A STUDY ON SOME GENERALIZED UNIFORM STRUCTURES IN L-TOPOLOGY

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Dedicated to my beloved parents

(M. L. Mitra & Kalyani Mitra)

Abstract

Uniform spaces occupy a very important place in the study of topological spaces. Uniform structures have also been extensively studied in the fuzzy setting over the last few decades. In this thesis we undertake the study of some generalizations of uniform structures in the L-valued fuzzy topological setting.

The primary objective of our study is to develop and study various generalizations of uniformity and quasi-uniformity in the L-topological setting. This includes the development of the notions of semi-uniformity, semi-quasi-uniformity and the study of localization of uniform and quasi uniform structures in the same setting. It also includes a study of their relationship and applications as regards to various other important topological notions such as compactness, the metrization problem and other notions in the context of uniform spaces in L-topological spaces.

The notions of L-semi-uniformity and L-semi-quasi-uniformity are introduced generalizing Hutton's uniformity and quasi-uniformity respectively. L-semi-quasi-uniform spaces are shown to be interior spaces and condition for an L-semi-quasi-uniform space to be L-topological is obtained. A notion of generalized continuous functions on L-semi-uniform spaces is defined and studied with respect to the relative interior spaces. The notion of L-semi p. metric as a generalization of Peng's p. metric is introduced and a metrizability result is obtained. The notions of completeness and totally boundedness are introduced in L-semi-uniform spaces

and in presence of totally boundedness, equivalence of completeness and compactness is examined. It is established that completeness are weakly hereditary and are preserved under L-semi-uniform isomorphism.

Further, the generalized uniformity structure in the form of L-local quasi-uniformity is introduced as a generalization of L-quasi-uniformity. The question of compatibility of L-local quasi-uniformity and L-topology has been examined. Notions of generalized uniformly continuous type of functions are introduced and examined in relation to uniform continuous functions. A Condition for L-pseudo quasi-metrizability is provided.

Another generalized uniform structure in the form of L-local uniformity is introduced via localization of the triangle axiom of Hutton's uniformity. The structure is then examined in relation to various existing notions including separation axioms, completeness, compactness and totally boundedness. Various notions are developed on the same and metrizability results are presented.

The relationship among these notions that have been developed is presented along with various examples and counterexamples on the newly developed notions to present a fairly complete framework of the generalized uniform structures. In a nutshell, our work has tried to fill some of the vacant spaces in the theory of fuzzy uniform spaces. In the concluding part we have outlined the implication of our work and suggest some future directions of work that can be undertaken as a sequel of our work.

DECLARATION

I, **Dipak Kumar Mitra**, hereby declare that the subject matter in this thesis entitled **A study on some generalized uniform structures in L-topology** is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/institute.

This thesis is being submitted to the Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

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CERTIFICATE

This is to certify that the thesis entitled **A study on some generalized uniform structures in L-topology** submitted to the School of Science and Technology of Tezpur University in partial fulfillment for the award of the degree of Doctor of Philosophy in Mathematical Sciences is a record of research work carried out by **Mr. Dipak Kumar Mitra** under my supervision and guidance.

All help received by him from various sources have been duly acknowledged.

No part of this thesis has been submitted elsewhere for award of any other degree.

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LIST OF SYMBOLS AND NOTATIONS

SYMBOL/NOTATION	MEANING
L	A complete, completely distributive lattice with an order reversing involution $'$
1_L (0_L)	The largest (resp. smallest) element of L
$M(L)$	The set of all the molecules in L
L^X	The family of all L-sets on X
$\underline{0}(\underline{1})$	The constant L-set taking each element of X to 0_L (resp. 1_L)
x_α	An L-fuzzy point with support x and value α, $\alpha \in L$
$\mathrm{Pt}(L^X)$	The set of all L-fuzzy points on X
x_α^*	Dual point of $x_\alpha \in \mathrm{Pt}(L^X)$
$f^\rightarrow$ $(f^\leftarrow)$	An L-fuzzy (resp. reverse) map induced by ordinary map f
$(L^X, \mathbb{F})$	An L-topological space with L-topology $\mathbb{F}$

A° ($\bar{A}$)	The interior (resp. closure) of the set A in $(L^X, \mathbb{F})$
$md_{\mathbb{R}}(L)$	The set of all monotonically decreasing mappings $\lambda \in L^{\mathbb{R}}$
$md_I(L)$	$\{\lambda \in md_{\mathbb{R}}(L) \mid t < 0 \Rightarrow \lambda(t) = 1, t > 1 \Rightarrow \lambda(t) = 0\}$
$\mathbb{R}[L]$ ($I[L]$)	Family of all the equivalent classes in $md_{\mathbb{R}}(L)$ (resp. $md_I(L)$)
$I(L)$	The L-topological space $(L^{I[L]}, \mathscr{T}_L^I)$
$\mathscr{T}_L^I$	Standard topology of $I(L)$
$\mathscr{S}_L^I$	Standard subbase of $I(L)$
$x_\alpha \ll A$	x_α is quasi-coincident with A, $A \in L^X$
$A\hat{q}B$	A quasi-coincides with B, $A, B \in L^X$
$\mathscr{Q}(x_\alpha)$	The family of all Q-nbd at x_α in $(L^X, \mathbb{F})$
$S \to x_\alpha$	The net S is converges to x_α
$\mathscr{F} \to x_\alpha$	The filter $\mathscr{F}$ is converges to x_α
$\mathscr{F} \rightsquigarrow x_\alpha$	x_α is in the cluster set of $\mathscr{F}$
$\mathscr{F} \propto x_\alpha$	x_α is a cluster point of $\mathscr{F}$
***L*-TOP**	The category of all L-topological spaces and L-continuous functions as morphisms

CHAPTER 1
INTRODUCTION

1.1 Background

1.1.1 Uniform structures in topological spaces

Uniform spaces play a very important role as a bridge between metric spaces and general topological spaces. Uniform spaces are those topological spaces which equipped with an additional structure which is used to define uniform properties such as completeness, uniform continuity and uniform convergence. They generalize metric spaces and topological groups and therefore underlie most of analysis. In point-set topology uniform spaces can be defined in three ways: the entourage approach, the pseudo-metric approach and the uniform cover approach, which were formulated by different mathematicians at different times. Study and applications of uniform spaces occupy a very important place in the study of topological spaces.

1.1.2 Generalized uniform structures

A wide variety of important classes of spaces occur as uniform spaces. Further, many generalizations of uniform spaces have been developed leading to a broad spectrum of theory and applications in related fields. In uniform spaces, the concept of uniformity and related notions of metric spaces were developed to a fair degree of success, though the space was felt to be equipped with fairly strong conditions. So it was, therefore, felt necessary at some point of time to weaken the conditions to develop more general spaces wherein results in uniform spaces could possibly be developed. As a result various generalization of uniform spaces such as quasi-uniform spaces, locally uniform spaces, locally quasi-uniform spaces, semi-uniform space and semi-quasi-uniform spaces were developed. Many important results were obtained in these spaces, which firmly placed them in the larger framework of topological spaces.

These uniform structures satisfy the following relation:

uniformity $\Rightarrow$ local uniformity $\Rightarrow$ semi-uniformity $\Rightarrow$ semi-quasi-uniformity

$\searrow$ $\searrow$ $\nearrow$

quasi-uniformity $\Rightarrow$ local quasi-uniformity.

We present an outline of the development of these generalized forms of uniform structures:

Pu and Pu in [56] defined a semi-quasi-uniform space to be a pair $(X, \mathscr{U})$ consisting of a set X and a filter $\mathscr{U}$ in $X \times X$ such that $(x, x) \in U$ for each $x \in X$ and $U \in \mathscr{U}$ (also introduced by Császár in [8] as pseudo-uniform space). A semi-quasi-uniform space $(X, \mathscr{U})$ determines a closure space (X, c) obtained by defining the neighbourhood filter of $x \in X$ to be equal to $\{U[x]: U \in \mathscr{U}\}$, i.e., by letting $cA = \{x \in X: U[x] \cap A \neq \varnothing$ for $U \in \mathscr{U}\}$. The authors also formulated a necessary and sufficient condition on $\mathscr{U}$ in order that (X, c) as above to be a topological space. Subsequently, various results on uniform spaces have been generalized to semi-quasi-uniform spaces [56]. In [57] the same authors proved analogues for semi-quasi-uniform spaces of two classical extension theorems from the theory of uniform topological spaces in semi-quasi-uniform spaces. Further, Steiner and Steiner in [67] developed the theory of semi-uniform spaces (semi-uniformizable= regular Hausdorff) as regards to the problem of completion.

The notion of localized form of uniform spaces was developed as locally uniform space by Williams [75]. The axioms for a local uniformity are obtained from those for a uniformity by localizing the triangle axiom, viz.: for each point x and each entourage U there is an entourage V such that $V \circ V[x] \subset U[x]$. It was shown that a topological space has a compatible local uniformity if and only if it is regular. In order to carry over the concepts of Cauchy filters and complete spaces to the local setting, Williams introduced the concept of a neighbourhood local uniformity (or NLU), i.e., a local uniformity such that for each entourage U and each point x there exists an entourage V such that $V[x] \times V[x] \subset U$. If $\mathscr{U}$ is a local uniformity on X, then there is a strongest NLU $\mathscr{V}$ which is contained in $\mathscr{U}$ such that $\mathscr{U}, \mathscr{V}$ are compatible. If $\mathscr{V}$ is a local uniformity, then for natural number $n \geq 2$, $\mathscr{V}^n = \{U \subset X \times X:$ there is a $V \in \mathscr{V}$ such that $V^n \subset U\}$ is an NLU with the same topology. Two local uniformities $\mathscr{V}$

and $\mathscr{W}$ are weakly equivalent if, for some natural numbers n, m, $\mathscr{V}^n \subset \mathscr{W}$ and $\mathscr{W}^m \subset \mathscr{V}$. Concepts of (weakly) uniform map, (weakly) uniform isomorphism etc. are then introduced and it is shown that a weakly uniform map is continuous with respect to the associated topologies. It is also shown that if a local uniformity $\mathscr{U}$ on a set X has a nested base, then the set of all neighborhoods of the diagonal in $X \times X$ is a uniformity. A locally uniform space with a countable base has a pseudo-metrizable topology. Also, every locally uniform space with a nested base has a paracompact topology. The well-known Nagata metrization theorem is then proved in a generalized form. The author also proved that there is a unique compatible NLU for a compact regular space. Vasudevan and Goel [69] introduced the concept of a locally uniform cover of a locally uniform space and proved that the collection of all locally uniform covers has two characteristic properties which, when assumed for a collection μ of covers of a nonempty set X, will generate a base for a local uniformity on X whose local uniform covers are precisely the members of μ. Further, Papatriantafillou [53] proved a generalized form of Ascoli's theorem for locally uniform spaces. Lindgren and Fletcher [41] introduced the concept of local quasi-uniformity and obtained a necessary and sufficient condition for a space to admit such a structure with a countable base. Hicks and Huffman [26] considered other aspects of locally quasi-uniform spaces and proved that a topological space is uniquely locally quasi-uniformizable if and only if the underlying topology is finite. Hušek in [32] provided a relation between proximity spaces, semi-uniform spaces and closure spaces in details. Any closure space is shown to be topological if and only if it has a semi-uniformity which is feebly transitive; every locally compact topological space has the coarsest uniformizable proximity inducing the closure function.

1.1.3 Fuzzy topology

Lotfi A. Zadeh [76] introduced fuzzy sets in 1965. Since then, fuzzy set theory has entered into many disciplines of science and technology as well as into many branches of humanities to become a highly fruitful and fast growing interdisciplinary area of research. Fuzzy sets are generalizations of sets, the basic mathematical objects on which modern mathematics is based. This way, fuzzy sets provide a more general framework for mathematical theories, pure and applied alike. It has therefore initiated new researches in the fields of category theory, topology, algebra, analysis, graph theory, theory of generalized measure and integrals etc.

General topology is one of the earliest branches of mathematics which applied fuzzy set theory systematically. The combination and synthesis of ideas, notions and methods of fuzzy set theory with general topology has resulted in fuzzy topology as a new branch of mathematics. It was in 1968 that C. L. Chang [7] grafted the notion of a fuzzy set into general topology and attempted to develop the basic topological notions for such spaces . Since the early eighties, the intensity of research in the areas of fuzzy topology has increased sharply, resulting in several thousand research papers. After Chang, several other definitions of a fuzzy topological space were put forward in an attempt of either generalizing the notion or in an attempt to remove several pathologies existent in the prevailing notions. Following Goguen's generalization of fuzzy sets to L-fuzzy sets [17], Chang's definition was later generalized to L-fuzzy topological spaces. Other important approaches are due to Lowen, Shostak et al. [64].

In the nearly four decades of existence fuzzy topology has made considerable advances. Several benefits have come out:

- the categories of fuzzy topology have been able to fill up several major deficiencies in the category **TOP** of (classical) topological spaces [61].

- proofs used in fuzzy topology compared to original techniques at times turn out to be more effective, conceptual as well as simple due to use of stratification methods; providing stronger results with better proofs.

- brings closer to other branches of classical mathematics. Use of results of L-topology has given characterizations of complete distributive lattices leading to closer relationship between algebra and analysis.

- use of results of L-topology being a theory developed in lattice has yielded analytic characterizations of complete distributive lattices leading to closer relationship between algebra and analysis [44].

- fuzzy topology has also found substantial applications in upcoming fields such as digital topology and image processing with applications to GIS ([10], [63], [34], [9], [11]).

It may at the same time be pointed that building up of various notions in fuzzy topology on the lines of general topology has not been easy. Due to the existence of stratum structures, it is often more difficult to establish a theory in fuzzy topology. For the same reason one single notion in general topology usually generates several counterparts in fuzzy topology having their respective advantages and shortcomings. It times it may look unrealistic to choose or establish one best analogous notion. There have been efforts directed to unify different notions with mixed degree of success.

1.1.4 Uniform structures in fuzzy setting

Uniform spaces on various categories of fuzzy topological spaces have been studied by several authors including, Höhle, Koetz, Hutton, Katsarsas, Lowen, Hu Cheng Ming et. al. [27, 37, 29, 33, 45, 50].

Almost a decade after the introduction of fuzzy topological spaces by Chang in 1968, Hutton [29] generalized the notions of quasi-uniformity and uniformities in topological spaces to the setting of L-valued topological spaces, showing that every fuzzy topological space is quasi-uniformizable and that every completely regular fuzzy topological space is uniformizable. Lowen [45] introduced an independent notion of fuzzy uniform space and derived a fuzzy topology from his notion of fuzzy uniformity. Lowen further proved that the category of uniform spaces is injected in the category of fuzzy uniform spaces and established some basic results on the associated fuzzy topology. Further, Lowen and Wuyts [46, 47] contributed significantly on completeness, pre-compactness. completion and ultracompactness of Lowen Uniform spaces.

Katsaras [33] introduced and studied some of the properties of n-completely regular fuzzy neighborhood structures. It was shown that a fuzzy neighborhood structure N is n-completely regular if and only if it is induced by a fuzzy uniformity in the sense of Lowen. He also introduced fuzzy L-quasi-uniform spaces and proved that every fuzzy neighborhood structure is induced by some fuzzy L-quasi-uniform structure. Burton in [5] defined the notion of a precompact fuzzy set in a fuzzy uniform space using the notion of a Cauchy prefilter in a fuzzy uniform space. Burton et. al. [6] also extended the notion of a uniform space in a intuitive way by defining the degree to which a subset is an element of a uniformity. The category of Lowen fuzzy uniform spaces was shown to be isomorphic to the category of these spaces. Various characterization theorems are obtained and it is shown that this provides a 'good' extension of the notion of a complete subset of a uniform space. The theory of complete fuzzy sets in a fuzzy uniform space is developed and this theory generalizes the theory of complete subsets of a uniform space. The generalized uniform space analogues of various fuzzy uniform space notions are characterized. Soetens and Wuyts [65] introduced fuzzy uniform spaces by means of coverings and generalizing a classical result, proved that these uniformities are equivalent to the diagonal fuzzy uniformities defined by Lowen. They also studied a number of functors between the categories of fuzzy uniform spaces and uniform spaces, both in the diagonal and the covering setting. Zhu [14] studied (quasi-) uniformly continuous mappings and (quasi-) uniform structures on products of fuzzy (quasi-) uniform spaces. Liang [39] introduced the concepts of fuzzy α-Cauchy net, α-completeness and completeness, and obtained results concerning productive and closed hereditary α-completeness. In another notable work, Hu Cheng Ming [50] provided detailed discussion on the concepts of fuzzy uniform continuity, fuzzy product uniformities, fuzzy embeddings and fuzzy completeness.

In a fairly recent work, Garcîa et. al. [60] introduced uniform spaces in a unifying framework of GL-monoid to include both the categories of Lowen uniformity and Hutton types uniformities.

1.2 Motivation and Objectives

The brief sketch of the development of the theory of uniform spaces and their generalizations in various categories of fuzzy topological spaces indicates the rapid development in the theory of uniform spaces in the fuzzy setting. However, a careful examination of the development indicates a somewhat scattered development unlike their counterpart in the general topological setting. Moreover several important concepts namely, *semi-uniformity*, *semi-quasi-uniformity*, *local uniformity*, *local quasi-uniformity* remained unexplored and unutilized. This therefore indicates a void in the development of the theory of generalization of uniform spaces in the categories of fuzzy topological spaces.

The primary objective of our study is to introduce and study various generalizations of uniformity and quasi-uniformity in the setting of L-topological spaces. This includes the development of the notions of semi-uniformity, semi-quasi-uniformity and the study of localization of uniform and quasi-uniform structures in the same setting. A study of the mutual relationship of the newly introduced notions and relationship with the existing notions by means of construction of suitable examples and counter-examples are taken up. Further, an extensive study of the relationship and applications of these generalized uniform structures vis-a-vis various other important topological notions such as compactness, totally boundedness and convergence structures have been taken up. The pseudo-metrization problem has also been taken up in the context of these generalized forms of uniform structures. The study of deviations from general uniform structures and finding reasons thereof would appear to indicate a better understanding of the subject.

In brief, this study makes a concerted attempt to develop a theory of certain generalized uniform structures and related concepts in the L-valued topological setting comprising the category of fixed basis L-topological spaces.

1.3 Overview of the thesis

1.3.1 Organization and Chapterwise outline

The thesis is organized as follows:

Chapter 1: This is an introductory chapter in which a brief sketch of the background of the work embodied in the thesis is presented. The motivational aspects, basic objectives and an outline of the work carried out is also included in this chapter.

Chapter 2: In this chapter basic definitions and results that are used in the subsequent chapters are provided as preliminaries.

Chapter 3: We build up the generalized uniform structures namely L-semi-quasi-uniformity and L-semi-uniformity in context of the category **L-TOP**. These are generalizations of Hutton's quasi-uniformity and uniformity respectively. Several results of uniform spaces including the problem of metrization are developed in generalized form. Examples of an L-semi-quasi-uniformity that is not Hutton's quasi-uniformity as well as L-semi-uniformity are provided. A sufficient condition under which the generating interior space will be L-topological is obtained. L-semi-uniformly continuous functions are shown as continuous with respect to the induced interior spaces. Various results on L-semi-uniformly continuous functions are obtained. The notion of L-semi-pseudo-metric as a generalization of Peng's p. metric is introduced and it has been established that every L-semi-uniform space with countable base is L-semi-pseudo-metrizable. Further, the important topological notions of completeness and totally boundedness are introduced in L-semi-uniform spaces. We have also shown that in a totally bounded space, completeness is equivalent to compactness.

A part of the work presented in this chapter appeared in "*On L-semi-pseudo metrization*", in the journal of International Mathematical Forum [24]. Selected portion of this chapter was presented at the 31st Linz Seminar on Fuzzy Set Theory [25].

Chapter 4: In this chapter we have introduced L-local quasi-uniformity in terms of L-semi-quasi-uniformity as a generalization of Hutton's quasi-uniformity, also in the category **L-TOP**. We have taken up the question of compatibility of L-local quasi-uniformity and L-topology and it has been shown that every L-local quasi-uniformity generates an L-topology and conversely every L-topology is generated by an L-local quasi-uniformity. Further, the notion of L-weakly quasi-uniformly continuous functions is

introduced and it is shown that every L-weakly quasi-uniformly continuous functions are L-fuzzy continuous. We have also proved a metrization theorem in this setting showing that every L-locally quasi-uniform space with countable base is L-pseudo quasi-metrizable.
A part of the work presented in this chapter is appearing in "*L - locally quasi-uniform spaces*," in the Journal of The Indian Academy of Mathematics [21]. The notion of fuzzy locally quasi-uniform spaces has also been established in Chang's category of fuzzy topological spaces in [18], submitted for publication.

Chapter 5: In this chapter we have taken up the L-local quasi-uniformities containing the inverse of each of its member and called these spaces as L-local uniformity. The notion of L-local uniformity generalizes Hutton's uniformity which provides more insight into the placement of the generalized uniform structures in relation to stronger topological axioms in terms of uniformization results. Results on compactness in relation to totally boundedness and complete regularity are also obtained in this chapter. In the last section of this chapter, we take up the problem of metrizability of L-locally uniform spaces. By showing that an $L-$topological space admits a uniformity with a countable base if it admits an $L-$local uniformity with a countable base, we have established that every L-locally uniform space with countable base is L-pseudo-metrizable.
Part of the work presented in this chapter is being published in "*L - locally uniform spaces*", in the Journal of Fuzzy Mathematics [23]. An independent $L = I$ version was present in an earlier paper entitled "Fuzzy locally uniform spaces" in the same journal [22].

Chapter 6: The problem of completion of spaces has always occupied topologists. We take up this problem in the context of L-locally uniform spaces. We have introduced the notion of strong completeness in a subclass of L-locally uniform spaces. Various results on the hereditary property, unimorphic invariance and productivity of strong completeness are being presented in this chapter. We have further introduced a relationship of strong completeness and compactness analogous to the classical results.

Some results in this chapter are presented in the paper entitled "*Strong completeness in L - locally uniform spaces*" [20], submitted for publication.

Chapter 7: In the last chapter a brief observation in the form of a conclusion is presented. The directions of future research work that can be undertaken based on the work embodied in the thesis are briefly outlined.

A bibliography which includes all the references is given at the end of the thesis.

CHAPTER 2

PRELIMINARIES

2.1 Preliminaries: Definitions and Results

2.1.1 Lattice theoretic background

In the sequel, we provide a few basic lattice theoretic definitions and results which are utilized in the subsequent chapters. Most lattice theoretic definitions and results cited here can be found in [44], [3] or [16].

Definition 2.1.1. Let P be a set and '$\leq$' is a relation on P.

Then P is said to be *partially ordered set*(poset for short) with respect to the relation '$\leq$', if '$\leq$' satisfies the following conditions:

(po1) Reflexive: $a \leq a$ for every $a \in P$.

(po2) Antisymmetric: For any a, $b \in P$, $a \leq b$ and $b \leq a$ imply $a = b$.

(po3) Transitive: $a \leq b$ and $b \leq c$ in P imply $a \leq c$.

Definition 2.1.2. Let L be a poset and $A \subseteq L$. An element $x \in L$ is called

1) The *join* of A, denoted by $\bigvee A$ or $\sup A$, if,

i) x is an upper bound of A,

ii) if y is an upper bound for A, then $x \leq y$.

If A is finite, we shall call $\bigvee A$ (if it exists) a *finite join*. If A consists of two elements a and b, then we will write $a \bigvee b$.

2) The *meet* of A, denoted by $\bigwedge A$ or $\inf A$, if,

i) x is a lower bound of A,

ii) if y is a lower bound of A, then $y \leq x$.

If A is finite, call $\bigwedge A$ (if it exists) a *finite meet*. If A consist of two elements a and b, then we will write $a \bigwedge b$.

Remark 2.1.1. By the antisymmetry of partial order, a join or a meet in a poset, if it exists, must be unique.

Definition 2.1.3. Let L be a poset. It shall be called a *join-semilattice* if every join for a finite subset of L exists; particularly, the *smallest element* exists as the join of empty subset.

L is called a *meet-semilattice* if every meet for a finite subset of L exists; particularly, the *largest element* exists as the meet of empty subset.

L is called a *lattice*, if it is both a *join-semilattice* and a *meet-semilattice.*

Remark 2.1.2. A lattice will always be nonempty; it can although consist of only one element- the smallest element coincides with the largest one. These lattices are not considered in our investigation. Therefore, in the sequel, every lattice is always assumed to possess at least two elements: the smallest element 0_L and the largest element 1_L.

Remark 2.1.3. Some authors [3, 16. 70] defined semilattices and lattices by join of and meet of two elements (equivalently, nonempty finite subsets) but without the smallest element and the largest element. Since, smallest elements and largest elements are always needed in our investigation, we adopt the preceding definition.

Definition 2.1.4. A poset L is called a *complete join-semilattice* if every join for an arbitrary subset of L exists; particularly, the *smallest element* exists as the join of empty subset.

L is called a *complete meet-semilattice* if every meet for an arbitrary subset of L exists; particularly, the *largest element* exists as the meet of empty subset.

L is called *complete lattice*, if it is both a *complete join-semilattice* and a *complete meet-semilattice.*

Proposition 2.1.1. *Let L be a poset, then the following conditions are equivalent:*

(i) *L is a complete lattice.*

(ii) *L has the smallest element and $\forall\ A \subseteq L,\ A \neq \phi,\ \bigvee A$ exists in L.*

(iii) *L has the largest element and $\forall\ A \subseteq L,\ A \neq \phi,\ \bigwedge A$ exists in L.*

(iv) *L is a lattice and $\forall\ A \subseteq L,\ A \neq \phi,\ \bigvee A$ exists in L.*

(v) *L is a lattice and $\forall\ A \subseteq L,\ A \neq \phi.\ \bigwedge A$ exists in L.*

Definition 2.1.5. Let L be a complete lattice. L is called *infinitely distributive*, if L satisfies both following two conditions (IFD1) and (IFD2), called the *1st infinitely distributive law* and *2nd infinitely distributive law* respectively:

(IFD1) $\forall\ a \in L,\ \ \forall B \subseteq L,\ \ a \bigwedge \bigvee B = \bigvee_{b \in B}(a \bigwedge b),$

(IFD2) $\forall\ a \in L,\ \ \forall B \subseteq L,\ \ a \bigvee \bigwedge B = \bigwedge_{b \in B}(a \bigvee b).$

Remark 2.1.4. The 1st infinitely distributive law is not equivalent to 2nd infinitely distributive law.

Proposition 2.1.2. *Let L be a complete lattice. Then*

(i) *L satisfies the 1st infinitely distributive law (IFD1) if and only if*

$\forall\ A,\ B \subseteq L,\quad \bigvee A \bigwedge \bigvee B = \bigvee_{a\in A\ b\in B}(a \bigwedge b)$.

(ii) *L satisfies the 2nd infinitely distributive law (IFD2) if and only if*

$\forall\ A,\ B \subseteq L,\quad \bigwedge A \bigvee \bigwedge B = \bigwedge_{a\in A,\, b\in B}(a \bigvee b)$.

