A STUDY ON RAMANUJAN-TYPE SERIES FOR $1/\pi$

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Dedicated to my family

(Maa & Deuta)

Abstract

In this thesis, we study Ramanujan-type series for $1/\pi$ and $1/\pi^2$. Using certain representations for Eisenstein series in Ramanujan's second notebook, which were not used by previous mathematicians in deriving series for $1/\pi$, we obtain several well known series for $1/\pi$. In this process, we also rediscover an explicit well known value of Eisenstein series and certain quotients of theta functions recorded by Ramanujan in his first notebook. Again, using certain representations for Eisenstein series, we derive some new as well as known Ramanujan-type series for $1/\pi$ corresponding to Ramanujan's singular moduli x_n, for some non integral values of n. In the process we also find the values of singular moduli $x_{3/2}$, $x_{5/2}$, $x_{9/2}$ and $x_{5/3}$. We also find new hypergeometric-like series for $1/\pi^2$ arising from Ramanujan's theory of elliptic functions to alternative base 3. Here we find two series representations for $1/\pi^2$ which are perfect analogues of two of Ramanujan's 17 series for $1/\pi$ recorded by him in his epic paper "Modular equations and approximations to π". On page 212 of his lost notebook, Ramanujan defined a new class invariant λ_n and recorded/intended to record several values for λ_n. In 2001, Chan, Liaw and Tan found a new class of series for $1/\pi$ associated with λ_n. Using certain representations for Eisenstein series, identities for hypergeometric series and the class invariant λ_n, we derive a general series for $1/\pi^2$ and in particular we find six new series for $1/\pi^2$ corresponding to six values of λ_n.

DECLARATION BY THE CANDIDATE

I, Narayan Nayak, hereby declare that the subject matter in this thesis entitled, **"A Study on Ramanujan-type Series for $1/\pi$"**, is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/institute.

This thesis is being submitted to the Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

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CERTIFICATE OF THE SUPERVISOR

This is to certify that the thesis entitled **"A Study on Ramanujan-type Series for $1/\pi$"**, submitted to the School of Sciences of Tezpur University in partial fulfilment for the award of the degree of Doctor of Philosophy in Mathematical Sciences is a record of research work carried out by **Mr. Narayan Nayak** under my supervision and guidance.

All help received by him from various sources have been duly acknowledged.

No part of this thesis has been submitted elsewhere for award of any other degree.

Place: Tezpur

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Table of Contents

Chapter 1

Introduction

In Section 13 of his famous paper "Modular equations and approximations to π" [51], [54, p. 36], Ramanujan recorded three series representations for $1/\pi$ given in the following theorem.

Theorem 1.0.1. *If* $A_n := \frac{(\frac{1}{2})_n^3}{n!^3}, \qquad n \geq 0,$

where $(a)_0 := 1, \qquad (a)_n := a(a+1)(a+2)...(a+n-1), \quad n \geq 1,$ *then*

$$\frac{4}{\pi} = \sum_{n=0}^{\infty}(6n+1)A_n\frac{1}{4^n}, \tag{1.0.1}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty}(42n+5)A_n\frac{1}{2^{6n}}, \tag{1.0.2}$$

$$\frac{32}{\pi} = \sum_{n=0}^{\infty}\left((42\sqrt{5}+30)n+5\sqrt{5}-1\right)A_n\frac{1}{2^{6n}}\left(\frac{\sqrt{5}-1}{2}\right)^{8n}. \tag{1.0.3}$$

Again, at the beginning of Section 14 of the same paper [51], [54, p. 37], Ramanujan wrote, "There are corresponding theories in which q is replaced by one or other of the functions"

$$q_r := q_r(x) := \exp\left(-\pi\csc(\pi/r)\frac{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;1-x)}{{}_2F_1(\frac{1}{r},\frac{1-r}{r};1;x)}\right), \tag{1.0.4}$$

where $r = 3$, 4, or 6, and where the hypergeometric functions ${}_2F_1$, is defined by

$${}_2F_1(a_1,a_2;b_1;x) := \sum_{n=0}^{\infty}\frac{(a_1)_n \cdot (a_2)_n}{(b_1)_n}\frac{x^n}{n!}, \; |x| < 1,$$

where a_1, a_2 and b_1 are complex numbers, except that b_1 can not be a non-positive integer.

Ramanujan then offered fourteen further series representations for $1/\pi$. Of which, ten belong to the quartic theory, i.e., for $r = 4$; two belong to the cubic theory, i.e., for $r = 3$; and two belong to the sextic theory, i.e., for $r = 6$. Ramanujan never returned to the "corresponding theories" in his published papers, but on six pages in his second notebook [52] he recorded without proofs some of his results of these theories, which were first proved by B. C. Berndt, S. Bhargava, and F. G. Garvan [16] in 1995 and can also be found in Berndt's book [15, Chapter 33]. These theories are now known as *"Ramanujan's theories of elliptic functions to alternative bases* 3, 4, *or* 6" or *"Ramanujan's elliptic functions in the theories of signatures* 3, 4, *or* 6."

All these 17 series for $1/\pi$ can be found in Ramanujan's notebooks [52], which were written prior to his departure for England. In particular, (1.0.1) – (1.0.3) can be found on page 355 in his second notebook and the remaining 14 series found in his third notebook [52, p. 378]. In 1928, S. Chowla [36], [37], [38, pp. 87-91, 116-119] gave the first published proof of a general series reoresentation for $1/\pi$ and used it to derive (1.0.1). In 1987, J. M. Borwein and P. B. Borwein [22] proved all these 17 formulas. They also discovered many further series for $1/\pi$ [23], [24], [27], [25], [26], where D. H. Bailey also coauthored [27]. At about the same time, D. V. Chudnovsky and G. V. Chudnovsky [40] also independently proved several of Ramanujan's series representations for $1/\pi$ and established several new series. Further particular series representations for $1/\pi$ as well as some general formulas have subsequently been derived by H. H. Chan and W. -C. Liaw [31], B. C. Berndt and Chan [18], Berndt, Chan and Liaw [20], Chan, Liaw and V. Tan [32], Chan, S. H. Chan and Z. -G. Liu [29], Chan and K. P. Loo [33], and Chan and H. Verrill [35]. J. Guillera [43]–[47] discovered "experimentally" some beautiful series for $1/\pi$ as well as for $1/\pi^2$ and proved some of his series by using the WZ-method. Further work has been accomplished by W. Zudilin [57]–[61]. More recently, N.

D. Baruah and Berndt [6], returned to Ramanujan's ideas expressed in Section 13 of his paper [51], [54, p. 36]. They used those ideas in conjunction with twelve identities for Eisenstein series recorded without proofs in Section 10 of [51], [54, pp. 33-34], and further identities of this type to reprove 13 of Ramanujan's 17 identities from [51], as well as to establish many new series representations for $1/\pi$. In particular, they rely on Ramanujan's initial ideas more so than the previous authors. Again in [4], they utilized Ramanujn's cubic and quartic theories to establish five of Ramanujan's 17 formulas in addition to some new representations. We refer to a paper by Baruah, Berndt and Chan [7] for a survey on Ramanujan-type series for $1/\pi$ up untill their paper has appeared. Most recently, further particular series representations for $1/\pi$ have subsequently been derived by S. Cooper [41], [42], W. Chu [39], Chan, Y. Tanigawa, Y. Yang, and Zudilin [34], Chan and Cooper [30], N. D. Bagis, M. L. Glasser [3] and Liu [48], [49]. In [55], Z. -W. Sun provided 170 conjectural series for $1/\pi$ and other constants.

In this thesis we derive some known as well as new series for $1/\pi$ and $1/\pi^2$. We emphasize on Ramanujan's original ideas in [51], [54, p. 36].

