) as

$$\sum_{n=0}^{\infty} p_{15}(3n+2)q^n$$
$$= 3^5 q(q^3;q^3)_\infty^{15} + 90(q^3;q^3)_\infty^3 (q^9;q^9)_\infty^{12} \left(4q^3W^2(q^3) - 3q + W^{-1}(q^3)\right)^4. \tag{4.3.29}$$

Collecting the terms involving q^{3n+1} in (4.3.29), then dividing the resulting identity by q and replacing q^3 by q, we find that

$$\sum_{n=0}^{\infty} p_{15}(3^2n+5)q^n = 1107(q;q)_\infty^{15} - 21870q(q^3;q^3)_\infty^{12}(q;q)_\infty^3. \tag{4.3.30}$$

Using (4.1.6) in (4.3.30), we obtain

$$\sum_{n=0}^{\infty} p_{15}(3^2n+5)q^n$$
$$= 1107\sum_{n=0}^{\infty} p_{15}(n)q^n - 21870q(q^3;q^3)_\infty^{12}(q^9;q^9)_\infty^3 \left(4q^3W^2(q^3) - 3q + W^{-1}(q^3)\right). \tag{4.3.31}$$

Extracting the terms involving q^{3n+2} in (4.3.31), we find that

$$\sum_{n=0}^{\infty} p_{15}(3^3n+23)q^n = 1107\sum_{n=0}^{\infty} p_{15}(3n+2)q^n + 65610(q;q)_\infty^{12}(q^3;q^3)_\infty^3. \tag{4.3.32}$$

Eliminating $(q;q)_\infty^{12}(q^3;q^3)_\infty^3$ between (4.3.28) and (4.3.32), we deduce that

$$\sum_{n=0}^{\infty} p_{15}(3^3n+23)q^n = 1836\sum_{n=0}^{\infty} p_{15}(3n+2)q^n - 3^{13}\sum_{n=0}^{\infty} p_{15}(n)q^{3n+1}. \tag{4.3.33}$$

Equating the coefficients of q^{3n+1} in (4.3.33), we arrive at (4.3.9) to complete the proof.

□

Remark 4.3.1. *Equating coefficients of* q^{5n+j}, $j = 0, 1, 2, 3$ *in* (4.3.12), *we obtain two terms recursion relations*

$$p_{24}(5^2n + 5j + 4) = 4830p_{24}(5n + j). \tag{4.3.34}$$

The identities involving p_6, p_{12}, p_{18}, p_4, *and* p_2 *analogous to the above are*

$$\begin{aligned}
p_6\left(5^2n + 5j + 1\right) &= -6p_6(5n + j) \text{ with } j = 0, 2, 3, 4,\\
p_{12}\left(5^2n + 5j + 2\right) &= 54p_{12}(5n + j) \text{ with } j = 0, 1, 3, 4,\\
p_{18}\left(5^2n + 5j + 3\right) &= -510p_{18}(5n + j) \text{ with } j = 0, 1, 2, 4,\\
p_4\left(7^2n + 7j + 1\right) &= -4p_4(7n + j) \text{ with } j = 0, 2, 3, 4, 5, 6,\\
p_2\left(13^2n + 13j + 1\right) &= -2p_2(13n + j) \text{ with } j = 0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,\\
p_{12}\left(3^2n + 3j + 1\right) &= -12p_{12}(3n + j) \text{ with } j = 0, 2.
\end{aligned}$$

Since $p_{24}(n) = \tau(n + 1)$, we immediately arrive at the following corollary from (4.3.4) and (4.3.34).

Corollary 4.3.2. *We have*

$$\tau(5^2n) = 4830\tau(5n) - 5^{11}\tau(n),$$

$$\tau(5^2n + 5j + 5) = 4830\tau(5n + j + 1), \ j = 0, 1, 2, 3.$$

Note that, the first identity in the above corollary is a special case for $p = 5$ of Ramanujan's famous conjecture [50]

$$\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}), \tag{4.3.35}$$

first proved by Mordell in [43].

Remark 4.3.2. *Equating the coefficients of q^{3n} and q^{3n+2} in (4.3.27), we obtain the two term recursion relations*

$$p_9(3^4n+3) = -12p_9(3^2n),$$

$$p_9(3^4n+57) = -12p_9(3^2n+6).$$

Similarly, equating the coefficients of q^{3n} and q^{3n+2} in (4.3.33), we have

$$p_{15}(3^4n+23) = 1836p_{15}(3^2n+2),$$

$$p_{15}(3^4n+77) = 1836p_{15}(3^2n+8).$$

Also equating the coefficients of q^{3n} in (4.3.31), we find that

$$p_{15}(3^3n+5) = 1107p_{15}(3n).$$

4.4 More Identities for $p_{24}(n)$ and $\tau(n)$

Theorem 4.4.1. *If k, n are nonnegative integers, then*

$$p_{24}(2^{k+2}n+2^{k+2}-1) = a_kp_{24}(2n+1)+b_kp_{24}(n), \tag{4.4.1}$$

where a_k and b_k are given recursively as

$$a_k = -24a_{k-1}+b_{k-1}, \qquad b_k = -2048a_{k-1},$$

with $a_0 = -24$ and $b_0 = -2048$.

Proof. We prove (4.4.1) by induction on k. Employing (4.1.7) and (4.1.8), we have

$$\sum_{n=0}^{\infty} p_{24}(n)q^n = (q^2;q^2)_\infty^{12}(\varphi(q^4)-2q\psi(q^8))^{12}. \tag{4.4.2}$$

Expanding the right hand side of (4.4.2) binomially and extracting the terms involving q^{2n+1}, we obtain

$$\sum_{n=0}^{\infty} p_{24}(2n+1)q^n = -(q;q)_\infty^{12}(12\cdot 2\varphi^{11}(q^2)\psi(q^4) + 220\cdot 2^3 q\varphi^9(q^2)\psi^3(q^4)$$
$$+ 792\cdot 2^5 q^2\varphi^7(q^2)\psi^5(q^4) + 792\cdot 2^7 q^3\varphi^5(q^2)\psi^7(q^4)$$
$$+ 220\cdot 2^9 q^4\varphi^3(q^2)\psi^9(q^4) + 12\cdot 2^{11}q^5\varphi(q^2)\psi^{11}(q^4))$$
$$= -(q;q)_\infty^{12}(3\cdot 2^3\varphi(q^2)\psi(q^4)(\varphi^{10}(q^2) + 2^{10}q^5\psi^{10}(q^4))$$
$$+ 220\cdot 2^3 q\varphi^3(q^2)\psi^3(q^4)(\varphi^6(q^2) + 2^6 q^3\psi^6(q^4))$$
$$+ 792\cdot 2^5 q^2\varphi^5(q^2)\psi^5(q^4)(\varphi^2(q^2) + 2^2 q\psi^4(q^4))). \tag{4.4.3}$$

From (4.1.14) and (4.1.15), we find that

$$\varphi^{10}(q^2) + 2^{10}q^5\psi^{10}(q^4) = \Big(\varphi^2(q^2) + 4q\psi^2(q^4)\Big)\Big(\big(\varphi^2(q^2) + 4q\psi^2(q^4)\big)^4$$
$$- 20q\varphi^2(q^2)\psi^2(q^4)\big(\varphi^2(q^2) + 4q\psi^2(q^4)\big)^2 + 80q^2\varphi^4(q^2)\psi^4(q^4)\Big), \tag{4.4.4}$$

$$\varphi^6(q^2) + 2^6q^3\psi^6(q^4) = (\varphi^2(q^2) + 4q\psi^2(q^4))((\varphi^2(q^2) + 4q\psi^2(q^4))^2 - 12q\varphi^2(q^2)\psi^2(q^4). \tag{4.4.5}$$

Employing (4.1.9) and (4.1.11) in (4.4.4) and (4.4.5), we deduce that

$$\varphi^{10}(q^2) + 2^{10}q^5\psi^{10}(q^4) = \varphi^2(q)(\varphi^8(q) - 20q\psi^8(q) + 80q^2\psi^8(q^2)), \tag{4.4.6}$$
$$\varphi^6(q^2) + 2^6q^3\psi^6(q^4) = \varphi^2(q)(\varphi^4(q) - 12q\psi^4(q^2)). \tag{4.4.7}$$