Definition 2.1.6. Let L be a complete lattice. L is called *completely distributive*, if L satisfies following two conditions (CD1) and (CD2), called *completely distributive law*:

$\forall\{\{a_{i,j} \mid j \in J_i\} \mid i \in I\} \subseteq \mathscr{P}(L) \setminus \{\phi\},\ I \neq \phi,$

(CD1) $\bigwedge_{i\in I}(\bigvee_{j\in J_i} a_{i,j}) = \bigvee_{\varphi\in\Pi_{i\in I}J_i}(\bigwedge_{i\in I} a_{i,\varphi(i)}),$

(CD2) $\bigvee_{i\in I}(\bigwedge_{j\in J_i} a_{i,j}) = \bigwedge_{\varphi\in\Pi_{i\in I}J_i}(\bigvee_{i\in I} a_{i,\varphi(i)}).$

Remark 2.1.5. For $I = \{0,\ 1\}$, $J_0 = \{0\}$ in (CD1) and (CD2). So, (CD1)$\Rightarrow$ (IFD1), (CD2)$\Rightarrow$ (IFD2) and hence completely distributive lattice must be infinitely distributive.

But the converse is not true in general.

Theorem 2.1.3. *A completely lattice L satisfies (CD1) if and only if it satisfies (CD2).*

Proposition 2.1.4. *Let $\{P_t \mid t \in \Lambda\}$ be a family of posets. Then the relation $\leq$ on $\Pi_{t\in\Lambda}P_t$ defined by,*

$$\alpha,\ \beta \in \Pi_{t\in\Lambda}P_t,\quad \alpha \leq \beta \Leftrightarrow \forall\ t \in \Lambda,\ \alpha(t) \leq \beta(t)$$

is reflexive, antisymmetric and transitive.

Theorem 2.1.5. *Let $\{L_t \mid t \in \Lambda\}$ be a family of posets. Then $\Pi_{t\in\Lambda} L_t$ is a completive distributive lattice $\Leftrightarrow \forall\ t \in \Lambda$, L_t is a completive distributive lattice.*

Definition 2.1.7. Let L be a lattice, $\alpha \in L$. α is called *join-irreducible*, if $\alpha < 1_L$ and $\forall\ a,\ b \in L$, $\alpha = a \bigvee b \Rightarrow \alpha = a$ or $\alpha = b$.

A join-irreducible element of L is called *molecule* in L.

The set of all the molecules in L is denoted by $M(L)$.

Remark 2.1.6. Every element in a completely distributive lattice can be represented as a join of molecules.

Definition 2.1.8. Let L be a complete lattice. Define a relation $\preceq$ in L as follows:

$\forall\ a,\ b \in L$, $a \preceq b$ if and only if $\forall\ S \subseteq L$, $b \leq \bigvee S \Rightarrow\ \exists\, s \in S$ such that $a \leq s$. $\forall\ a \in L$, denote $\beta_L(a) = \{b \in L \mid b \preceq a\}$, $\beta^*_L(a) = M(\beta_L(a))$; or denote them respectively by $\beta(a)$ and $\beta^*(a)$ for short.

$\forall\ a \in L$. $D \subseteq \beta(a)$ is called a minimal set of a, if $\bigvee D = a$.

Theorem 2.1.6. *Let L be a complete lattice, then following conditions are equivalent:*

(i) *L is completely distributive lattice.*

(ii) $\forall\ a \in L$, *$\beta(a)$ is a minimal set of a.*

(iii) $\forall\ a \in L$, *$\beta^*(a)$ is a minimal set of a.*

Theorem 2.1.7. *Let L be a complete lattice. Then*

(i) *$\beta : L \to \mathscr{P}(L)$ is an arbitrary join preserving mapping, i.e., for every $A \subseteq L$,*

$$\beta(\bigvee A) = \bigvee_{a\in A} \beta(a).$$

(ii) $\beta^* : L \to \mathscr{P}(M(L))$ *is an arbitrary join preserving mapping, i.e., for every* $A \subseteq L$,

$$\beta^*(\bigvee A) = \bigvee_{a \in A} \beta^*(a).$$

Where $\mathscr{P}(L)$ *and* $\mathscr{P}(M(L))$ *are respectively power set of* L *and* $M(L)$.

Definition 2.1.9. A set D equipped with a relation $\leq$ is said to be *down-directed*, if for any finite $D_0 \subseteq D$, there is $d_0 \in D$ such that $d_0 \leq d$ for every $d \in D_0$.

Definition 2.1.10. Let L be a lattice, then a mapping $' : L \to L$ is called *order reversing involution* if

$\forall\ a, b \in L.\ a \leq b \Rightarrow b' \leq a'$ and $(a')' = a$.

Proposition 2.1.8. *Let* L *be a lattice with order reversing involution* $'$. *Then following equalities hold if all the joins and meets involved exist:*

(DM1) $(\bigvee_i a_i)' = \bigwedge_i a_i'$.

(DM2) $(\bigwedge_i a_i)' = \bigvee_i a_i'$.

Definition 2.1.11. A completely distributive lattice L is called a *fuzzy lattice* or an F-lattice for short, if L has an order-reversing involution $' : L \to L$.

2.1.2 *L*-Fuzzy sets and *L*-topology

Throughout the thesis, we consider $(L, \leq, \bigwedge, \bigvee, ')$ as a fuzzy lattice with order reversing involution $'$; 0_L and 1_L are respectively inf and sup in L.

For simplicity, we shall denote it by L.

Here we provide some basic definitions and results of L-fuzzy sets and L-fuzzy topology, which can be found in [44], [29] or [31].

Definition 2.1.12. Let X be a non empty arbitrary set and L a complete lattice. Let L^X will denote the collection of all mappings $A : X \to L$. Then any member of L^X is called an *L-fuzzy set.* L^X here is called an *L-fuzzy space.*

The L-fuzzy sets $x_\alpha : X \to L$ defined by $x_\alpha(y) = 0_L$ if $x \neq y$ and $x_\alpha(y) = \alpha$ if $x = y$ are the *L-fuzzy points.*

The set of all L-fuzzy points on X is denoted by $\mathrm{Pt}(L^X)$.

The mappings $A : X \to L$ and $B : X \to L$ defined by $A(x) = 1_L,\ \forall x \in X$ and $B(x) = 0_L,\ \forall x \in X$ are denoted by $\underline{1}$ and $\underline{0}$ respectively. Clearly, $\underline{1}$ and $\underline{0}$ will act as the largest and smallest element respectively on the fuzzy lattice L^X.

Definition 2.1.13. For any $A,\ B \in L^X$, the *union* $A \bigcup B$ and *intersection* $A \bigcap B$ of A and B are respectively defined as

$(A \bigcup B)(x) = A(x) \bigvee B(x), \quad \forall\, x \in X.$

$(A \bigcap B)(x) = A(x) \bigwedge B(x), \quad \forall\, x \in X.$

Proposition 2.1.9. *Let L^X be an L-fuzzy space. Then*

(i) *L^X is a complete lattice and for any $\mathscr{A} \subseteq L^X$, we have the following:* $\forall\, x \in X,\ (\bigvee_{A \in \mathscr{A}} A)(x) = \bigvee_{A \in \mathscr{A}} A(x)$ *and* $(\bigwedge_{A \in \mathscr{A}} A)(x) = \bigwedge_{A \in \mathscr{A}} A(x)$.

(ii) *L is distributive* $\Leftrightarrow$ *L^X is distributive.*

(iii) *L satisfies* IFD1 $\Leftrightarrow$ *L^X satisfies* IFD1.

(iv) *L satisfies* IFD2 $\Leftrightarrow$ *L^X satisfies* IFD2.

(v) *L is completely distributive* $\Leftrightarrow$ *L^X is completely distributive.*

Definition 2.1.14. Let X be a nonempty ordinary set and L a fuzzy lattice with order-reversing involution $'$. Let $' : L^X \to L^X$ be an operation on L^X, called *pseudo-complementary operation*, defined by,

$A'(x) = (A(x))', \forall x \in X, \forall A \in L^X.$

Proposition 2.1.10. *Let X be nonempty ordinary set and L a fuzzy lattice with order-reversing involution $'$. Then the pseudo-complementary operation $' : L^X \to L^X$ is an order-reversing involution.*

Definition 2.1.15. $A \subseteq B$ iff $A(x) \leq B(x)$ and *complement* of A is defined as $A'(x) = A(x)'$.

For any $x_\alpha \in \mathrm{Pt}(L^X)$ and $A \in L^X$, $x_\alpha \in A$ if and only if $\alpha < A(x)$ and $x_\alpha \subseteq A$ if and only if $\alpha \leq A(x)$. In particular for any $y_\beta \in \mathrm{Pt}(L^X)$, $x_\alpha \subseteq y_\beta$ if and only if $x = y$ and $\alpha \leq \beta$.

Definition 2.1.16. For any ordinary mapping $f : X \to Y$, the induced *L-fuzzy mapping* $f^\to : L^X \to L^Y$ and its *L-fuzzy reverse mapping* $f^\leftarrow : L^Y \to L^X$ respectively are defined as.

$f^\to(A)(y) = \bigvee\{A(x) \mid x \in X,\ f(x) = y\},\ \forall A \in L^X,\ \forall y \in Y.$

$f^\leftarrow(B)(x) = B(f(x)).\ \forall B \in L^Y,\ \forall x \in X$

Symbol $f^\to$ and $f^\leftarrow$ always denote $f^\to$ to be the L-fuzzy mapping induced from an ordinary mapping f and $f^\leftarrow$ is the L-fuzzy reverse mapping of $f^\to$.

Theorem 2.1.11. *Let L^X and L^Y be L-fuzzy spaces, $f : X \to Y$ an ordinary mapping. Then*

(i) *$f^\to$ is injective if and only if f is injective.*

(ii) *$f^\to$ is surjective if and only if f is surjective.*

(iii) $f^{\rightarrow}$ *is bijective if and only if* f *is bijective.*

Theorem 2.1.12. *Let* L^X *and* L^Y *be L-fuzzy spaces,* $f : X \to Y$ *and* $g : Y \to Z$ *be ordinary mappings. Then*

(i) $g^{\rightarrow}f^{\rightarrow} = (gf)^{\rightarrow}$.

(ii) $f^{\leftarrow}g^{\leftarrow} = (gf)^{\leftarrow}$.

Theorem 2.1.13. *Let* L^X, L^Y *be L-fuzzy space,* $f : X \to Y$ *an ordinary mapping. Then*

(i) $A \subseteq f^{\leftarrow}f^{\rightarrow}(A), \quad \forall\, A \in L^X.$

(ii) $f^{\rightarrow}f^{\leftarrow}(B) \subseteq B, \quad \forall\, B \in L^X.$

(iii) $f^{\rightarrow}(A) = f^{\rightarrow}f^{\leftarrow}f^{\rightarrow}(A), \quad \forall\, A \in L^X.$

(iv) $f^{\leftarrow}f^{\rightarrow}f^{\leftarrow}(B) = f^{\leftarrow}(B), \quad \forall\, B \in L^X.$

Proposition 2.1.14. *Let* X, Y *be nonempty ordinary sets,* L *a fuzzy lattice and* $f : X \to Y$ *be an ordinary mapping. Then* $\forall A \in L^X$, $(f^{\rightarrow}(A))' \subseteq f^{\rightarrow}(A')$ *and* $f^{\leftarrow}(A') = (f^{\leftarrow}(A))'$, *where* $'$ *is the pseudo-complementary operation.*

Definition 2.1.17. Let $\mathbb{F}$ be a subset of L^X, then $\mathbb{F}$ is called an *L-fuzzy topology* on L^X or call *L-topology* for short, if $\mathbb{F}$ is closed under finite intersection and arbitrary union. The elements of $\mathbb{F}$ are called *L-open sets* and their complements are the *L-closed sets.* The pair $(L^X, \mathbb{F})$ is called an *L-topological space.*

Throughout the thesis, the L-open and L-closed sets are referred to as open and closed sets respectively.

Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two L-topologies on L^X. Then $\mathbb{F}_1$ is called *coarser* than $\mathbb{F}_2$ or $\mathbb{F}_2$ is called *finer* than $\mathbb{F}_1$, if $\mathbb{F}_1 \subseteq \mathbb{F}_2$.

Definition 2.1.18. Let $(L^X, \mathbb{F})$ be an L-topological space. Then

(i) A nonempty subfamily $\mathscr{B}$ of $\mathbb{F}$ is called *base* for $\mathbb{F}$, if

$$\mathbb{F} = \{\bigcup \mathscr{A} \mid \mathscr{A} \subseteq \mathscr{B}\}.$$

(ii) A nonempty subfamily $\mathscr{C}$ of $\mathbb{F}$ is called *subbase* for $\mathbb{F}$, if

$$\{\bigcap \mathscr{D} \mid \mathscr{D} \in [\mathscr{C}]^{<\omega} \setminus \{\phi\}\}$$

is a base for $\mathbb{F}$.

Definition 2.1.19. Let $(L^X, \mathbb{F})$ be an L-topological space and $A \in L^X$.

(i) The *interior* of A, denoted by A° is defined as $A^\circ = \bigcup\{G \mid G \in \mathbb{F}$ and $G \subseteq A\}$.

(ii) The *closure* of A, denoted by $\bar{A}$ is defined as $\bar{A} = \bigcap\{F \mid F' \in \mathbb{F}$ and $A \subseteq F\}$.

Theorem 2.1.15. *Let $(L^X, \mathbb{F})$ be an L-topological space. Then*

(i) $\underline{0}^\circ = \underline{0}$, $\underline{1}^\circ = \underline{1}$.

(ii) $A^\circ \subseteq A, \quad \forall\, A \in L^X$.

(iii) $A^{\circ\circ} = A^\circ, \quad \forall\, A \in L^X$.

(iv) $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ, \quad \forall\, A,\, B \in L^X$.

(v) $(A \bigcap B)^\circ = A^\circ \bigcap B^\circ, \quad \forall\, A,\, B \in L^X$.

Theorem 2.1.16. *Let $(L^X, \mathbb{F})$ be an L-topological space. Then*

(i) $\bar{\underline{0}} = \underline{0}$, $\bar{\underline{1}} = \underline{1}$.

(ii) $A \subseteq \bar{A}. \quad \forall\, A \in L^X$.

(iii) $\bar{\bar{A}} = \bar{A}, \quad \forall\, A \in L^X$.

(iv) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}, \quad \forall\, A,\, B \in L^X$.

(v) $\overline{(A \bigcup B)} = \bar{A} \bigcup \bar{B}, \quad \forall\, A,\, B \in L^X$.

Theorem 2.1.17. *Let* $(L^X, \mathbb{F})$ *be an L-topological space. Then*

(i) $(A^\circ)' = \overline{A'}, \quad \forall\ A \in L^X.$

(ii) $(A')^\circ = (\bar{A})', \quad \forall\ A \in L^X.$

(iii) $A^\circ = (\overline{A'})', \quad \forall\ A \in L^X.$

(iv) $\bar{A} = ((A')^\circ)', \quad \forall\ A \in L^X.$

Proposition 2.1.18. *Let* X *be a nonempty ordinary set and* L *a fuzzy lattice. Let* $\imath\ .\ L^X \to L^X$ *be a map satisfies the following axioms.*

(IO1) $\imath(\underline{1}) = \underline{1}.$

(IO2) $\imath(A) \subseteq A, \quad \forall\ A \in L^X.$

(IO3) $\imath(A \bigcap B) = \imath(A) \bigcap \imath(B), \quad \forall\ A,\ B \in L^X.$

(IO4) $\imath(\imath(A)) = A, \quad \forall\ A \in L^X.$

Then $\mathbb{F} = \{A \in L^X \mid \imath(A) = A\}$ *is an L-topology and* $\imath(A) = A^\circ$.

Proposition 2.1.19. *Let* X *be a nonempty ordinary set and* L *a fuzzy lattice. Let* $c : L^X \to L^X$ *be a map satisfies the following axioms.*

(CO1) $c(\underline{0}) = \underline{0}.$

(CO2) $A \subseteq c(A), \quad \forall\ A \in L^X.$

(CO3) $c(A \bigcup B) = c(A) \bigcup c(B). \quad \forall\ A,\ B \in L^X.$

(CO4) $c(c(A)) = A. \quad \forall\ A \in L^X.$

Then $\mathbb{F} = \{A \in L^X \mid c(A') = A'\}$ *is an L-topology and* $c(A) = \bar{A}$.

Definition 2.1.20. Let $(L^X, \mathbb{F}_1)$ and $(L^Y, \mathbb{F}_2)$ be two L-topological spaces. Then a mapping $f^{\to} : L^X \to L^Y$ is called an *L-fuzzy continuous mapping* or call *continuous mapping* for short iff for any $V \in \mathbb{F}_2$ implies $f^{\leftarrow}(V)$ is in $\mathbb{F}_1$.

Definition 2.1.21. Let $(L^X, \mathbb{F}_1)$ and $(L^Y . \mathbb{F}_2)$ be two L- topological spaces and $f^{\to} : L^X \to L^Y$ be a mapping. Then

(i) $f^{\to}$ is called an *open*, if it maps every open subset in $(L^X, \mathbb{F}_1)$ as an open one in $(L^Y, \mathbb{F}_2)$, i.e. for any $G \in \mathbb{F}_1$ implies $f^{\to}(G) \in \mathbb{F}_2$.

(ii) $f^{\to}$ is called *closed*, if it maps every closed subset in $(L^X, \mathbb{F}_1)$ as a closed one in $(L^Y, \mathbb{F}_2)$, i.e. for any $F \in \mathbb{F}_1'$ implies $f^{\to}(F) \in \mathbb{F}_2'$.

Definition 2.1.22. Let $(L^X, \mathbb{F}_1)$ and $(L^Y, \mathbb{F}_2)$ be two L- topological spaces, then $f^{\to} : L^X \to L^Y$ is called an *L-fuzzy homeomorphism*, if it is bijective, continuous and open.

Definition 2.1.23. Let $\{(L^{X_t}, \mathbb{F}_t) \mid t \in \Lambda\}$ be a family of L-topological spaces, where Λ is the index set. Denote $X = \Pi_{t\in\Lambda} X_t$.

For every $t \in \Lambda$, let $\pi_t : X \to X_t$ be the ordinary projection, we define the *projection* from L-fuzzy space L^X to L-fuzzy space L^{X_t} as

$$\pi_t^{\to} : L^X \to L^{X_t}.$$

We define the *product topology* of L-topologies $\{\mathbb{F}_t \mid t \in \Lambda\}$ on L^X, denoted by $\Pi_{t\in\Lambda}\mathbb{F}_t$, as the L-topology $\mathbb{F}$ on L^X generated by the subbase

$$\{\pi_t^{\leftarrow}(G_t) \mid G_t \in \mathbb{F}_t,\, t \in \Lambda\}.$$

and call the L-topological space $(L^X, \mathbb{F})$ the *product space* of L-topological spaces $\{(L^{X_t}, \mathbb{F}_t) \mid t \in \Lambda\}$, denote it by $\Pi_{t\in\Lambda}(L^{X_t}, \mathbb{F}_t)$.

Theorem 2.1.20. *Let $\{(L^{X_t}, \mathbb{F}_t) \mid t \in \Lambda\}$ be a family of L-topological spaces and $(L^X, \mathbb{F})$ be their product space. Then*

(i) *For every $t \in \Lambda$, projection $\pi_t^{\to} : (L^X, \mathbb{F}) \to (L^{X_t}, \mathbb{F}_t)$ is continuous.*

(ii) *The product topology $\mathbb{F}$ is just the coarsest L-topology on L^X which makes every projection $\pi_t^{\to}$ continuous.*

Definition 2.1.24. Let $(L^X, \mathbb{F})$ be an L-topological space. Then $(L^X, \mathbb{F})$ is said to be *regular*, if for every $G \in \mathbb{F}$ and $x_\alpha \subseteq G$, there is $A \in \mathbb{F}$ such that $x_\alpha \subseteq A \subseteq \bar{A} \subseteq G$.

Theorem 2.1.21. *Product of regular L-topological spaces is regular.*

Definition 2.1.25. Let L be a complete lattice with order-reversing involution $' : L \to L$.

$\lambda \in L^{\mathbb{R}}$ is called *monotonically increasing*, if

$$s,\ t \in \mathbb{R},\ s \leq t \Rightarrow \lambda(s) \leq \lambda(t).$$

$\lambda \in L^{\mathbb{R}}$ is called *monotonically decreasing*, if

$$s,\ t \in \mathbb{R},\ s \leq t \Rightarrow \lambda(s) \geq \lambda(t).$$

Denote the family of all the monotonically decreasing mappings $\lambda \in L^{\mathbb{R}}$ fulfilling the following conditions by $md_{\mathbb{R}}(L)$:

$$\bigvee_{t \in \mathbb{R}} \lambda(t) = 1, \quad \bigwedge_{t \in \mathbb{R}} \lambda(t) = 0.$$

Denote the family of all the elements in $md_{\mathbb{R}}(L)$ fulfilling the following conditions by $md_I(L)$:

$$t < 0 \Rightarrow \lambda(t) = 1, \quad t > 1 \Rightarrow \lambda(t) = 0.$$

For every $\lambda \in md_{\mathbb{R}}(L)$ and every $t \in \mathbb{R}$, we define

$$\lambda(t-) = \bigwedge_{s<t} \lambda(s). \quad \lambda(t+) = \bigvee_{s>t} \lambda(s).$$

Define an equivalence relation $\sim$ on $md_{\mathbb{R}}(L)$ as follows:

$$\forall\, \lambda,\ \mu \in md_{\mathbb{R}}(L),\ \ \lambda \sim \mu \Leftrightarrow\ \forall\, t \in \mathbb{R},\ \lambda(t-) = \mu(t-),\ \lambda(t+) = \mu(t+).$$

For every $\lambda \in md_{\mathbb{R}}(L)$, let $[\lambda]$ denotes the equivalent class in $md_{\mathbb{R}}(L)$ with respect to $\sim$ and containing λ; i.e.

$$[\lambda] = \{\mu \in md_{\mathbb{R}}(L) \mid \mu \sim \lambda\}.$$

We denote the family of all the equivalent classes in $md_{\mathbb{R}}(L)$ with respect to $\sim$ by $\mathbb{R}[L]$. For convenience, for every $\lambda \in md_{\mathbb{R}}(L)$, we identify λ with every μ such that $\mu \sim \lambda$ and still use λ to denote $[\lambda]$.

For every $t \in \mathbb{R}$, we define $L_t,\ R_t \in L^{\mathbb{R}[L]}$ as follows:

$$\forall\ \lambda \in \mathbb{R}[L], \quad L_t(\lambda) = \lambda(t-)', \quad R_t(\lambda) = \lambda(t+).$$

We restrict the equivalence relation $\sim$ defined above on $md_I(L)$, still denoting the corresponding equivalent classes in $md_I(L)$ by $[\lambda]$ for every $\lambda \in md_I(L)$ and denote the family of all the corresponding equivalent classes in $md_I(L)$ by $I[L]$. Also denote the restrictions of $L_t,\ R_t : \mathbb{R}[L] \to L$ on $I[L]$ by $L_t,\ R_t$ for every $t \in \mathbb{R}$ respectively.

Denote $(L^{I[L]},\ \mathscr{T}_L^I)$ by $I(L)$, where $\mathscr{T}_L^I$ is called the *standard topology* of $I(L)$ generated by the subbase

$$\mathscr{S}_L^I = \{L_t,\ R_t \in L^{I[L]} \mid t \in \mathbb{R}\}$$

and $\mathscr{S}_L^I$ is called the *standard subbase* of $I(L)$.

Definition 2.1.26. Let $(L^X,\ \mathbb{F})$ be an L-topological space. $(L^X,\ \mathbb{F})$ is called *completely regular*, if for every $G \in \mathbb{F}$, there exists a family $\{G_t \mid t \in \Lambda\} \subseteq \mathbb{F}$ such that $\bigcup_{t\in\Lambda} G_t = G$ and for every $t \in \Lambda$, there exists a continuous mapping $f_t^{\rightarrow} : (L^X,\ \mathbb{F}) \to I(L)$ such that

$$G_t \subseteq f_t^{\leftarrow}(L_1') \subseteq f_t^{\leftarrow}(R_0) \subseteq G.$$

Definition 2.1.27. For any x_α, A, $B \in L^X$, x_α is said to be *quasi-coincident* with A, denoted as $x_\alpha \ll A$ if $x_\alpha \nsubseteq A'$ i.e., $\alpha \nleq A'(x)$.

A is called *quasi-coincident* with B at y if $A(y) \nleq B'(y)$. A is called *quasi-coincident* with B, denoted as $A\widehat{q}B$, if A quasi-coincides with B at some $y \in X$.

Definition 2.1.28. Let $x_\alpha \in \mathrm{Pt}(L^X)$. Then an L-fuzzy set U is said to be a *quasi-coincident neighborhood* (Q-nbd) at x_α in an L-topological space $(L^X, \mathbb{F})$, if there is $G \in \mathbb{F}$ such that $x_\alpha \ll G \subseteq U$.

The family of all Q-nbd at x_α in an L-topological space $(L^X, \mathbb{F})$ is denoted by $\mathscr{Q}(x_\alpha)$.

Definition 2.1.29. A subfamily $\mathscr{A} \subseteq \mathscr{Q}(x_\alpha)$ is called a *Q-nbd base* of x_α, if for every $U \in \mathscr{Q}(x_\alpha)$, $\exists\, V \in \mathscr{A}$ s.t. $V \subseteq U$.

Theorem 2.1.22. *Let $(L^X, \mathbb{F})$ be an L-topological space. Then for any $x_\alpha \in M(L^X)$, $\mathscr{Q}(x_\alpha)$ is a down-directed set in L^X and $\underline{0} \notin \mathscr{Q}(x_\alpha)$.*

Theorem 2.1.23. *Let $(L^X, \mathbb{F})$ be an L-topological space and $A \in L^X$. Then an L-fuzzy point $x_\alpha \in \bar{A}$ iff each Q-nbd at x_α is quasi-coincident with A.*

2.1.3 Convergence structures in L-topology

Here we provide some definitions and results regarding nets and filters in L-topology that will be required in our subsequent chapters. All the definitions and results cited here can be found in [44], [30]or [59].

Definition 2.1.30. Let X be a non-empty ordinary set and D a directed set. Then every mapping $S : D \to X$ is called a *net* in X and D the *index set* of S.

In an L-fuzzy space L^X, a net $S : D \to \mathrm{Pt}(L^X)$ is called a *net* in L^X. In particular, a net $S : D \to M(L^X)$ is called a *molecule net* in L^X.

Let S be a net in an L-topological space $(L^X, \mathbb{F})$ and $x_\alpha \in \mathrm{Pt}(L^X)$. Then S is said to be *convergent to* x_α, denoted by $S \to x_\alpha$, if for any $U \in \mathscr{Q}(x_\alpha)$ there is $n_0 \in D$ such that $S(n)\widehat{q}U$, $\forall n \geq n_0$.

Definition 2.1.31. A non-empty sub collection $\mathscr{F}$ of L^X is said to be a *filter* in an L-topological space, if:

F1) $\underline{0} \notin \mathscr{F}$.

F2) $U_1, U_2 \in \mathscr{F} \Rightarrow U_1 \bigcap U_2 \in \mathscr{F}$.