The thesis consists of five chapters including the introductory chapter. In the following few paragraphs, we briefly outline the work contained in the next chapters.

Define Ramanujan's Eisenstein series

$$P(q) := 1 - 24\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}, \qquad |q| < 1. \tag{1.0.5}$$

In Section 10 of his paper [51], [54, pp. 33–34], Ramanujan recorded without proofs twelve representations for

$$f_n(q) := nP(q^{2n}) - P(q^2), \tag{1.0.6}$$

corresponding to twelve values of n, namely, $n = 2, 3, 4, 5, 7, 11, 15, 17, 19, 23, 31$, and 35. He also recorded the representations for $n = 2$ and 4 in Chapter 17 and for the remaining ten values and for $n = 9$ and $n = 25$ in Chapter 21 of his second notebook [52]. Berndt [14] proved all these representations. Berndt's proofs of some of the representations for $f_n(q)$ follow a hint given by Ramanujan at the beginning of the Chapter 21, but unfortunately Berndt was not able to use Ramanujan's idea, or any idea with which Ramanujan might have been familiar, to prove several of these representations . Thus, Berndt resorted to the theory of modular forms to prove those representations. As we already mentioned above, by employing the aforementioned representations for $f_n(q)$ with $q = e^{-\pi/\sqrt{n}}$, a transformation formula for $P(q^2)$ and certain hypergeometric identities, Baruah and Berndt [6], established 13 of Ramanujan's 17 identities from [51] as well as derived many new series representations for $1/\pi$. In Chapter 21 of his second notebook, Ramanujan [52, Vol. II] also recorded without proofs three representations for

$$h_n(q) := nP(-q^n) - P(-q), \tag{1.0.7}$$

corresponding to $n = 3$, 5, and 7 along with the representations for $h_9(-q)$ and $h_{25}(-q)$. Proofs of all these results can be found in Berndt's book [14, pp. 473–480]. Trivially, we have $P(-q) = -P(q) + 6P(q^2) - 4P(q^4)$. But there is no trivial way to transform the representations for $f_n(q)$ into $h_n(q)$. One might speculate why Ramanujan was interested in such representations for $h_n(\pm q)$. One reason could be, as in the cases of $f_n(q)$, Ramanujan might have tried to find series for $1/\pi$ arising from these representations as well. In Chapter 2, we prove several known series for $1/\pi$ by employing the above mentioned representations for $h_n(\pm q)$ along with a transformation formula for $P(-q)$ and certain hypergeometric series identities.

Chapter 3 is devoted to finding series for $1/\pi$ corresponding to Ramanujan's singular moduli x_n, for some non integral values of n.

Next, set

$$x_n := x(e^{-\pi\sqrt{n}}) \quad \text{and} \quad z_n := z(e^{-\pi\sqrt{n}}), \tag{1.0.8}$$

where $x(q)$ and $z(q)$ are defined in (2.2.4) and (2.2.2) respectively. The numbers x_n are the so-called *singular moduli*. Ramanujan recorded the values of many singular moduli in his notebooks [52]. We refer to [21] and [15, Chapter 34] for the proofs and commentaries on Ramanujan's singular moduli.

Although experts like J. M. and P. B. Borwein might have known it but it has been noticed that Ramanujan's singular moduli x_n for non integral values of n were not used by the previous authors to derive Ramanujan-type series for $1/\pi$. In Chapter 3, we prove several new as well as known series for $1/\pi$ by using Ramanujan's singular moduli x_n, for some non integral values of n along with the ideas developed by the previous authors. For example, corresponding to the singular modulus $x_{5/3}$, we find the new series

$$\frac{96}{\pi} = \sum_{k=0}^{\infty}\{(42\sqrt{5}-30)k+5\sqrt{5}+1\}A_k\frac{1}{2^{6k}}\left(\frac{\sqrt{5}+1}{2}\right)^{8k},$$

where

$$A_k := \frac{(\frac{1}{2})_k^3}{k!^3}.$$

The main difference of our method with that of [4], [6] and [8] is that, here we need to use two different representations of $f_n(q)$ for certain q.

In Chapter 4, we find series for $1/\pi^2$ arising from Ramanujan's theory of elliptic functions to alternative base 3.

As mentioned earlier, J. Guillera [43]-[47], discovered some beautiful series for $1/\pi$ as well as for $1/\pi^2$. He proved some of his series with the help of the WZ-method, and "experimentally"discovered several other series for $1/\pi^2$. For example, using WZ- method, he proved that

$$\frac{8}{\pi^2} = \sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k^5}{k!^5}\left(20k^2+8k+1\right)\left(-\frac{1}{4}\right)^k,$$

$$\frac{128}{\pi^2} = \sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k^5}{k!^5}\left(820k^2+180k+13\right)\left(-\frac{1}{1024}\right)^k.$$

Using a hypergeometric series transformation arising from the quadratic transformation $z \rightarrowtail -4z/(1-z)^2$ and two series of Guillera above, W. Zudilin [58], produced the two new series

$$\frac{10\sqrt{5}}{\pi^2} = \sum_{k=0}^{\infty} U_k \frac{(4k)!}{k!^2(2k)!} \frac{18k^2 - 10k - 3}{(2^8 5^2)^k},$$

$$\frac{5^4 41\sqrt{41}}{\pi^2} = \sum_{k=0}^{\infty} U_k \frac{(4k)!}{k!^2(2k)!} \frac{1046529k^2 + 227104k + 16032}{(5^4 41^2)^k},$$

where the sequence of integers

$$U_k = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \binom{2k-2n}{k-n} 2^{4(k-n)}, \qquad k = 0, 1, 2, \ldots,$$

satisfies the recurrence relation

$$(k+1)^3 U_{k+1} - 8(2k+1)(8k^2+8k+5)U_k + 4096k^3 U_{k-1} = 0 \qquad k = 1, 2, \ldots$$

Also in [59], Zudilin derived an algorithm for producing Ramanujan-Guillera - type series for $1/\pi^2$ from known ones for $1/\pi$. Motivated by the work of Guillera, Baruah and Berndt [5] extended Ramanujan's ideas to derive hypergeometric-like series representations for $1/\pi^2$. They used the classical theory to prove these. In Chapter 4, we use cubic theory to derive hypergeometric-like series representations for $1/\pi^2$. In the process, we find two series representations for $1/\pi^2$ which are analogues of two of Ramanujan's 17 series for $1/\pi$ recorded by him in his epic paper [51]. For example

$$\frac{3}{4\pi^2} = \sum_{k=0}^{\infty} \{225k^2 + 81k + 8\} U_k \left(\frac{2}{27}\right)^{k+2}$$

is an analogue of

$$\frac{27}{4\pi} = \sum_{k=0}^{\infty} (15k+2) \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(k!)^3} \left(\frac{2}{27}\right)^k$$

where

$$U_k = \sum_{n=0}^{k} \frac{(\frac{1}{3})_n (\frac{2}{3})_n (\frac{1}{2})_n (\frac{1}{3})_{k-n} (\frac{2}{3})_{k-n} (\frac{1}{2})_{k-n}}{(n!)^3((k-n)!)^3}.$$

On page 212 of his lost notebook, Ramanujan [53] recorded or intended to record several values of λ_n, which is defined as

$$\lambda_n = \frac{e^{(\pi/2)\sqrt{n/3}}}{3\sqrt{3}} \frac{f^6(e^{-\pi\sqrt{n/3}})}{f^6(e^{-\pi\sqrt{3n}})} = \frac{1}{3\sqrt{3}} \frac{\eta^6\left(\frac{1+i\sqrt{n/3}}{2}\right)}{\eta^6\left(\frac{1+i\sqrt{3n}}{2}\right)},$$

where $f(-q)$ and $\eta(\tau)$ are defined by

$$f(-q) := \prod_{k=1}^{\infty}(1-q^k), \quad |q|<1,$$

and

$$\eta(\tau) := e^{2\pi\imath\tau/24}\prod_{k=1}^{\infty}(1-e^{2\pi\imath k\tau}) =: q^{1/24}f(-q),$$

where $q = e^{2\pi\imath\tau}$ and $\operatorname{Im}\tau > 0$.