Using (4.1.11), (4.4.6), and (4.4.7) in (4.4.3) and simplifying, we have

$$\sum_{n=0}^{\infty} p_{24}(2n+1)q^n$$
$$= -2^3(q;q)_\infty^{12}\psi^4(q)(3\varphi^8(q) + 160q\psi^8(q) + 768q^2\psi^8(q^2)),$$
$$= -2^3(q;q)_\infty^{12}\psi^4(q)(3((\varphi^4(q) - 16q\psi^4(q^2))^2 + 32q\varphi^4(q)\psi^4(q^2)) + 160q\psi^8(q)). \tag{4.4.8}$$

We further simplify (4.4.8) by employing (4.1.10), (4.1.11), (4.1.7), and (2.2.3) to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p_{24}(2n+1)q^n &= -3.2^3(q;q)_\infty^{24} - 2^{11}q(q^2;q^2)_\infty^{24} \\ &= -3.2^3\sum_{n=0}^{\infty} p_{24}(n)q^n - 2^{11}\sum_{n=0}^{\infty} p_{24}(n)q^{2n+1}. \end{aligned} \tag{4.4.9}$$

Equating the coefficients of q^{2n+1} in (4.4.9), we find that

$$p_{24}(4n+3) = -3.2^3 p_{24}(2n+1) - 2^{11}p_{24}(n).$$

This proves (4.4.1) for $k=0$.

To complete the induction step, we assume that (4.4.1) is true for $k=t$, which is equivalent to

$$\begin{aligned}\sum_{n=0}^{\infty} p_{24}(2^{t+1}n + 2^{t+1} - 1)q^n &= a_t\sum_{n=0}^{\infty} p_{24}(n)q^n + b_t\sum_{n=0}^{\infty} p_{24}(n)q^{2n+1} \\ &= a_t(q;q)_\infty^{24} + b_t q(q^2;q^2)_\infty^{24} \\ &= a_t(q;q^2)_\infty^{24}(q^2;q^2)_\infty^{24} + b_t q(q^2;q^2)_\infty^{24}. \end{aligned} \tag{4.4.10}$$

Replacing q by $-q$ in (4.4.10) and subtracting the resulting identity from (4.4.10), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} p_{24}(2^{t+1}n + 2^{t+1} - 1)q^n - \sum_{n=0}^{\infty} p_{24}(2^{t+1}n + 2^{t+1} - 1)(-q)^n \\ &= -a_t(q^2;q^2)_\infty^{24}((-q;q^2)_\infty^{24} - (q;q^2)_\infty^{24}) + 2b_t q(q^2;q^2)_\infty^{24} \\ &= -a_t(q^2;q^2)_\infty^{24}(((-q;q^2)_\infty^{8} - (q;q^2)_\infty^{8})^3 + 3(q^2;q^4)_\infty^{8}((-q;q^2)_\infty^{8} - (q;q^2)_\infty^{8})) \\ &\quad + 2b_t q(q^2;q^2)_\infty^{24}. \end{aligned} \tag{4.4.11}$$

Now, employing (2.2.2) and (2.2.3) in (4.1.10), we obtain

$$(-q;q^2)_\infty^{8} - (q;q^2)_\infty^{8} = 16q(-q^2;q^2)_\infty^{8}. \tag{4.4.12}$$

Using (4.4.12) in (4.4.11), we have

$$\sum_{n=0}^{\infty} p_{24}(2^{t+1}n + 2^{t+1} - 1)q^n - \sum_{n=0}^{\infty} p_{24}(2^{t+1}n + 2^{t+1} - 1)(-q)^n$$
$$= -a_t(q^2;q^2)_\infty^{24}(4096q^3(-q^2;q^2)_\infty^{24} + 48q) + 2b_t q(q^2;q^2)_\infty^{24}$$
$$= 2(b_t - 24a_t)q(q^2;q^2)_\infty^{24} - 4096a_t q^3(q^4;q^4)_\infty^{24}. \tag{4.4.13}$$

Extracting the terms in (4.4.13) having odd powers of q, we find that

$$\sum_{n=0}^{\infty} p_{24}(2^{t+2}n + 2^{t+2} - 1)q^n = (b_t - 24a_t)(q;q)_\infty^{24} - 2048a_t q(q^2;q^2)_\infty^{24}$$
$$= a_{t+1}\sum_{n=0}^{\infty} p_{24}(n)q^n + b_{t+1}\sum_{n=0}^{\infty} p_{24}(n)q^{2n+1}. \tag{4.4.14}$$

Equating the coefficients of q^{2n+1} in (4.4.14), we arrive at

$$p_{24}(2^{t+3}n + 2^{t+3} - 1) = a_{t+1}p_{24}(2n+1) + b_{t+1}p_{24}(n),$$

which completes the induction step to finish the proof. □

Remark 4.4.1. *The first equality in* (4.4.9) *can be rewritten, after multiplying both sides by q, as*

$$q(q;q)_\infty^{24} - q(-q;-q)_\infty^{24} = -48q^2(q^2;q^2)_\infty^{24} - 2^{12}q^4(q^4;q^4)_\infty^{24},$$

from which it can be easily deduced that

$$\tau(2k) = -24\tau(k) - 2^{11}\tau\left(\frac{k}{2}\right), \tag{4.4.15}$$

where k is any integer and $\tau(x) = 0$ if x is not an integer.

The above two identities were written by Ramanujan in an incomplete and unpublished manuscript in two parts on the partition function $p(n)$ and tau function $\tau(n)$ published along with Ramanujan's Lost Notebook [54]. For additional details and commentary on Ramanujan's manuscript, we refer to Berndt and Ono's beautiful exposition [13].

Note that, when $k = 2^n$ in (4.4.15), then it reduces to the special case with $p = 2$ of Ramanujan-Mordell's identity (4.3.35).

Remark 4.4.2. *Rewriting (4.4.1) as*

$$\sum_{n=0}^{\infty} p_{24}(2^{k+1}n + 2^{k+1} - 1)q^n = a_k \sum_{n=0}^{\infty} p_{24}(n)q^n + b_k \sum_{n=0}^{\infty} p_{24}(n)q^{2n+1}, \tag{4.4.16}$$

and equating the coefficients of q^{2n} in (4.4.16), we obtain

$$p_{24}(2^{k+2}n + 2^{k+1} - 1) = a_k p_{24}(2n). \tag{4.4.17}$$

We can show, inductively, that $a_k \equiv 0 (\text{mod } 2^{3k+3})$ for $k \geq 0$. Therefore, from (4.4.17), we conclude that

$$p_{24}(2^{k+2}n + 2^{k+1} - 1) \equiv 0 (\text{mod } 2^{3k+3}).$$

Since $\tau(n+1) = p_{24}(n)$, from (4.4.1), we obtain

$$\tau(2^{k+2}n) = a_k \tau(2n) + b_k \tau(n),$$

and that from (4.4.17) and the multiplicativity of $\tau(n)$, we conclude that

$$\tau(2^{k+1}) = a_k.$$

4.5 Congruences for $p_r(n)$ with $r < 0$

In this section, we derive some congruences for $p_r(n)$, when $r = -2, \ -4$ and -8.

Theorem 4.5.1. We have

$$p_{-4}(4n+3) \equiv 0 \ (\text{mod } 2^3), \tag{4.5.1}$$

$$p_{-8}(4n+3) \equiv 0 \ (\text{mod } 2^6), \tag{4.5.2}$$

$$p_{-8}(8n+7) \equiv 0 \ (\text{mod } 2^9), \tag{4.5.3}$$

$$p_{-2}(5n+2) \equiv p_{-2}(5n+3) \equiv p_{-2}(5n+4) \equiv 0 \ (\text{mod } 5), \tag{4.5.4}$$

$$p_{-4}(5n+3) \equiv p_{-4}(5n+4) \equiv 0 \ (\text{mod } 5), \tag{4.5.5}$$

Note that Farkas and Kra [22, p. 405] proved (4.5.2) and several other similar congruences. The remaining congruences in the above theorem seem to be new.