F3) $U \in \mathscr{F}$ and $V \in L^X$ such that $U \subseteq V$ then $V \in \mathscr{F}$.

$\mathscr{F}$ is said to be *proper* in $(L^X, \mathbb{F})$, if $\mathscr{F} \neq L^X$.

Definition 2.1.32. A subfamily $\mathscr{B}$ of L^X is called a *filter base* in an L-topological space, if

B1) $\underline{0} \notin \mathscr{B}$

B2) for any $U, V \in \mathscr{B}$, there exists $W \in \mathscr{B}$ such that $W \subseteq U \bigcap V$.

Definition 2.1.33. Let $x_\alpha \in L^X$ and $\mathscr{F}$ be a filter in an L-topological space $(L^X, \mathbb{F})$, then $\mathscr{F}$ is said to be *convergent to* x_α, denoted by $\mathscr{F} \to x_\alpha$, if for any $U \in \mathscr{Q}(x_\alpha)$ there exists $F \in \mathscr{F}$ such that $F \subseteq U$, that is, $\mathscr{Q}(x_\alpha) \subseteq \mathscr{F}$.

An L-fuzzy point x_α is called a *cluster point* of $\mathscr{F}$, denoted by $\mathscr{F} \propto x_\alpha$, if for every $U \in \mathscr{Q}(x_\alpha)$ and $F \in \mathscr{F}$, $U \bigcap F \neq \underline{0}$.

Proposition 2.1.24. *Let $(L^X, \mathbb{F})$ be an L-topological space, S and $\mathscr{F}$ respectively molecular net and proper filter in $(L^X, \mathbb{F})$.*

Let $\mathscr{F}(S) = \{A \in L^X \mid \exists\, n_0 \in D$ s.t. $S(n)\widehat{q}A,\ \forall\, n \in D\}$.

Let

$$D(\mathscr{F}) = \{(x_\alpha, A) \in M(L^X) \times \mathscr{F} \mid x_\alpha \widehat{q} A \in \mathscr{F}\}$$

and $\leq$ be a relation defined by,

$$\forall\,(x_\alpha, A), (y_\beta, B) \in D(\mathscr{F}),\ (x_\alpha, A) \leq (y_\beta, B) \Leftrightarrow B \subseteq A.$$

Let $S(\mathscr{F}) : D(\mathscr{F}) \to M(L^X)$ be a mapping defined by,

$$S(\mathscr{F})(x_\alpha, A) = x_\alpha,\ \forall\,(x_\alpha, A) \in D(\mathscr{F}).$$

Then

(i) *$\mathscr{F}(S)$ is a proper filter in $(L^X, \mathbb{F})$.*

(ii) *$D(\mathscr{F})$ equipped with $\leq$ is a directed set.*

(iii) *$S(\mathscr{F})$ is a molecular net in $(L^X, \mathbb{F})$.*

Theorem 2.1.25. [44] *Let $(L^X, \mathbb{F})$ be an L-topological space, $x_\alpha \in \mathrm{Pt}(L^X)$, S and $\mathscr{F}$ be respectively molecular net and proper filter in $(L^X, \mathbb{F})$. Then*

(i) $S \to x_\alpha \Leftrightarrow \mathscr{F}(S) \to x_\alpha$.

(ii) $\mathscr{F} \to x_\alpha \Leftrightarrow S(\mathscr{F}) \to x_\alpha$.

Theorem 2.1.26. *Let $\{(L^{X_t}, \mathbb{F}_t) \mid t \in \Lambda\}$ be a family of L-topological spaces and $(L^X, \mathbb{F})$ be their product space, where Λ is the index set. Then a net S in the product space $(L^X, \mathbb{F})$ converges to an L-fuzzy point e iff, for each $t \in \Lambda$, the net $\pi_t^{\to} \circ S = \{\pi_t^{\to}(S_n) \mid n \in D\}$ in $(L^{X_t}, \mathbb{F}_t)$ converges to the L-fuzzy point $\pi_t^{\to}(e)$.*

Definition 2.1.34. Let $(L^X. \mathbb{F})$ be an L-topological space and $\mathscr{F}$, $\mathscr{G}$ be proper filters in $(L^X, \mathbb{F})$. Then $\mathscr{G}$ is said to be *finer* than $\mathscr{F}$ or $\mathscr{F}$ is *coarser* than $\mathscr{G}$, if $\mathscr{F} \subseteq \mathscr{G}$.

Theorem 2.1.27. *Let $(L^X, \mathbb{F}_1)$ and $(L^Y, \mathbb{F}_2)$ be L-topological spaces. Then an L-fuzzy function $f^{\rightarrow} : L^X \to L^Y$ is continuous if and only if for any filter $\mathscr{F}$ converges on L^X implies $f^{\rightarrow}(\mathscr{F})$ converges on L^Y.*

Theorem 2.1.28. *Let $(L^X. \mathbb{F})$ be an L-topological space. Let $A \in L^X$ and $x_\alpha \in M(L^X)$. Then $x_\alpha \in \bar{A}$ if and only if there is a proper filter $\mathscr{F}$ in $(L^X, \mathbb{F})$ such that $A' \notin \mathscr{F}$, $\mathscr{F} \to x_\alpha$.*

Theorem 2.1.29. *Let $(L^X, \mathbb{F})$ be an L-topological space and $x_\alpha \in \mathrm{Pt}(L^X)$. If $\mathscr{Q}(x_\alpha)$ is a down-directed set, then for every proper filter $\mathscr{F}$ in $(L^X, \mathbb{F})$, $\mathscr{F} \propto x_\alpha$ if and only if $\mathscr{F}$ has a finer proper filter $\mathscr{G}$ such that $\mathscr{G} \to x_\alpha$.*

Definition 2.1.35. Let $A \in L^X$, then a filter *relative to A* is a filter $\mathscr{F}$ such that for any $F \subseteq A$ implies $F \notin \mathscr{F}$.

With respect to the partially ordering by set inclusion, every filter $\mathscr{F}$ relative to A is contained in a maximal filter relative to A.

Obviously every filter $\mathscr{F}$ is a filter relative to $\underline{0}$.

Definition 2.1.36. Let $\mathscr{F}$ and $\mathscr{G}$ be nonempty subsets of L^X, then the pair $(\mathscr{F}, \mathscr{G})$ is called a *filter pair* if both $\mathscr{F}$ and $\mathscr{G}' = \{G' \mid G \in \mathscr{G}\}$ are filters such that for any $F \in \mathscr{F}$ and $G \in \mathscr{G}'$. $F \nsubseteq G$.

Lemma 2.1.30. *Let $(\mathscr{F}_\aleph, \mathscr{G})$ be a maximal filter pair and $A, B \in L^X$ such that $A \bigcup B \in \mathscr{F}_\aleph$. Then either $A \in \mathscr{F}_\aleph$ or $B \in \mathscr{F}_\aleph$.*

Definition 2.1.37. A subset $\mathscr{F}$ of L^X is said to satisfy the finite intersection property or the *F. I. P. relative to* an L-fuzzy set G if $F_1, ..., F_n \in \mathscr{F} \Rightarrow \bigcap_{i=1}^{n} F_i \nsubseteq G$.

Obviously every subset $\mathscr{F}$ of L^X which satisfy the F. I. P. relative to G is contained in a filter relative to G.

Theorem 2.1.31. *Let $(L^X, \mathbb{F})$ be an L-topological spaces. Then the following conditions are equivalent.*

(i) $A = \bar{A}$.

(ii) *For every proper filter $\mathscr{F}$ in $(L^X, \mathbb{F})$ and every $x_\alpha \in M(L^X)$ such that $A' \notin \mathscr{F}$, $\mathscr{F} \to x_\alpha \Rightarrow x_\alpha \in A$.*

2.1.4 Compactness in *L*-topology

We consider the notion of compactness and the related results in the sense of Hutton.

Definition 2.1.38. [30] Let $(L^X, \mathbb{F})$ be an L-topological space. Then a *cover* $\mathscr{C}$ of an L-fuzzy set A is a non empty collection of L-fuzzy sets such that $A \subseteq \bigcup_{B \in \mathscr{C}} B$.

A cover $\mathscr{C}$ of an L-fuzzy set A is said to be an *open cover*, if $\mathscr{C} \subseteq \mathbb{F}$.

A non empty subcollection $\mathscr{D}$ of $\mathscr{C}$ is said to be a subcover of A, if $\mathscr{D}$ is also a cover of A.

Definition 2.1.39. [30] Let $(L^X, \mathbb{F})$ be an L-topological spaces. Then $(L^X, \mathbb{F})$ is said to be *compact* if it satisfies any of following equivalent statements:

(i) Every open cover $\mathscr{C}$ of each closed set A has a finite subcover.

(ii) Every collection of closed sets $\mathscr{F}$ satisfying the F. I. P. relative to an open set G satisfies $\bigcap_{F\in\mathscr{F}} F \not\subseteq G$.

(iii) Every closed-filter $\mathscr{F}$ relative to an open set G satisfies $\bigcap_{F\in\mathscr{F}} F \not\subseteq G$.

(iv) Every maximal closed-filter $\mathscr{F}$ relative to an open set G satisfies $\bigcap_{F\in\mathscr{F}} F \not\subseteq G$.

Theorem 2.1.32. *Every compact regular L-topological space is completely regular.*

The following result follows from theorem 13 [30]:

Theorem 2.1.33. *Product of compact spaces is compact.*

Remark 2.1.7. It is a well known fact that compactness in the sense of Chang [7] is not productive.

2.1.5 Hutton's uniformity

Definition 2.1.40. [29] Let $\mathscr{U}^*$ be the collection of all maps $U : L^X \to L^X$ which satisfy:

(s1) $\Delta \subseteq U$.

Where Δ is a mapping $\Delta : L^X \to L^X$ such that

$\Delta(A) = A, \quad \forall\ A \in L^X.$

(s2) $U(\bigcup_\lambda V_\lambda) = \bigcup_\lambda U(V_\lambda), \quad V_\lambda \in L^X.$

For any $U,\ V \in \mathscr{U}^*$, $U \circ V$ is the composition of functions and $U \subseteq V \Leftrightarrow U(A) \subseteq V(A),\ \forall\ A \in L^X$.

Obviously, $\Delta \circ U = U = U \circ \Delta$.

Remark 2.1.8. For any $U \in \mathscr{U}^*$, we say $(x_\alpha, y_\beta) \in U \Leftrightarrow y_\beta \in U(x_\alpha)$, where $x_\alpha,\ y_\beta \in L^X$.

Lemma 2.1.34. [29] *For any* $U, V \in \mathscr{U}^*$ *and* $A \in L^X$,
$(U \bigcup V)(A) = \bigcap_{B \bigcup C = A}(U(B) \bigcup V(C))$.

Definition 2.1.41. [29] For any $U \in \mathscr{U}^*$, $U^r(x_\alpha) = \bigcap\{y_\beta \mid U(y'_\beta) \subseteq x'_\alpha\}$. If $U = U^r$, then U is said to be symmetric.

Proposition 2.1.35. [29] *For any* $U \in \mathscr{U}^*$ *and* $A,\ B \in L^X$,

(i) $U(A) \subseteq B \Leftrightarrow U^r(B') \subseteq A'$.

(ii) $U^r \in \mathscr{U}^*$.

(iii) $(U^r)^r = U$.

Also, for any $V \in \mathscr{U}^*$, *we have the following:*

(iv) $U \subseteq V \Leftrightarrow U^r \subseteq V^r$,

(v) $(U \circ V)^r = V^r \circ U^r$.

(vi) $(U \bigcap V)^r = V^r \bigcap U^r$.

Definition 2.1.42. [29] An *L-quasi uniformity* on L^X is a non empty subset $\mathscr{U}$ of $\mathscr{U}^*$ such that:

Q1) If $V \in \mathscr{U}^*$ such that $U \subseteq V$ for some $U \in \mathscr{U}$, then $V \in \mathscr{U}$.

Q2) $U \cap V \in \mathscr{U}$, whenever $U, V \in \mathscr{U}$.

Q3) For any $U \in \mathscr{U}$, there exists $V \in \mathscr{U}$, such that $V \circ V \subseteq U$.

The pair $(L^X, \mathscr{U})$ is called an *L-quasi-uniform space.*

Definition 2.1.43. [29] A nonempty subset $\mathscr{S}$ of $\mathscr{U}^*$ is said to be a *subbase* for some L-quasi-uniformity if it satisfies the axiom (Q3).

Definition 2.1.44. [29] A nonempty subset $\mathscr{B}$ of $\mathscr{U}^*$ is said to be *base* for some L-quasi-uniformity if it satisfies the axiom (Q3) and the following:

(Q2′) For any U_1, $U_2 \in \mathscr{B}$ there is $V \in \mathscr{B}$ such that $V \subseteq U_1$ and $V \subseteq U_2$.

Definition 2.1.45. [29] An L-quasi-uniformity $\mathscr{U}$ is said to be *L-uniformity* if it satisfies the following:

(Q4) $U \in \mathscr{U}$ implies $U^r \in \mathscr{U}$,

or, equivalently

(Q4′) $\mathscr{U}$ has a base of symmetric members.

Theorem 2.1.36. [29] *Every L-quasi-uniformity generates an L-topology.*

Definition 2.1.46. [29] Let $(L^X, \mathscr{U})$ be an L-quasi-uniform space. Then for any $U \in \mathscr{U}$ and $A \in L^X$, $U(A)$ is a neighborhood of A in the L-topology generated by $\mathscr{U}$.

Lemma 2.1.37. [29] *Let $(L^X, \mathbb{F})$ be an L-topological space. Let $U_i \in \mathscr{U}^*$ and $A \in L^X$ such that $U_i(A)$ is a neighborhood of A, $i = 1, 2$. Then $(U_1 \bigcap U_2)(A)$ is a neighborhood of A.*

Theorem 2.1.38. [29] *Every L-topology is generated by an L-quasi uniformity.*

Definition 2.1.47. [29] Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be L-quasi-uniform spaces (L-uniform spaces respectively). A map $f^{\rightarrow} : L^X \rightarrow L^Y$ is said to be *L-quasi-uniformly continuous* (*L-uniformly continuous respectively*), if for every $V \in \mathscr{V}$, there exists $U \in \mathscr{U}$ such that $\widehat{f^{\rightarrow}}(U) \subseteq V$. Where $\widehat{f^{\rightarrow}}(x_\alpha, y_\beta) = (f^{\rightarrow}(x_\alpha), f^{\rightarrow}(y_\beta))$, that is, for $A \in L^X$, $U(A) \subseteq f^{\leftarrow}(V)(f^{\rightarrow}(A))$.

Theorem 2.1.39. [29] *L-quasi-uniformly continuous functions on L-quasi-uniform spaces are continuous with respect to the relative L-topologies.*

Theorem 2.1.40. [29] *Let $(L^X, \mathbb{F})$ be an L-topological space. Then $(L^X, \mathbb{F})$ is generated by an L-uniformity if and only if $(L^X, \mathbb{F})$ is completely regular.*

Theorem 2.1.41. [2] *Let $(L^X, \mathbb{F})$ be a compact completely regular L-topological space. Then there is a unique L-uniformity which generates the L-topology $\mathbb{F}$.*

2.1.6 Fuzzy metric

Fuzzy Metric in Hutton's sense is defined in [44] as follows:

Definition 2.1.48. Let $(L^X, \mathscr{U})$ be an L-uniform space (L-quasi-uniform space, respectively). Then $(L^X, \mathscr{U})$ is called an *L-pseudo metric space* (*L-pseudo quasi-metric space*, respectively), if $\mathscr{U}$ has a countable base.

To describe pseudo quasi-metric from angle of "distance", Erceg in [13] introduced a "distance" function between every pair of L-fuzzy subsets to define an L-pseudo quasi-metric:

Definition 2.1.49. A mapping $P : L^X \times L^X \to [0, +\infty]$ is called an *L-pseudo quasi-metric* on L^X, if P fulfills the following conditions (EM1)-(EM4):

(EM1) $A \neq \underline{0} \Rightarrow P(\underline{0}, A) = +\infty$,

$P(A, \underline{0}) = P(A, A) = 0$.

(EM2) $P(A, B) \leq P(A, C) + P(C, B)$.

(EM3) (i) $A \subseteq B \Rightarrow P(B, C) \leq P(A, C)$.

(ii) $P(A, \bigcup_{\lambda\in\Lambda} B_\lambda) = \bigcup_{\lambda\in\Lambda} P(A, B_\lambda)$.

(EM4) If $P(A_\lambda, C) < r \Rightarrow C \subseteq B$ for every $C \in L^X$ and for every $\lambda \in \Lambda$, then the following implication holds for every $D \in L^X$: $P(\bigcup_{\lambda\in\Lambda} A_\lambda, D) < r \Rightarrow D \subseteq B$.

Definition 2.1.50. Let P be an L-pseudo quasi-metric on L^X and for every $s > 0$, $f_s : L^X \to L^X$ be a mapping defined as:

$\forall\ A \in L^X,\ f_s(A) = \bigcup\{C \in L^X \mid P(A, C) < s\}$.

Then $\mathscr{D}_P^* = \{f_s \mid s > 0\}$ is called the *associated neighborhood mappings* of P.

Definition 2.1.51. Let P be an L-pseudo quasi-metric on L^X and $\mathscr{D}_P^*$ be the family of the associated neighborhood mappings of P. Then P is called an L-pseudo metric on L^X, if for every $f_s \in \mathscr{D}_P^*$ satisfies the following condition (EM5):

(EM5) $f_s = f_s^r$.

In [44], the authors stated two theorems viz. theorems 14.1.11 14.1.13 to show that each of (Erceg's) pseudo quasi-metric determines a Hutton's pseudo quasi-metric and conversely.

2.1.7 Pointwise characterizations of fuzzy metric

Both L-pseudo quasi-metric in Hutton's sense and L-pseudo quasi-metric in Erceg's sense are based on L-fuzzy subsets but not L-fuzzy points as ordinary metric. In particular, this structure makes the geometric intuition of the symmetry of a metric dubious. To offset this point, Liang in [40] introduced a pointwise description of an L-pseudo quasi-metric. In [43] Liu proved the Fuzzy Urysohn Metrization Theorem with the imbedding theory. The L-topology generated by this kind of pseudo quasi-metric is coincident with the one generated by Erceg's pseudo quasi-metric, but they are not coincident. To be precise, Liang's distance function is not a restriction of the one of Erceg on the set of molecules. This problem was solved by Peng in [54]. Equivalently he simplified Erceg's definition of L-pseudo quasi-metric and then restricted the distance function on the set of molecules to obtain a pointwise characterization in [55].

Lemma 2.1.42. [44] *Let (L^X, P) be an L-pseudo quasi-metric space. $\{f_s \mid s > 0\}$ be the family of the associate neighborhood mappings of P. Then*

(i) $P(A, B) < s \Rightarrow B \subseteq f_s(A) \Rightarrow P(A, B) \leq s$.

(ii) $x_\alpha \in \beta * (f_s(A)) \Rightarrow P(A, x_\alpha) < s$.

Theorem 2.1.43. *A mapping $P : L^X \times L^X \to [0, +\infty]$ is an L-pseudo quasi-metric (L-pseudo metric, respectively) on L^X if P satisfies the following conditions* (SEM1)-(SEM3) ((SEM1)-(SEM4), *respectively*) :

(SEM1) $B \subseteq A \Rightarrow P(A, B) = \underline{0}$,

$B \neq \underline{0} \Rightarrow P(\underline{0}, B) = +\infty$.

(SEM2) $P(A, B) \leq P(A, C) + P(C, B)$.

(SEM3) $A, B \neq \underline{0} \Rightarrow P(A, B) = \bigcup_{x_\alpha \in \beta^*(B)} \bigcap_{y_\beta \in \beta^*(A)} P(y_\beta, x_\alpha)$.

(SEM4) "$P(A, C) < r \Rightarrow C \subseteq B$" $\Leftrightarrow$ "$P(B', D) < r \Rightarrow D \subseteq A'$".

Remark 2.1.9. In [44], it was shown that the L-pseudo quasi-metric (L-pseudo metric, respectively) and L-fuzzy pointwise pseudo quasi-metric (L-fuzzy pointwise pseudo quasi-metric, respectively) are equivalent.

CHAPTER 3

L-SEMI-QUASI-UNIFORM AND L-SEMI-UNIFORM SPACES

3.1 Introduction

In the first chapter we presented a sketch on the development of uniform spaces on various categories of fuzzy topological spaces and it was pointed that various important notions namely, *semi-uniformity*, *semi-quasi-uniformity* were not developed and remained unexplored. At the same time it is undeniable that these notions occupy a very important place in classical topology. For the filling up of this void in the development of the theory of generalization of uniform spaces in the categories of fuzzy topological spaces we have in this chapter developed two kinds of generalized uniform structures namely L-semi-quasi-uniformity and L-semi-uniformity in the category **L-TOP** as a generalization of Hutton's quasi-uniformity and uniformity respectively.

In the first section we introduce the notions of L-semi-quasi-uniformity and L-semi-uniformity and build up these notions in terms of various properties. In the second section we have introduced the notion of L-semi-pseudo-metric as a generalization of Hutton's pseudo metric offering a treatment of the problem of metrization. In the last section we have considered the notions of completeness, compactness and totally boundedness and several results therein are presented.

3.2 *L*-Semi-quasi-uniform structures

The following operator shall play an important role in the subsequent part:

Definition 3.2.1. Let X be a nonempty ordinary set and L be a fuzzy lattice.
Let $i : L^X \to L^X$ be a mapping on L^X.
Then 'i' is called an *interior operator* on L^X, if it fulfills the following conditions:

(IO1) $i(\underline{1}) = \underline{1}$.

(IO2) $i(A) \subseteq A, \ \forall A \in L^X$.

(IO3) $i(A \bigcap B) = i(A) \bigcap i(B). \ \forall A, B \in L^X$.

L^X together with an interior operator 'i' shall be called an *interior space.*

For any $A \in L^X$, we shall call $(i(A'))'$ is the *closure* of A with respect to the interior operator 'i' [denoted by $c(A)$] and A is called *closed* or *open* with respect to that interior operator according as $A = c(A)$ or $A = i(A)$ respectively.

Obviously, for any interior operator 'i' and $A \in L^X$, we have A is open with respect to 'i' iff A' is closed with respect to that interior operator.

An interior operator 'i' shall be called an *L-topological interior operator* if in addition it satisfies the following:

(IO4) $i(i(A)) = i(A), \ \forall A \in L^X$.

We shall call a closure operator is *L-topological closure operator* iff the related interior operator is an L-topological interior operator.

Definition 3.2.2. An *L-semi-quasi-uniformity* $\mathscr{U}$ on L^X is a non empty subfamily of $\mathscr{U}^*$ satisfying the following:

(SQ1) $U \bigcap V \in \mathscr{U}, \ \ \forall U, V \in \mathscr{U}$.

(SQ2) $U \in \mathscr{U}$ and $V \in \mathscr{U}^*$ such that $U \subseteq V$ then $V \in \mathscr{U}$.

The pair $(L^X, \mathscr{U})$ is called an *L-semi-quasi-uniform space.*
Obviously, every L-quasi-uniformity is an L-semi-quasi-uniformity.

Definition 3.2.3. A non empty subfamily $\mathscr{B}$ of $\mathscr{U}^*$ is called a *base* for some L-semi-quasi-uniformity $\mathscr{U}$, if for any $U \in \mathscr{U}$, there is $B \in \mathscr{B}$ such that $B \subseteq U$.

A non empty subfamily $\mathscr{B}$ of $\mathscr{U}^*$ is a base for some L-semi-quasi-uniformity $\mathscr{U}$, if it satisfies the following:

(SQ1′) For any U, $V \in \mathscr{B}$, there is $W \in \mathscr{B}$ such that $W \subseteq U \bigcap V$.

Definition 3.2.4. An L-semi-quasi-uniformity $\mathscr{U}$ on L^X is said to be an L-semi-uniformity if $\mathscr{U}$ has a base $\mathscr{B}$ such that:

(SQ3) For any $B \in \mathscr{B}$ implies $B^r \in \mathscr{B}$.

The pair $(L^X, \mathscr{U})$ is then called an L-semi-uniform space.

Clearly, every L-uniformity is an L-semi-uniformity.

Also, the collection of symmetric members of $\mathscr{U}$ is a base for $\mathscr{U}$.

Definition 3.2.5. If $\mathscr{U}$ and $\mathscr{V}$ are L-semi-uniformities on L^X. Then $\mathscr{V}$ is called *coarser* (*weaker*) than $\mathscr{U}$ or $\mathscr{U}$ is called *finer* (*stronger*) than $\mathscr{V}$ iff $\mathscr{V} \subseteq \mathscr{U}$.

We now provide examples to show that

(i) An L-semi-quasi-uniformity need not be an L-semi-uniformity.

(ii) An L-semi-quasi-uniformity need not be an L-quasi-uniformity.

Example 3.2.1. Let $L = \{0\} \bigcup([0, 1] \bigcap\{\frac{1}{n}\}_{n\in\mathbb{N}})$ and $X = \{x\}$. Then L is a fuzzy lattice with the general order relation.

For each $n \in \mathbb{N}$, let

$B_n =$

$\Delta_{L^X} \bigcup\{(x_{1/2}, x_{1/i}) \mid n \leq i\} \bigcup\{(x_{1/i}, x_{1/i+1}) \mid n \leq i\} \bigcup\{(x_{1/i}, x_1) \mid n \leq i\}$, where $\Delta_{L^X} = \{(x_\alpha, x_\alpha) \mid x_\alpha \in L^X\}$.

Let $\mathscr{B} = \{B_n \mid n \in \mathbb{N}\}$.

For each $B_n \in \mathscr{B}$, let $U_{B_n} : L^X \to L^X$ be a mapping defined by,

$$U_{B_n}(x_\alpha) = \bigcup\{x_\beta \mid (x_\alpha, x_\beta) \in B_n\}, \quad \forall\, x_\alpha \in L^X.$$

Then clearly,

1) $\mathscr{U}_{\mathscr{B}} \subseteq \mathscr{U}^*$.

2) $U_{B_{n+1}} \subseteq U_{B_n}, \quad \forall\, n \in \mathbb{N}$.

Therefore, $U_{B_n} \bigcap U_{B_m} = U_{B_l} \in \mathscr{U}_{\mathscr{B}}$,

where $l = \min\{m, n\}$ and $\mathscr{U}_{\mathscr{B}} = \{U_{B_n} \mid B_n \in \mathscr{B}\}$.

Hence, $\mathscr{U}_{\mathscr{B}}$ is a base for some L-semi-quasi-uniformity.

We shall now show that $\mathscr{U}_{\mathscr{B}}$ is not a base for L-semi-uniformity.

Obviously, for each $n \in \mathbb{N}$, $U^{\tau}_{B_n} = U_{B_n^{-1}}$,

where $B_n^{-1} = \Delta_{L^X} \bigcup\{(x_{1/i}, x_{1/2}) \mid n \leq i\} \bigcup\{(x_{1/i+1}, x_{1/i}) \mid n \leq i\} \bigcup\{(x_1, x_{1/i}) \mid n \leq i\}$.