In particular, for $n = 1, 9, 17, 25, 33, 41, 49, 57, 65, 81, 89, 73, 97, 121, 169, 193,$ $217, 241, 265, 289,$ and 361 Ramanujan recorded or intended to record the values of λ_n. All these values of λ_n and several new values were established by Berndt, Chan, Kang, and Zhang [19] by using modular j-invariant, modular equations, Kronecker's limit formula, and an empirical approach. In [32], Chan, W.-C. Liaw, and V. Tan derived six series representation for $1/\pi$ corresponding to six values of λ_n, namely, $n = 3, 17, 25, 41, 49, 89$. Motivated by their work, in Chapter 5 we derive a general series representation for $1/\pi^2$ and used it derive six series for $1/\pi^2$, which are analogous of the six series given by Chan, Liaw, and Tan.

Chapter 2

Series for $1/\pi$ arising from certain representations for Eisenstein series in Ramanujan's second notebook

2.1 Introduction

As mentioned in the introductory chapter, in Chapter 21 of his second notebook, Ramanujan [52, Vol. II] recorded without proofs three representations for

$$h_n(q) := nP(-q^n) - P(-q), \tag{2.1.1}$$

corresponding to $n = 3$, 5, and 7 along with the representations for $h_9(-q)$ and $h_{25}(-q)$. In this chapter, we prove several known series for $1/\pi$ by employing the above mentioned representations for $h_n(\pm q)$ along with a transformation formula for $P(-q)$ and certain hypergeometric series identities. In certain cases, we have found proofs of some series for $1/\pi$ which are simpler/shorter than those in [6]. In the process, we have also rediscovered an explicit well-known value of Eisenstein series (see (2.4.10)) and certain quotients of theta functions (see (2.7.13) and (2.8.11)) recorded by Ramanujan in his first notebook [52]. Further remarks on

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Note: The contents of this chapter appeared in Ramanujan Rediscovered, RMS Lecture Note Series [8].

these can be found in appropriate places of Sections 2.4–2.8.

2.2 Definitions, preliminary results, and notation

If

$$q = \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)} \right), \tag{2.2.1}$$

then one of the fundamental results in the theory of elliptic functions [14, p. 101, Entry 6] is given by

$$z(q) := \varphi^2(q) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right), \tag{2.2.2}$$

where Ramanujan's theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{k=-\infty}^{\infty} q^{k^2}, \quad |q| < 1 \tag{2.2.3}$$

It is also well-known, for example, from Entries 10(i) and 11(iii) of [14, pp. 122–123], that

$$x(q) = 16q \frac{\psi^4(q^2)}{\varphi^4(q)}, \tag{2.2.4}$$

where $\varphi(q)$ is as defined in (2.2.3) and Ramanujan's special theta function $\psi(q)$ is defined by

$$\psi(q) = \sum_{k=0}^{\infty} q^{k(k+1)/2}, \quad |q| < 1 \tag{2.2.5}$$

In the sequel, we often write x and z for $x(q)$ and $z(q)$, respectively.

We also need Ramanujan's theta function

$$f(-q) := \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = \prod_{j=1}^{\infty} (1 - q^j), \tag{2.2.6}$$

where the later equality in (2.2.6) is Euler's pentagonal number theorem. Note that, if $q = \exp(2\pi i \tau)$, with $\text{Im}\ \tau > 0$, then $f(-q) = q^{-1/24}\eta(\tau)$, where $\eta(\tau)$ is the classical Dedekind eta-function.

In his second notebook, Ramanujan [52], [14, p. 124, Entry 12(i)] recorded the identity

$$f(q) = \sqrt{z}\left\{\frac{x(1-x)}{16q}\right\}^{1/24} \tag{2.2.7}$$

If we replace q by q^n in the above, then the same representation also holds with x replaced by $\beta = x(q^n)$ and z replaced by $z(q^n) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)$. We will use such representations in Sections 2.7 and 2.8.

We end this introduction by defining a modular equation of degree n. Let n denote a fixed natural number, and assume that

$$n\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha)} = \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \beta)}. \tag{2.2.8}$$

Then a modular equation of degree n is a relation between α and β induced by (2.2.8). The corresponding multiplier $m(\alpha, \beta)$ is defined by

$$m(\alpha, \beta) = \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \beta)}. \tag{2.2.9}$$

2.3 Main ideas

In brief, we combine two different relations between $P(-q)$ and $P(-q^n)$, for certain positive integers n, along with a Clausen's formula, to obtain the series representations for $1/\pi$. The development of the method for deriving series for $1/\pi$ described in this section is analogous to that in [6].

First we recall the following Clausen's formulas from Theorems 5.7(i)–(vi) and Formula (5.5.9) in [22, pp. 180-181]. Let

$$A_k := \frac{(\frac{1}{2})_k^3}{k!^3}, \quad B_k := \frac{(\frac{1}{4})_k(\frac{1}{2})_k(\frac{3}{4})_k}{k!^3}, \quad C_k := \frac{(\frac{1}{6})_k(\frac{1}{2})_k(\frac{5}{6})_k}{k!^3}, \tag{2.3.1}$$

and set

$$X := 4x(1-x), \quad Y := \frac{4x}{(1-x)^2}, \quad U := \frac{x^2}{4(1-x)}, \quad V := \frac{4\sqrt{x}\,(1-x)}{(1+x)^2},$$
$$W := \frac{2\sqrt{X}}{1-X}, \quad L := \frac{27X^2}{(4-X)^3} \quad \text{and} \quad M := \frac{27X}{(1-4X)^3}.$$

Then

$$z^2 = {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;X\right) = \sum_{k=0}^{\infty} A_k X^k, \quad 0 \le x \le \frac{1}{2}, \tag{2.3.2}$$

$$= \frac{1}{1-x}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;-Y\right) = \frac{1}{1-x}\sum_{k=0}^{\infty}(-1)^k A_k Y^k, \quad 0 \le x \le 3-2\sqrt{2}, \tag{2.3.3}$$

$$= \frac{1}{\sqrt{1-x}}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;-U\right) = \frac{1}{\sqrt{1-x}}\sum_{k=0}^{\infty}(-1)^k A_k U^k, \quad 0 \le x \le 2\sqrt{2}-2, \tag{2.3.4}$$

$$= \frac{1}{1+x}\,{}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};1,1;V^2\right) = \frac{1}{1+x}\sum_{k=0}^{\infty} B_k V^{2k}, \quad 0 \le x \le 3-2\sqrt{2}, \tag{2.3.5}$$

$$= \frac{1}{1-2x}\,{}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};1,1;-W^2\right) = \frac{1}{1-2x}\sum_{k=0}^{\infty}(-1)^k B_k W^{2k}, \quad 0 \le x \le \frac{1-2^{1/4}\sqrt{2-\sqrt{2}}}{2}, \tag{2.3.6}$$

$$= \frac{2}{\sqrt{4-X}}\,{}_3F_2\left(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;L\right) = \frac{2}{\sqrt{4-X}}\sum_{k=0}^{\infty} C_k L^k, \quad 0 \le x \le \frac{1}{2}, \tag{2.3.7}$$

$$= \frac{1}{\sqrt{1-4X}}\,{}_3F_2\left(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;-M\right) = \frac{1}{\sqrt{1-4X}}\sum_{k=0}^{\infty}(-1)^k C_k M^k, \quad 0 \le x \le \frac{1}{2}. \tag{2.3.8}$$