Proof. Employing (3.3.6) and (3.3.7) in (4.1.9), we find that

$$\frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty^4(q^4;q^4)_\infty^4} = \frac{(q^4;q^4)_\infty^{10}}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^4} + 4q\frac{(q^8;q^8)_\infty^4}{(q^4;q^4)_\infty^2}. \tag{4.5.6}$$

From (4.5.6), it is clear that

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-4}(n)q^n &= \frac{1}{(q;q)_\infty^4} \\ &= \frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}.\end{aligned} \tag{4.5.7}$$

Extracting the terms involving odd powers of q in (4.5.7), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-4}(2n+1)q^n &= 4\frac{(q^2;q^2)_\infty^2(q^4;q^4)_\infty^4}{(q;q)_\infty^{10}} \\ &= 4\frac{(q^4;q^4)_\infty^{14}(-q;q^2)_\infty^{10}}{(q^2;q^2)_\infty^{18}}.\end{aligned} \tag{4.5.8}$$

Replacing q by $-q$ in (4.5.8) and subtracting the resulting identity from (4.5.8), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} p_{-4}(2n+1)q^n - \sum_{n=0}^{\infty} p_{-4}(2n+1)(-q)^n \\ &= 4\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{18}}((-q;q^2)_\infty^{10} - (q;q^2)_\infty^{10}) \\ &= 4\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{18}}((-q;q^2)_\infty^{2} - (q;q^2)_\infty^{2}) \\ &\times (((-q;q^2)_\infty^2 - (q;q^2)_\infty^2)^4 + 5(q^2;q^4)_\infty^2((-q;q^2)_\infty^2 - (q;q^2)_\infty^2)^2 + 5(q^2;q^4)_\infty^4).\end{aligned} \tag{4.5.9}$$

Now, writing [10, p. 40, Entry 25(ii)] in terms of q-products with the aid of (2.2.2) and (2.2.3), we find that

$$(-q;q^2)_\infty^2 - (q;q^2)_\infty^2 = 4q\frac{(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty(q^8;q^8)_\infty}. \tag{4.5.10}$$

Using (4.5.10) in (4.5.9) and then equating the coefficients of q^{2n+1}, we arrive at (4.5.1).

Next, from (4.5.6), we have

$$\sum_{n=0}^{\infty} p_{-8}(n)q^n = \frac{1}{(q;q)_\infty^8}$$
$$= \left(\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}\right)^2. \tag{4.5.11}$$

Equating the terms involving odd powers of q on both sides of (4.5.11), we obtain

$$\sum_{n=0}^{\infty} p_{-8}(2n+1)q^n = 8\frac{(q^2;q^2)_\infty^{16}}{(q;q)_\infty^{24}} = 8\frac{(q^4;q^4)_\infty^{24}(-q;q^2)_\infty^{24}}{(q^2;q^2)_\infty^{32}}. \tag{4.5.12}$$

Replacing q by $-q$ in (4.5.12) and subtracting the resulting identity from (4.5.12), we get

$$\sum_{n=0}^{\infty} p_{-8}(2n+1)q^n - \sum_{n=0}^{\infty} p_{-8}(2n+1)q^n$$
$$= 8\frac{(q^4;q^4)_\infty^{24}}{(q^2;q^2)_\infty^{32}}((-q;q^2)_\infty^{24} - (q;q^2)_\infty^{24})$$
$$= 8\frac{(q^4;q^4)_\infty^{24}}{(q^2;q^2)_\infty^{32}}(((-q;q^2)_\infty^{8} - (q;q^2)_\infty^{8})^3 + 3(q^2;q^4)_\infty^8((-q;q^2)_\infty^{8} - (q;q^2)_\infty^{8})). \tag{4.5.13}$$

With the help of (4.4.12), we write (4.5.13) as

$$\sum_{n=0}^{\infty} p_{-8}(2n+1)q^n - \sum_{n=0}^{\infty} p_{-8}(2n+1)q^n = 8\frac{(q^4;q^4)_\infty^{24}}{(q^2;q^2)_\infty^{32}}\left(4096q^3(-q^2;q^2)_\infty^{24} + 48q\right). \tag{4.5.14}$$

Extracting the terms involving q^{2n+1} in (4.5.14), we obtain

$$\sum_{n=0}^{\infty} p_{-8}(4n+3)q^n = 64\frac{(q^2;q^2)_\infty^{24}}{(q;q)_\infty^{32}}(256q(-q;q)_\infty^{24} + 3), \tag{4.5.15}$$

which yields (4.5.2).

Again, manipulating the q-products, we can rewrite (4.5.15) as

$$\sum_{n=0}^{\infty} p_{-8}(4n+3)q^n = \frac{64}{(q^2;q^2)_\infty^8}\left(256q\frac{(-q;q^2)_\infty^{56}}{(q^2;q^4)_\infty^{56}} + 3\frac{(-q;q^2)_\infty^{32}}{(q^2;q^4)_\infty^{32}}\right), \tag{4.5.16}$$

Replacing q by $-q$ in (4.5.16) and subtracting the resulting identity from (4.5.16), we get

$$\sum_{n=0}^{\infty} p_{-8}(4n+3)q^n - \sum_{n=0}^{\infty} p_{-8}(4n+3)(-q)^n$$
$$= \frac{64}{(q^2;q^2)_\infty^8}\left(\frac{256q}{(q^2;q^4)_\infty^{56}}((-q;q^2)_\infty^{56} + (q;q^2)_\infty^{56}) + \frac{3}{(q^2;q^4)_\infty^{32}}((-q;q^2)_\infty^{32} - (q;q^2)_\infty^{32})\right)$$
$$= \frac{64}{(q^2;q^2)_\infty^8}\left(\frac{256q}{(q^2;q^4)_\infty^{56}}(X^7+Y^7) + \frac{3}{(q^2;q^4)_\infty^{32}}(X^4-Y^4)\right), \tag{4.5.17}$$

where $X = (-q;q^2)_\infty^8$ and $Y = (q;q^2)_\infty^8$.

Now, employing [10, p. 40. Entry 25(v)], we find that

$$\begin{aligned} X+Y &= (-q;q^2)_\infty^8 + (q;q^2)_\infty^8 \\ &= ((-q;q^2)_\infty^4 - (q;q^2)_\infty^4)^2 + 2(q^2;q^4)_\infty^4 \\ &= 64q^2\frac{(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^4(q^4;q^8)_\infty^4} + 2(q^2;q^4)_\infty^4. \end{aligned} \tag{4.5.18}$$

In view of (4.5.18) and (4.1.16) we conclude that X^7+Y^7 and $(X+Y)((X+Y)^2-2XY)$ are functions of q^2, say, $U(q^2)$ and $V(q^2)$ respectively. Therefore, using (4.1.17) and (4.4.12) in (4.5.17), we obtain

$$\sum_{n=0}^{\infty} p_{-8}(4n+3)q^n - \sum_{n=0}^{\infty} p_{-8}(4n+3)(-q)^n = \frac{64}{(q^2;q^2)_\infty^8}\left(\frac{256qU(q^2)}{(q^2;q^4)_\infty^{56}} + \frac{48qV(q^2)}{(q^2;q^4)_\infty^{40}}\right). \tag{4.5.19}$$

Extracting the coefficients of q^{2n+1} in (4.5.19), we arrive at (4.5.3).

Again, from (4.1.4), we have

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n$$
$$= \frac{1}{(q;q)_\infty^2}$$
$$= \frac{(q^{25};q^{25})_\infty^{12}}{(q^5;q^5)_\infty^{12}}\Big(F^4(q^5) + qF^3(q^5) + 2q^2F^2(q^5) + 3q^3F(q^5) + 5q^4 - 3q^5F^{-1}(q^5)$$
$$+ 2q^6F^{-2}(q^5) - q^7F^{-3}(q^5) + q^8F^{-4}(q^5)\Big)^2. \tag{4.5.20}$$

Expanding the last expression of (4.5.20) and extracting the terms involving q^{5n+2}, q^{5n+3}, and q^{5n+4}, we easily deduce (4.5.4).