But then $B_n^{-1} \notin \mathscr{B}, \quad \forall n \in \mathbb{N}$.

This implies that $U^r_{B_n} \notin \mathscr{U}_{\mathscr{B}}, \quad \forall n \in \mathbb{N}$.

Hence, $\mathscr{U}_{\mathscr{B}}$ is not a base for an L-semi-uniformity.

Remark 3.2.1. In the above example we observe that for any $n \in \mathbb{N}$, $U_{B_n} \circ U_{B_n}(x_{1/2}) \nsubseteq U_{B_2}(x_{1/2})$. Thus for U_{B_2}, there is no $n \in \mathbb{N}$ such that $U_{B_n} \circ U_{B_n} \subseteq U_{B_2}$.

Hence, $\mathscr{U}_{\mathscr{B}}$ is also not a base for an L-quasi-uniformity.

Theorem 3.2.1. *Let $(L^X, \mathscr{U})$ be an L-semi-quasi-uniform space and $\mathscr{B}$ be any base for $\mathscr{U}$. Then the mapping* $\text{int} : L^X \to L^X$ *defined by,* $\text{int}(A) = \bigcup\{x_\alpha \mid \exists\ V \in \mathscr{B} \text{ s.t. } V(x_\alpha) \subseteq A\}$, *is an interior operator on L^X.*

Proof. (IO1) Clearly $\text{int}(\underline{1}) = \underline{1}$ and (IO2) $\text{int}(A) \subseteq A$.

Finally, (IO3) $\text{int}(A \bigcap B) = \text{int}(A) \bigcap \text{int}(B)$ is fulfilled due to (SQ1). □

Every L-semi-quasi-uniformity therefore generates an interior space.

Further, for any L-semi-uniform space $(L^X, \mathscr{U})$, since the interior space is generated by 'int'. So, in particular we note that for any $x_\alpha \in \text{Pt}(L^X)$, the collection $\mathscr{N}_{x_\alpha} = \{U(x_\alpha) \mid U \in \mathscr{U}\}$ is the neighborhood system at x_α in the generating interior space.

If the family $\{\mathscr{N}_{x_\alpha} \mid x_\alpha \in \text{Pt}(L^X)\}$ is a neighborhood system for some L-topology $\mathbb{F}$, we say that $\mathbb{F}$ is the L-topology generated by $\mathscr{U}$.

Lemma 3.2.2. *Let $(L^X, \mathscr{U})$ be an L-semi-uniform space and* $\text{cl} : L^X \to L^X$ *be a mapping such that for any $A \in L^X$,* $\text{cl}(A) = \bigcap\{V(A) \mid V \in \mathscr{U}\}$. *Then* $\text{cl}(A) = (\text{int}(A'))'$, $\forall\ A \in L^X$.

Proof. Let $\mathscr{B}$ be a base for $\mathscr{U}$ consisting of symmetric members of $\mathscr{U}$.

Now, $\text{int}(A') = \bigcup\{x_\alpha \mid \exists U \in \mathscr{B} \text{ s.t. } U(x_\alpha) \subseteq A'\}$.
$= \bigcup\{\bigcup\{x_\alpha \mid U(x_\alpha) \subseteq A'\}, U \in \mathscr{B}\}$.
$= \bigcup\{[U^r(A)]' \mid U \in \mathscr{B}\}$.

Hence, $[\text{int}(A')]' = \bigcap\{U^r(A) \mid U \in \mathscr{B}\}$.
$= \bigcap\{U(A) \mid U \in \mathscr{B}\}$,
since $U \in \mathscr{B} \Rightarrow U^r = U$.

$\Rightarrow \text{cl}(A) = \bigcap\{U(A) \mid U \in \mathscr{B}\}$.
$= \bigcap\{V(A) \mid V \in \mathscr{U}\}$,
since $\mathscr{B}$ is a base for $\mathscr{U}$. □

Lemma 3.2.3. *Let $(L^X, \mathscr{U})$ be an L-semi-quasi-uniform space. Then the interior operator* 'int' *defined in theorem 3.2.1 is an L-topological interior operator under the following condition:*
"For any $U \in \mathscr{U}$ and $x_\alpha \in \text{Pt}(L^X)$, there exists $V \in \mathscr{U}$ such that to each $y_\beta \subseteq V(x_\alpha)$ there corresponds $W \in \mathscr{U}$ with $W(y_\beta) \subseteq U(x_\alpha)$."

Proof. Let $(L^X, \mathscr{U})$ be an L-semi-quasi-uniform space satisfying the given condition. For any $A \in L^X$, let $x_\alpha \subseteq \text{int}(A)$.
Then there exists $U \in \mathscr{U}$ such that $U(x_\alpha) \subseteq A$.
Let $V \in \mathscr{U}$ such that for any $y_\beta \subseteq V(x_\alpha)$, there is $W \in \mathscr{U}$ such that $W(y_\beta) \subseteq U(x_\alpha)$.
By $(s1)$, $x_\alpha \subseteq V(x_\alpha)$.
But for $x_\alpha \subseteq V(x_\alpha)$ we may choose $W \in \mathscr{U}$ such that $W(x_\alpha) \subseteq U(x_\alpha)$.
This implies $x_\alpha \subseteq \text{int}(\text{int}(A))$.
Since the other inclusion follows by $(IO2)$ in theorem 3.2.1, we have $\text{int}(A) = \text{int}(\text{int}(A))$. □

Now by proposition 2.1.18, we have the following:

Theorem 3.2.4. *Every L-semi-quasi-uniformity with the condition given in lemma 3.2.3 generates an L-topological space.*

Theorem 3.2.5. *Let* $(L^X, \mathscr{U})$ *be an L-semi-uniform space. Then for any* $A \in L^X$, $\mathrm{cl}(A)$ *satisfies the following:*

(CO1) $\mathrm{cl}(\underline{0}) = \underline{0}$.

(CO2) $A \subseteq \mathrm{cl}(A)$.

(CO3) $\mathrm{cl}(A \bigcup B) = \mathrm{cl}(A) \bigcup \mathrm{cl}(B), \quad \forall B \in L^X$.

Proof. Since by lemma 3.2.2, we have $\mathrm{cl}(A) = (\mathrm{int}(A'))', \quad \forall A \in L^X$. Therefore,

(CO1) $\mathrm{cl}(\underline{0}) = (\mathrm{int}(\underline{0}'))' = (\mathrm{int}(\underline{1}))'$. Then by (IO1) $\mathrm{cl}(\underline{0}) = (\underline{1})' = \underline{0}$.

(CO2) $(\mathrm{int}(A'))' = \mathrm{cl}(A) \Rightarrow ((A'))' \subseteq \mathrm{cl}(A)$, by (IO2).

Therefore, $A \subseteq \mathrm{cl}(A)$.

(CO3) $\mathrm{cl}(A \bigcup B) = (\mathrm{int}(A \bigcup B)')' = (\mathrm{int}(A' \bigcap B'))'$.

Then, by (IO3), $\mathrm{cl}(A \bigcup B) = (\mathrm{int}(A') \bigcap \mathrm{int}(B'))'$.

This implies that $\mathrm{cl}(A \bigcup B) = (\mathrm{int}(A'))' \bigcup (\mathrm{int}(B'))' = \mathrm{cl}(A) \bigcup \mathrm{cl}(B)$.

□

Remark 3.2.2. Obviously, 'cl' on L^X is an L-topological iff it satisfies the following axiom:

(CO4) $\mathrm{cl}(\mathrm{cl}(A)) = A, \quad \forall A \in L^X$.

Definition 3.2.6. Let L^X and L^Y be interior spaces with respect to the interior operators $\imath_X$ and $\imath_Y$ respectively. Then a function $f^{\rightarrow} : L^X \rightarrow L^Y$ is said to be *continuous* with respect to the interior operators iff for each

$x_\alpha \in \mathrm{Pt}(L^X)$ and each neighborhood V of $f^\to(x_\alpha)$ with respect to the interior operator i_Y, there is a neighborhood U of x_α with respect to the interior operator i_X such that $f^\to(U) \subseteq V$.

Definition 3.2.7. Let L^X and L^Y be interior spaces with respect to the interior operators i_X and i_Y respectively. Then a function $f^\to : L^X \to L^Y$ is said to be a *homeomorphism* with respect to the interior operators iff $f^\to$ is bijective and both $f^\to$ and $f^\leftarrow$ are continuous with respect to the interior operators.

Definition 3.2.8. Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be L-semi-uniform spaces. A function $f^\to : (L^X, \mathscr{U}) \to (L^Y, \mathscr{V})$ is called *L-semi-uniformly continuous* iff for every $V \in \mathscr{V}$, there exists $U \in \mathscr{U}$ such that $\widehat{f^\to}(U) \subseteq V$, where $\widehat{f^\to}(x_\alpha, y_\beta) = (f^\to(x_\alpha), f^\to(y_\beta))$.

The function $f^\to$ is said to be an *L-semi-uniformly isomorphism* iff $f^\to$ is bijective and both $f^\to$ and $f^\leftarrow$ are L-semi-uniformly continuous.

Now since $\widehat{f^\to}(U) \subseteq V$ implies $\widehat{f^\to}(U)(f^\to(x_\alpha)) \subseteq V(f^\to(x_\alpha))$, $\forall\, x_\alpha \in \mathrm{Pt}(L^X)$, therefore we have the following:

Theorem 3.2.6. *L-semi-uniformly continuous functions on L-semi-uniform spaces are continuous with respect to the relative interior spaces.*

Proof. Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be L-semi-uniform spaces.
Let $f^\to : (L^X, \mathscr{U}) \to (L^Y, \mathscr{V})$ be L-semi-uniformly continuous and $\mathrm{int}_{\mathscr{U}}$ and $\mathrm{int}_{\mathscr{V}}$ be the respectively interior operators generated by $\mathscr{U}$ and $\mathscr{V}$.
For $x_\alpha \in \mathrm{Pt}(L^X)$ and for each neighborhood N of $f^\to(x_\alpha)$ in the interior space generated by $\mathscr{V}$, we may choose $V \in \mathscr{V}$ and $U \in \mathscr{U}$ so that

$V(f^{\rightarrow}(x_\alpha)) \subseteq N$ and $\widehat{f^{\rightarrow}}(U) \subseteq V$

Therefore, $f^{\rightarrow}(U(x_\alpha)) = \widehat{f^{\rightarrow}}(U)(f^{\rightarrow}(x_\alpha)) \subseteq V(f^{\rightarrow}(x_\alpha)) \subseteq N$. □

Corollary 3.2.7. *Every L-semi-uniformly isomorphism is a homeomorphism.*

Theorem 3.2.8. *Composition of two L-semi-uniformly continuous is an L-semi-uniformly continuous.*

Proof. Straightforward. □

3.3 *L*-Semi-pseudo-metrization

The problem of metrization has occupied an important place in the study of uniform spaces. Having developed the theory of semi-uniform spaces, we proceed to discuss the problem of metrization (semi-pseudo-metrization) in the same context.

Definition 3.3.1. A mapping $P : L^X \times L^X \rightarrow [0, +\infty]$ is said to be an *L-semi-pseudo-metric* on L^X if P satisfies the axioms (SEM1), (SEM3), (SEM4) and the following:

(SEM5) $A \subseteq B \Rightarrow P(B, C) \leq P(A. C)$

The pair (L^X, P) is called an *L-semi-pseudo-metric space.*

Now by theorem 2.1.43, we may conclude that L-semi-pseudo-metric generalizes the notion of L-pseudo metric.

Theorem 3.3.1. *Every L-semi-pseudo-metric generates an L-semi-uniformity.*

Proof. Let (L^X, P) be an L-semi-pseudo-metric space and for any $s > 0$, $f_s : L^X \to L^X$ be a mapping defined by,

$$f_s(A) = \bigcup\{C \in L^X \mid P(A, C) < s\}. \tag{3.3.1}$$

Let $\mathscr{B} = \{f_s(A) \mid s > 0\}$.

We claim that $\mathscr{B}$ is a base for some L-semi-uniformity.

(s1) Since by (SEM1), we have $P(A, A) = \underline{0}$, therefore

$$A \subseteq f_s(A), \quad \forall A \in L^X.$$

This implies that $\Delta \subseteq f_s$.

(s2) (i) By (SEM5) we have,

$\bigcup_{\lambda\in\Lambda} f_s(A_\lambda) \subseteq f_s(\bigcup_{\lambda\in\Lambda} A_\lambda)$, where Λ is an index set.

(ii) Conversely, suppose $\bigcup_{\lambda\in\Lambda} f_s(A_\lambda) \subseteq B$, then $f_s(A_\lambda) \subseteq B, \quad \forall\lambda \in \Lambda$ and hence by 3.3.1, we get

$C \in L^X, \lambda \in \Lambda, P(A_\lambda, C) < s \Rightarrow C \subseteq B.$

Now by (SEM3) in view of theorem 2.1.43, we get

$D \in L^X, P(\bigcup_{\lambda\in\Lambda} A_\lambda, D) < s \Rightarrow D \subseteq B.$

So, it follows from 3.3.1, that $f_s(\bigcup_{\lambda\in\Lambda} A_\lambda) \subseteq B$.

This now implies that $f_s(\bigcup_{\lambda\in\Lambda} A_\lambda) \subseteq \bigcup_{\lambda\in\Lambda} f_s(A_\lambda)$.

Hence, $f_s(\bigcup_{\lambda\in\Lambda} A_\lambda) = \bigcup_{\lambda\in\Lambda} f_s(A_\lambda)$.

(SQ1′) From the construction of f_s, we have, for any p, q with $p < q$ implies $f_p \subseteq f_q$, therefore for any $f_s, f_t \in \mathscr{B}$, let $p = \min\{s, t\}$, then $f_p \subseteq f_s \bigcap f_t$.

(SQ3) By theorem 2.1.43, we have $f_s^T = f_s$.

Thus, $\mathscr{B}$ is a base for some L-semi-uniformity. □

Since every L-semi-uniformity generates an interior space, therefore we have the following corollary:

Corollary 3.3.2. *Every L-semi-pseudo-metric generates an interior operator.*
In that generating space for any $x_\alpha \in \mathrm{Pt}(L^X)$, $\{f_s(x_\alpha) \mid s > 0\}$ is the neighborhood system at x_α.

The following lemma follows from lemma 2.1.42.

Lemma 3.3.3. *Let (L^X, P) be an L-semi-pseudo-metric space. Then for any $s > 0$, the mapping f_s, defined in theorem 3.3.1 satisfies the following:*

(i) $P(A, B) < s \Rightarrow B \subseteq f_s(A) \Rightarrow P(A, B) \leq s$.

(ii) $x_\alpha \in \beta^*(f_s(A)) \Rightarrow P(A, x_\alpha) < s$.

Definition 3.3.2. We shall say that an L-semi-uniform space $(L^X, \mathscr{U})$ is *L-semi-pseudo-metrizable* if there is an L-semi-pseudo-metric that generates $\mathscr{U}$.

We now proceed the following theorem which is the main result in this section:

Theorem 3.3.4. *An L-semi-uniform space $(L^X, \mathscr{U})$ is an L-semi-pseudo-metric space iff $\mathscr{U}$ has a countable base.*

Proof. Let $(L^X, \mathscr{U})$ be an L-semi-uniform space.
Let $\mathscr{B} = \{U_n \mid n \in \mathbb{N}\}$ be a countable base for $\mathscr{U}$ of symmetric members.
For any $r > 0$, let $\phi_r : L^X \to L^X$ be a mapping defined by,

$\forall A \in L^X$, if $\frac{1}{2^n} < r \leq \frac{1}{2^{n-1}}$, then $\phi_r(A) = U_n(A)$

and if $r > 1$, then $\phi_r(A) = \underline{1}$ or $\underline{0}$ according as $A \neq \underline{0}$ or $A = \underline{0}$.

Then, $\phi_r \in \mathscr{U}^*$.

For every $r > 0$, let $f_r : L^X \to L^X$ be a mapping defined by,

$$f_r = \bigcup\{\phi_{r_0} \circ \ldots \circ \phi_{r_k} \mid \sum_{i=0}^{k} r_k = r,\ \forall\, i \leq k.\ r_i > 0,\ k < \omega\}.$$

Since $\mathscr{U}^*$ is closed under compositions and arbitrary unions, therefore $f_r \in \mathscr{U}^*, \forall\, r > 0$.

Obviously, $\{f_r \mid r > 0\}$ is a base for $\mathscr{U}$.

Let $P : L^X \times L^X \to [0,\, \infty]$ be a mapping defined by,

$P(A,\, B) = \bigwedge\{r \mid B \subseteq f_r(A)\}$, where we assume that $\bigwedge \Phi = +\infty$.

Then P fulfils (SEM1).

(SEM3)By theorem 2.1.7(ii), for arbitrary $A,\, B \neq \underline{0}$,

$$\beta^*(f_r(A)) = \beta^*(f_r(\bigcup \beta^*(A))).$$

$$= \beta^*(\bigcup_{x_\alpha \in \beta^*(A)} f_r(x_\alpha)).$$

$$= \bigcup_{x_\alpha \in \beta^*(A)} \beta^*(f_r(x_\alpha)).$$

Now, $P(A,\, B) < r \Rightarrow B \subseteq f_r(A)$, by lemma 3.3.3(i).

$\Rightarrow \quad \forall\, x_\alpha \in \beta^*(B),\ x_\alpha \in \beta^*(f_r(A))$.

$\Rightarrow \quad \forall\, x_\alpha \in \beta^*(B),\ \exists\, y_\beta \in \beta^*(A),\ x_\alpha \in \beta^*(f_r(y_\beta))$.

$\Rightarrow \quad \forall\, x_\alpha \in \beta^*(B),\ \exists\, y_\beta \in \beta^*(A),\ P(y_\beta,\, x_\alpha) < r$, by lemma 3.3.3(ii).

$\Rightarrow \bigcup_{x_\alpha \in \beta^*(B)} \bigcap_{y_\beta \in \beta^*(A)} P(y_\beta,\, x_\alpha) < r$.

Again, $\bigcup_{x_\alpha \in \beta^*(B)} \bigcap_{y_\beta \in \beta^*(A)} P(y_\beta,\, x_\alpha) < r$

$\Rightarrow \forall\, x_\alpha \in \beta^*(B),\ \exists\, y_\beta(x_\alpha) \in \beta^*(A), P(y_\beta(x_\alpha),\, x_\alpha) < r$,

where $y_\beta(x_\alpha)$ denote an L-fuzzy point corresponding to x_α.

$\Rightarrow \forall\, x_\alpha \in \beta^*(B),\ \exists\, y_\beta(x_\alpha) \in \beta^*(A),\ x_\alpha \subseteq f_r(y_\beta(x_\alpha))$.

$\Rightarrow B = \bigcup \beta^*(B) \subseteq \bigcup_{x_\alpha \in \beta^*(B)} f_r(y_\beta(x_\alpha)) = f_r(\bigcup_{x_\alpha \in \beta^*(B)} y_\beta(x_\alpha))$.

$\subseteq f_r(\bigcup \beta^*(A)) = f_r(A)$.

$\Rightarrow P(A, B) < r$, by lemma 3.3.3(i).

Hence, $P(A, B) < r \Leftrightarrow \bigcup_{x_\alpha \in \beta^*(B)} \bigcap_{y_\beta \in \beta^*(A)} P(y_\beta, x_\alpha) < r$.

So, $P(A, B) = \bigcup_{x_\alpha \in \beta^*(B)} \bigcap_{y_\beta \in \beta^*(A)} P(y_\beta, x_\alpha)$.

(SEM4) Since U_n's are symmetric, therefore $\phi_s^r = \phi_s, \quad \forall\ s > 0$.

This implies $(\phi_{r_0} \circ \ldots \circ \phi_{r_k})^r = \phi_{r_0}^r \circ \ldots \circ \phi_{r_k}^r = \phi_{r_0} \circ \ldots \circ \phi_{r_k}$.

Now, $f_s^r = (\bigcup\{\phi_{r_0} \circ \ldots \circ \phi_{r_k} \mid \sum_{i=0}^{k} r_k = s,\ \forall\ i \leq k,\ r_i > 0,\ k < \omega\})^r$.

$= \bigcup\{(\phi_{r_0} \circ \ldots \circ \phi_{r_k})^r \mid \sum_{i=0}^{k} r_k = s,\ \forall\ i \leq k.\ r_i > 0,\ k < \omega\}$.

$= \bigcup\{(\phi_{r_0} \circ \ldots \circ \phi_{r_k}) \mid \sum_{i=0}^{k} r_k = s,\ \forall\ i \leq k,\ r_i > 0,\ k < \omega\}$.

$= f_r$.

So, $f_s^r = f_s$.

Now for any $C \in L^X$, let $P(A, C) < r \Rightarrow C \subseteq B$.

Since $f_s(A) = \bigcup\{E \in L^X \mid P(A, E) < r\}$, therefore $f_s(A) \subseteq B$.

This implies that $f_s^r(B') \subseteq A'$.

Hence, $f_s(B') \subseteq A'$, as $f_s^r = f_s$.

Now, if for any $D \in L^X$, $P(B', D) < r$.

Then as $f_s(B) = \bigcup\{F \in L^X \mid P(B', F) < r\}$, so $D \subseteq f_s(B')$.

We then have, $D \subseteq A'$.

Thus we get "$P(A, C) < r \Rightarrow C \subseteq B$" $\Rightarrow$ "$P(B', D) < r \Rightarrow D \subseteq A'$".

The other way implication may similarly be proved.

Hence, P fulfils (SEM4).

(SEM5) Since $A \subseteq B$ implies $f_r(A) \subseteq f_r(B)$, therefore for any $C \in L^X$,

we get $\{r \mid C \subseteq f_r(A)\} \subseteq \{r \mid C \subseteq f_r(B)\}$.

This implies $\bigwedge\{r \mid C \subseteq f_r(B)\} \leq \bigwedge\{r \mid C \subseteq f_r(A)\}$.

Hence, $P(B, C) \leq P(A, C)$.

Finally, we shall show that $\mathscr{U}(P) = \mathscr{U}$, where $\mathscr{U}(P)$ is the L-semi-uniformity generated by P.

For any $s > 0$, let $\psi_s : L^X \to L^X$ be a mapping such that

$\forall\ A \in L^X, \quad \psi_s(A) = \bigcup\{C \in L^X \mid P(A, C) < s\}$.

Let $\mathscr{B} = \{\psi_s \mid s > 0\}$.

Then as shown in theorem 3.3.1, $\mathscr{B}$ is a base for $\mathscr{U}(P)$.

Now, for any $s > 0$, $\psi_s(A) = \bigcup\{C \in L^X \mid P(A, C) < s\}$ implies that

$\psi_s(A) = \bigcup\{C \in L^X \mid C \subseteq f_s(A)\}$, from the construction of 'P'.

$= f_s(A)$, since for any $D \in L^X$, we have $D = \bigcup_{E \subseteq D} E$.

So, $\psi_s = f_s, \quad \forall\ s > 0$.

Hence, $\mathscr{U}(P) = \mathscr{U}$.

Conversely, let the L-semi-uniformity $\mathscr{U}$ is generated by an L-semi-pseudo-metric 'P'.

Then for any $r \in \mathbb{Q}_+$, where $\mathbb{Q}_+ = \{r \in \mathbb{Q} \mid r > 0\}$,

let $\psi_r : L^X \to L^X$ be a mapping defined by.

$\forall\ A \in L^X, \quad \psi_s(A) = \bigcup\{C \in L^X \mid P(A, C) < s\}$.

Then, $\{\psi_r \mid r \in \mathbb{Q}_+\}$ is the required base. □

3.4 Completeness and Compactness

In this section by characterizing completeness and compactness in terms of Cauchy ultrafilter and ultrafilter respectively, we have established that in a totally bounded L-semi-uniform space, the notions of completeness and compactness are equivalent. In this section we consider those L-semi-uniformities $\mathscr{U}$ on L^X for which $x_\alpha \in U(x_\alpha)$, $\forall\ U \in \mathscr{U}$, $\forall\ x_\alpha \in \mathrm{Pt}(L^X)$.

Definition 3.4.1. Let '$\imath$' be an interior operator on L^X. Then for any $x_\alpha \in \mathrm{Pt}(L^X)$, we shall call an L-fuzzy set N as a *neighborhood* (nbd) at x_α with respect to '$\imath$', if there is $G \in L^X$ such that $\imath(G) \nsubseteq x_\alpha$ and $\imath(G) \subseteq N$. The family of all nbd at x_α in the interior space is denoted by, $\mathscr{N}_\imath(x_\alpha)$. We shall call an L-fuzzy set F as a *quasi-coincident neighborhood* (Q-nbd) at x_α with respect to '$\imath$', if there is an L-fuzzy set B such that $x_\alpha \ll \imath(B)$ and $B \subseteq F$. The family of all Q-nbd at x_α in the interior space is denoted by, $\mathscr{Q}_\imath(x_\alpha)$.

Definition 3.4.2. Let '$\imath$' be an interior operator on L^X. Then a subfamily $\mathscr{A}$ of L^X is said to be a

(i) nbd base if $\mathscr{A} \subseteq \mathscr{N}_\imath(x_\alpha)$ and for every $N \in \mathscr{N}_\imath(x_\alpha)$, $\exists\ A \in \mathscr{A}$ such that $A \subseteq N$.

(ii) Q-nbd base if $\mathscr{A} \subseteq \mathscr{Q}_\imath(x_\alpha)$ and for every $F \in \mathscr{Q}_\imath(x_\alpha)$, $\exists\ B \in \mathscr{A}$ such that $B \subseteq F$.

Definition 3.4.3. For any $x_\alpha \in L^X$ we define its *dual point* as an L-fuzzy point x^*_α such that

$$x^*_\alpha(y) = \begin{cases} \alpha' & \text{if } y = x \\ 0_L & \text{if } y \neq x \end{cases}$$

In view of theorem 2.1.23, we have the following:

Theorem 3.4.1. *Let '$\imath$' be an interior operator on L^X and $A \in L^X$. Then an L-fuzzy point $x_\alpha \in (\imath(A'))'$ iff each neighborhood of its dual point x^*_α is quasi-coincident with A.*

Following definitions are adopted to an interior space:

Definition 3.4.4. A non empty sub collection $\mathscr{F}$ of L^X is said to be a *filter* in an interior space, if:

F1) $\underline{0} \notin \mathscr{F}$.

F2) $U, V \in \mathscr{F} \Rightarrow U \bigcap V \in \mathscr{F}$.

F3) $U \in \mathscr{F}$ and $G \in L^X$ such that $U \subseteq G$ then $G \in \mathscr{F}$.

Definition 3.4.5. A subfamily $\mathscr{B}$ of L^X is called a *filter base* in an interior space, if:

B1) $\underline{0} \notin \mathscr{B}$.

B2) for any $U, V \in \mathscr{B}$, there exists $W \in \mathscr{B}$ such that $W \subseteq U \bigcap V$.

Definition 3.4.6. A filter $\mathscr{F}$ is said to be *closed* with respect to some interior operator '$\imath$', if for any $F \in \mathscr{F}$ implies $F = c(F)$.