Next, we find a representation for $P(-q)$ in terms of x and z. Let

$$y = \pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)},$$

so that, by (2.2.1), $q = e^{-y}$. Thus, from (1.0.5), we obtain

$$\begin{aligned} P(-q) &= 1 - 24\sum_{k=1}^{\infty}\frac{(-1)^k k q^k}{1-(-1)^k q^k} \\ &= 1 - 24\sum_{k=1}^{\infty}\frac{k}{(-1)^{-k}e^{ky}-1} \\ &= 1 - 24\frac{d}{dy}\sum_{k=1}^{\infty}\log(1-(-1)^k e^{-ky}) \\ &= 1 - 24\frac{d}{dy}\log\prod_{k=1}^{\infty}(1-(-1)^k e^{-ky}). \end{aligned} \tag{2.3.9}$$

Using (2.2.6) in (2.3.9), we find that

$$\begin{aligned} P(-q) &= 1 - 24\frac{d}{dy}\log f(e^{-y}) \\ &= -8\frac{d}{dy}\log\left\{e^{-y/8}f^3(e^{-y})\right\}. \end{aligned} \tag{2.3.10}$$

Transcribing the expression inside the curly brackets in (2.3.10) with the aid of (2.2.7), we arrive at

$$\begin{aligned} P(-q) &= -8\frac{d}{dy}\log\left\{z^{3/2}2^{-1/2}(x(1-x))^{1/8}\right\} \\ &= -\frac{d}{dy}\log\left\{x(1-x)z^{12}\right\}. \end{aligned} \tag{2.3.11}$$

Now, recall from Entry 9(i) of [14, p. 120] that

$$\frac{dy}{dx} = -\frac{1}{x(1-x)z^2}. \tag{2.3.12}$$

From (2.3.11) and (2.3.12), we deduce that

$$\begin{aligned} P(-q) &= x(1-x)z^2\frac{d}{dx}\log\left\{x(1-x)z^{12}\right\} \\ &= (1-2x)z^2 + 12x(1-x)z\frac{dz}{dx}, \end{aligned} \tag{2.3.13}$$

which is the desired representation for $P(-q)$ in terms of x and z.

Now, differentiating (2.3.2) with respect to x, we find that

$$2z\frac{dz}{dx} = \sum_{k=0}^{\infty} A_k k X^{k-1} \cdot 4(1-2x). \tag{2.3.14}$$

Employing (2.3.2) and (2.3.14) in (2.3.13), we deduce that

$$P(-q) = (1-2x)\sum_{k=0}^{\infty}(6k+1)A_k X^k. \tag{2.3.15}$$

It can be shown that [2, pp. 378], x_n and z_n satisfy the transformation formulas

$$1 - x_n = x_{1/n} \quad \text{and} \quad z_{1/n} = \sqrt{n}z_n. \tag{2.3.16}$$

Setting $q = e^{-\pi\sqrt{n}}$ in (2.3.15), we deduce that

$$P(-e^{-\pi\sqrt{n}}) = (1-2x_n)\sum_{k=0}^{\infty}(6k+1)A_k X_n^k, \tag{2.3.17}$$

where $X_n = 4x_n(1 - x_n)$.

Similarly, differentiating each of (2.3.3)–(2.3.8) with respect to x, and proceeding as above, we can find that

$$P(-e^{-\pi\sqrt{n}}) = \frac{1}{1-x_n}\sum_{k=0}^{\infty}\{6(1+x_n)k+1+4x_n\}(-1)^k A_k Y_n^k, \tag{2.3.18}$$

$$= \frac{1}{\sqrt{1-x_n}}\sum_{k=0}^{\infty}\{6(2-x_n)k+1+x_n\}(-1)^k A_k U_n^k, \tag{2.3.19}$$

$$= \frac{1}{4(1+x_n)^2}\sum_{k=0}^{\infty}\{24(x_n^2-6x_n+1)k-6X_n + 4(1+x_n)\sqrt{1-X_n}\}B_k V_n^{2k}, \tag{2.3.20}$$

$$= \frac{1}{1-X_n}\sum_{k=0}^{\infty}\{6(1+X_n)k+1+2X_n\}(-1)^k B_k W_n^{2k}, \tag{2.3.21}$$

$$= \frac{4\sqrt{1-X_n}}{(4-X_n)\sqrt{4-X_n}}\sum_{k=0}^{\infty}\{3(8+X_n)k+2+X_n\}C_k L_n^k, \tag{2.3.22}$$

$$= \frac{(1+8X_n)\sqrt{1-X_n}}{(1-4X_n)\sqrt{1-4X_n}}\sum_{k=0}^{\infty}(6k+1)(-1)^k C_k M_n^k, \tag{2.3.23}$$

where

$$Y_n := \frac{4x_n}{(1-x_n)^2}, \quad U_n := \frac{x_n^2}{4(1-x_n)}, \quad V_n := \frac{4\sqrt{x_n}(1-x_n)}{(1+x_n)^2}, \tag{2.3.24}$$

$$W_n := \frac{2\sqrt{X_n}}{1-X_n}, \quad L_n := \frac{27X_n^2}{(4-X_n)^3} \quad \text{and} \quad M_n := \frac{27X_n}{(1-4X_n)^3}. \tag{2.3.25}$$

Next, we determine a transformation formula for $P(-q)$. If $\alpha, \beta > 1$ with $\alpha\beta = \pi^2$, then Ramanujan's function $f(q)$ defined in (2.2.6) satisfies the transformation formula [14, p. 43, Entry 27 (iv)]

$$e^{-\alpha/24}\alpha^{1/4}f(e^{-\alpha}) = e^{-\beta/24}\beta^{1/4}f(e^{-\beta}). \tag{2.3.26}$$

Taking logarithms of both sides of (2.3.26), we find that

$$-\frac{\alpha}{24}+\frac{\log\alpha}{4}+\sum_{k=0}^{\infty}\log\left(1+(-1)^k e^{-(k+1)\alpha}\right) = -\frac{\beta}{24}+\frac{\log\beta}{4}+\sum_{k=0}^{\infty}\log\left(1+(-1)^k e^{-(k+1)\beta}\right) \tag{2.3.27}$$

Differentiating both sides of (2.3.27) with respect to α, we obtain

$$-\frac{1}{24}+\frac{1}{4\alpha}+\sum_{k=1}^{\infty}\frac{k(-e^{-\alpha})^k}{1-(-e^{-\alpha})^k}=\frac{\beta}{24\alpha}-\frac{1}{4\alpha}-\sum_{k=1}^{\infty}\frac{(k\beta/\alpha)(-e^{-\beta})^k}{1-(-e^{-\beta})^k}. \tag{2.3.28}$$

Multiplying both sides of (2.3.28) by 24α, rearranging, and then employing the definition of $P(q)$ from (1.0.5), we deduce that

$$12-\alpha P(-e^{-\alpha})=\beta P(-e^{-\beta}). \tag{2.3.29}$$

Setting $\alpha=\pi/\sqrt{n}$, so that $\beta=\pi\sqrt{n}$, in (2.3.29), we arrive at

$$nP(-e^{-\pi\sqrt{n}})+P(-e^{-\pi/\sqrt{n}})=\frac{12\sqrt{n}}{\pi}, \tag{2.3.30}$$

which is the desired transformation formula for $P(-q)$.