Finally, from (4.1.4), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{-4}(n)q^n \\
&= \frac{1}{(q;q)_\infty^4} \\
&= \frac{(q^{25};q^{25})_\infty^{24}}{(q^5;q^5)_\infty^{24}} \Big(F^4(q^5) + qF^3(q^5) + 2q^2F^2(q^5) + 3q^3F(q^5) + 5q^4 - 3q^5F^{-1}(q^5) \\
&\quad + 2q^6F^{-2}(q^5) - q^7F^{-3}(q^5) + q^8F^{-4}(q^5) \Big)^4. \qquad (4.5.21)
\end{aligned}$$

Equating the terms involving q^{5n+3} and q^{5n+4} on both sides of (4.5.21), we arrive at (4.5.5). □

Chapter 5

On Representations of a Number as Sums of Polygonal Numbers

5.1 Introduction

Jacobi's celebrated two-square theorem is as follows.

Theorem 5.1.1. ([32]). *Let $r\{\square + \square\}(n)$ denote the number of representations of n as a sum of two squares and $d_{i,j}(n)$ denote the number of positive divisors of n congruent to i modulo j. Then*

$$r\{\square + \square\}(n) = 4(d_{1,4}(n) - d_{3,4}(n)). \tag{5.1.1}$$

Simple proofs of Theorem 5.1.1 can be seen in [15] and [28]. Similar representation theorems involving squares and triangular numbers were found by Dirichlet, Lorenz, Legendre, and Ramanujan. For example, another classical result due to Lorenz [30, Theorem 3] is stated below.

Theorem 5.1.2. *Let $r\{l\square + m\square\}(n)$ denote the number of representations of n as a sum of l times of a square and m times of a square. Then*

$$r\{\square + 3\square\}(n) = 2(d_{1,3}(n) - d_{2,3}(n)) + 4(d_{4,12}(n) - d_{8,12}(n)). \tag{5.1.2}$$

Recent contributions of Hirschhorn [30, 31] have been discussed in the introductory chapter. In this chapter, we present twenty five more identities involving squares, triangular numbers, pentagonal numbers, heptagonal numbers, octagonal numbers, decagonal numbers, hendecagonal numbers, dodecagonal numbers, and octadecagonal numbers which are obtained by using Ramanujan's theta-function.

For $k \geq 3$, the n^{th} k-gonal number $F_k(n)$ is given by

$$F_k := F_k(n) = \frac{(k-2)n^2 - (k-4)n}{2}.$$

By allowing the domain for $F_k(n)$ to be the set of all integers, we see that the generating function $G_k(q)$ of $F_k(n)$ is given by

$$G_k(q) = \sum_{n=-\infty}^{\infty} q^{F_k} = \sum_{n=-\infty}^{\infty} q^{\frac{(k-2)n^2 - (k-4)n}{2}}.$$

We note an exception for the case $k = 3$. We observe that $G_3(q)$ generates each triangular number twice while $G_6(q)$ generates them only once. As such we take $G_6(q)$ as the generating function for triangular numbers instead of $G_3(q)$. We further observe that

$$G_k(q) = f(q, q^{k-3}), \tag{5.1.3}$$

where $f(a, b)$ is Ramanujan's general theta function. In view of (5.1.3), the respective generating functions of squares, triangular numbers, pentagonal numbers, heptagonal numbers, octagonal numbers, decagonal numbers, hendecagonal numbers, dodecagonal numbers, and octadecagonal numbers are

$$\begin{aligned}
G_4(q) &= f(q, q) = \varphi(q), \\
G_6(q) &= f(q, q^3) = \psi(q), \\
G_5(q) &= f(q, q^2), \\
G_7(q) &= f(q, q^4),
\end{aligned}$$

$$G_8(q) = f(q, q^5),$$

$$G_{10}(q) = f(q, q^7),$$

$$G_{11}(q) = f(q, q^8),$$

$$G_{12}(q) = f(q, q^9),$$

and

$$G_{18}(q) = f(q, q^{15}).$$

In Section 5.2, we give dissections of $\varphi(q)$, $\psi(q)$, $G_5(q)$, and $G_{12}(q)$ and recall some identities established in [30] and [31]. In the remaining five sections, we successively present sets of identities involving decagonal numbers, hendecagonal numbers, dodecagonal numbers, heptagonal numbers, and octadecagonal numbers.

5.2 Preliminary Results

Let $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$ for each integer n. Then we have [10, p. 48, Entry 31]

$$f(a, b) = f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{5.2.1}$$

Replacing a by q^a and b by q^b in (5.2.1), we find that

$$f(q^a, q^b) = \sum_{r=0}^{n-1} q^{\left(\frac{a+b}{2}\right)r^2+\left(\frac{a-b}{2}\right)r} f\left(q^{\left(\frac{a+b}{2}\right)n^2+(a+b)nr+\left(\frac{a-b}{2}\right)n}, q^{\left(\frac{a+b}{2}\right)n^2-(a+b)nr-\left(\frac{a-b}{2}\right)n}\right). \tag{5.2.2}$$

Setting $a = b = 1$ and then letting $n = 3, 5, 8$ in (5.2.2), we obtain

$$\varphi(q) = \varphi(q^9) + 2qG_8(q^3), \tag{5.2.3}$$

$$\varphi(q) = \varphi(q^{25}) + 2qA(q^5) + 2q^4G_{12}(q^5), \tag{5.2.4}$$

$$\varphi(q) = \varphi(q^{64}) + 2qB(q^{16}) + 2q^4\psi(q^{32}) + 2q^9G_{10}(q^{16}) + 2q^{16}\psi(q^{128}), \tag{5.2.5}$$

respectively, where $A(q) = f(q^3, q^7)$ and $B(q) = f(q^3, q^5)$.

Setting $a = 1$, $b = 3$ and then putting $n = 2, 4, 6$ in (5.2.2), we deduce that

$$\psi(q) = B(q^2) + qG_{10}(q^2), \tag{5.2.6}$$

$$\psi(q) = f(q^{28}, q^{36}) + qf(q^{20}, q^{44}) + q^3 f(q^{12}, q^{52}) + q^6 G_{18}(q^4), \tag{5.2.7}$$

and
$$\psi(q) = f(q^{66}, q^{78}) + qB(q^{18}) + q^3 f(q^{42}, q^{102}) + q^6 f(q^{30}, q^{114}) + q^{10}G_{10}(q^{18}) + q^{15}G_{26}(q^6), \tag{5.2.8}$$

respectively.

Setting $a = 1$, $b = 0$ and then choosing $n = 3, 5$ in (5.2.2) and noting that $\psi(q) = \frac{1}{2}f(1, q)$, we obtain

$$\psi(q) = G_5(q^3) + q\psi(q^9) \tag{5.2.9}$$

and
$$\psi(q) = C(q^5) + qG_7(q^5) + q^3\psi(q^{25}), \tag{5.2.10}$$

respectively, where $C(q) = f(q^2, q^3)$.