Definition 3.4.7. Let $x_\alpha \in \mathrm{Pt}(L^X)$ and $\mathscr{F}$ be a filter, then $\mathscr{F}$ is said to be *converges to* x_α with respect to some interior operator '$\imath$', denoted by $\mathscr{F} \to x_\alpha$, if for any $U \in \mathscr{Q}_\imath(x_\alpha)$ there exists $F \in \mathscr{F}$ such that $F \subseteq U$, that is, $\mathscr{Q}_\imath(x_\alpha) \subseteq \mathscr{F}$.

Definition 3.4.8. *Cluster set* of $\mathscr{F}$ with respect to some interior operator '$\imath$', is given by,

$\bigcap\{\mathrm{cl}(F) \mid F \in \mathscr{F}\}$.

For any $x_\alpha \in \mathrm{Pt}(L^X)$, if x_α is in the cluster set of $\mathscr{F}$, then we denote it by, $\mathscr{F} \rightsquigarrow x_\alpha$.

Remark 3.4.1. Cluster set of a filter with respect to an L-topological space was defined by Hutton in an analogous way.

Remark 3.4.2. If x_α is in the cluster set of $\mathscr{F}$ with respect to some interior operator '$\imath$', then for any $F \in \mathscr{F}$, $x_\alpha \subseteq (\imath(F'))'$.
But $x_\alpha \subseteq (\imath(F'))' \Rightarrow \alpha \leq (\imath(F'))'(x) \Rightarrow (\imath(F'))'(x) \not< \alpha \Rightarrow \alpha' \not< \imath(F')(x)$.

Now $\alpha' \nleq \imath(F')(x) \Rightarrow x^*_\alpha \notin \imath(F') \Rightarrow G \nsubseteq F'$, $\forall\, G \in \mathscr{N}_\imath(x^*_\alpha)$.

This implies that $G\widehat{q}F$, $\forall G \in \mathscr{N}_\imath(x^*_\alpha)$.

But for any $A,\ B \in L^X$, $A\widehat{q}B$ implies $A \bigcap B \neq \underline{0}$.

For if $A\widehat{q}B$, then there exists $x \in X$ such that $A(x) \nleq B'(x)$.

So, $0_L < A(x)$ and $B'(x) < 1_L$.

Hence, $0_L < A(x)$ and $0_L < B(x)$.

We then have, $A(x) \bigwedge B(x) \neq 0_L$. So, $A \bigcap B \neq \underline{0}$.

Hence, $G \bigcap F \neq \underline{0}$, $\forall G \in \mathscr{N}_\imath(x^*_\alpha)$.

Again since G is a nbd at x^*_α iff G is a Q-nbd at x_α.

Therefore, this definition implies the definition given in 2.1.33.

Definition 3.4.9. Let '$\imath$' be an interior operator on L^X. A subset $\mathscr{F}$ of L^X is said to satisfy the *F. I. P. relative to an open set G* with respect to the interior operator '$\imath$' if $F_1, ..., F_n \in \mathscr{F} \Rightarrow \bigcap_{i=1}^n F_i \nsubseteq G$.

Obviously, every subset $\mathscr{F}$ of L^X which satisfy the F. I. P. relative to G is contained in a filter relative to G.

The following result follows from theorem 2.1.27.

Theorem 3.4.2. *Let $\imath_X$ and $\imath_Y$ be two interior operators on L^X and L^Y respectively. Then a function $f^\rightarrow : L^X \rightarrow L^Y$ is continuous with respect to the interior operators if and only if for any filter $\mathscr{F}$ converges on L^X with respect to the interior operator $\imath_X$ implies $f^\rightarrow(\mathscr{F}) = \{f^\rightarrow(F) \mid F \in \mathscr{F}\}$ converges on L^Y with respect to the interior operator $\imath_Y$.*

In view of theorem 2.1.28, we obtain the following:

Theorem 3.4.3. *Let ' $\imath$ ' be an interior operator on L^X and $A \in L^X$. Then $x_\alpha \in (\imath(A'))'$ iff there is a filter $\mathscr{F}$ relative to A' such that $\mathscr{F} \rightarrow x_\alpha$,* *with respect to that interior operator '$\imath$'.*

Definition 3.4.10. Let $A \in L^X$. We shall call the maximal filter (partially ordered by set inclusion) $\mathscr{F}_{\text{и}}$ relative to A as an *ultrafilter relative to A.*

If $A = \underline{0}$, then we simply call $\mathscr{F}_{\text{и}}$ as an *ultrafilter.*

Theorem 3.4.4. *Let '$\imath$' be an interior operator and $\mathscr{F}$ be a filter on L^X. Let $x_\alpha \in \mathrm{Pt}(L^X)$ such that $\mathscr{F} \to x_\alpha$ with respect to '$\imath$'. Then $\mathscr{F} \rightsquigarrow x_\alpha$.*

Proof. Let F be any member of $\mathscr{F}$.
Now we consider the following two cases.
Case I. Let $F' \notin \mathscr{F}$. Then, $\mathscr{F}$ is a filter relative to F' and $\mathscr{F} \to x_\alpha$.
So, by theorem 3.4.3, $x_\alpha \in (\imath(F'))'$.
Case II. Let $F' \in \mathscr{F}$. Let N be any Q-nbd at x_α. Then $N \in \mathscr{F}$ and hence $F' \bigcap N \neq \underline{0}$. Then, there exists $y_\beta \in F$ such that $y^*_\beta \in N$.
This implies that $N(y) \nleq (F(y))'$ and hence $N\widehat{q}F$.
Therefore, by theorem 2.1.23, $x_\alpha \in (\imath(F'))'$.
Thus, in either case $x_\alpha \in (\imath(F'))', \ \forall\, F \in \mathscr{F}$.
Hence, $\mathscr{F} \rightsquigarrow x_\alpha$. □

The following result can be obtained from theorems 2.1.29 and 3.4.4.

Theorem 3.4.5. *Let '$\imath$' be an interior operator and $\mathscr{F}_{\text{и}}$ be an ultrafilter on L^X. Then $\mathscr{F}_{\text{и}} \rightsquigarrow x_\alpha$ iff $\mathscr{F}_{\text{и}} \to x_\alpha$.*

In view of lemma 2.1.30, we have the following:

Lemma 3.4.6. *For any ultrafilter $\mathscr{F}_{\text{и}}$ and A, $B \in L^X$ such that $A \bigcup B \in \mathscr{F}_{\text{и}}$, either $A \in \mathscr{F}_{\text{и}}$ or $B \in \mathscr{F}_{\text{и}}$.*

Definition 3.4.11. Let 'i' be an interior operator on L^X, an *open cover* $\mathscr{C}$ of an L-fuzzy set A is a collection of open sets with respect to the interior operator 'i' such that $A \subseteq \bigcup_{G\in\mathscr{C}} G$.

In view of definition 2.1.39, we adopt the following definition for an interior space.

Definition 3.4.12. An interior space is said to be *compact* if it satisfies any of the following equivalent conditions:

(1) Every open cover $\mathscr{C}$ of a closed set has a finite subcover.

(2) Every collection of closed sets $\mathscr{F}$ satisfying the F. I. P. relative to an open set G has $\bigcap_{F\in\mathscr{F}} F \nsubseteq G$.

We now state the following lemma:

Lemma 3.4.7. *For any* $U \in \mathscr{U}^*$ *and* $x_\alpha, y_\beta \in \mathrm{Pt}(L^X)$ *we get,*
$y_\beta \subseteq U(x_\alpha)$ *iff* $x_\alpha \subseteq U^r(y_\beta)$.

Proof. By proposition 2.1.35 (iii), $(U^r)^r = U$.
So, we need to prove only one way implication.
Here, $U^r(y_\beta) = \bigcap\{z_\gamma \mid U(z'_\gamma) \subseteq y'_\beta\}$.
Let $y_\beta \subseteq U(x_\alpha)$.
Then $[U(x_\alpha)]' \subseteq y'_\beta$.
Let $A : X \to L$ be a mapping defined by,

$$\forall\ z\in X,\quad A(z)=\begin{cases}\gamma & \text{if } U(z'_\gamma)\subseteq y'_\beta,\\ 0_L & \text{if } U(z'_\gamma)\nsubseteq y'_\beta.\end{cases}$$

Then $U^r(y_\beta) = \bigcap A$.
Let $B : X \to L$ be a mapping defined by,

$$\forall\ w\in X,\quad B(w)=\begin{cases}\eta & \text{if } U(w_\eta)\subseteq [U(x_\alpha)]',\\ 0_L & \text{if } U(w_\eta)\nsubseteq [U(x_\alpha)]'.\end{cases}$$

Let w be any element of X.

Then, $B'(w) = \eta \Rightarrow B(w) = \eta'$.

$\Rightarrow U(w'_\eta) \subseteq [U(x_\alpha)]'$.

$\Rightarrow U(w'_\eta) \subseteq y'_\beta$.

$\Rightarrow A(w) = \eta$.

Therefore, $B' \subseteq A$ and hence $\bigcup A' \subseteq \bigcup B$.

Again $b_\mu \subseteq B \Rightarrow U(b_\mu) \subseteq [U(x_\alpha)]' \Rightarrow U(b_\mu) \subseteq x'_\alpha \Rightarrow b_\mu \subseteq x'_\alpha$.

$\Rightarrow \bigcup B \subseteq x'_\alpha \Rightarrow \bigcup A' \subseteq x'_\alpha \Rightarrow x_\alpha \subseteq \bigcap A$.

Hence, $x_\alpha \subseteq U^r(y_\beta)$. □

Theorem 3.4.8. *Let $(L^X, \mathscr{U})$ be an L-semi-uniform space and 'int' be the induced interior operator on L^X. Then the respective interior space is compact iff every ultrafilter relative to an open set with respect to 'int' is convergent.*

Proof. Let the space be compact and $\mathscr{F}_u$ be a ultrafilter relative to an open set G on the space.

Then by theorem 3.2.5, $\mathscr{F} = \{\text{cl}(F) \mid F \in \mathscr{F}_u\}$ is a collection of closed sets satisfying F. I. P. relative to the open set G.

Consequently, by compactness, we have $\bigcap_{F \in \mathscr{F}_u} \text{cl}(F) \nsubseteq G$.

This implies that there is some $x_\alpha \in \text{Pt}(L^X)$ such that $x_\alpha \subseteq \bigcap_{F \in \mathscr{F}_u} \text{cl}(F)$.

So by theorem 3.4.5, we have $\mathscr{F}_u \to x_\alpha$.

Conversely, let $\mathscr{F}$ be a collection of closed sets satisfying F. I. P. relative to an open set G.

Let $\mathscr{F}^*$ be a filter relative to the open set G and containing $\mathscr{F}$.

Then, $\bigcap_{F^* \in \mathscr{F}^*} F^* \subseteq \bigcap_{F \in \mathscr{F}} F$.

Let $\mathscr{F}_u$ be an ultrafilter relative to the open set G.

We then have,

$$\bigcap_{F_{\varkappa}\in\mathscr{F}_{\varkappa}} F_{\varkappa} \subseteq \bigcap_{F^*\in\mathscr{F}^*} F^* \subseteq \bigcap_{F\in\mathscr{F}} F \tag{3.4.1}$$

Let $\mathscr{F}_{\varkappa} \to x_\alpha$.

Then, $\mathscr{Q}(x_\alpha) = \{U(x^*_\alpha) \mid U \in \mathscr{U}\} \subseteq \mathscr{F}_{\varkappa}$.

Now let U be any symmetric member of $\mathscr{U}$.

Let $F_{\varkappa}$ be any member of $\mathscr{F}_{\varkappa}$.

Then $U(x^*_\alpha) \in \mathscr{F}_{\varkappa}$ implies $F_{\varkappa} \bigcap U(x^*_\alpha) \neq \underline{0}$.

This implies there exists $y_\beta \subseteq F_{\varkappa}$ such that $y_\beta \subseteq U(x^*_\alpha)$.

This further implies $x^*_\alpha \subseteq U^r(y_\beta) = U(y_\beta)$, by lemma 3.4.7.

But $y_\beta \subseteq F_{\varkappa}$ implies $U(y_\beta) \subseteq U(F_{\varkappa})$.

Hence, for any symmetric member U of $\mathscr{U}$, we get $x^*_\alpha \subseteq U(F_{\varkappa})$.

Again since the collection of symmetric members of $\mathscr{U}$ is a base for $\mathscr{U}$, therefore by lemma 3.2.2, we get $x^*_\alpha \subseteq \mathrm{cl}(F_{\varkappa}) = F_{\varkappa}, \ \forall\, F_{\varkappa} \in \mathscr{F}_{\varkappa}$.

Hence, we get,

$$x^*_\alpha \subseteq \bigcap_{F_{\varkappa}\in\mathscr{F}_{\varkappa}} F_{\varkappa} \tag{3.4.2}$$

Now if $x^*_\alpha \subseteq G$, then there is $U \in \mathscr{U}$ such that $U(x^*_\alpha) \subseteq G$, as G is open.

But then $G \in \mathscr{F}_{\varkappa}$ and this contradicts the fact that $\mathscr{F}_{\varkappa}$ is relative to G.

So $x^*_\alpha \nsubseteq G$.

This implies $\bigcap_{F_{\varkappa}\in\mathscr{F}_{\varkappa}} F_{\varkappa} \nsubseteq G$, by 3.4.2.

Thus by 3.4.1, we have $\bigcap_{F\in\mathscr{F}} F \nsubseteq G$.

Hence, the space is compact. □

Definition 3.4.13. A filter $\mathscr{F}$ in an L-semi-uniform space $(L^X, \mathscr{U})$ is said to be *Cauchy* if for each $U \in \mathscr{U}$, $\exists\, x_\alpha \in \mathrm{Pt}(L^X)$ and $F \in \mathscr{F}$ such that $F \subseteq U(x_\alpha)$.

Definition 3.4.14. An L-semi-uniform space $(L^X, \mathscr{U})$ is said to be *complete* if and only if for every Cauchy filter $\mathscr{F}$ relative to an open set with respect to the interior operator generated by $\mathscr{U}$, $\bigcap_{F \in \mathscr{F}} \mathrm{cl}(F) \neq \underline{0}$.

The following result follows from theorem 3.4.5.

Theorem 3.4.9. *An L-semi-uniform space $(L^X . \mathscr{U})$ is complete iff every Cauchy ultrafilter relative to an open set with respect to the interior operator generated by $\mathscr{U}$ is convergent.*

Theorem 3.4.10. *Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be L-semi-uniform spaces and let $f^{\rightarrow} : L^X \rightarrow L^Y$ be L-semi-uniformly continuous. If $\mathscr{F}$ is a Cauchy filter in $(L^X . \mathscr{U})$, then $f^{\rightarrow}(\mathscr{F})$ is Cauchy filter in $(L^Y, \mathscr{V})$.*

Proof. Let $\mathscr{F}$ be a Cauchy filter on L^X.
Let $V \in \mathscr{V}$. Since $f^{\rightarrow} : L^X \rightarrow L^Y$ is L-semi-uniformly continuous, therefore there exists $U \in \mathscr{U}$ such that $\widehat{f^{\rightarrow}}(U) \subseteq V$.
Now, $\mathscr{F}$ is a Cauchy filter on L^X.
Hence, there exists $F \in \mathscr{F}$ and $x_\alpha \in \mathrm{Pt}(L^X)$ such that $F \subseteq U(x_\alpha)$.
Then, $f^{\rightarrow}(F) \subseteq V(f^{\rightarrow}(x_\alpha))$.
Hence, $f^{\rightarrow}(\mathscr{F})$ is a Cauchy filter on $(L^Y, \mathscr{V})$. □

Theorem 3.4.11. *Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be two L-semi-uniform spaces and $f^{\rightarrow} : L^X \rightarrow L^Y$ be an L-semi-uniformly isomorphism, then $(L^X, \mathscr{U})$ is complete iff $(L^Y, \mathscr{V})$ is so.*

Proof. Let $(L^Y, \mathscr{V})$ be complete and $\mathscr{F}$ be a Cauchy filter on L^X relative to an open set G.
Let $V \in \mathscr{V}$, then by theorem 3.4.10, $f(\mathscr{F})$ is a Cauchy filter on $(L^Y, \mathscr{V})$.

Again, since $f^{\leftarrow}$ is L-semi-uniformly continuous, therefore by theorem 3.2.6, $f^{\leftarrow}$ is continuous and so $f^{\rightarrow}$ is open.
Hence, $f^{\rightarrow}(G)$ is open in L^Y.
Also, as $G \subseteq f^{\leftarrow}(f^{\rightarrow}(G))$ and $G \notin \mathscr{F}$, therefore $f^{\rightarrow}(\mathscr{F})$ is a Cauchy filter relative to the open set $f^{\rightarrow}(G)$.
Thus, $f^{\rightarrow}(\mathscr{F})$ is convergent on $(L^Y, \mathscr{V})$, it being complete.
But by corollary 3.2.7, $f^{\rightarrow}$ is a homeomorphism.
Consequently, $\mathscr{F}$ converges on $(L^X, \mathscr{U})$, by theorem 3.4.2.
Hence, $(L^X, \mathscr{U})$ is complete. □

Definition 3.4.15. Let $(L^X, \mathscr{U})$ be an L-semi-uniform space and $A \in L^X$.

Let for any $U \in \mathscr{U}$, $U_A : L^X \to L^X$ be a mapping such that

$$U_A(x_\alpha) = \begin{cases} U(x_\alpha) & \text{if } x_\alpha \subseteq A. \\ \underline{0} & \text{if } x_\alpha \nsubseteq A. \end{cases}$$

Then, $\mathscr{U}_A = \{U_A \mid U \in \mathscr{U}\}$ is an L-semi-uniformity on A, which we call a *sub L-semi-uniformity* on A and $(A, \mathscr{U}_A)$ to be the *subspace*.

$\mathscr{U}_A$ is called *open* or *closed sub L-semi-uniformity* provided $A = \text{int}_{\mathscr{U}_A}(A)$ or $A = (\text{int}_{\mathscr{U}_A}(A'))'$ respectively, where $\text{int}_{\mathscr{U}_A}$ is the interior operator generated by $\mathscr{U}_A$.

Theorem 3.4.12. *Every closed sub L-semi-uniformity in a complete L-semi-uniform space is complete.*

Proof. Let $(L^X, \mathscr{U})$ be a complete L-semi-uniform space and $A \in L^X$ such that $A = (\text{int}_{\mathscr{U}}(A'))'$, where $\text{int}_{\mathscr{U}}$ is the interior operator generated by $\mathscr{U}$.

Let $\mathscr{F}_{\kappa} = \{F \mid F \subseteq A\}$ be a Cauchy ultrafilter relative to an open set B with respect to $\text{int}_{\mathscr{U}_A}$, where $\text{int}_{\mathscr{U}_A}$ is the interior operator generated by $\mathscr{U}_A$.

Now if $B' \in \mathscr{F}_{\kappa}$, then from the definition of $\mathscr{F}_{\kappa}$, we get, $B' \subseteq A$.

But $B' \subseteq A$ implies $A' \subseteq B$ and consequently, $A' \notin \mathscr{F}_{\kappa}$, as $\mathscr{F}_{\kappa}$ is a filter relative to B.

Also if, $B' \notin \mathscr{F}_{\kappa}$, then as $B \subseteq A$ implies $A' \subseteq B'$, so $A' \notin \mathscr{F}_{\kappa}$.

Thus, in either case $\mathscr{F}_{\kappa}$ is an ultrafilter in $(L^X, \mathscr{U})$ relative to A'.

Now, since for any $U \in \mathscr{U}$ we get, $U_A \subseteq U$, therefore $\mathscr{F}_{\kappa}$ is also Cauchy in $(L^X, \mathscr{U})$.

Thus, $\mathscr{F}_{\kappa}$ is a Cauchy ultrafilter in $(L^X, \mathscr{U})$ relative to the open set A' and consequently there exists $x_\alpha \in L^X$ such that $\mathscr{F}_{\kappa} \to x_\alpha$.

But as $A = (\text{int}_{\mathscr{U}}(A'))'$, so by theorem 3.4.3, $x_\alpha \in A$.

Hence, $(A, \mathscr{U}_A)$ is complete. □

Definition 3.4.16. An L-semi-uniform space $(L^X, \mathscr{U})$ is said to be *totally bounded* if for any $U \in \mathscr{U}$ there is finite $A \subseteq Pt(L^X)$ such that $\underline{1} = U(A) = \bigcup\{U(x_\alpha) \mid x_\alpha \in A\}$.

This generalizes the notion of totally boundedness of G. Artico and R. Moresco [1].

Theorem 3.4.13. *In a totally bounded space $(L^X, \mathscr{U})$, every ultrafilter is a Cauchy filter.*

Proof. Let $\mathscr{F}_{\kappa}$ be a ultrafilter and $U \in \mathscr{U}$.

By totally boundedness there is a finite $A \subseteq Pt(L^X)$ such that

$$\underline{1} = U(A) = \bigcup\{U(x_\alpha) \mid x_\alpha \in A\}.$$

But as $\underline{1} \in \mathscr{F}_{\kappa}$, therefore by lemma 3.4.6,

$U(x_\alpha) \in \mathscr{F}_{\kappa}$, for some $x_\alpha \in A$. □

Theorem 3.4.14. *Let $(L^X, \mathscr{U})$ be an L-semi-uniform space, then the space is compact iff*

(i) *$(L^X, \mathscr{U})$ is totally bounded and*

(ii) *$(L^X, \mathscr{U})$ is complete.*

Proof. Let $(L^X, \mathscr{U})$ be a compact space.

(i) Let $U \in \mathscr{U}$ be any and 'cl' be the closure operator generated by $\mathscr{U}$. Then $\{\text{int}(U(x_\alpha)) \mid x_\alpha \in L^X\}$ is an open cover of $\underline{1}$.

Since $\text{cl}(\underline{1}) = \underline{1}$, therefore by compactness, for this open cover there is a finite $A \subseteq Pt(L^X)$ such that $\underline{1} = \bigcup\{\text{int}(U(x_\alpha)) \mid x_\alpha \in A\}$.

Hence, $(L^X, \mathscr{U})$ is totally bounded.

(ii) Follows from the theorems 3.4.8 and 3.4.9.

Conversely, let the space be totally bounded and complete.

Then, by theorems 3.4.13, 3.4.9 and 3.4.8 the space is compact. □

CHAPTER 4

L - LOCALLY QUASI-UNIFORM SPACES

4.1 Introduction

In the previous chapter we have developed the notions of L-semi-quasi-uniformity and L-semi-uniformity in the category **L-TOP** as generalizations of Hutton's quasi-uniformity and uniformity respectively. Various results of uniform spaces concerning completeness, compactness, metrizability and uniformly continuous functions have been developed in generalized form in the previous chapter. It is, however, observed that the generating spaces in the above context are interior spaces, which are generalizations of L-topological spaces. So, it becomes pertinent to investigate whether there are other generalized uniformity structures in the same category weaker than Hutton's quasi-uniformity and uniformity in which generating spaces are L-topological and whether the results of uniform spaces could be developed therein. While looking for an answer to the above question, we have introduced the notion of L-local quasi-uniformity as a generalization of Hutton's quasi-uniformity. The compatibility of L-local quasi-uniformity and L-topology is then examined. L-weakly quasi-uniformly continuous functions which are introduced as a generalization of L-quasi-uniformly continuous functions are examined for continuity with respect to the induced L-topological spaces. In order to show that the notion of L-locally quasi-uniform spaces lies between L-semi-quasi-uniform spaces and Hutton's quasi-uniform spaces, suitable examples are provided. Further, the problem of metrization of the introduced generalized form of uniformity is considered and a satisfactory answer has been provided.

4.2 *L*-Locally quasi-uniform space

Definition 4.2.1. An L-semi-quasi-uniformity $\mathscr{U}$ on L^X is said to be an *L-local quasi-uniformity* if $\mathscr{U}$ has a base $\mathscr{B}$ satisfying the following:

(LQ) For any $U \in \mathscr{B}$ and for any $x_\alpha \in \mathrm{Pt}(L^X)$ there exists $V \in \mathscr{B}$ such that $V \circ V(x_\alpha) \subseteq U(x_\alpha)$.

The pair $(L^X, \mathscr{U})$ is then called an *L-locally quasi-uniform space.*

Clearly, every L-quasi-uniformity is an L-local quasi-uniformity.

We now provide examples to show that

(i) An L-local quasi-uniformity need not be an L-quasi-uniformity.

(ii) An L-semi-quasi-uniformity need not be an L-local quasi-uniformity.

Example 4.2.1. (i) Let $L = \{0\} \bigcup([0,1] \bigcap\{\frac{1}{n}\}_{n \in \mathbb{N}})$ and $X = \{x\}$. Then L is a fuzzy lattice with the general order relation.

For each $n \in \mathbb{N}$, let

$B_n =$

$\Delta_{L^X} \bigcup\{(x_{1/i}, x_{1/2}) \mid n \leq i\} \bigcup\{(x_{1/i+1}, x_{1/i}) \mid n \leq i\} \bigcup\{(x_1, x_{1/i}) \mid n \leq i\}$,

where $\Delta_{L^X} = \{(x_\alpha, x_\alpha) \mid x_\alpha \in L^X\}$.

Let $\mathscr{B} = \{B_n \mid n \in \mathbb{N}\}$.

For each $B_n \in \mathscr{B}$, let $U_{B_n} : L^X \to L^X$ be a mapping defined by,

$$U_{B_n}(x_\alpha) = \bigcup\{x_\beta \mid (x_\alpha, x_\beta) \in B_n\}, \quad \forall\, x_\alpha \in L^X.$$

Then,

1) $\mathscr{U}_{\mathscr{B}} \subseteq \mathscr{U}^*$.

2) $U_{B_{n+1}} \subseteq U_{B_n}, \quad \forall\, n \in \mathbb{N}$.

Therefore, $U_{B_n} \bigcap U_{B_m} = U_{B_l} \in \mathscr{U}_{\mathscr{B}}$,

where $l = \min\{m, n\}$ and $\mathscr{U}_{\mathscr{B}} = \{U_{B_n} \mid B_n \in \mathscr{B}\}$.

Hence, $\mathscr{U}_{\mathscr{B}}$ is a base for some L-semi-quasi-uniformity.

A) We now show that $\mathscr{U}_{\mathscr{B}}$ is a base for an L-local quasi-uniformity.

For fixed $n \in \mathbb{N}$, we have

$U_{B_n}(x_{1/2}) = U_{B_n} \circ U_{B_n}(x_{1/2}) = \bigcup\{x_{1/2}\} \; \forall\, n \in \mathbb{N}$.

$U_{B_1}(x_1) = U_{B_1} \circ U_{B_1}(x_1) = \bigcup\{x_{1/2}, x_1, x_{1/2}, \ldots\}$.

$U_{B_n}(x_1) = U_{B_n} \circ U_{B_n}(x_1) = \bigcup\{x_1, x_{1/n}, x_{1/(n+1)}, \ldots\}, \quad \forall\, n > 1$.

Fix $n \in \mathbb{N}$ and $k > 1$, then

$U_{B_{k+1}} \circ U_{B_{k+1}}(x_{1/k}) = \{x_{1/k}\}$.

Hence, $U_{B_{k+1}} \circ U_{B_{k+1}}(x_{1/k}) \subseteq U_{B_n}(x_{1/k}), \quad \forall\, n \in \mathbb{N}$.