Note that, if we set $n=1$ in (2.3.30), then we arrive at the evaluation

$$P(-e^{-\pi})=\frac{6}{\pi}. \tag{2.3.31}$$

By combining (2.3.30) and the representations for $h_n(q):=nP(-q^n)-P(-q)$ with $q=e^{-\pi/\sqrt{n}}$, for each $n=3,\ 5,\ 7,\ 9,$ and 25 along with the identities (2.3.2)-(2.3.8), we can derive certain known series representations for $1/\pi$. We present these in the remaining sections. In the cases for $n=5,\ 9,$ and 25, one can also get a few more series for $1/\pi$, that are not so elegant. So we do not record them.

2.4 Series corresponding to x_3

Theorem 2.4.1. *If A_k, B_k, and C_k, $k\geq 0$, are defined by* (2.3.1), *then*

$$\frac{4}{\pi}=\sum_{k=0}^{\infty}(6k+1)A_k\frac{1}{4^k}, \tag{2.4.1}$$

$$\frac{1}{\pi}=\sum_{k=0}^{\infty}(-1)^k\{(15\sqrt{3}-24)k+6\sqrt{3}-10\}A_k2^k(\sqrt{3}-1)^{6k}, \tag{2.4.2}$$

$$\frac{4\sqrt{2}}{\pi}=\sum_{k=0}^{\infty}(-1)^k\{(30-6\sqrt{3})k+7-3\sqrt{3}\}A_k\frac{(2-\sqrt{3})^{3k}}{2^{4k}}, \tag{2.4.3}$$

$$\frac{8\sqrt{2}}{\pi} = \sum_{k=0}^{\infty}\{(85\sqrt{3}-135)k+8\sqrt{3}-12\}B_k\left(\frac{(8\sqrt{2})}{51\sqrt{3}-75}\right)^{2k+1} \tag{2.4.4}$$

$$\frac{5\sqrt{5}}{2\pi\sqrt{3}} = \sum_{k=0}^{\infty}(11k+1)C_k\left(\frac{4}{125}\right)^{k}. \tag{2.4.5}$$

The identities (2.4.1) and (2.4.5) are due to Ramanujan [51, Eqs. (28) and (33), respectively], [54, pp. 36–37], [52, pp. 355, 378], [53, p. 370]. The remaining identities are due to Baruah and Berndt [6]. To prove Theorem 2.4.1, Baruah and Berndt [6] calculated and used the identity $f_3(e^{-\pi/\sqrt{3}}) = 3P(e^{-2\pi\sqrt{3}}) - P(e^{-2\pi/\sqrt{3}}) = \frac{3\sqrt{3}}{2}z_3^2$. In the following proof, we calculated and used (2.4.8), which makes our calculations somewhat shorter. *Proof of* (2.4.1). First of all, we derive an expression for $h_3(e^{-\pi/\sqrt{3}}) \stackrel{!}{=} 3P(-e^{-\pi\sqrt{3}}) - P(-e^{-\pi/\sqrt{3}})$.

Recall from [14, p. 473, Entry 6(i)], that

$$1+12\sum_{k=1}^{\infty}\frac{k(-q)^k}{1-(-q)^k} - 36\sum_{k=1}^{\infty}\frac{k(-q)^{3k}}{1-(-q)^{3k}} = \varphi^2(q)\varphi^2(q^3)\big((x(q)x(q^3))^{1/4} - \{(1-x(q))(1-x(q^3))\}^{1/4}\big)^2. \tag{2.4.6}$$

With the aid of (1.0.5) and (2.2.2) we rewrite (2.4.6) in the form

$$h_3(q) = 3P(-q^3) - P(-q) = 2z(q)z(q^3)\big((x(q)x(q^3))^{1/4} - \{(1-x(q))(1-x(q^3))\}^{1/4}\big)^2 \tag{2.4.7}$$

Now we set $q = e^{-\pi/\sqrt{3}}$, so that, by (2.3.16), $x(q) = x(e^{-\pi/\sqrt{3}}) = 1 - x(e^{-\pi\sqrt{3}}) = 1 - x_3$, $x(q^3) = x_3$ and $z(q) = z(e^{-\pi/\sqrt{3}}) = \sqrt{3}z(e^{-\pi\sqrt{3}}) = \sqrt{3}z_3$. We therefore deduce from (2.4.7) that

$$h_3(e^{-\pi/\sqrt{3}}) = 3P(-e^{-\pi\sqrt{3}}) - P(-e^{-\pi/\sqrt{3}}) = 0. \tag{2.4.8}$$

Next, setting $n=3$ in (2.3.30), we obtain

$$3P(-e^{-\pi\sqrt{3}}) + P(-e^{-\pi/\sqrt{3}}) = \frac{12\sqrt{3}}{\pi}. \tag{2.4.9}$$

From (2.4.8) and (2.4.9), we readily deduce

$$P(-e^{-\pi\sqrt{3}}) = \frac{2\sqrt{3}}{\pi}, \tag{2.4.10}$$

which is a well-known result (See, for example, Berndt's [13, Proposition 2.13]). Chan, Liaw and Tan [32] also established (2.4.10) and wrongly noted this value to be new.

Now, setting $n = 3$ in (2.3.17), we find that

$$P(-e^{-\pi\sqrt{3}}) = (1-2x_3)\sum_{k=0}^{\infty}(6k+1)A_kX_3^k. \tag{2.4.11}$$

From (2.4.10) and (2.4.11), we obtain

$$\frac{2\sqrt{3}}{\pi} = (1-2x_3)\sum_{k=0}^{\infty}(6k+1)A_kX_3^k. \tag{2.4.12}$$

But, from [15, p. 290, Theorem 9.9], we note that

$$x_3 = \frac{2-\sqrt{3}}{4}, \tag{2.4.13}$$

so that

$$X_3 = 4x_3(1-x_3) = \frac{1}{4}. \tag{2.4.14}$$

Employing (2.4.13) and (2.4.14) in (2.4.12), we complete the proof of (2.4.1).

Proof of (2.4.2). Setting $n = 3$ in (2.3.18), we find that

$$P(-e^{-\pi\sqrt{3}}) = \frac{1}{1-x_3}\sum_{k=0}^{\infty}\{6(1+x_3)k+1+4x_3\}(-1)^kA_kY_3^k, \tag{2.4.15}$$

where, by (2.3.24),

$$Y_3 = \frac{4x_3}{(1-x_3)^2}. \tag{2.4.16}$$

Using (2.4.13) in (2.4.16) and (2.4.15), we obtain

$$P(-e^{-\pi\sqrt{3}}) = \sum_{k=0}^{\infty}\{(90-48\sqrt{3})k+36-20\sqrt{3}\}(-1)^kA_k\left(2(\sqrt{3}-1)^6\right)^k. \tag{2.4.17}$$

From (2.4.17) and (2.4.10), we easily arrive at (2.4.2).

Proof of (2.4.3). Setting $n = 3$ in (2.3.19), we find that

$$P(-e^{-\pi\sqrt{3}}) = \frac{1}{\sqrt{1-x_3}}\sum_{k=0}^{\infty}\{6(2-x_3)k+1+x_3\}(-1)^kA_kU_3^k, \tag{2.4.18}$$

where, by (2.3.24),

$$U_3 := \frac{x_3^2}{4(1-x_3)}. \tag{2.4.19}$$

Using (2.4.13) in (2.4.19) and (2.4.18), we arrive at

$$P(-e^{-\pi\sqrt{3}}) = \sum_{k=0}^{\infty} \left\{ \frac{3}{2}(\sqrt{6}-\sqrt{2})(6+\sqrt{3})k + \frac{1}{4}(\sqrt{6}-\sqrt{2})(6-\sqrt{3}) \right\} \times (-1)^k A_k \frac{(2-\sqrt{3})^{3k}}{2^{4k}}. \tag{2.4.20}$$

We readily deduce (2.4.3) from (2.4.20) and (2.4.10).