Furthermore, setting $a = 1$, $b = 2$ and $n = 3$ in (5.2.2), we find that

$$G_5(q) = f(q^{12}, q^{15}) + qf(q^6, q^{21}) + q^2 G_{11}(q^3). \tag{5.2.11}$$

Again setting $a = 1$, $b = 9$ and $n = 2$ in (5.2.2), we obtain

$$G_{12}(q) = A(q^4) + qG_7(q^8). \tag{5.2.12}$$

We require identities deduced in [30] and [31] which we mention below. The first seven of these identities appear in [30] as equations (1.1), (1.3), (1.4), (1.5), (1.11), (1.12), and (1.14), respectively, while the last six identities appear in [31] as equation (1.2), (1.3), (1.4), (1.6), (1.13), and (1.14), respectively. The notations of the type $r\{lF_i + mF_j\}(n)$, that are used throughout the sequel, denote the number of representations of n as a sum of l times a polygonal number F_i and m times a polygonal number F_j. Note that $r\{2\Box + \triangle\}(n)$ that appears in (5.2.14) is $r\{2F_4 + F_6\}(n)$. However, we have kept the former notation in such cases which involve squares or triangular numbers.

$$r\{\triangle+\triangle\}(n)=d_{1,4}(4n+1)-d_{3,4}(4n+1), \tag{5.2.13}$$

$$r\{2\square+\triangle\}(n)=d_{1,4}(8n+1)-d_{3,4}(8n+1), \tag{5.2.14}$$

$$r\{\triangle+4\triangle\}(n)=\frac{1}{2}(d_{1,4}(8n+5)-d_{3,4}(8n+5)), \tag{5.2.15}$$

$$r\{\triangle+2\triangle\}(n)=\frac{1}{2}(d_{1,8}(8n+3)+d_{3,8}(8n+3)-d_{5,8}(8n+3) \\ -d_{7,8}(8n+3)), \tag{5.2.16}$$

$$r\{6\square+\triangle\}(n)=d_{1,3}(8n+1)-d_{2,3}(8n+1), \tag{5.2.17}$$

$$r\{\triangle+12\triangle\}(n)=\frac{1}{2}(d_{1,3}(8n+13)-d_{2,3}(8n+13)), \tag{5.2.18}$$

$$r\{3\triangle+4\triangle\}(n)=\frac{1}{2}(d_{1,3}(8n+7)-d_{2,3}(8n+7)), \tag{5.2.19}$$

$$r\{\triangle+4F_5\}(n)=d_{1,24}(24n+7)+d_{19,24}(24n+7)-d_{5,24}(24n+7) \\ -d_{23,24}(24n+7), \tag{5.2.20}$$

$$r\{3\triangle+F_5\}(n)=d_{1,12}(12n+5)-d_{11,12}(12n+5), \tag{5.2.21}$$

$$r\{3\triangle+2F_5\}(n)=d_{1,8}(24n+11)-d_{7,8}(24n+11), \tag{5.2.22}$$

$$r\{6\triangle+F_5\}(n)=d_{1,8}(24n+19)-d_{7,8}(24n+19), \tag{5.2.23}$$

$$r\{3\square+F_5\}(n)=d_{1,8}(24n+1)+d_{3,8}(24n+1)-d_{5,8}(24n+1) \\ -d_{7,8}(24n+1), \tag{5.2.24}$$

$$r\{3\square+4F_5\}(n)=d_{1,8}(6n+1)+d_{3,8}(6n+1)-d_{5,8}(6n+1)-d_{7,8}(6n+1). \tag{5.2.25}$$

5.3 Identities involving decagonal numbers

Theorem 5.3.1. *We have*

$$r\{\square+3F_{10}\}(n)=d_{1,3}(16n+27)-d_{2,3}(16n+27), \tag{5.3.1}$$

$$r\{2\triangle+3F_{10}\}(n)=\frac{1}{2}(d_{1,3}(16n+31)-d_{2,3}(16n+31)), \tag{5.3.2}$$

$$r\{2\triangle+F_{10}\}(n)=\frac{1}{2}(d_{1,4}(16n+13)-d_{3,4}(16n+13)), \tag{5.3.3}$$

$$r\{\square+F_{10}\}(n)=d_{1,4}(16n+9)-d_{3,4}(16n+9), \tag{5.3.4}$$

$$r\{6\triangle + F_{10}\}(n) = \frac{1}{2}(d_{1,3}(16n+21) - d_{2,3}(16n+21)), \tag{5.3.5}$$

$$r\{3\square + F_{10}\}(n) = d_{1,3}(16n+9) - d_{2,3}(16n+9), \tag{5.3.6}$$

$$r\{F_8 + F_{10}\}(n) = \frac{1}{2}(d_{1,3}(48n+43) - d_{2,3}(48n+43)), \tag{5.3.7}$$

$$r\{F_5 + 3F_{10}\}(n) = d_{1,8}(48n+83) - d_{7,8}(48n+83), \tag{5.3.8}$$

$$\begin{aligned} r\{2F_5 + F_{10}\}(n) &= d_{1,24}(48n+31) + d_{19,24}(48n+31) \\ &\quad - d_{5,24}(48n+31) - d_{23,24}(48n+31), \end{aligned} \tag{5.3.9}$$

$$\begin{aligned} r\{\triangle + F_{10}\}(n) &= \frac{1}{2}(d_{1,8}(16n+11) + d_{3,8}(16n+11) \\ &\quad - d_{5,8}(16n+11) - d_{7,8}(16n+11)). \end{aligned} \tag{5.3.10}$$

Proof. The identity (5.2.17) is equivalent to

$$\varphi(q^6)\psi(q) = \sum_{n\geq 0}(d_{1,3}(8n+1) - d_{2,3}(8n+1))q^n. \tag{5.3.11}$$

Using (5.2.8) in (5.3.11), we have

$$\begin{aligned} &\varphi(q^6)(f(q^{66},q^{78}) + qB(q^{18}) + q^3f(q^{42},q^{102}) + q^6f(q^{30},q^{114}) \\ &\quad + q^{10}G_{10}(q^{18}) + q^{15}G_{26}(q^6)) = \sum_{n\geq 0}(d_{1,3}(8n+1) - d_{2,3}(8n+1))q^n. \end{aligned} \tag{5.3.12}$$

Extracting the terms involving q^{6n+4} in (5.3.12) and then dividing the resulting identity by q^4 and replacing q^6 by q, we find that

$$q\varphi(q)G_{10}(q^3) = \sum_{n\geq 0}(d_{1,3}(48n+33) - d_{2,3}(48n+33))q^n. \tag{5.3.13}$$

Equating the coefficients of q^{n+1} on both sides of (5.3.13) and noting that $d_{1,3}(48n+33) = d_{1,3}(16n+11)$ and $d_{2,3}(48n+33) = d_{2,3}(16n+11)$, we arrive at (5.3.1).

The identity (5.2.18) is equivalent to

$$\psi(q)\psi(q^{12}) = \frac{1}{2}\sum_{n\geq 0}(d_{1,3}(8n+13) - d_{2,3}(8n+13))q^n. \tag{5.3.14}$$

With the aid of (5.2.8), we rewrite (5.3.14) as

$$\psi(q^{12})(f(q^{66},q^{78})+qB(q^{18})+q^3f(q^{42},q^{102})+q^6f(q^{30},q^{114})$$
$$+q^{10}G_{10}(q^{18})+q^{15}G_{26}(q^6))=\frac{1}{2}\sum_{n\geq 0}(d_{1,3}(8n+13)-d_{2,3}(8n+13))q^n. \quad (5.3.15)$$

Collecting the terms in (5.3.15) in which the power of q is congruent to 4 modulo 6, we find that

$$q\psi(q^2)G_{10}(q^3)=\frac{1}{2}\sum_{n\geq 0}(d_{1,3}(48n+45)-d_{2,3}(48n+45))q^n. \quad (5.3.16)$$

Equating the coefficients of q^{n+1} on both sides of (5.3.16) and noting that $d_{1,3}(48n+45)=d_{1,3}(16n+15)$ and $d_{2,3}(48n+45)=d_{2,3}(16n+15)$, we arrive at (5.3.2).

The identity (5.1.1) is equivalent to

$$\varphi^2(q)=1+4\sum_{n\geq 1}(d_{1,4}(n)-d_{3,4}(n))q^n. \quad (5.3.17)$$

If we invoke (5.2.5), (5.3.17) becomes

$$(\varphi(q^{64})+2qB(q^{16})+2q^4\psi(q^{32})+2q^9G_{10}(q^{16})+2q^{16}\psi(q^{128}))^2$$
$$=1+4\sum_{n\geq 1}(d_{1,4}(n)-d_{3,4}(n))q^n. \quad (5.3.18)$$

Now, we extract those terms in (5.3.18) where the powers of q are congruent to 13 modulo 16, divide the resulting identity by q^{13} and replace q^{16} by q, to obtain

$$\psi(q^2)G_{10}(q)=\frac{1}{2}\sum_{n\geq 0}(d_{1,4}(16n+13)-d_{3,4}(16n+13))q^n,$$

which readily yields (5.3.3).