B) We now show that $\mathscr{U}_{\mathscr{B}}$ is not a base for an L-quasi-uniformity.

Fix $U_{B_l} \in \mathscr{U}_{\mathscr{B}}$ and the pair $(x_{1/(n'+2)}, x_{1/n'})$.

For any $n' \in \mathbb{N}$, we have,

$x_{1/(n'+1)} \subseteq U_{B_{n'}}(x_{1/(n'+2)})$ and $x_{1/n'} \subseteq U_{B_{n'}}(x_{1/(n'+1)})$.

This implies $x_{1/n'} \subseteq U_{B_{n'}} \circ U_{B_{n'}}(x_{1/(n'+2)})$, $\forall\ n' \in \mathbb{N}$.

However, $x_{1/n'} \not\subseteq U_{B_l}(x_{1/(n'+2)})$.

Thus, for $U_{B_l} \in \mathscr{U}_{\mathscr{B}}$ there is no $U_{B_{n'}} \in \mathscr{U}_{\mathscr{B}}$ such that $U_{B_{n'}} \circ U_{B_{n'}} \subseteq U_{B_l}$.

(ii) In order to show that L-semi-quasi-uniformity does not implies L-local quasi-uniformity, we refer to example 3.2.1.

We have shown that $\mathscr{U}_{\mathscr{B}}$ defined in example 3.2.1 is a base for an L-semi-quasi-uniformity.

However, $\mathscr{U}_{\mathscr{B}}$ is not a base for an L-local quasi-uniformity.

Observe that for any $n \in \mathbb{N}$, we have,

$x_{1/m} \subseteq U_{B_n}(x_{1/2})$ and $x_1 \subseteq U_{B_n}(x_{1/m})$.

But $U_{B_2}(x_{1/2}) = \bigcup\{x_{1/2}, x_{1/2}, x_{1/3}, ...\}$.

Thus $x_1 \subseteq U_{B_n} \circ U_{B_n}(x_{1/2})$, $\forall n \in \mathbb{N}$, but $x_1 \not\subseteq U_{B_2}(x_{1/2})$.

Hence, for $U_{B_2} \in \mathscr{U}_{\mathscr{B}}$ and $x_{1/2} \in L^X$ there is no $U_{B_n} \in \mathscr{U}_{\mathscr{B}}$ such that $U_{B_n} \circ U_{B_n}(x_{1/2}) \subseteq U_{B_2}(x_{1/2})$.

Definition 4.2.2. A non void subfamily $\mathscr{S}$ of $\mathscr{U}^*$ is called a *subbase* for some L-local quasi-uniformity $\mathscr{U}$, if the collection of non empty intersection of finitely many members of $\mathscr{S}$ is a base for $\mathscr{U}$.

A subset $\mathscr{S}$ of $\mathscr{U}^*$ is a subbase for some L-local quasi-uniformity if it satisfies the following condition:

for any $U \in \mathscr{S}$ and $x_\alpha \in \mathrm{Pt}(L^X)$, $\exists V \in \mathscr{S}$ s.t. $V \circ V(x_\alpha) \subseteq U(x_\alpha)$.

We now proceed to define *L-locally quasi-uniformizability* and examine conditions leading to the same.

Definition 4.2.3. An L-topology is said to be *L-locally quasi-uniformizable* if there is an L-local quasi-uniformity $\mathscr{U}$ that generates the L-topology.

Theorem 4.2.1. *Every L-local quasi-uniformity is an L-semi-quasi-uniformity along with the condition given in lemma 3.2.3.*

Proof. Straightforward.

Thus, by lemma 3.2.3, we have the following:

Lemma 4.2.2. *Let $(L^X, \mathscr{U})$ be an L-locally quasi-uniform space. Then the interior operator 'int' defined in theorem 3.2.1 is an L-topological interior operator.*

Now, by proposition 2.1.18 we have the following:

Theorem 4.2.3. *Every L-local quasi-uniform space generates an L-topological space.*

For an L-locally quasi-uniform space $(L^X. \mathscr{U})$, the generating L-topology on L^X is the L-topology generated by int, where int : $L^X \to L^X$ is a function defined by, $\text{int}(A) = \bigcup\{x_\alpha \mid \exists\, V \in \mathscr{B} \text{ s.t. } V(x_\alpha) \subseteq A\}$.
Hence, we note that, in particular, for any $x_\alpha \in \text{Pt}(L^X)$, $\{U(x_\alpha) \mid U \in \mathscr{U}\}$ is the neighborhood system at x_α in $(L^X, \mathscr{U})$.

The following result shows that the converse of theorem 4.2.3 is also true.

Theorem 4.2.4. *Every L- topology is L- locally quasi-uniformizable.*

Proof. Let $(L^X, \mathbb{F})$ be an L-topological space. For any $G \in \mathbb{F}$, let $U_G : L^X \to L^X$ be a mapping defined by,

$$U_G(x_\alpha)=\begin{cases} \underline{1} & \text{if} \quad x_\alpha \nsubseteq G, \\ G & \text{if} \quad x_\alpha \subseteq G. \end{cases}$$

Then, for any x_α and $G \in \mathbb{F}$, we get either $U_G(x_\alpha) = \underline{1}$ or $U_G(x_\alpha) = G$. Thus, in either case we get $x_\alpha \subseteq U_G(x_\alpha)$.

Hence, for any $G \in \mathbb{F}$, we get $\Delta \subseteq U_G$.

Also for any $G \in \mathbb{F}$, we get

$$U_G(\bigcup_{x_\alpha \in \mathrm{Pt}(L^X)} x_\alpha)=\begin{cases} \underline{1} & \text{if} \quad \bigcup_{x_\alpha \in \mathrm{Pt}(L^X)} x_\alpha \nsubseteq G, \\ G & \text{if} \quad \bigcup_{x_\alpha \in \mathrm{Pt}(L^X)} x_\alpha \subseteq G. \end{cases} \tag{4.2.1}$$

and

$$\bigcup_{x_\alpha \in \mathrm{Pt}(L^X)} U_G(x_\alpha)=\begin{cases} \underline{1} & \text{if } \exists\ x_\alpha \in \mathrm{Pt}(L^X) \text{ s.t. } x_\alpha \nsubseteq G, \\ G & \text{if } \forall\ x_\alpha \in \mathrm{Pt}(L^X),\ x_\alpha \subseteq G. \end{cases} \tag{4.2.2}$$

Since the conditions in the right sides of 4.2.1 and 4.2.2 are equivalent respectively.

Therefore,

$$U_G(\cup_{x_\alpha \in \mathrm{Pt}(L^X)} x_\alpha) = \cup_{x_\alpha \in \mathrm{Pt}(L^X)} U_G(x_\alpha). \tag{4.2.3}$$

Since for any $A \in L^X$, we have $A = \bigcup_{x_\alpha \subseteq A} x_\alpha$.

Therefore, by 4.2.3, we have the following:

$$U_G(A)=\begin{cases} \underline{1} & \text{if} \quad A \nsubseteq G, \\ G & \text{if} \quad A \subseteq G. \end{cases}$$

Now

$$U_G(\bigcup_\lambda A_\lambda)=\begin{cases} \underline{1} & \text{if} \quad \bigcup_\lambda A_\lambda \nsubseteq G, \\ G & \text{if} \quad \bigcup_\lambda A_\lambda \subseteq G. \end{cases} \tag{4.2.4}$$

and

$$\bigcup_\lambda U_G(A_\lambda)=\begin{cases}\underline{1} & \text{if } \exists\ \lambda \text{ s.t. } A_\lambda \not\subseteq G,\\ G & \text{if } \forall\ \lambda,\ A_\lambda \subseteq G.\end{cases} \tag{4.2.5}$$

By 4.2.4 and 4.2.5 we have, $U_G(\bigcup_\lambda A_\lambda) = \bigcup_\lambda U_G(A_\lambda)$.

Again for any x_α,

$$(U_G \circ U_G)(x_\alpha)=\begin{cases}\underline{1} & \text{if } U_G(x_\alpha)\not\subseteq G,\\ G & \text{if } U_G(x_\alpha)\subseteq G.\end{cases}$$

But $U_G(x_\alpha) \not\subseteq G$ if and only if $x_\alpha \not\subseteq G$.

So $U_G(x_\alpha) \subseteq G$ if and only if $x_\alpha \subseteq G$.

Therefore, $U_G \circ U_G = U_G$. $\forall\ G \in \mathbb{F}$.

Hence, $\{U_G \mid G \in \mathbb{F}\}$ forms a subbase for some L-local quasi-uniformity $\mathscr{U}$.

It remains now to show that $\mathscr{U}(\mathbb{F}) = \mathbb{F}$, where $\mathscr{U}(\mathbb{F})$ is the L-topology generated by $\mathscr{U}$.

Let $A \in \mathbb{F}$. Then $U_A(A) = A$ and hence for any $x_\alpha \subseteq A$ implies $U_A(x_\alpha) = A$.

Since $\text{int}(A) = \bigcup\{y_\beta \mid \exists\ G \in \mathbb{F} \text{ s.t. } U_G(y_\beta) \subseteq A\}$, therefore $\text{int}(A) = A$ and hence $A \in \mathscr{U}(\mathbb{F})$.

Conversely, let $A \in \mathscr{U}(\mathbb{F})$, we now consider the following cases.

Case I. Let $A = \underline{1}$, then obviously $A \in \mathbb{F}$.

Case II. Let $A \neq \underline{1}$.

Since for any x_α and G_1, $G_2 \in \mathbb{F}$, we have,

$U_{G_1}(x_\alpha) \bigcap U_{G_2}(x_\alpha) = U_{G_1 \cap G_2}(x_\alpha)$.

Also, as $\mathbb{F}$ is closed under finite intersection.

Therefore, by proposition 2.1.9(i), we get

$\text{int}(A) = \bigcup\{y_\beta \mid \exists\, G \in \mathbb{F} \text{ s.t. } U_G(y_\beta) \subseteq A\}$.

But as $A \neq \underline{1}$, therefore $U_G(y_\beta) \subseteq A$ implies $U_G(y_\beta) \neq \underline{1}$.

This further implies $G \subseteq A$ and $y_\beta \subseteq G$.

Thus $\text{int}(A) = \bigcup\{y_\beta \mid \exists\, G \in \mathbb{F} \text{ s.t. } y_\beta \subseteq G \text{ and } G \subseteq A\}$.

$= \bigcup\{\bigcup\{y_\beta \mid y_\beta \subseteq G\} \mid G \in \mathbb{F} \text{ and } G \subseteq A\}$.

$= \bigcup\{G \mid G \in \mathbb{F} \text{ and } G \subseteq A\}$.

But $A \in \mathbb{F}(\mathscr{U})$ implies $A = \text{int}(A)$.

Therefore, $A = \bigcup\{G \mid G \in \mathbb{F} \text{ and } G \subseteq A\}$.

Hence, $A \in \mathbb{F}$ as $\mathbb{F}$ is closed under arbitrary unions.

Remark 4.2.1. $\mathscr{U}(\mathbb{F}) \neq \mathbb{F}_{\text{cl}}$, where $\mathbb{F}_{\text{cl}}$ is the L-topology generated by the closure operator 'cl' defined in lemma 3.2.2.

Definition 4.2.4. If $\mathscr{U}$ is an L-local quasi-uniformity. Then for each $n \in \mathbb{N}$ and $V \in \mathscr{U}$, let $V^2 = V \circ V$ and $V^{n+1} = V^n \circ V$.

Theorem 4.2.5. *If $\mathscr{U}$ is an L-local quasi-uniformity. Then for each $n \in \mathbb{N}$,*

$$\mathscr{U}^n = \{U \in \mathscr{U}^* \mid \exists\, V \in \mathscr{U} \quad s.t. \quad V^n \subseteq U\}$$

is an L-local quasi-uniformity with the same L-topology generated by $\mathscr{U}$.

Proof. It suffices to prove the theorem for $\mathscr{U}^2$.

Clearly, $\mathscr{U}^2$ is an L-semi-quasi-uniformity.

Now, for $W \in \mathscr{U}^2$ and $x_\alpha \in \text{Pt}(L^X)$, $\exists\, U,\ V \in \mathscr{U}$ so that $U^4(x_\alpha) \subseteq V(x_\alpha)$ and $V^2 \subseteq W$.

But $U \in \mathscr{U}$ implies $U^2 \in \mathscr{U}^2$ as $U \subseteq U^2$ for any $U \in \mathscr{U}$.

Also, $(U^2 \circ U^2)(x_\alpha) \subseteq V(x_\alpha) \subseteq V^2(x_\alpha) \subseteq W(x_\alpha)$.
Hence, $\mathscr{U}^2$ is an L-local quasi-uniformity.
Now by the definition of $\mathscr{U}^2$, the relative L-topology of $\mathscr{U}^2$ is weaker than that of $\mathscr{U}$. Again as $U \in \mathscr{U}$ implies $U^2 \in \mathscr{U}^2$, so it is also stronger. $\square$

Definition 4.2.5. Two L-local quasi-uniformities $\mathscr{V}$ and $\mathscr{U}$ are *weakly equivalent* if for some $n, m \in \mathbb{N}$, $\mathscr{V}^n \subseteq \mathscr{U}$ and $\mathscr{U}^m \subseteq \mathscr{V}$.

In view of theorem 4.2.5, we observe that two weakly equivalent L-local quasi-uniformities generates the same L-topology.
We proceed to define the following generalized form of L-quasi-uniformly continuous functions.

Definition 4.2.6. Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be L-locally quasi-uniform spaces. A function $f^{\rightarrow} : (L^X, \mathscr{U}) \rightarrow (L^Y, \mathscr{V})$ shall be called *L-weakly quasi-uniformly continuous* iff for some $n \in \mathbb{N}$, $f^{\rightarrow} : (L^X, \mathscr{U}) \rightarrow (L^Y, \mathscr{V}^n)$ is L-quasi-uniformly continuous.

In analogy to theorem 2.1.39 for L-quasi-uniformly continuous functions, we have the following theorem:

Theorem 4.2.6. *L-weakly quasi-uniformly continuous functions on L-locally quasi-uniform spaces are continuous with respect to the relative L-topologies.*

Proof. Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be L-locally quasi-uniform spaces.
Let for some $n \in \mathbb{N}$, $f^{\rightarrow} : (L^X, \mathscr{U}) \rightarrow (L^Y, \mathscr{V}^n)$ be L-quasi-uniformly continuous. For each x_α and each neighborhood N of $f(x_\alpha)$, we may

choose $V \in \mathscr{V}$ and $U \in \mathscr{U}$ so that $V^n(f^{\rightarrow}(x_\alpha)) \subseteq N$ and $\widehat{f^{\rightarrow}}(U) \subseteq V^n$, where $\widehat{f^{\rightarrow}}(x_\alpha, y_\beta) = (f^{\rightarrow}(x_\alpha), f^{\rightarrow}(y_\beta))$.
Then, $f^{\rightarrow}(U(x_\alpha)) = \widehat{f^{\rightarrow}}(U)(f^{\rightarrow}(x_\alpha)) \subseteq V^n(f^{\rightarrow}(x_\alpha)) \subseteq N$. □

4.3 *L*-Pseudo quasi-metrizability

In this section we take up the problem of L-pseudo quasi-metrization. Our consideration of L-pseudo quasi-metric is in the sense of Hutton's pseudo quasi-metric, which is equivalent to Erceg and Peng's pseudo quasi-metric as per our discussion in the preliminaries chapter.

Definition 4.3.1. An L-locally quasi-uniform space $(L^X, \mathscr{U})$ is said to be *L-pseudo quasi-metrizable* if there is an L-pseudo quasi-metric which generates the L-local quasi-uniformity $\mathscr{U}$.

We now state the following lemma, which will be used in the next theorem.

Lemma 4.3.1. *Let $\{V_n \mid n \in \mathbb{N}\}$ be a countable base for an L-local quasi-uniformity $\mathscr{U}$, then there exists a countable base $\{U_n \mid n \in \mathbb{N}\}$ of $\mathscr{U}$ such that $U_{n+1} \subseteq U_n$ for $n \in \mathbb{N}$.*

Proof. Let $\mathscr{B}_1 = \{V_n \mid n \in \mathbb{N}\}$ be a base for some L-local quasi-uniformity.

Let $\mathscr{B}_2 = \{U_n \mid n \in \mathbb{N}\}$, where $U_1 = V_1, U_2 = U_1 \bigcap V_2, U_3 = U_2 \bigcap V_3, ..., U_n = U_{n-1} \bigcap V_n$.

Then, $U_n \subseteq U_{n-1}$, $\forall\, U_n \in \mathscr{B}_2$.

Also, since for each $n \in \mathbb{N}$, U_n is a finite intersection of members of $\mathscr{B}_1$, therefore $U_n \in \mathscr{U}^*$, $\forall\, U_n \in \mathscr{B}_2$.

(SQ1′) Let U_n, $U_m \in \mathscr{B}_2$ such that $m \leq n$, then $U_m \subseteq U_n$ and consequently $U_m \subseteq U_m \bigcap U_n$.

Finally, let U_n be any member of $\mathscr{B}_2$.

Then, there is $V_k \in \mathscr{B}_1$ such that $V_k \subseteq V_1 \bigcap V_2 \bigcap V_3 \bigcap \ldots \bigcap V_n = U_n$.

Again, for any L-fuzzy point x_α and $V_k \in \mathscr{B}_1$ there is $V_l \in \mathscr{B}_1$ such that $V_l \circ V_l(x_\alpha) \subseteq V_k(x_\alpha)$.

This implies $U_l \circ U_l(x_\alpha) \subseteq U_n(x_\alpha)$.

□

Theorem 4.3.2. *An L-topological space $(L^X, \mathbb{F})$ admits an L-quasi-uniformity with a countable base if it admits an L-local quasi-uniformity $\mathscr{U}$ with a countable base.*

Proof. Let $\mathscr{V}$ be an L-local quasi-uniformity on L^X and $< V_n >$ be its base. Without loss of generality we may assume that for any $n \in \mathbb{N}$, $V_{n+1} \subseteq V_n$. Let $U_1 = V_1$ and for each $2 \leq n$, $U_n : L^X \to L^X$ be a mapping such that

$$U_n(\bigcup_{x_\alpha} x_\alpha) = \bigcup_{x_\alpha} V_m(x_\alpha),$$

where $m = \min_{k \in \mathbb{N} \setminus \{1,2,\ ,n-1\}} k$, s.t. $V_k^2(x_\alpha) \subseteq U_{n-1}(x_\alpha)$.

Then clearly,

i) $x_\alpha \subseteq U_n(x_\alpha)$.

ii) $U_n(\bigcup_{x_\alpha} x_\alpha) = \bigcup_{x_\alpha} U_n(x_\alpha)$.

iii) Again for any $x_\alpha \in \mathrm{Pt}(L^X)$, we have $U_n^2(x_\alpha) = V_m^2(x_\alpha)$,

where $m = \min_{k \in \mathbb{N} \setminus \{1,2,\ ,n-1\}} k$, s.t. $V_k^2(x_\alpha) \subseteq U_{n-1}(x_\alpha)$.

But $m = \min_{k \in \mathbb{N} \setminus \{1,2,\ ,n-1\}} k$, s.t. $V_k^2(x_\alpha) \subseteq U_{n-1}(x_\alpha)$.

$\Rightarrow V_m^2(x_\alpha) \subseteq U_{n-1}(x_\alpha)$.

$\Rightarrow U_n^2(x_\alpha) \subseteq U_{n-1}(x_\alpha)$.

So, for any $n \in \mathbb{N}$, we have $U_n^2 \subseteq U_{n-1}$.

Hence, $\{U_n \mid n \in \mathbb{N}\}$ is a subbase for some L-quasi-uniformity $\mathscr{U}$ on L^X.

Also, from the construction of U_n, we get that for any $m \in \mathbb{N}$ and $x_\alpha \in \mathrm{Pt}(L^X)$ there exists $n \in \mathbb{N}$ s.t. $U_n(x_\alpha) \subseteq V_m(x_\alpha)$ and vice versa.

Hence, $\mathscr{U}$ generates the same L- topology on L^X. $\square$

CHAPTER 5

L-LOCALLY UNIFORM SPACES

5.1 Introduction

In continuation of the work in the preceding chapter, we now introduce the notion of L-local uniformities as L-local quasi-uniformities which contain the inverse of each of its member. This special class shall provide more insight into the generalized uniform structures in relation to stronger topological axioms in terms of uniformization results. Suitable examples are provided to show that L-local uniformity lies between Hutton's uniformity and L-local quasi-uniformity. Results on compactness in relation to totally boundedness and complete regularity in the setting of L-locally uniform spaces are also obtained in this chapter.

In the last section of this chapter, we take up the problem of metrizability of L-locally uniform spaces. A satisfactory result in terms of countability axioms is provided in this section.

5.2 L-Locally uniform spaces

In this section, we define L-locally uniform spaces and show that every L-locally uniform spaces generates an L-topology and that the generating L-topology is regular. We also show that every regular L- topology is generated by an L-local uniformity. We finally show that every L-weakly uniformly continuous functions are L-fuzzy continuous.

Definition 5.2.1. An L-semi-uniformity $\mathscr{U}$ on L^X is said to be an *L-local uniformity* if $\mathscr{U}$ has a base $\mathscr{B}$ satisfying the axiom (LQ) as in definition 4.2.1.
The pair $(L^X, \mathscr{U})$ is then called an *L-locally uniform space.*

Let, $\mathscr{U}$ and $\mathscr{V}$ be two L-local uniformities on L^X. Then $\mathscr{V}$ is called *coarser* (*weaker*) than $\mathscr{U}$ or $\mathscr{U}$ is called *finer* (*stronger*) than $\mathscr{V}$ iff $\mathscr{V} \subseteq \mathscr{U}$.

Clearly, every L-uniformity is an L-local uniformity and every L-local uniformity is an L-local quasi-uniformity.
We now provide examples to show that
(i) An L-local uniformity need not be an L-uniformity.
(ii) An L-local quasi-uniformity need not be an L-local uniformity.
(iii) An L-semi-uniformity need not be an L-local uniformity.

Example 5.2.1. (i) Let $L = \{0\} \bigcup([0,1] \bigcap \{\frac{1}{n}\}_{n\in\mathbb{N}})$ and $X = \{x\}$. Then L is a fuzzy lattice with the general order relation.
For each $n \in \mathbb{N}$, let
$B_n = \Delta_{L^X} \bigcup\{(x_{1/2}, x_{1/i}) \mid n \leq i\} \bigcup\{(x_{1/i+1}, x_{1/i}) \mid n \leq i\}$,

where $\Delta_{L^X} = \{(x_\alpha, x_\alpha) \mid x_\alpha \in L^X\}$.

Let $\mathscr{B} = \{C_n \in \{B_n, B_n^{-1}\} \mid n \in \mathbb{N}\}$,

Where $B_n^{-1} = \Delta_{L^X} \bigcup\{(x_{1/i}, x_{1/2}) \mid n \leq i\} \bigcup\{(x_{1/i}, x_{1/i+1}) \mid n \leq i\}$.

For each $C_n \in \mathscr{B}$, let $U_{C_n} : L^X \to L^X$ be a mapping defined by,

$U_{C_n}(x_\alpha) = \bigcup\{x_\beta \mid (x_\alpha, x_\beta) \in C_n\}, \quad \forall\, x_\alpha \in L^X$.

Then,

1) $\mathscr{U}_{\mathscr{B}} \subseteq \mathscr{U}^*$.

2) $U_{C_n}^r = U_{C_n^{-1}}, \quad \forall\, n \in \mathbb{N}$.

3) $U_{C_{n+1}} \subseteq U_{C_n}, \quad \forall\, n \in \mathbb{N}$.

Therefore, $U_{C_n} \bigcap U_{C_m} = U_{C_l} \in \mathscr{U}_{\mathscr{B}}$,

where $l = \min\{m, n\}$ and $\mathscr{U}_{\mathscr{B}} = \{U_{C_n} \mid C_n \in \mathscr{B}\}$.

Hence, $\mathscr{U}_{\mathscr{B}}$ is a base for some L-semi-uniformity.

A) We now show that $\mathscr{U}_{\mathscr{B}}$ is a base for an L-local uniformity.

For each $n \in \mathbb{N}$, we have:

$U_{B_n}(x_{1/2}) = U_{B_n} \circ U_{B_n}(x_{1/2}) = \bigcup\{x_{1/2}, x_{1/n}, x_{1/(n+1)}, \ldots\}$

Now let $k \geq 1$ and $n \geq 1$, we have:

$U_{B_k} \circ U_{B_k}(x_{1/k}) = \{x_{1/k}\}$, hence $U_{B_k} \circ U_{B_k}(x_{1/k}) \subseteq U_{B_n}(x_{1/k})$.

Also, for any $n \in \mathbb{N}$, we have:

$U_{B_n^{-1}}(x_{1/2}) = U_{B_n^{-1}} \circ U_{B_n^{-1}}(x_{1/2}) = \{x_{1/2}\}$.

Now for fix $n, k \in \mathbb{N}$, we have:

$U_{B_{k+1}^{-1}} \circ U_{B_{k+1}^{-1}}(x_{1/k}) = \{x_{1/k}\}$, hence $U_{B_{k+1}^{-1}} \circ U_{B_{k+1}^{-1}}(x_{1/k}) \subseteq U_{B_n^{-1}}(x_{1/k})$.

B) We now show that $\mathscr{U}_{\mathscr{B}}$ is not a base for an L-uniformity.

Fix $U_{B_l} \in \mathscr{U}_{\mathscr{B}}$ and the pair $(x_{1/(n'+2)}, x_{1/n'})$.

For any $n' \in \mathbb{N}$, we have,

$x_{1/(n'+1)} \subseteq U_{B_{n'}}(x_{1/(n'+2)})$ and $x_{1/n'} \subseteq U_{B_{n'}}(x_{1/(n'+1)})$.

This implies that $x_{1/n'} \subseteq U_{B_{n'}} \circ U_{B_{n'}}(x_{1/(n'+2)}), \quad \forall\, n' \in \mathbb{N}$.

However, $x_{1/n'} \nsubseteq U_{B_l}(x_{1/(n'+2)})$.

Thus, for $U_{B_l} \in \mathscr{U}_{\mathscr{B}}$ there is no $U_{B_{n'}} \in \mathscr{U}_{\mathscr{B}}$ such that $U_{B_{n'}} \circ U_{B_{n'}} \subseteq U_{B_l}$.

(ii) In order to show that an L-local quasi-uniformity need not be an L-local uniformity, we refer to example 4.2.1.

We have shown that $\mathscr{U}_{\mathscr{B}}$ as constructed in example 4.2.1 is a base for L-local quasi-uniformity, we now show that $\mathscr{U}_{\mathscr{B}}$ is not a base for L-local uniformity.

Observe that for any $n \in \mathbb{N}$,

$B_n^{-1} = \Delta_{L^X} \bigcup\{(x_{1/2},\, x_{1/i}) \mid n \leq i\} \bigcup\{(x_{1/i},\, x_{1/i+1}) \mid n \leq i\} \bigcup\{(x_{1/i},\, x_1) \mid n \leq i\}$.