Proof of (2.4.4). Setting $n = 3$ in (2.3.20), we obtain

$$P(-e^{-\pi\sqrt{3}}) = \frac{1}{4(1+x_3)^2} \sum_{k=0}^{\infty} \{24(x_3^2 - 6x_3 + 1)k - 6X_3 + 4(1+x_3)\sqrt{1-X_3}\} B_k V_3^{2k}, \tag{2.4.21}$$

where, again by (2.3.24),

$$V_3 := \frac{4\sqrt{x_3}(1-x_3)}{(1+x_3)^2}. \tag{2.4.22}$$

Using (2.4.13) in (2.4.22) and (2.4.21), we obtain

$$P(-e^{-\pi\sqrt{3}}) = \frac{1}{121} \sum_{k=0}^{\infty} \{(320\sqrt{3} - 170)k + 36\sqrt{3} - 4\} B_k \left(\frac{8\sqrt{2}}{51\sqrt{3} - 75} \right)^{2k}. \tag{2.4.23}$$

From (2.4.23) and (2.4.10), we derive (2.4.4).

Proof of (2.4.5). Setting $n = 3$ in (2.3.22), we find that

$$P(-e^{-\pi\sqrt{3}}) = \frac{4\sqrt{1-X_3}}{(4-X_3)(\sqrt{4-X_3})} \sum_{k=0}^{\infty} \{3(8+X_3)k + 2 + X_3\} C_k L_3^k, \tag{2.4.24}$$

where, by (2.3.25),

$$L_3 = \frac{27X_3^2}{(4-X_3)^3}. \tag{2.4.25}$$

Employing (2.4.14) in (2.4.25) and (2.4.24), we obtain

$$P(-e^{-\pi\sqrt{3}}) = \frac{12}{5\sqrt{5}} \sum_{k=0}^{\infty} (11k+1) C_k \left(\frac{4}{125} \right)^k. \tag{2.4.26}$$

From (2.4.26) and (2.4.10), we deduce (2.4.4) to complete the proof of the theorem.

2.5 Series corresponding to x_7

Theorem 2.5.1. *If A_k, B_k, and C_k, $k \geq 0$, are defined by* (2.3.1), *then*

$$\frac{16}{\pi} = \sum_{k=0}^{\infty} (42k+5) A_k \frac{1}{2^{6k}}, \tag{2.5.1}$$

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} (-1)^k \{(255\sqrt{7} - 672)k + 112\sqrt{7} - 296\} A_k (32 - 12\sqrt{7})^{3k}, \tag{2.5.2}$$

$$\frac{8\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} (-1)^k \{(102\sqrt{7} - 210)k + 35\sqrt{7} - 89\} A_k \left(\frac{8 - 3\sqrt{7}}{4}\right)^{3k}, \tag{2.5.3}$$

$$\frac{29241}{\pi} = \sum_{k=0}^{\infty} \{(76160 - 455\sqrt{7})k + 784\sqrt{7} + 6728\} B_k \left(\frac{8\sqrt{2}(325 + 119\sqrt{7})}{29241}\right)^{2k}, \tag{2.5.4}$$

$$\frac{9\sqrt{7}}{\pi} = \sum_{k=0}^{\infty} (-1)^k (65k+8) B_k \left(\frac{16}{63}\right)^{2k}, \tag{2.5.5}$$

$$\frac{85\sqrt{85}}{18\pi\sqrt{3}} = \sum_{k=0}^{\infty} (133k+8) C_k \left(\frac{4}{85}\right)^{3k+1} \tag{2.5.6}$$

$$\frac{5\sqrt{15}}{\pi} = \sum_{k=0}^{\infty} (-1)^k (63k+8) C_k \left(\frac{4}{5}\right)^{3k}. \tag{2.5.7}$$

The identities (2.5.1) and (2.5.6) are due to Ramanujan [51, Eqs. (29) and (34), respectively], [54, pp. 36–37], [52, pp. 355, 378], [53, p. 370] and (2.5.5) is due to Berndt, Chan, and Liaw [20]. The other four identities are due to Baruah and Berndt [6]. The level of difficulty in the following proof is the same as in [6].

Proof of (2.5.1). At first, we derive an expression for $h_7(e^{-\pi/\sqrt{3}}) = 7P(-e^{-\pi\sqrt{3}}) - P(-e^{-\pi/\sqrt{3}})$. Recall from [14, p. 473, Entry 6(iii)], that

$$1 + 4\sum_{k=1}^{\infty} \frac{k(-q)^k}{1-(-q)^k} - 28\sum_{k=1}^{\infty} \frac{k(-q)^{7k}}{1-(-q)^{7k}} = \varphi^2(q)\varphi^2(q^7)\big(\{x(q)x(q^7)\}^{1/4} + \{(1-x(q))(1-x(q^7))\}^{1/4}\big)^2. \tag{2.5.8}$$

Using (1.0.5) and (2.2.2) in (2.5.8), we find that

$$h_7(q) = 7P(-q^7) - P(-q) = 6z(q)z(q^7)\big((x(q)x(q^7))^{1/4} + \{(1-x(q))(1-x(q^7))\}^{1/4}\big)^2 \tag{2.5.9}$$

Now we set $q = e^{-\pi/\sqrt{7}}$, so that, by (2.3.16), $x(q) = x(e^{-\pi/\sqrt{7}}) = 1 - x(e^{-\pi\sqrt{7}}) = 1 - x_7$, $x(q^7) = x_7$, $z(q) = z(e^{-\pi/\sqrt{7}}) = \sqrt{7}z(e^{-\pi\sqrt{7}}) = \sqrt{7}z_7$, and $z(q^7) = z(e^{-\pi\sqrt{7}}) = z_7$. Thus, we deduce from (2.5.9) that

$$h_7(e^{-\pi/\sqrt{7}}) = 7P(-e^{-\pi\sqrt{7}}) - P(-e^{-\pi/\sqrt{7}}) = 24\sqrt{7}\sqrt{x_7(1-x_7)}z_7^2. \tag{2.5.10}$$

But, from [15, p. 290, Theorem 9.9], we have

$$x_7 = \frac{8 - 3\sqrt{7}}{16}, \tag{2.5.11}$$

and consequently,

$$X_7 = 4x_7(1 - x_7) = \frac{1}{64}. \tag{2.5.12}$$

Using (2.5.11) in (2.5.10), we find that

$$7P(-e^{-\pi\sqrt{7}}) - P(-e^{-\pi/\sqrt{7}}) = \frac{3}{2}\sqrt{7}z_7^2. \tag{2.5.13}$$

Again, setting $n = 7$, in (2.3.30), we obtain

$$7P(-e^{-\pi\sqrt{7}}) + P(-e^{-\pi/\sqrt{7}}) = \frac{12\sqrt{7}}{\pi}. \tag{2.5.14}$$

From (2.5.13) and (2.5.14), we arrive at

$$P(-e^{-\pi\sqrt{7}}) = \frac{12\sqrt{7}}{14\pi} + \frac{3\sqrt{7}}{28}z_7^2. \tag{2.5.15}$$

Next, setting $n = 7$ in (2.3.17), we find that

$$P(-e^{-\pi\sqrt{7}}) = (1 - 2x_7)\sum_{k=0}^{\infty}(6k+1)A_k X_7^k. \tag{2.5.16}$$

Using (2.5.11) and (2.5.12) in (2.5.16), we obtain

$$P(-e^{-\pi\sqrt{7}}) = \frac{3\sqrt{7}}{8}\sum_{k=0}^{\infty}(6k+1)A_k\frac{1}{2^{6k}}. \tag{2.5.17}$$

From (2.5.15), (2.5.17), and (2.3.2), we easily deduce (2.5.1).