Further, we collect the terms involving q^{16+2} in (5.3.18) to deduce that

$$B^2(q)+qG_{10}^2(q)=\sum_{n\geq 0}(d_{1,4}(16n+2)-d_{3,4}(16n+2))q^n. \quad (5.3.19)$$

Noting the fact that $d_{1,4}(16n+2) = d_{1,4}(8n+1)$ and $d_{3,4}(16n+2) = d_{3,4}(8n+1)$, we obtain from (5.3.19),

$$B^2(q) + qG_{10}^2(q) = \sum_{n\geq 0}(d_{1,4}(8n+1) - d_{3,4}(8n+1))q^n. \tag{5.3.20}$$

Using [10, p. 46, Entry 30(v),(vi)], we have

$$B^2(q) = f^2(q^3,q^5) = f(q^6,q^{10})\varphi(q^8) + 2q^3 f(q^2,q^{14})\psi(q^{16}) \tag{5.3.21}$$

and
$$G_{10}^2(q) = f^2(q,q^7) = f(q^2,q^{14})\varphi(q^8) + 2qf(q^6,q^{10})\psi(q^{16}). \tag{5.3.22}$$

Replacing $B^2(q)$ and $G_{10}^2(q)$ in (5.3.20) by the right sides of (5.3.21) and (5.3.22) respectively, and then factorizing, we find that

$$(\varphi(q^8) + 2q^2\psi(q^{16}))(f(q^6,q^{10}) + qf(q^2,q^{14})) = \sum_{n\geq 0}(d_{1,4}(8n+1) - d_{3,4}(8n+1))q^n. \tag{5.3.23}$$

Setting $a = b = n = 2$ in (5.2.2), we obtain

$$\varphi(q^2) = \varphi(q^8) + 2q^2\psi(q^{16}). \tag{5.3.24}$$

Employing (5.3.24) in (5.3.23), we get

$$\varphi(q^2)(B(q^2) + qG_{10}(q^2)) = \sum_{n\geq 0}(d_{1,4}(8n+1) - d_{3,4}(8n+1))q^n. \tag{5.3.25}$$

Equating odd parts in (5.3.25), we find that

$$\varphi(q)G_{10}(q) = \sum_{n\geq 0}(d_{1,4}(16n+9) - d_{3,4}(16n+9))q^n,$$

which readily yields (5.3.4).

The identity (5.1.2) is equivalent to

$$\begin{aligned}\varphi(q)\varphi(q^3) &= 1 + 2\sum_{n\geq 1}(d_{1,3}(n) - d_{2,3}(n))q^n + 4\sum_{n\geq 1}(d_{4,12}(n) - d_{8,12}(n))q^n \\ &= 1 + 2\sum_{n\geq 1}(d_{1,3}(n) - d_{2,3}(n))q^n + 4\sum_{n\geq 1}(d_{1,3}(n) - d_{2,3}(n))q^{4n}. \end{aligned} \tag{5.3.26}$$

Employing (5.2.5) in (5.3.26), we have

$$\begin{aligned}&(\varphi(q^{64})+2qB(q^{16})+2q^4\psi(q^{32})+2q^9G_{10}(q^{16})+2q^{16}\psi(q^{128}))\\&\quad\times(\varphi(q^{192})+2q^3B(q^{48})+2q^{12}\psi(q^{96})+2q^{27}G_{10}(q^{48})+2q^{48}\psi(q^{384}))\\&=1+2\sum_{n\geq1}(d_{1,3}(n)-d_{2,3}(n))q^n+4\sum_{n\geq1}(d_{1,3}(n)-d_{2,3}(n))q^{4n}.\end{aligned}\tag{5.3.27}$$

Extracting the terms in (5.3.27) involving q^{16n+5}, then dividing the resulting identity by q^5 and replacing q^{16} by q, we find that

$$q\psi(q^6)G_{10}(q)=\frac{1}{2}\sum_{n\geq0}(d_{1,3}(16n+5)-d_{2,3}(16n+5))q^n,$$

from which we easily deduce (5.3.5).

Again using (5.2.3) in (5.3.13), we have

$$q(\varphi(q^9)+2qG_8(q^3))G_{10}(q^3)=\sum_{n\geq0}(d_{1,3}(16n+11)-d_{2,3}(16n+11))q^n.\tag{5.3.28}$$

Separating the terms involving q^{3n+1} and q^{3n+2} in (5.3.28), we obtain

$$\varphi(q^3)G_{10}(q)=\sum_{n\geq0}(d_{1,3}(48n+27)-d_{2,3}(48n+27))q^n\tag{5.3.29}$$

and

$$2G_8(q)G_{10}(q)=\sum_{n\geq0}(d_{1,3}(48n+43)-d_{2,3}(48n+43))q^n,\tag{5.3.30}$$

respectively. Now, the identities (5.3.6) and (5.3.7) follow easily from (5.3.29) and (5.3.30), respectively.

The identity (5.2.22) is equivalent to

$$\psi(q^3)G_5(q^2)=\sum_{n\geq0}(d_{1,8}(24n+11)-d_{7,8}(24n+11))q^n.\tag{5.3.31}$$

Invoking (5.2.6) in (5.3.31), we have

$$(B(q^6)+q^3G_{10}(q^6))G_5(q^2)=\sum_{n\geq0}(d_{1,8}(24n+11)-d_{7,8}(24n+11))q^n.\tag{5.3.32}$$

Isolating the terms involving q^{2n+1} in (5.3.32), we obtain

$$qG_{10}(q^3)G_5(q) = \sum_{n\geq 0}(d_{1,8}(48n+35) - d_{7,8}(48n+35))q^n. \tag{5.3.33}$$

Comparing the coefficients of q^{n+1} on both sides of (5.3.33), we arrive at (5.3.8).

The identity (5.2.20) is equivalent to

$$\begin{aligned}&\psi(q)G_5(q^4)\\&= \sum_{n\geq 0}(d_{1,24}(24n+7) + d_{19,24}(24n+7) - d_{5,24}(24n+7) - d_{23,24}(24n+7))q^n.\end{aligned} \tag{5.3.34}$$

Using (5.2.6) in (5.3.34), we have

$$\begin{aligned}&(B(q^2) + qG_{10}(q^2))G_5(q^4)\\&= \sum_{n\geq 0}(d_{1,24}(24n+7) + d_{19,24}(24n+7) - d_{5,24}(24n+7) - d_{23,24}(24n+7))q^n.\end{aligned} \tag{5.3.35}$$

Extracting the terms having odd powers of q in (5.3.35), we obtain

$$\begin{aligned}&G_{10}(q)G_5(q^2)\\&= \sum_{n\geq 0}(d_{1,24}(48n+31) + d_{19,24}(48n+31) - d_{5,24}(48n+31) - d_{23,24}(48n+31))q^n,\end{aligned}$$

which readily yields (5.3.9).

The identity (5.2.16) is equivalent to

$$\psi(q)\psi(q^2) = \frac{1}{2}\sum_{n\geq 0}(d_{1,8}(8n+3) + d_{3,8}(8n+3) - d_{5,8}(8n+3) - d_{7,8}(8n+3))q^n. \tag{5.3.36}$$

With the help of (5.2.6), we rewrite (5.3.36) as

$$\begin{aligned}&(B(q^2) + qG_{10}(q^2))\psi(q^2)\\&= \frac{1}{2}\sum_{n\geq 0}(d_{1,8}(8n+3) + d_{3,8}(8n+3) - d_{5,8}(8n+3) - d_{7,8}(8n+3))q^n.\end{aligned} \tag{5.3.37}$$