So $U_{B_2^{-1}}(x_{1/2}) = \bigcup\{x_{1/2},\, x_{1/2}.\, x_{1/3}, \ldots\}$.

But for any $n \in \mathbb{N}$, we have:

$(x_{1/2},\, x_{1/m}),\, (x_{1/m},\, x_1) \in B_n^{-1}$.

This implies $x_1 \in U_{B_n^{-1}} \circ U_{B_n^{-1}}(x_{1/2}),\ \forall n \in \mathbb{N}$, but $x_1 \notin U_{B_2^{-1}}(x_{1/2})$.

(iii) Let $L = \{0\} \bigcup([0,1] \bigcap\{\frac{1}{n}\}_{n\in\mathbb{N}})$ and $X = \{x\}$.

Then, L is a fuzzy lattice with the general order relation.

For each $n \in \mathbb{N}$, let

$B_n = \Delta_{L^X} \bigcup\{(x_{1/2},\, x_{1/i}) \mid n \leq i\} \bigcup\{(x_{1/i},\, x_{1/i+1}) \mid n \leq i\} \bigcup\{(x_{1/i}.\, x_1) \mid n \leq i\}$,

where $\Delta_{L^X} = \{(x_\alpha,\, x_\alpha) \mid x_\alpha \in L^X\}$.

Let $\mathscr{B} = \{C_n \in \{B_n,\, B_n^r\} \mid n \in \mathbb{N}\}$,

where for each $n \in \mathbb{N}$, $B_n^{-1} = \{(x_\alpha,\, x_\beta) \mid (x_\beta,\, x_\alpha) \in B_n\}$.

For each $C_n \in \mathscr{B}$, let $U_{C_n} : L^X \to L^X$ be a mapping defined by,

$U_{C_n}(x_\alpha) = \bigcup\{x_\beta \mid (x_\alpha, x_\beta) \in C_n\}, \quad \forall\ x_\alpha \in L^X.$

Then,

1) $\mathscr{U}_{\mathscr{B}} \subseteq \mathscr{U}^*$.

2) $U^r_{C_n} = U_{C_n^{-1}}, \quad \forall\ n \in \mathbb{N}$.

3) $U_{C_{n+1}} \subseteq U_{C_n}, \quad \forall\ n \in \mathbb{N}$.

Therefore, $U_{C_n} \bigcap U_{C_m} = U_{C_l} \in \mathscr{U}_{\mathscr{B}}$,

where $l = \min\{m, n\}$ and $\mathscr{U}_{\mathscr{B}} = \{U_{C_n} \mid C_n \in \mathscr{B}\}$.

Hence, $\mathscr{U}_{\mathscr{B}}$ is a base for some L-semi-uniformity.

Now for any $n \in \mathbb{N}$, we have, $(x_{1/2}, x_{1/m}), (x_{1/m}, x_1) \in B_n$.

But $U_{B_2}(x_{1/2}) = \bigcup\{x_{1/2}, x_{1/2}, x_{1/3}, ...\}$.

This implies that $x_1 \in U_{B_n} \circ U_{B_n}(x_{1/2})$, $\forall n \in \mathbb{N}$, but $x_1 \notin U_{B_2}(x_{1/2})$.

Hence, $\mathscr{U}_{\mathscr{B}}$ is not a base for an L-local uniformity.

Definition 5.2.2. For any $V \in \mathscr{U}^*$, let $V^{2r} = V^r \circ V^r$ and for each $n \in \mathbb{N}$, let $V^{(n+1)r} = V^{nr} \circ V^r$.

Theorem 5.2.1. *If $(L^X, \mathscr{U})$ is an L-locally uniform space and $n \in \mathbb{N}$, then for any $U \in \mathscr{U}$, $\exists\ V \in \mathscr{U}$ s.t. $\forall\ x_\alpha \in \mathrm{Pt}(L^X)$, $V^n(x_\alpha) \subseteq U(x_\alpha)$.*

Proof. Follows from definition. □

Since an L-local quasi-uniformity is a generalization of the notion of L-local uniformity, therefore in view of theorem 4.2.3, we have the following theorem:

Theorem 5.2.2. *Every L-locally uniform space is L-topological.*

The L- topology generated by an L- local uniformity $\mathscr{U}$ is the L-topology generated by int, where int : $L^X \to L^X$ is a function defined by, $\text{int}(A) = \bigcup\{x_\alpha \mid \exists\, V \in \mathscr{B} \text{ s.t. } V(x_\alpha) \subseteq A\}$.
Hence, in particular, we note that for any $x_\alpha \in \text{Pt}(L^X)$, $\{U(x_\alpha) \mid U \in \mathscr{U}\}$ is the neighborhood system at x_α in $(L^X, \mathscr{U})$.
In the following theorem, we provide a condition under which an L-topological semi-uniformity will be an L-local uniformity.

Theorem 5.2.3. *Let $\mathscr{V}$ be an L-local uniformity on L^X If $\mathscr{U}$ is an L-topological semi-uniformity on L^X stronger than $\mathscr{V}$ and generating the same L-topology, then $\mathscr{U}$ is an L-local uniformity.*

Proof. Let $U \in \mathscr{U}$ be any and $\mathbb{F}(\mathscr{U})$ be the L-topology generated by $\mathscr{U}$.
Then, for any $x_\alpha \in \text{Pt}(L^X)$, $U(x_\alpha)$ is a neighborhood of x_α.
Since $\mathbb{F}(\mathscr{U}) = \mathbb{F}(\mathscr{V})$, where $\mathbb{F}(\mathscr{V})$ is the L-topology generated by $\mathscr{V}$.
Hence, there exists $V \in \mathscr{V}$ such that $V(x_\alpha) \subseteq U(x_\alpha)$.
But since $\mathscr{V}$ is an L-local uniformity, therefore for $V(x_\alpha)$, there is $W \in \mathscr{V}$ such that $W \circ W(x_\alpha) \subseteq V(x_\alpha)$.
This implies that $W \circ W(x_\alpha) \subseteq U(x_\alpha)$.
Also, as $\mathscr{U}$ is stronger than $\mathscr{V}$, therefore $W \in \mathscr{V}$. Hence, $W \in \mathscr{U}$.
So, for any $U \in \mathscr{U}$ and $x_\alpha \in \text{Pt}(L^X)$, there is $W \in \mathscr{U}$ such that

$W \circ W(x_\alpha) \subseteq U(x_\alpha)$.

Hence, $\mathscr{U}$ is an L-local uniformity. □

Theorem 5.2.4. *Let $\mathscr{U}$ and $\mathscr{V}$ be two L-local uniformities on L^X such that $\mathscr{U}$ is finer than $\mathscr{V}$. Then the L-topology generated by $\mathscr{U}$ is also finer than the L-topology generated by $\mathscr{V}$.*

Proof. Straightforward. □

Theorem 5.2.5. *If $\mathscr{U}$ is an L-local uniformity, then for each $n \in \mathbb{N}$,*

$$\mathscr{U}^n = \{U \in \mathscr{U}^* \mid \exists V \in \mathscr{U} \quad \text{s.t.} \quad V^n \subseteq U\}$$

is an L-local uniformity with the same L-topology generated by $\mathscr{U}$.

Proof. Follows from proposition 2.1.35 (iv), (v) and theorem 4.2.5. □

Theorem 5.2.6. *Let $(L^X, \mathscr{U})$ be an L-locally uniform space. Then for any $A \in L^X$,*

$$\bar{A} = \bigcap\{V(A) \mid V \in \mathscr{U}\}.$$

Proof. Let $(L^X, \mathscr{U})$ be an L-locally uniform space and cl : $L^X \to L^X$ be a mapping defined by,

$$\text{cl}(A) = \bigcap\{V(A) \mid V \in \mathscr{B}\}, \ \forall \ A \in L^X.$$

Then by lemma 3.2.2, $\text{cl}(A) = (\text{int}(A'))'$, $\forall \ A \in L^X$.

Since every L-local uniformity is an L-local quasi-uniformity, therefore by lemma 4.2.2, 'int' is an L-topological interior operator and consequently 'cl' is an L-topological closure operator.

But as $(\text{int}(A'))' = \text{cl}(A)$, $\forall \ A \in L^X$.

Therefore, $\delta_i = \delta_c$, where δ_i and δ_c are the L-topologies generated by 'int' and 'cl' operators respectively.

Now by proposition 2.1.19, we have

$\bar{A} = \bigcap\{V(A) \mid V \in \mathscr{U}\}$. □

We now proceed to define *L-locally uniformizability* and examine conditions leading to the same.

Definition 5.2.3. An L-topology is called *L-locally uniformizable* if there is an L-local uniformity $\mathscr{U}$ which generates the L-topology.

Theorem 5.2.7. *The L-topology of an L-locally uniform space is regular. Conversely, for any regular L-topological space, the set of all neighborhoods of Δ is an L-local uniformity which generates the L-topology.*

Proof. Let $(L^X, \mathscr{U})$ be an L-locally uniform space.
Now for any $x_\alpha \in \mathrm{Pt}(L^X)$ and neighborhood $U(x_\alpha)$, $U \in \mathscr{U}$, let $V \in \mathscr{U}$ is such that $V^2(x_\alpha) \subseteq U(x_\alpha)$.
Then, by theorem 5.2.6, $\overline{V(x_\alpha)} \subseteq V(V(x_\alpha)) \subseteq U(x_\alpha)$.
Hence, the space is regular.
Now let the L-topological space $(L^X, \mathbb{F})$ be regular and $\mathscr{U}$ be the collection of all neighborhoods of Δ.
Then, $\mathscr{U}$ is an L-semi-uniformity generating the L-topology $\mathbb{F}$.
Also, by regularity, for any $U \in \mathscr{U}$ and $x_\alpha \in \mathrm{Pt}(L^X)$, there exists an L-fuzzy set B s.t. $x_\alpha \subseteq \mathrm{int}B \subseteq \overline{B} \subseteq U(x_\alpha)$.
Define $W_B : L^X \to L^X$ as follows:

$$W_B(A) = \underline{1} \text{ if } A \nsubseteq B,$$
$$= B \text{ otherwise.}$$

Then, $W_B \in \mathscr{U}$.
Also, since $x_\alpha \subseteq B$, hence we have

$$W_B(W_B(x_\alpha)) = W_B(B) = B \subseteq \overline{B} \subseteq U(x_\alpha). \qquad \square$$

Corollary 5.2.8. *Let $\{(L^{X_t}, \mathbb{F}_t) \mid t \in \Lambda\}$ be a family of L-topological spaces such that for each $t \in \Lambda$, $(L^{X_t}, \mathbb{F}_t)$ is L-locally uniformizable.*

Then the product topology of L-topologies $\{\mathbb{F}_t \mid t \in \Lambda\}$ *on* L^X *is also L-locally uniformizable, where* $X = \Pi_{t\in\Lambda} X_t$.

Definition 5.2.4. Two L-local uniformities $\mathscr{V}$ and $\mathscr{U}$ are *weakly equivalent* if for some $n, m \in \mathbb{N}$, $\mathscr{V}^n \subseteq \mathscr{U}$ and $\mathscr{U}^m \subseteq \mathscr{V}$.

From the theorem 5.2.5, we observe that two weakly equivalent L-local uniformities generate the same L-topology.

We now give the following definition.

Definition 5.2.5. Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be L-locally uniform spaces, a function $f^{\rightarrow} : (L^X, \mathscr{U}) \rightarrow (L^Y, \mathscr{V})$ is said to be *L-weakly uniformly continuous* iff for some $n \in \mathbb{N}$, $f^{\rightarrow} : (L^X, \mathscr{U}) \rightarrow (L^Y, \mathscr{V}^n)$ is uniformly continuous.

Theorem 5.2.9. *L-weakly uniformly continuous functions on L-locally uniform spaces are continuous with respect to the relative L-topologies.*

Proof. Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be L-locally uniform spaces.
Let for some $n \in \mathbb{N}$, $f^{\rightarrow} : (L^X, \mathscr{U}) \rightarrow (L^Y, \mathscr{V}^n)$ be uniformly continuous.
Then for each x_α and each neighborhood N of $f^{\rightarrow}(x_\alpha)$,
we may choose $V \in \mathscr{V}$ and $U \in \mathscr{U}$ so that

$V^n(f^{\rightarrow}(x_\alpha)) \subseteq N$ and $\widehat{f^{\rightarrow}}(U) \subseteq V^n$,

where $\widehat{f^{\rightarrow}}(x_\alpha, y_\beta) = (f^{\rightarrow}(x_\alpha), f^{\rightarrow}(y_\beta))$.

Then, $f^{\rightarrow}(U(x_\alpha)) = \widehat{f^{\rightarrow}}(U)(f^{\rightarrow}(x_\alpha)) \subseteq V^n(f^{\rightarrow}(x_\alpha)) \subseteq N$. □

Definition 5.2.6. Let $\{(L^{X_t}, \mathscr{U}_t) \mid t \in \Lambda\}$ be a family of L-locally uniform spaces, where Λ is the index set.

Let $X = \Pi_{t\in\Lambda}X_t$.

The *product L-local uniformity* on L^X is defined as the coarsest L-local uniformity such that for every $t \in \Lambda$. projection $\pi_t^{\rightarrow} : L^X \rightarrow L^{X_t}$ is L-weakly uniformly continuous.

The following theorem is now obvious.

Theorem 5.2.10. *The L-topology generated by the product L-local uniformity is the product topology.*

5.3 Compactness and Totally boundedness

Having established that every L-locally uniformity generates an L-topology, we proceed to investigate the possible role of the notions of compactness and totally boundedness in L-local uniform spaces. We proceed to show among other results that every compact L-locally uniform space is totally bounded.

Definition 5.3.1. An L-locally uniform space $(L^X, \mathscr{U})$ is said to be *totally bounded* if for any $U \in \mathscr{U}$ there is finite $A \subseteq Pt(L^X)$ such that $\underline{1} = U(A) = \bigcup\{U(x_\alpha) \mid x_\alpha \in A\}$.

This is a generalization of the same notion of Artico and Moresco[1].

Theorem 5.3.1. *Every compact L-locally uniform space $(L^X, \mathscr{U})$ is totally bounded.*

Proof. Let $(L^X, \mathscr{U})$ be a compact space. Then for any $U \in \mathscr{U}$, the collection

$\{\text{int } U(x_\alpha) \mid x_\alpha \in \text{Pt}(L^X)\}$ is an open cover of $\underline{1}$.

Since $\underline{1}$ is closed.

Therefore, by compactness there exists a collection of finite L-fuzzy points μ_i, $1 \leq i \leq n$, $n \in \mathbb{N}$ s.t. $\underline{1} = \bigcup_{i=1}^{n} \text{int}U(\mu_i)$.

Therefore, $\underline{1} = \bigcup_{i=1}^{n} U(\mu_i)$.

Hence, $(L^X, \mathscr{U})$ is totally bounded. □

Now we establish the following lemmas:

Lemma 5.3.2. *Let $(L^X, \mathscr{U})$ be an L-locally uniform space and $\mathbb{F}$ be the L-topology generated by $\mathscr{U}$. Let $U \in \mathscr{U}$, such that U is open. Then*

(i) *U^r is open.*

(ii) *$U(x_\alpha)$ is open for any x_α.*

(iii) *$U^r(x_\alpha)$ is open for any x_α.*

Proof. i) Let $(x_\alpha, y_\beta) \in U^r$.

Then $(x_\alpha, y_\beta) \in U^r \Rightarrow (y_\beta, x_\alpha) \in U$. [By lemma 3.4.7.]

Since U is open and $(y_\beta, x_\alpha) \in U$.

Therefore, there exist $V_1, V_2 \in \mathscr{U}$ s.t. $V_1(y_\beta) \times V_2(x_\alpha) \subseteq U$.

Let $V = V_1 \bigcap V_2$. Then $V \in \mathscr{U}$ and $V(y_\beta) \times V(x_\alpha) \subseteq U$.

Now, $(z_\gamma, w_\eta) \in V(x_\alpha) \times V(y_\beta)$

$\Rightarrow w_\eta \in V(y_\beta)$, $z_\gamma \in V(x_\alpha)$.

$\Rightarrow (w_\eta, z_\gamma) \in U$.

$\Rightarrow (z_\gamma, w_\eta) \in U^r$.

So, $V(x_\alpha) \times V(y_\beta) \subseteq U^r$ whenever $(x_\alpha, y_\beta) \in U^r$.

Hence, U^r is open.

ii) Let U be open.

To show that $U(x_\alpha)$ is open, it suffices to show that

for any $y_\beta \in U(x_\alpha)$, $\exists V \in \mathscr{U}$ s.t. $V(y_\beta) \subseteq U(x_\alpha)$.

Now $y_\beta \in U(x_\alpha) \Rightarrow (x_\alpha, y_\beta) \in U$.

$\Rightarrow \exists\ V_1, V_2 \in \mathscr{U}$ s.t. $V_1(x_\alpha) \times V_2(y_\beta) \subseteq U$. [As U is open.]

Let $V = V_1 \bigcap V_2$.

Then $V \in \mathscr{U}$ and $V(x_\alpha) \times V(y_\beta) \subseteq U$.

Now for $z_\gamma \in V(y_\beta)$, we get,

$(x_\alpha, z_\gamma) \in V(x_\alpha) \times V(y_\beta) \Rightarrow (x_\alpha, z_\gamma) \in U \Rightarrow z_\gamma \in U(x_\alpha)$.

So, $V(y_\beta) \subseteq U(x_\alpha)$.

Hence, $U(x_\alpha)$ is open.

iii) Follows from i) and ii) as $U^r \in \mathscr{U}$ whenever $U \in \mathscr{U}$. □

Lemma 5.3.3. *For any $U \in \mathscr{U}^*$, $U \circ U = \bigcup_{z_\gamma} U^r(z_\gamma) \times U(z_\gamma)$.*

Proof. For any $(x_\alpha, y_\beta) \in U \circ U$ we have,

$(x_\alpha, y_\beta) \in U \circ U \Leftrightarrow (x_\alpha, z_\gamma) \in U$, $(z_\gamma, y_\beta) \in U$ for some $z_\gamma \in L^X$.

$\Leftrightarrow (z_\gamma, x_\alpha) \in U^r$, $(z_\gamma, y_\beta) \in U$ for some $z_\gamma \in L^X$, by lemma 3.4.7.

$\Leftrightarrow (x_\alpha, y_\beta) \in U^r(z_\gamma) \times U(z_\gamma)$ for some $z_\gamma \in L^X$.

$\Leftrightarrow (x_\alpha, y_\beta) \in \bigcup_{z_\gamma} U^r(z_\gamma) \times U(z_\gamma)$. □

Theorem 5.3.4. *Let $(L^X, \mathscr{U})$ be an L-locally uniform space. Then for any $U \in \mathscr{U}$, $U \circ U$ is open whenever U is open.*

Proof. Follows from the lemmas 5.3.3, 5.3.2, as arbitrary union of open sets is open. □

Lemma 5.3.5. *Let $(L^X, \mathscr{U})$ be a compact L-locally uniform space. Then for any $U \in \mathscr{U}$, there exists $n \in \mathbb{N}$ s.t. $U \circ U = \bigcup_{i=1}^{n} U^r(A_i) \times U(A_i)$, where $A_i = U(\mu_i)$, $1 \leq i \leq n$ and μ_i's are L- fuzzy points.*

Proof. Let $(L^X, \mathscr{U})$ be a compact space.
Then by theorem 5.3.1, for any $U \in \mathscr{U}$, we have $\underline{1} = U(\bigcup_{i=1}^{n} \mu_i) = \bigcup_{i=1}^{n} U(\mu_i)$, for some $n \in \mathbb{N}$, where μ_i's are L- fuzzy points.
Let $A_i = U(\mu_i)$ $1 \leq i \leq n$, then since by axiom (s2), we have $U(A_i) = U(\bigcup_{x_\alpha \in A_i} x_\alpha) = \bigcup_{x_\alpha \in A_i} U(x_\alpha)$.
So from lemma 5.3.3, we have

$$U \circ U = \bigcup_{x_\alpha \in \mathrm{Pt}(L^X)} U^r(x_\alpha) \times U(x_\alpha) = \bigcup_{i=1}^{n} U^r(A_i) \times U(A_i). \quad \square$$

Theorem 5.3.6. *Let $(L^X, \mathscr{U})$ be a compact L-locally uniform space. Then for any $U \in \mathscr{U}$, $U \circ U$ is closed whenever U is closed.*

Proof. For some $n \in \mathbb{N}$, we have from lemma 5.3.5,

$U \circ U = \bigcup_{i=1}^{n} U^r(A_i) \times U(A_i)$, where $A_i = U(\mu_i)$, $1 \leq i \leq n$ and μ_i are L- fuzzy points.

$= \bigcup_{i=1}^{n} [\pi^{-1}(U^r(A_i)) \bigcap \pi^{-1}(U(A_i))]$.

$\Rightarrow (U \circ U)' = \bigcap_{i=1}^{n} [[\pi^{-1}(U^r(A_i))]' \bigcup [\pi^{-1}(U(A_i))]']$.

$= \bigcap_{i=1}^{n} [\pi^{-1}(U^{r'}(A_i)) \bigcup \pi^{-1}(U'(A_i))]$.

Now U' is open as U is closed.

Hence, by lemma 5.3.2, $(U \circ U)'$ is open as π is continuous and consequently $U \circ U$ is closed. $\square$

Theorem 5.3.7. *Let $(L^X, \mathbb{F})$ be a compact regular L-topological space, then there is a unique L-local uniformity which generates the L-topology of L^X.*

Proof. Since the L-topological space is regular, therefore let $\mathscr{U}$ be an L-local uniformity generating the L-topology $\mathbb{F}$.

Also by theorem 2.1.32, $\mathbb{F}$ is completely regular.

So by theorem 2.1.41, it suffices to show that $\mathscr{U}$ is an L-uniformity.

Let $U \in \mathscr{U}$ and $x_\alpha \in \mathrm{Pt}(L^X)$.

Then there exists $W \in \mathscr{U}$ such that $\mathrm{int}W(\mathrm{int}W(x_\alpha)) \subseteq \mathrm{int}(U(x_\alpha))$.

Now by regularity, we may choose V_1, $V_2 \in \mathscr{U}$ such that

$\mathrm{cl}V_1(\mathrm{int}W(x_\alpha)) \subseteq \mathrm{int}W(\mathrm{int}W(x_\alpha))$ and $\mathrm{cl}V_2(x_\alpha) \subseteq \mathrm{int}W(x_\alpha)$.

Let $V_3 = V_1 \bigcap V_2$ and $V = \mathrm{cl}V_3 = \mathrm{cl}V_1 \bigcap \mathrm{cl}V_2$.

Then $V_3 \in \mathscr{U}$ and $\mathrm{cl}V = V$ such that $V(V(x_\alpha)) \subseteq \mathrm{int}(U(x_\alpha))$.

Hence, $V^2(x_\alpha) \subseteq \mathrm{int}(U(x_\alpha)) \subseteq U(x_\alpha)$.

Thus for any $U \in \mathscr{U}$ and for each $x_\alpha \in \mathrm{Pt}(L^X)$, we may let

$V_{x_\alpha}(= V_3) \in \mathscr{U}$ such that $\mathrm{cl}V_{x_\alpha}^2(x_\alpha) \subseteq \mathrm{int}(U(x_\alpha))$.

Then, $\forall y_\beta$, $(\bigcap\{\mathrm{cl}V_{x_\alpha}^2 \mid x_\alpha \in L^X\})(y_\beta) \subseteq \mathrm{cl}V_{y_\beta}^2(y_\beta) \subseteq \mathrm{int}(U(y_\beta))$.

This implies that $\bigcap\{\mathrm{cl}V_{x_\alpha}^2 \mid x_\alpha \in L^X\} \subseteq \mathrm{int}(U)$.

By theorem 5.3.6, each $\mathrm{cl}V_{x_\alpha}^2$ is closed.

Also by theorem 2.1.33, the corresponding product L-topology is compact.

Hence, $\exists\ n \in \mathbb{N}$ such that $\bigcap_{i=1}^n \mathrm{cl}V_{x_{\alpha_i}}^2 \subseteq U$.

This implies that $\bigcap_{i=1}^n V_{x_{\alpha_i}}^2 \subseteq U$.

Since $(\bigcap_{i=1}^n V_{x_{\alpha_i}})^2 \subseteq \bigcap_{i=1}^n V_{x_{\alpha_i}}^2$, we have $(\bigcap_{i=1}^n V_{x_{\alpha_i}})^2 \subseteq U$.

Also, $\bigcap_{i=1}^n V_{x_{\alpha_i}} \in \mathscr{U}$.

Hence, $\mathscr{U}$ is an L-uniformity. □

Corollary 5.3.8. *Let* $(L^X, \mathbb{F}_1)$ *be a compact regular* L*-topological space* *and $(L^Y, \mathbb{F}_2)$ be a regular L-topological space. Then any continuous mapping $f^{\rightarrow} : L^X \rightarrow L^Y$ is an L-weakly uniformly continuous.*

5.4 *L*-Pseudo metrizability in *L*-locally uniform spaces

In this section we show that an L-locally uniform space having countable base is L-pseudo metrizable, which extends a classical theorem of general topology given by James [75].

Definition 5.4.1. An L-locally uniform space $(L^X, \mathscr{U})$ is said to be L-*pseudo metrizable* if there is an L-pseudo metric which generates the L-local uniformity $\mathscr{U}$.

We require the following lemma:

Lemma 5.4.1. *Let $\{V_n \mid n \in \mathbb{N}\}$ be a countable base for an L-local uniformity $\mathscr{U}$, then there exists a countable base $\{U_n \mid n \in \mathbb{N}\}$ of $\mathscr{U}$ such that $U_{n+1} \subseteq U_n$ for $n \in \mathbb{N}$.*

Proof. Let $\mathscr{B}_1 = \{V_n \mid n \in \mathbb{N}\}$ be a base for some L-local uniformity.
Let $\mathscr{B}_2 = \{U_n \mid n \in \mathbb{N}\}$, where
$U_1 = V_1, U_2 = U_1 \bigcap V_2, U_3 = U_2 \bigcap V_3, ..., U_n = U_{n-1} \bigcap V_n$.
Then as in proof of lemma 4.3.1, we have
$\mathscr{B}_2$ is a base for some L-local quasi-uniformity satisfying the following:

(i) $U_n \subseteq U_{n-1}, \ \forall U_n \in \mathscr{B}_2$.

(ii) For any $U_n \in \mathscr{B}_2$ there exists $V_k \in \mathscr{B}_1$ such that $V_k \subseteq U_n$.

Finally,

(SQ3) For any $U_n \in \mathscr{B}_2$, we have $U_n = V_1 \bigcap V_2 \bigcap V_3 \bigcap ... \bigcap V_n$.

So, by proposition 2.1.35(vi), we get $U_n^r = V_n^r \bigcap \ldots \bigcap V_3^r \bigcap V_2^r \bigcap V_1^r$.
This implies that $U_n^r \in \mathscr{B}_2, \ \forall U_n \in \mathscr{B}_2$.