Proof of (2.5.2). Setting $n = 7$ in (2.3.18), we find that

$$P(-e^{-\pi\sqrt{7}}) = \frac{1}{1 - x_7}\sum_{k=0}^{\infty}\{6(1+x_7)k + 1 + 4x_7\}(-1)^k A_k Y_7^k. \tag{2.5.18}$$

Using (2.5.11) in (2.5.18), we obtain

$$P(-e^{-\pi\sqrt{7}}) = \sum_{k=0}^{\infty}\{6(255 - 96\sqrt{7})k + 4(159 - 60\sqrt{7})\}(-1)^k A_k Y_7^k, \tag{2.5.19}$$

where $Y_7 = (32 - 12\sqrt{7})^3$.

Again, using (2.3.3) and (2.5.11) in (2.5.15), we obtain

$$P(-e^{-\pi\sqrt{7}}) = \frac{12\sqrt{7}}{14\pi} + \frac{12\sqrt{7}}{7}\sum_{k=0}^{\infty}(-1)^k A_k Y_7^k. \tag{2.5.20}$$

From (2.5.19) and (2.5.20), we arrive at (2.5.2).

Proof of (2.5.3). Setting $n = 7$ in (2.3.19), we find that

$$P(-e^{-\pi\sqrt{7}}) = \frac{1}{\sqrt{1-x_7}}\sum_{k=0}^{\infty}\{6(2 - x_7)k + 1 + x_7\}(-1)^k A_k U_7^k. \tag{2.5.21}$$

Using (2.5.11) in (2.5.21), we obtain

$$P(-e^{-\pi\sqrt{7}}) = \frac{3}{4\sqrt{2}}\sum_{k=0}^{\infty}\{6(17 - 5\sqrt{7})k + 31 - 11\sqrt{7}\}(-1)^k A_k U_7^k, \tag{2.5.22}$$

where $U_7 = \left(\frac{8-3\sqrt{7}}{4}\right)^3$.

But, employing (2.3.4) and (2.5.11) in (2.5.15), we also obtain

$$P(-e^{-\pi\sqrt{7}}) = \frac{12\sqrt{7}}{14\pi} + \frac{3\sqrt{7}}{7\sqrt{2}}(3 - \sqrt{7})\sum_{k=0}^{\infty}(-1)^k A_k U_7^k. \tag{2.5.23}$$

From (2.5.22) and (2.5.23), we easily deduce (2.5.3).

Proof of (2.5.4). Setting $n = 7$ in (2.3.20), we find that

$$P(-e^{-\pi\sqrt{7}}) = \frac{1}{4(1+x_7)^2}\sum_{k=0}^{\infty}\{24(x_7^2 - 6x_7 + 1)k - 6X_7 + 4(1+x_7)\sqrt{1-X_7}\}B_k V_7^{2k}. \tag{2.5.24}$$

Using (2.5.11) in (2.5.24), we obtain

$$P(-e^{-\pi\sqrt{7}}) = \frac{1}{29241}\sum_{k=0}^{\infty}\{6(10880\sqrt{7} - 455)k + 6756 + 8112\sqrt{7}\}B_k V_7^{2k}, \tag{2.5.25}$$

where $V_7 = \dfrac{8\sqrt{2}(325 + 119\sqrt{7})}{29241}$.

Again, using (2.3.5) and (2.5.11) in (2.5.15), we obtain

$$P(-e^{-\pi\sqrt{7}}) = \frac{12\sqrt{7}}{14\pi} + \frac{4\sqrt{7}}{399}(8 + \sqrt{7}) \sum_{k=0}^{\infty} (-1)^k B_k V_7^{2k}. \tag{2.5.26}$$

From (2.5.25) and (2.5.26), we arrive at (2.5.4).

Proof of (2.5.5). Setting $n = 7$, in (2.3.21), we find that

$$P(-e^{-\pi\sqrt{7}}) = \frac{1}{1 - X_7} \sum_{k=0}^{\infty} \{6(1 + X_7)k + 1 + 2X_7\}(-1)^k B_k W_7^{2k}. \tag{2.5.27}$$

Using (2.5.11) in (2.5.27), we obtain

$$P(-e^{-\pi\sqrt{7}}) = \frac{1}{21} \sum_{k=0}^{\infty} \{130k + 22\}(-1)^k B_k W_7^{2k}, \tag{2.5.28}$$

where $W_7 = \dfrac{16}{63}$.

But, with the aid of (2.3.6) and (2.5.11) we can rewrite (2.5.15) in the form

$$P(-e^{-\pi\sqrt{7}}) = \frac{12\sqrt{7}}{14\pi} + \frac{2}{7} \sum_{k=0}^{\infty} (-1)^k B_k V_7^{2k}. \tag{2.5.29}$$

From (2.5.28) and (2.5.29), we arrive at (2.5.5).

Proof of (2.5.6). Setting $n = 7$, in (2.3.22), we have

$$P(-e^{-\pi\sqrt{7}}) = \frac{4\sqrt{1 - X_7}}{(4 - X_7)(\sqrt{4 - X_7})} \sum_{k=0}^{\infty} \{3(8 + X_7)k + 2 + X_7\} C_k L_7^k. \tag{2.5.30}$$

Using (2.5.11) in (2.5.30), we obtain

$$P(-e^{-\pi\sqrt{7}}) = \frac{12\sqrt{7}}{85\sqrt{85}\sqrt{3}} \sum_{k=0}^{\infty} (513k + 43) C_k L_7^k, \tag{2.5.31}$$

where $L_7 = \left(\dfrac{4}{85}\right)^3$.

On the other hand, using (2.3.7) and (2.5.11) in (2.5.15), we obtain

$$P(-e^{-\pi\sqrt{7}}) = \frac{12\sqrt{7}}{14\pi} + \frac{24\sqrt{7}}{14\sqrt{255}} \sum_{k=0}^{\infty} (-1)^k B_k L_7^{2k}. \tag{2.5.32}$$

From (2.5.31) and (2.5.32), we arrive at (2.5.6).

Proof of(2.5.7). Setting $n = 7$, in (2.3.23), we find that

$$P(-e^{-\pi\sqrt{7}}) = \frac{(1+8X_7)\sqrt{1-X_7}}{(1-4X_7)(\sqrt{1-4X_7})}\sum_{k=0}^{\infty}(6k+1)(-1)^k C_k M_7^k. \tag{2.5.33}$$

Using (2.5.11) in (2.5.33), we obtain

$$P(-e^{-\pi\sqrt{7}}) = \frac{9\sqrt{7}}{5\sqrt{15}}\sum_{k=0}^{\infty}(6k+1)(-1)^k C_k M_7^k, \tag{2.5.34}$$

where $M_7 = \left(\frac{4}{5}\right)^3$

Using (2.3.8) and (2.5.11) in (2.5.15), we obtain

$$P(-e^{-\pi\sqrt{7}}) = \frac{12\sqrt{7}}{14\pi} + \frac{6\sqrt{7}}{14\sqrt{15}}\sum_{k=0}^{\infty}(-1)^k C_k M_7^{2k}. \tag{2.5.35}$$

From (2.5.34) and (2.5.35), we arrive at (2.5.7).

2.6 Series corresponding to x_5

Theorem 2.6.1. *If A_k and B_k, $k \geq 0$, are defined by* (2.3.1), *then*

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}\{4\sqrt{5}k + \sqrt{5} - 1\}A_k(\sqrt{5}-2)^{2k+1/2}, \tag{2.6.1}$$

$$\frac{8}{\pi} = \sum_{k=0}^{\infty}(-1)^k\Big[2\{(15+5\sqrt{5})\sqrt{\sqrt{5}+1} - 7\sqrt{10} - 5\sqrt{2}\}k + (9+3\sqrt{5})\sqrt{\sqrt{5}+1} - 7\sqrt{2} - 5\sqrt{10}\Big]A_k\left(\frac{\sqrt{5}-1}{4}\right)^{3k}\left(\frac{\sqrt{5}+1}{2} - \sqrt{\frac{\sqrt{5}+1}{2}}\right)^{6k}, \tag{2.6.2}$$

$$\frac{8}{\pi} = \sum_{k=0}^{\infty}(-1)^k(20k+3)B_k\frac{1}{4^k}. \tag{2.6.3}$$

The last identity was recorded by Ramanujan [51, Eq. (35)], [54, p. 38]. The remaining two identities are due to Baruah and Berndt [6].