Collecting the terms involving q^{2n+1} in (5.3.37), we obtain

$$G_{10}(q)\psi(q) = \frac{1}{2}\sum_{n\geq 0}(d_{1,8}(16n+11)+d_{3,8}(16n+11)-d_{5,8}(16n+11)-d_{7,8}(16n+11))q^n,$$

from which we easily arrive at (5.3.10). □

5.4 Identities involving hendecagonal numbers

Theorem 5.4.1. *We have*

$$r\{\triangle+F_{11}\}(n) = d_{1,12}(36n+29) - d_{11,12}(36n+29), \tag{5.4.1}$$

$$r\{\triangle+2F_{11}\}(n) = d_{1,8}(72n+107) - d_{7,8}(72n+107), \tag{5.4.2}$$

$$r\{2\triangle+F_{11}\}(n) = d_{1,8}(72n+67) - d_{7,8}(72n+67), \tag{5.4.3}$$

$$r\{\square+F_{11}\}(n) = d_{1,8}(72n+49) + d_{3,8}(72n+49) - d_{5,8}(72n+49) - d_{7,8}(72n+49), \tag{5.4.4}$$

$$r\{\square+4F_{11}\}(n) = d_{1,8}(18n+49) + d_{3,8}(18n+49) - d_{5,8}(18n+49) - d_{7,8}(18n+49), \tag{5.4.5}$$

$$r\{F_{10}+F_{11}\}(n) = d_{1,8}(144n+179) - d_{7,8}(144n+179). \tag{5.4.6}$$

Proof. The identity (5.2.21) is equivalent to

$$\psi(q^3)G_5(q) = \sum_{n\geq 0}(d_{1,12}(12n+5) - d_{11,12}(12n+5))q^n. \tag{5.4.7}$$

With the aid of (5.2.11), we write (5.4.7) as

$$\psi(q^3)(f(q^{12},q^{15}) + qf(q^6,q^{21}) + q^2G_{11}(q^3)) = \sum_{n\geq 0}(d_{1,12}(12n+5) - d_{11,12}(12n+5))q^n. \tag{5.4.8}$$

Extracting the terms involving q^{3n+2} in (5.4.8), we obtain

$$\psi(q)G_{11}(q) = \sum_{n\geq 0}(d_{1,12}(36n+29) - d_{11,12}(36n+29))q^n,$$

which readily yields (5.4.1).

The identity (5.2.22) is equivalent to

$$\psi(q^3)G_5(q^2) = \sum_{n\geq 0}(d_{1,8}(24n+11) - d_{7,8}(24n+11))q^n. \tag{5.4.9}$$

Invoking (5.2.11) in (5.4.9), we find that

$$\begin{aligned}&\psi(q^3)(f(q^{24},q^{30}) + q^2 f(q^{12},q^{42}) + q^4 G_{11}(q^6))\\ &= \sum_{n\geq 0}(d_{1,8}(24n+11) - d_{7,8}(24n+11))q^n.\end{aligned} \tag{5.4.10}$$

Equating the terms involving q^{3n+1} in (5.4.10), we obtain

$$q\psi(q)G_{11}(q^2) = \sum_{n\geq 0}(d_{1,8}(72n+35) - d_{7,8}(72n+35))q^n, \tag{5.4.11}$$

from which we deduce (5.4.2).

The identity (5.2.23) is equivalent to

$$\psi(q^6)G_5(q) = \sum_{n\geq 0}(d_{1,8}(24n+19) - d_{7,8}(24n+19))q^n. \tag{5.4.12}$$

Using (5.2.11) in (5.4.12), we have

$$\begin{aligned}&\psi(q^6)(f(q^{12},q^{15}) + qf(q^6,q^{21}) + q^2 G_{11}(q^3))\\ &= \sum_{n\geq 0}(d_{1,8}(24n+19) - d_{7,8}(24n+19))q^n.\end{aligned} \tag{5.4.13}$$

If we collect the terms involving q^{3n+2} in (5.4.13), then we get

$$\psi(q^2)G_{11}(q) = \sum_{n\geq 0}(d_{1,8}(72n+67) - d_{7,8}(72n+67))q^n,$$

which gives the identity (5.4.3).

The identity (5.2.24) is equivalent to

$$\varphi(q^3)G_5(q)$$
$$= \sum_{n\geq 0}(d_{1,8}(24n+1) + d_{3,8}(24n+1) - d_{5,8}(24n+1) - d_{7,8}(24n+1))q^n. \quad (5.4.14)$$

With the aid of (5.2.11), we rewrite (5.4.14) as

$$\varphi(q^3)(f(q^{12},q^{15}) + qf(q^6,q^{21}) + q^2G_{11}(q^3))$$
$$= \sum_{n\geq 0}(d_{1,8}(24n+1) + d_{3,8}(24n+1) - d_{5,8}(24n+1) - d_{7,8}(24n+1))q^n. \quad (5.4.15)$$

Extracting the terms involving q^{3n+2} in (5.4.15), we obtain

$$\varphi(q)G_{11}(q)$$
$$= \sum_{n\geq 0}(d_{1,8}(72n+49) + d_{3,8}(72n+49) - d_{5,8}(72n+49) - d_{7,8}(72n+49))q^n,$$

which readily yields (5.4.4).

The identity (5.2.25) is equivalent to

$$\varphi(q^3)G_5(q^4) = \sum_{n\geq 0}(d_{1,8}(6n+1) + d_{3,8}(6n+1) - d_{5,8}(6n+1) - d_{7,8}(6n+1))q^n. \quad (5.4.16)$$

Using (5.2.11) in (5.4.16), we have

$$\varphi(q^3)(f(q^{48},q^{60}) + q^4f(q^{24},q^{84}) + q^8G_{11}(q^{12}))$$
$$= \sum_{n\geq 0}(d_{1,8}(6n+1) + d_{3,8}(6n+1) - d_{5,8}(6n+1) - d_{7,8}(6n+1))q^n. \quad (5.4.17)$$

Isolating the terms involving q^{3n+2} in (5.4.17), we find that

$$q^2\varphi(q)G_{11}(q^4)$$
$$= \sum_{n\geq 0}(d_{1,8}(18n+13) + d_{3,8}(18n+13) - d_{5,8}(18n+13) - d_{7,8}(18n+13))q^n,$$

from which we easily deduce (5.4.5).

Again employing (5.2.6) in (5.4.11), we obtain

$$q(B(q^2)+qG_{10}(q^2))G_{11}(q^2)=\sum_{n\geq 0}(d_{1,8}(72n+35)-d_{7,8}(72n+35))q^n. \quad (5.4.18)$$

Comparing the terms in (5.4.18) where the powers of q are even, we find that

$$qG_{10}(q)G_{11}(q)=\sum_{n\geq 0}(d_{1,8}(144n+35)-d_{7,8}(144n+35))q^n. \quad (5.4.19)$$

Equating the coefficients of q^{n+1} in (5.4.19), we arrive at (5.4.6). □

5.5 Identities involving dodecagonal numbers

Theorem 5.5.1. *We have*

$$r\{5\square+F_{12}\}(n)=d_{1,4}(5n+4)-d_{3,4}(5n+4), \quad (5.5.1)$$

$$r\{F_{12}+F_{12}\}(n)=d_{1,4}(5n+8)-d_{3,4}(5n+8), \quad (5.5.2)$$

$$r\{5\triangle+F_{12}\}(n)=\frac{1}{2}(d_{1,4}(20n+17)-d_{3,4}(20n+17)). \quad (5.5.3)$$

Proof. Employing (5.2.4) in (5.3.17), we find that

$$(\varphi(q^{25})+2qA(q^5)+2q^4G_{12}(q^5))^2=1+4\sum_{n\geq 1}(d_{1,4}(n)-d_{3,4}(n))q^n. \quad (5.5.4)$$

Extracting those terms in (5.5.4) in which the powers of q are congruent to 4 modulo 5, we obtain

$$\varphi(q^5)G_{12}(q)=\sum_{n\geq 0}(d_{1,4}(5n+4)-d_{3,4}(5n+4))q^n,$$

from which (5.5.1) follows.

Again, collecting the terms involving q^{5n+3} in (5.5.4), we get

$$qG_{12}^2(q)=\sum_{n\geq 0}(d_{1,4}(5n+3)-d_{3,4}(5n+3))q^n, \quad (5.5.5)$$

which immediately gives (5.5.2).