□

Theorem 5.4.2. *An L-topological space $(L^X, \mathbb{F})$ admits a uniformity with a countable base if it admits an L-local uniformity $\mathscr{U}$ with a countable base.*

Proof. Let $(L^X, \mathbb{F})$ be an L-topological space and $< V_n >$ be a base for an L-local uniformity such that $V_{n+1} \subseteq V_n, \ \ \forall \ n \in \mathbb{N}$.

Inductively we define a sequence U_n of neighborhoods of Δ as follows:

$U_1 = V_1^2 = \bigcup\{V_1^r(x_\alpha) \times V_1(x_\alpha) \mid x_\alpha \in L^X\}$,

For each $n \in \mathbb{N}$, let, $U_{n+1} : L^X \to L^X$ be a mapping s.t.

$U_{n+1}(x_\alpha) = \bigcup\{y_\beta \mid (x_\alpha, y_\beta) \in B_{n+1}\}$,

where $B_{n+1} = \{V_k^r(x_\alpha) \times V_k(x_\alpha) \mid x_\alpha \in \text{Pt}(L^X), \ k \in \mathbb{N} \text{ with } n+1 \leq k$ and $V_k^{3r}(x_\alpha) \times V_k^3(x_\alpha) \subseteq U_n\}$.

Then, $< U_n >$ is a base for some L-semi-uniformity.

Let $n \in \mathbb{N}$ and let $(a_\lambda, c_\gamma) \in U_{n+1}^2$.

Then, there exists an L-fuzzy point b_β such that $(a_\lambda, b_\beta) \in U_{n+1}$ and $(b_\beta, c_\gamma) \in U_{n+1}$.

Hence, there are L-fuzzy points $x_\mu, \ y_\eta$ and $k, \ m \in \mathbb{N}$ with $n+1 \leq k$ and $n+1 \leq m$ such that $(a_\lambda, b_\beta) \in V_k^r(x_\mu) \times V_k(x_\mu), \ \ (b_\beta, c_\gamma) \in V_m^r(y_\eta) \times V_m(y_\eta), \ \ V_k^{3r}(x_\mu) \times V_k^3(x_\mu) \subseteq U_n$ and $V_m^{3r}(y_\eta) \times V_m^3(y_\eta) \subseteq U_n$.

Let $l = min\{k, m\}$.

Then, $V_k, V_m \subseteq V_l$ and $V_k^r, V_m^r \subseteq V_l^r$ as $V_{n+1} \subseteq V_m, \ \ \forall \ n \in \mathbb{N}$.

Now $(a_\lambda, b_\beta) \in V_k^r(x_\mu) \times V_k(x_\mu) \Rightarrow a_\lambda \in V_l^r(x_\mu)$ and $b_\beta \in V_l(x_\mu)$

$\Rightarrow a_\lambda \in V_l^r(x_\mu)$ and $x_\mu \in V_l^r(b_\beta)$ [by lemma 3.4.7]

$\Rightarrow a_\lambda \in V_l^r(x_\mu$ ... ... (i)

and $x_\mu \in V_l^r(b_\beta)$... ... ... (ii).

Again, $(b_\beta, c_\gamma) \in V_m^r(y_\eta) \times V_m(y_\eta) \Rightarrow b_\beta \in V_l^r(y_\eta)$ ... ... (iii)

and $c_\gamma \in V_l(y_\eta)$ ... ... (iv).

Now from $(iii), (ii)$ and (i) we get,

$a_\lambda \in V_l^r(x_\mu) \subseteq V_l^r(V_l^r(b_\beta)) \subseteq V_l^r(V_l^r(V_l^r(y_\eta))) = V_l^r \circ V_l^r \circ V_l^r(y_\eta)$.

This implies that $a_\lambda \in V_l^{3r}(y_\eta)$.

Also by (iv) $c_\gamma \in V_l^3(y_\eta)$.

Hence, $(a_\lambda, c_\gamma) \in U_n$.

Since $U_n \subseteq V_n^2$ for each $n \in \mathbb{N}$, we now have that

$\{U_n \mid n \in \mathbb{N}\}$ is a base for a uniformity compatible with $\mathbb{F}$. □

CHAPTER 6

STRONG COMPLETENESS IN L-LOCALLY UNIFORM SPACES

6.1 Introduction

In the previous chapter we developed the notion of L-local uniformity as a generalization of Hutton's uniformity in the category L-**TOP**. After having established that every L-local uniformity generates an L-topology we have proved several results providing answers to the questions:
(i) Which L-topologies are generated by an L-local uniformity and which of them are generated by an unique L-local uniformity?
(ii) Which L-local uniformities are generated by an L-Pseudo Metric?
In this chapter we aim to develop an analogue for the notion of completeness in the context of L-locally uniform spaces. The problem of completeness has always occupied topologists. For this purpose we have introduced the notion of weak Cauchy filter as a generalization of Cauchy filter and developed the notion of strong completeness for an L-locally uniform

space. Its hereditary property, unimorphic invariance and interrelationship with compactness are discussed in this chapter. The productivity of this notion is also taken up towards the end of this chapter.

6.2 Basic assumptions

In this chapter, we consider only those L-local uniformities $\mathscr{U}$ on L^X which satisfies the condition $x_\alpha \in U(x_\alpha)$ instead of our earlier assumption that $x_\alpha \subseteq U(x_\alpha), \ \forall\ U \in \mathscr{U}, x_\alpha \in L^X$. In the process we have obtained a subclass of the class of the L-local uniformities developed in the earlier chapter. This change has been necessitated in order to accommodate the system of Q-nbd at a point x_α. The Q-nbd system, as may be noted, plays an important in the theory of convergence.

All the necessary results of L-locally uniform spaces are adopted here with the above difference, which can be established without any difficulty. Further, we shall also continue to call this subclass of L-locally uniform spaces as L-locally uniform spaces for sake of convenience.

6.3 Completeness and Strong completeness

Definition 6.3.1. Let $(L^X, \mathscr{U})$ be an L-uniform space, then a *Cauchy filter* $\mathscr{F}$ is a filter s.t. for any $U \in \mathscr{U}$, there exists $F \in \mathscr{F}$ such that $F \times F \subseteq U$.

Definition 6.3.2. Let $(L^X, \mathscr{U})$ be an L-locally uniform space and let $n \in \mathbb{N}$. A *weak Cauchy filter* is a filter $\mathscr{F}$ such that for any $U \in \mathscr{U}$, there exists $F \in \mathscr{F}$ such that $F \times F \subseteq U^n$ for some $n \in \mathbb{N}$.

Clearly, Cauchy filters are weak Cauchy filters of degree 1.

Cauchy filters in L-semi-uniform spaces shall coincide with weak Cauchy filters in an L-uniform space in view of the following theorem.

Theorem 6.3.1. *Let $(L^X, \mathscr{U})$ be an L-uniform space. Then the following are equivalent:*

(i) *$\mathscr{F}$ is a Cauchy filter.*

(ii) *For any $U \in \mathscr{U}$, there exists $F \in \mathscr{F}$ such that $F \times F \subseteq U^n$ for some $n \in \mathbb{N}$.*

(iii) *For each $U \in \mathscr{U}$, $\exists x_\alpha \in \mathrm{Pt}(L^X)$ and $F \in \mathscr{F}$ such that $F \subseteq U(x_\alpha)$.*

Proof. Straightforward. □

Theorem 6.3.2. *In a totally bounded L-locally uniform space $(L^X, \mathscr{U})$, every ultrafilter relative to an $\mathbb{F}(\mathscr{U})$-open set G is a weak Cauchy filter relative to G.*

Proof. Let $\mathscr{F}_H$ be an ultrafilter relative to an $\mathbb{F}(\mathscr{U})$-open set G.

Let $\mathscr{B} = \{U \in \mathscr{U} \mid U^r = U\}$.

Then $\mathscr{B}$ is a base for $\mathscr{U}$.

Let $U \in \mathscr{B}$.

By totally boundedness there is a finite $A \subseteq Pt(L^X)$ such that

$\underline{1} = U(A) = \bigcup\{U(x_\alpha) \mid x_\alpha \in A\}$.

But $\underline{1} \in \mathscr{F}_H$.

So, by lemma 3.4.6, there is $F \in \mathscr{F}_H$ such that

$F \subseteq U(x_\alpha)$ for some $x_\alpha \in A$.

This implies $F \times F \subseteq U(x_\alpha) \times U(x_\alpha) = U^r(x_\alpha) \times U(x_\alpha)$, since $U = U^r$.

Now by lemma 5.3.3, we get $F \times F \subseteq U^2$. □

Corollary 6.3.3. *In a totally bounded L-locally uniform space* $(L^X, \mathscr{U})$, *every ultrafilter is a weak Cauchy filter.*

Definition 6.3.3. An L-(locally) uniform space $(L^X, \mathscr{U})$ is said to be *(strongly) complete* if every (weak) Cauchy filter relative to an $\mathbb{F}(\mathscr{U})$-open set is convergent.

Theorem 6.3.4. *Every convergent filter in an L-locally uniform space is a weak Cauchy filter.*

Proof. Let $(L^X, \mathscr{U})$ be an L-locally uniform space and $\mathscr{F}$ be a filter such that for some $x_\alpha \in \text{Pt}(L^X)$, $\mathscr{F} \to x_\alpha$ in $(L^X, \mathbb{F}(\mathscr{U}))$.

Let $\mathscr{Q}(x_\alpha) = \{U(x_\alpha^*) \mid U \in \mathscr{U}\}$ and $\mathscr{Q}_{\mathscr{B}}(x_\alpha) = \{V(x_\alpha^*) \mid V \in \mathscr{B}\}$,

where $\mathscr{B} = \{U \in \mathscr{U} \mid U^r = U\}$.

Then, $\mathscr{Q}(x_\alpha)$ is Q-nbd system at x_α in $\mathbb{F}(\mathscr{U})$.

Since, for any $U \in \mathscr{U}$, there exists $V \in \mathscr{B}$ s.t. $V \subseteq U$.

This implies that $V(x_\alpha^*) \subseteq U(x_\alpha^*)$ and hence, $\mathscr{Q}_{\mathscr{B}}(x_\alpha)$ is a base for $\mathscr{Q}(x_\alpha)$.

Now, since, $\mathscr{F} \to x_\alpha$, therefore for $V(x_\alpha^*) \in \mathscr{Q}_{\mathscr{B}}(x_\alpha)$, there exists $F \in \mathscr{F}$ s.t. $F \subseteq V(x_\alpha^*)$.

Now, if $(y_\beta, z_\gamma) \subseteq F \times F$, then, $y_\beta \subseteq F$, $z_\gamma \subseteq F$.

Hence, $y_\beta \subseteq V(x_\alpha^*)$, $z_\gamma \subseteq V(x_\alpha^*)$.

But by lemma 3.4.7, $y_\beta \subseteq V(x_\alpha^*)$ implies that $x_\alpha^* \subseteq V^r(y_\beta) = V(y_\beta)$ as $V = V^r$.

Now, $x^*_\alpha \subseteq V(y_\beta)$ and $z_\gamma \subseteq V(x^*_\alpha)$ implies that $z_\gamma \subseteq V^2(y_\beta)$.

So, $(y_\beta, z_\gamma) \subseteq V^2 \subseteq U^2$.

Hence, for any $U \in \mathscr{U}$, there exists $F \in \mathscr{F}$ such that $F \times F \subseteq U^2$. □

Theorem 6.3.5. *Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be L-locally uniform spaces and let $f^\rightarrow : L^X \to L^Y$ be L-weakly uniformly continuous. If $\mathscr{F}$ is a weak Cauchy filter in $(L^X, \mathscr{U})$, then $f^\rightarrow(\mathscr{F})$ is weak Cauchy filter in $(L^Y, \mathscr{V})$.*

Proof. Let $\mathscr{F}$ be a weak Cauchy filter on L^X.

Let $V \in \mathscr{V}$. Since $f^\rightarrow : L^X \to L^Y$ is L-weakly uniformly continuous, therefore there exists $U \in \mathscr{U}$ such that $\widehat{f^\rightarrow}(U) \subseteq V^m$ for some $m \in \mathbb{N}$.

Since, $\mathscr{F}$ is a weak Cauchy filter on L^X.

Therefore, there exists $F \in \mathscr{F}$ such that $F \times F \subseteq U^n$, for some $n \in \mathbb{N}$.

This implies that $f^\rightarrow(F) \times f^\rightarrow(F) \subseteq V^{mn}$.

Hence, $f^\rightarrow(\mathscr{F})$ is a weak Cauchy filter on $(L^Y, \mathscr{V})$. □

Theorem 6.3.6. *Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be two L-locally uniform spaces and let $f^\rightarrow : L^X \to L^Y$ be an L-weakly uniformly isomorphism. Then $(L^X, \mathscr{U})$ is strongly complete iff $(L^Y, \mathscr{V})$ is so.*

Proof. Let $(L^Y, \mathscr{V})$ be L-strongly complete and $\mathscr{F}$ be a weak Cauchy filter on L^X relative to an open set G.

Let $V \in \mathscr{V}$, then by theorem 6.3.5, $f^\rightarrow(\mathscr{F})$ is a weak Cauchy filter on $(L^Y, \mathscr{V})$.

Again, since $f^\leftarrow$ is L-weakly uniform continuous, therefore by theorem 5.2.9, $f^\leftarrow$ is continuous and so $f^\rightarrow$ is open.

This implies that $f^\rightarrow(G)$ is open in L^Y.

Now as $G \notin \mathscr{F}$, hence $f^{\rightarrow}(\mathscr{F})$ is a weak Cauchy filter relative to the open set $f^{\rightarrow}(G)$.

Thus, $f^{\rightarrow}(\mathscr{F})$ is convergent on $(L^Y . \mathscr{V})$, being strongly complete.

But $f^{\rightarrow}$ is an L-fuzzy homeomorphism being an L-weakly uniform isomorphism and consequently, $\mathscr{F}$ converges on $(L^X, \mathscr{U})$.

Hence, $(L^X, \mathscr{U})$ is strongly complete. □

Definition 6.3.4. Let $(L^X, \mathscr{U})$ be an L-locally uniform space and let $A \in L^X$.

Let for any $U \in \mathscr{U}$, $U_A : L^X \to L^X$ be a mapping such that

$$U_A(x_\alpha) = \begin{cases} U(x_\alpha) & \text{if } x_\alpha \subseteq A \\ \underline{0} & \text{if } x_\alpha \nsubseteq A \end{cases}$$

Then, $\mathscr{U}_A = \{U_A \mid U \in \mathscr{U}\}$ is an L-local uniformity on A, what we shall call as *a sub L-local uniformity* on A and $(A, \mathscr{U}_A)$ to be the *subspace.*

$\mathscr{U}_A$ is called *open* or *closed* sub L-local uniformity provided $A \in \mathbb{F}(\mathscr{U})$ or $A' \in \mathbb{F}(\mathscr{U})$ respectively.

Theorem 6.3.7. *Every closed subspace in a strongly complete L-locally uniform space is strongly complete.*

Proof. Let $(L^X, \mathscr{U})$ be a strongly complete L-locally uniform space and $A \in L^X$ such that $A' \in \mathbb{F}(\mathscr{U})$.

Let $\mathscr{F} = \{F \mid F \subseteq A\}$ be a weak Cauchy filter relative to an open set B in $\mathbb{F}(\mathscr{U}_A)$, where $\mathbb{F}(\mathscr{U}_A)$ is the L-topology on A generated by $\mathscr{U}_A$.

Then, $B \notin \mathscr{F}$. If $B' \in \mathscr{F}$, then $B' \subseteq A$, as $\mathscr{F}$ is a filter on $(L^A, \mathbb{F}(\mathscr{U}_A))$.

We then have, $A' \subseteq B$ and consequently, $A' \notin \mathscr{F}$.

Also if, $B' \notin \mathscr{F}$, then as $B \subseteq A$ implies $A' \subseteq B'$, we have $A' \notin \mathbb{F}$.

Now, since for any $U \in \mathscr{U}$ we have $U_A{}^n \subseteq U^n$, so in either case $\mathscr{F}$ is a weak Cauchy filter in L^X relative to the open set A'.

Consequently, there exists an L-fuzzy point $x_\alpha \in \mathrm{Pt}(L^X)$ such that $\mathscr{F} \to x_\alpha$.

But as $A' \in \mathbb{F}(\mathscr{U})$, so by theorem 2.1.31, $x_\alpha \in A$.

Hence, $(A, \mathscr{U}_A)$ is strongly complete. □

Theorem 6.3.8. *In a compact L-locally uniform space $(L^X, \mathbb{F}(\mathscr{U}))$ every filter relative to an $\mathbb{F}(\mathscr{U})$-open set has a cluster point.*

Proof. Let $(L^X, \mathbb{F}(\mathscr{U}))$ be a compact L-locally uniform space and $\mathscr{F}$ be a filter relative to the $\mathbb{F}(\mathscr{U})$-open set G.

Then, $\mathscr{G} = \{\overline{F} \mid F \in \mathscr{F}\}$ is a closed-filter relative to the $\mathbb{F}(\mathscr{U})$-open set G. Now, by compactness we have, $\bigcap \mathscr{G} \subseteq G$.

This implies that there exists $x_\alpha \in \mathrm{Pt}(L^X)$ s.t. $x_\alpha \in \bigcap \mathscr{G}$.

Hence, x_α is a cluster point of $\mathscr{F}$ as $\bigcap \mathscr{G} = \bigcap \{\overline{F} \mid F \in \mathscr{F}\}$. □

Theorem 6.3.9. *Let $(L^X, \mathscr{U})$ be an L-locally uniform space in which every ultrafilter relative to an $\mathbb{F}(\mathscr{U})$-open set is convergent. Then $(L^X, \mathscr{U})$ is compact.*

Proof. Follows from theorem 3.4.8. □

Theorem 6.3.10. *Every compact L-locally uniform space is a strongly complete.*

Proof. Let $(L^X, \mathbb{F}(\mathscr{U}))$ be a compact L-locally uniform space.

Let $\mathscr{F}$ be a weak Cauchy filter relative to the $\mathbb{F}(\mathscr{U})$-open set G.

Then, for any $U \in \mathscr{U}$, there exists $F \in \mathscr{F}$ such that $F \times F \subseteq V^{n-1}$.

Now by theorem 6.3.8, there exists $x_\alpha \in \mathrm{Pt}(L^X)$ s.t. $x_\alpha \in \bigcap\{\overline{F} \mid F \in \mathbb{F}\}$.

For any $U \in \mathscr{U}$, choose $V \in \mathscr{U}$ s.t. $V^n(x^*_\alpha) \subseteq U(x^*_\alpha)$ and $F \times F \subseteq V^{n-1}$.

Also, since for any A, $B \in L^X$, $A\widehat{q}B$ implies that $A \cap B \neq \underline{0}$.

Therefore, for $F \in \mathscr{F}$, there exists $y_\beta \in \mathrm{Pt}(L^X)$ s.t. $y_\beta \in F \cap V(x^*_\alpha)$.

This implies that $y_\beta \in F$ and $y_\beta \in V(x^*_\alpha)$.

Since $y_\beta \in F$ and $F \times F \subseteq V^{n-1}$, we have $F \subseteq V^{n-1}(y_\beta)$.

This further implies that $F \subseteq U(x^*_\alpha)$ (as $V^{n-1}(y_\beta) \subseteq V^n(x^*_\alpha)$).

Hence, $U(x^*_\alpha) \in \mathscr{F}$, $\forall\ U \in \mathscr{U}$ and consequently, $\mathscr{F} \to x_\alpha$. □

Theorem 6.3.11. *Let $(L^X, \mathscr{U})$ be an L-locally uniform space, then the space is compact iff*

(i) *$(L^X, \mathscr{U})$ is totally bounded, and*

(ii) *$(L^X, \mathscr{U})$ is strongly complete.*

Proof. Let $(L^X, \mathscr{U})$ be an L-locally uniform space which is compact then by theorem 5.3.1,

(i) $(L^X, \mathscr{U})$ is totally bounded.

Also, by theorem 6.3.10,

(ii) $(L^X, \mathscr{U})$ is strongly complete.

Conversely, let $(L^X, \mathscr{U})$ be totally bounded and strongly complete.

Let $\mathscr{F}_u$ be an ultrafilter relative to an $\mathbb{F}(\mathscr{U})$-open set G.

Then, by theorem 6.3.2, $\mathscr{F}_u$ is a weak Cauchy filter relative to G.

So, by strong completeness, $\mathscr{F}_u$ is convergent.

Hence, by theorem 6.3.9, the space is compact. □

We proceed to prove that the product of strongly complete spaces is strongly complete.

Theorem 6.3.12. *A filter $\mathscr{F}$ converges to (x^{α}_{β}) in $\Pi L^{X^{\alpha}}$ iff $\pi^{\rightarrow}_{\alpha}(\mathscr{F})$ converges to $\pi^{\rightarrow}_{\alpha}((x^{\alpha}_{\beta}))$ in $L^{X^{\alpha}}$, $\forall\ \alpha$.*

Proof. Since, for any α, $\pi^{\rightarrow}_{\alpha}$ is L-weakly uniformly continuous.
Therefore, by theorem 5.2.9, for any α, $\pi^{\rightarrow}_{\alpha}$ is continuous and consequently by theorem 2.1.27, the first part follows.
The second part follows from theorems 2.1.25 and the theorem 2.1.26.

□

Theorem 6.3.13. *Product of strongly complete spaces is strongly complete.*

Proof. Let $\{(L^{X^{\alpha}}, \mathscr{U}^{\alpha}) \mid \alpha \in \Lambda\}$ be a collection of strongly complete spaces and let $\mathscr{U}$ be the product L-local uniformity on $L^X = \Pi_{\alpha\in\Lambda} L^{X^{\alpha}}$. Let $\mathbb{F}(\mathscr{U})$ be the L-topology on L^X generated by $\mathscr{U}$ and $\mathscr{F}$ be a weak Cauchy filter relative to an $\mathbb{F}(\mathscr{U})$-open set G'.

$$\text{Then, for any } B \in L^X \text{ such that } B \subseteq G', \text{ implies } B \notin \mathscr{F} \qquad (1)$$

Now, by theorem 6.3.5 for any $\alpha \in \Lambda$. $\pi^{\rightarrow}_{\alpha}(\mathscr{F})$ is a weak Cauchy filter on $L^{X^{\alpha}}$.
Let $C = \pi^{\rightarrow}_{\alpha}(G)$. By theorem 2.1.13(i), we have $G \subseteq \pi^{\leftarrow}_{\alpha}(\pi^{\rightarrow}_{\alpha}(G))$.
Then, $[\pi^{\leftarrow}_{\alpha}(\pi^{\rightarrow}_{\alpha}(G))]' \subseteq G'$. This implies $[\pi^{\leftarrow}_{\alpha}(C)]' \subseteq G'$.
Hence, $[\pi^{\leftarrow}_{\alpha}(C)]' \notin \mathscr{F}$ (By 1).
Again by proposition 2.1.14, $[\pi^{\leftarrow}_{\alpha}(C)]' = \pi^{\leftarrow}_{\alpha}(C')$, therefore $\pi^{\leftarrow}_{\alpha}(C') \notin \mathscr{F}$.
This implies that $C' \notin \pi^{\rightarrow}_{\alpha}(\mathscr{F})$.

For, if $C' \in \pi_\alpha^\rightarrow(\mathscr{F})$, then $C \in [\pi_\alpha^\rightarrow(\mathscr{F})]'$.

This then implies that $\pi_\alpha^\leftarrow(C) \in \pi_\alpha^\leftarrow([\pi_\alpha^\rightarrow(\mathscr{F})]') = [\pi_\alpha^\leftarrow(\pi_\alpha^\rightarrow(\mathscr{F}))]'$.

But as $\mathscr{F} \subseteq \pi_\alpha^\leftarrow(\pi_\alpha^\rightarrow(\mathscr{F}))$, we have, $[\pi_\alpha^\leftarrow(\pi_\alpha^\rightarrow(\mathscr{F}))]' \subseteq \mathscr{F}'$.

Therefore, $\pi_\alpha^\leftarrow(C) \in \mathscr{F}'$, implying $[\pi_\alpha^\leftarrow(C)]' \in \mathscr{F}$.

This contradicts the fact that $[\pi_\alpha^\leftarrow(C)]' = \pi_\alpha^\leftarrow(C') \notin \mathscr{F}$.

Now $C' = [\pi_\alpha^\rightarrow(G)]' \notin \pi_\alpha^\rightarrow(\mathscr{F})$.

Hence, $[[\pi_\alpha^\rightarrow(G)]']^o \notin \pi_\alpha^\rightarrow(\mathscr{F})$, as $[[\pi_\alpha^\rightarrow(G)]']^o \subseteq [\pi_\alpha^\rightarrow(G)]'$.

Hence, for any $F_\alpha \subseteq [[\pi_\alpha^\rightarrow(G)]']^o$, $F_\alpha \notin \pi_\alpha^\rightarrow(\mathscr{F})$.

Thus, $\pi_\alpha^\rightarrow(\mathscr{F})$ is a weak Cauchy filter relative to the open set $[[\pi_\alpha^\rightarrow(G)]']^o$.

So $\exists\ x_\beta^\alpha \in X^\alpha$ such that $\pi_\alpha^\rightarrow(\mathscr{F}) \to x_\beta^\alpha$.

Therefore, by theorem 6.3.12, $\mathscr{F} \to (x_\beta^\alpha)$.

Hence, $(L^X, \mathscr{U})$ is strongly complete. □

CHAPTER 7

CONCLUSION

7.1 Significance of the work

Work presented in this thesis provides a detailed study on the core aspects of a certain generalized uniform structures in the L-valued topological setting representing the category L-TOP. They are studied in the context of other existing forms of generalized structures and in relation amongst themselves in an attempt to provide a fairly complete framework consisting these generalized uniform structures.

The results embodying the findings of the thesis establish that not only various uniform properties can be lifted to a reasonable degree using a suitable platform of existing notions, but also many new results can be obtained in the wider framework of generalized uniform structures.

The generalized uniform structures presented in this thesis are found to satisfy the following relation:

L-uniformity $\Rightarrow$ L-local uniformity $\Rightarrow$ L-semi-uniformity $\Rightarrow$ L-semi-quasi-uniformity

$\quad\quad\Searrow \quad\quad\quad\quad\quad\quad\quad\quad \Searrow \quad\quad\quad\quad \Nearrow$

L-quasi-uniformity $\Rightarrow$ L-local quasi-uniformity.

This is akin to the relationship of analogous forms in the classical framework of general topological spaces, indicating the reasonability of the extension.

7.2 Future directions

Our work has opened up various avenues for further study on different aspects of generalized uniform structures that have been developed in this thesis.

We have outlined some of these directions, without being exhaustive:

- a study on categoric classification and comparison among various forms of generalized uniform structures.
- a study of their detailed application in various fields. This is well worth a detailed investigation, given the utility of classical uniform spaces in diverse areas.
- a study of various generalized for example the sum structures, with respect to the generalized framework of uniform spaces.
- to extend the evolved notions to the category **C−TOP** which allows both the underlying lattices and underlying sets to change from space to

space, where C is some category of lattice-theoretic structures.

Further, the work needs to be examined in the very general known as the category **HL – UNIF**, that is, on the framework of GL-monoids of García et. al. As this structure unifies both the Lowen and Hutton categories, investigation on this line is expected to yield rich and varied results.

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