Further, extracting the terms involving q^{5n+2} in (5.5.4), we find that

$$A^2(q) = \sum_{n\geq 0}(d_{1,4}(5n+2) - d_{3,4}(5n+2))q^n. \tag{5.5.6}$$

By [10, p. 46, Entry 30(v),(vi)], we have

$$A^2(q) = f^2(q^3, q^7) = A(q^2)\varphi(q^{10}) + 2q^3G_{12}(q^4)\psi(q^{20}). \tag{5.5.7}$$

From (5.5.6) and (5.5.7), we obtain

$$A(q^2)\varphi(q^{10}) + 2q^3G_{12}(q^4)\psi(q^{20}) = \sum_{n\geq 0}(d_{1,4}(5n+2) - d_{3,4}(5n+2))q^n. \tag{5.5.8}$$

Collecting the terms involving q^{4n+3} in (5.5.8), we find that

$$2G_{12}(q)\psi(q^5) = \sum_{n\geq 0}(d_{1,4}(20n+17) - d_{3,4}(20n+17))q^n,$$

which readily yields (5.5.3). □

5.6 Identities involving heptagonal numbers

Theorem 5.6.1. *We have*

$$r\{F_7 + F_7\}(n) = d_{1,4}(20n+9) - d_{3,4}(20n+9), \tag{5.6.1}$$

$$r\{5\Delta + F_7\}(n) = \frac{1}{2}(d_{1,4}(20n+17) - d_{3,4}(20n+17)), \tag{5.6.2}$$

$$r\{2F_{12} + F_7\}(n) = \frac{1}{2}(d_{1,4}(40n+73) - d_{3,4}(40n+73)). \tag{5.6.3}$$

Proof. With the aid of (5.2.12), we rewrite (5.5.5) as

$$q(A(q^4) + qG_7(q^8))^2 = \sum_{n\geq 0}(d_{1,4}(5n+3) - d_{3,4}(5n+3))q^n. \tag{5.6.4}$$

Extracting the terms involving q^{8n+3} in (5.6.4), we find that

$$G_7^2(q) = \sum_{n\geq 0}(d_{1,4}(40n+18) - d_{3,4}(40n+18))q^n. \tag{5.6.5}$$

Equating the coefficients of q^n in (5.6.5) and noting the fact that $d_{1,4}(40n+18) = d_{1,4}(20n+9)$ and $d_{3,4}(40n+18) = d_{3,4}(20n+9)$, we arrive at (5.6.1).

The identity (5.2.13) is equivalent to

$$\psi^2(q) = \sum_{n\geq 0}(d_{1,4}(4n+1) - d_{3,4}(4n+1))q^n. \tag{5.6.6}$$

Invoking (5.2.10) in (5.6.6), we obtain

$$(C(q^5) + qG_7(q^5) + q^3\psi(q^{25}))^2 = \sum_{n\geq 0}(d_{1,4}(4n+1) - d_{3,4}(4n+1))q^n. \tag{5.6.7}$$

Equating the terms involving q^{5n+4} in (5.6.7), we get

$$2G_7(q)\psi(q^5) = \sum_{n\geq 0}(d_{1,4}(20n+17) - d_{3,4}(20n+17))q^n. \tag{5.6.8}$$

Comparing the coefficients of q^n in (5.6.8), we easily arrive at (5.6.2).

Further, the identity (5.2.14) is equivalent to

$$\varphi(q^2)\psi(q) = \sum_{n\geq 0}(d_{1,4}(8n+1) - d_{3,4}(8n+1))q^n. \tag{5.6.9}$$

Using (5.2.4) and (5.2.10) in (5.6.9), we find that

$$\begin{aligned}&(\varphi(q^{50}) + 2q^2A(q^{10}) + 2q^8G_{12}(q^{10}))(C(q^5) + qG_7(q^5) + q^3\psi(q^{25}))\\ &= \sum_{n\geq 0}(d_{1,4}(8n+1) - d_{3,4}(8n+1))q^n.\end{aligned} \tag{5.6.10}$$

Now, we collect the terms involving q^{5n+4} in (5.6.10) to obtain

$$2qG_{12}(q^2)G_7(q) = \sum_{n\geq 0}(d_{1,4}(40n+33) - d_{3,4}(40n+33))q^n,$$

from which we deduce (5.6.3). □

5.7 Identities involving octadecagonal numbers

Theorem 5.7.1. *We have*

$$r\{F_5+F_{18}\}(n) = d_{1,24}(96n+151)+d_{19,24}(96n+151) - d_{5,24}(96n+151)-d_{23,24}(96n+151), \tag{5.7.1}$$

$$r\{\triangle+F_{18}\}(n) = \frac{1}{2}(d_{1,4}(32n+53)-d_{3,4}(32n+53)), \tag{5.7.2}$$

$$r\{3\triangle+F_{18}\}(n) = \frac{1}{2}(d_{1,3}(32n+61)-d_{2,3}(32n+61)). \tag{5.7.3}$$

Proof. The identity (5.2.20) is equivalent to

$$\psi(q)G_5(q^4) = \sum_{n\geq 0}(d_{1,24}(24n+7)+d_{19,24}(24n+7)-d_{5,24}(24n+7)-d_{23,24}(24n+7))q^n. \tag{5.7.4}$$

Employing (5.2.7) in (5.7.4), we get

$$(f(q^{28},q^{36})+qf(q^{20},q^{44})+q^3f(q^{12},q^{52})+q^6G_{18}(q^4))G_5(q^4) = \sum_{n\geq 0}(d_{1,24}(24n+7)+d_{19,24}(24n+7)-d_{5,24}(24n+7)-d_{23,24}(24n+7))q^n. \tag{5.7.5}$$

Extracting those terms in (5.7.5) in which the powers of q are congruent to 2 modulo 4, we obtain

$$qG_{18}(q))G_5(q) = \sum_{n\geq 0}(d_{1,24}(96n+55)+d_{19,24}(96n+55)-d_{5,24}(96n+55)-d_{23,24}(96n+55))q^n,$$

which gives the identity (5.7.1).

Again, the identity (5.2.15) is equivalent to

$$\psi(q)\psi(q^4) = \frac{1}{2}\sum_{n\geq 0}(d_{1,4}(8n+5)-d_{3,4}(8n+5))q^n. \tag{5.7.6}$$

Using (5.2.7) in (5.7.6), we have

$$(f(q^{28},q^{36})+qf(q^{20},q^{44})+q^3f(q^{12},q^{52})+q^6G_{18}(q^4))\psi(q^4)$$
$$=\frac{1}{2}\sum_{n\geq0}(d_{1,4}(8n+5)-d_{3,4}(8n+5))q^n. \tag{5.7.7}$$

We collect the terms involving q^{4n+2} in (5.7.7) to obtain

$$qG_{18}(q)\psi(q)=\frac{1}{2}\sum_{n\geq0}(d_{1,4}(32n+21)-d_{3,4}(32n+21))q^n,$$

from which we easily arrive at (5.7.2).

The identity (5.2.19) is equivalent to

$$\psi(q^3)\psi(q^4)=\frac{1}{2}\sum_{n\geq0}(d_{1,3}(8n+7)-d_{2,3}(8n+7))q^n. \tag{5.7.8}$$

With the help of (5.2.7) and (5.2.9), we rewrite (5.7.8) as

$$(f(q^{84},q^{108})+q^3f(q^{60},q^{132})+q^9f(q^{36},q^{156})+q^{18}G_{18}(q^{12}))(G_5(q^{12})+q^4\psi(q^{36}))$$
$$=\frac{1}{2}\sum_{n\geq0}(d_{1,3}(8n+7)-d_{2,3}(8n+7))q^n. \tag{5.7.9}$$

Extracting the terms involving q^{12n+10} in (5.7.9), we obtain

$$qG_{18}(q)\psi(q^2)=\frac{1}{2}\sum_{n\geq0}(d_{1,3}(96n+87)-d_{2,3}(96n+87))q^n. \tag{5.7.10}$$

Equating the coefficients of q^{n+1} on both sides of (5.7.10) and noting that $d_{1,3}(96n+87)=d_{1,3}(32n+29)$ and $d_{2,3}(96n+87)=d_{2,3}(32n+29)$, we deduce (5.7.3). $\square$

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