At first, we derive an expression for $h_5(e^{-\pi/\sqrt{5}}) = 5P(-e^{-\pi\sqrt{5}}) - P(-e^{-\pi/\sqrt{5}})$.

Recall from [14, p. 473, Entry 6(ii)], that

$$1+6\sum_{k=1}^{\infty}\frac{k(-q)^k}{1-(-q)^k}-30\sum_{k=1}^{\infty}\frac{k(-q)^{5k}}{1-(-q)^{5k}}$$
$$=\varphi^2(q)\varphi^2(q^5)(\sqrt{x(q)x(q^5)}+\sqrt{(1-x(q))(1-x(q^5))}\ )$$
$$\times\left(\frac{1}{2}\{1+\sqrt{x(q)x(q^5)}+\sqrt{(1-x(q))(1-x(q^5))}\}\right)^{1/2}, \tag{2.6.4}$$

Using (1.0.5) and (2.2.2) in (2.6.4), we find that

$$h_5(q)=5P(-q^5)-P(-q)=4z(q)z(q^5)\left(\sqrt{x(q)x(q^5)}+\sqrt{(1-x(q))(1-x(q^5))}\ \right)$$
$$\times\left(\frac{1}{2}\{1+\sqrt{x(q)x(q^5)}+\sqrt{(1-x(q))(1-x(q^5))}\}\right)^{1/2} \tag{2.6.5}$$

Now we set $q=e^{-\pi/\sqrt{5}}$, so that, by (2.3.16), $x(q)=x(e^{-\pi/\sqrt{5}})=1-x(e^{-\pi\sqrt{5}})=1-x_5$, $x(q^5)=x_5$, $z(q)=z(e^{-\pi/\sqrt{5}})=\sqrt{5}z(e^{-\pi\sqrt{5}})=\sqrt{5}z_5$, and $z(q^5)=z(e^{-\pi\sqrt{5}})=z_5$. Thus, we deduce from (2.5.9) that

$$h_5(e^{-\pi/\sqrt{5}})=5P(-e^{-\pi\sqrt{5}})-P(-e^{-\pi/\sqrt{5}})=8\sqrt{5}\sqrt{x_5(1-x_5)}(\frac{1}{2}+\sqrt{x_5(1-x_5)})^{1/2}z_5^2. \tag{2.6.6}$$

But, from [15, p. 290], we have

$$x_5=\frac{1}{2}-\left(\frac{\sqrt{5}-1}{2}\right)^{3/2}, \tag{2.6.7}$$

and consequently,

$$X_5=4x_5(1-x_5)=9-4\sqrt{5}. \tag{2.6.8}$$

Using (2.6.7) in (2.6.6), we find that

$$5P(-e^{-\pi\sqrt{5}})-P(-e^{-\pi/\sqrt{5}})=2\sqrt{10}\left(\sqrt{5}-2\right)\left(\sqrt{5}-1\right)^{1/2}z_5^2. \tag{2.6.9}$$

Proof of (2.6.1). Setting $n=5$, in (2.3.30), we obtain

$$5P(-e^{-\pi\sqrt{5}})+P(-e^{-\pi/\sqrt{5}})=\frac{12\sqrt{5}}{\pi}. \tag{2.6.10}$$

From (2.6.9) and (2.6.10), we arrive at

$$P(-e^{-\pi\sqrt{5}}) = \frac{12\sqrt{5}}{10\pi} + \frac{2}{\sqrt{10}}\left(\sqrt{5}-2\right)\left(\sqrt{5}-1\right)^{1/2} z_5^2. \tag{2.6.11}$$

Next, setting $n = 5$ in (2.3.17), we find that

$$P(-e^{-\pi\sqrt{5}}) = (1-2x_5)\sum_{k=0}^{\infty}(6k+1)A_k X_5^k. \tag{2.6.12}$$

Using (2.6.7) and (2.6.8) in (2.6.12), we obtain

$$P(-e^{-\pi\sqrt{5}}) = 2\sqrt{\sqrt{5}-2}\sum_{k=0}^{\infty}(6k+1)A_k(9-4\sqrt{5})^k. \tag{2.6.13}$$

From (2.6.11), (2.6.13), and (2.3.2), we easily deduce (2.6.1).

Proof of (2.6.2). Setting $n = 5$ in (2.3.18), we find that

$$P(-e^{-\pi\sqrt{5}}) = \frac{1}{1-x_5}\sum_{k=0}^{\infty}\{6(1+x_5)k+1+4x_5\}(-1)^k A_k Y_5^k. \tag{2.6.14}$$

Using (2.6.7) in (2.6.14), we obtain

$$P(-e^{-\pi\sqrt{5}}) = \sum_{k=0}^{\infty}\{6(35+16\sqrt{5}-8\sqrt{38+17\sqrt{5}})k$$
$$+86+40\sqrt{5}-20\sqrt{38+17\sqrt{5}}\}(-1)^k A_k Y_5^k, \tag{2.6.15}$$

where $Y_5 = \frac{4x_5}{(1-x_5)^2} = \left(\frac{\sqrt{5}-1}{4}\right)^3\left(\frac{\sqrt{5}+1}{2}-\sqrt{\frac{\sqrt{5}+1}{2}}\right)^6$, where, here and in some other instances in the sequel, the simplification of the complicated radicals are aided by S. Wolfram's advanced mathematical package *MATHEMATICA*.

Again, using (2.3.3) and (2.6.7) in (2.6.11), we obtain

$$P(-e^{-\pi\sqrt{5}}) = \frac{12\sqrt{5}}{10\pi} + \left(-4-\frac{4}{\sqrt{5}}+2\sqrt{\frac{22}{5}+2\sqrt{5}}\right)\sum_{k=0}^{\infty}(-1)^k A_k Y_5^k. \tag{2.6.16}$$

From (2.6.15) and (2.6.16), we arrive at (2.6.2).

Proof of (2.6.3). Setting $n = 5$ in (2.3.21), we find that

$$P(-e^{-\pi\sqrt{5}}) = \frac{1}{1-X_5}\sum_{k=0}^{\infty}\{6(1+X_5)k+1+2X_5\}(-1)^k B_k W_5^{2k}. \tag{2.6.17}$$

Using (2.6.7) in (2.6.17), we obtain

$$P(-e^{-\pi\sqrt{5}}) = \frac{1}{4}\sum_{k=0}^{\infty}\left(12\sqrt{5}k + 3\sqrt{5} - 2\right)(-1)^k B_k \left(\frac{1}{4}\right)^k \tag{2.6.18}$$

Again, using (2.3.6) and (2.6.7) in (2.6.11), we obtain

$$P(-e^{-\pi\sqrt{5}}) = \frac{12\sqrt{5}}{10\pi} + \frac{3\sqrt{5}-5}{10}\sum_{k=0}^{\infty}(-1)^k B_k \left(\frac{1}{4}\right)^k \tag{2.6.19}$$

From (2.6.18) and (2.6.19), we arrive at (2.6.3).

2.7 Series corresponding to x_9

Theorem 2.7.1. *If A_k, and B_k, $k \geq 0$, are defined by* (2.3.1), *then*

$$\frac{12^{1/4}}{\pi} = \sum_{k=0}^{\infty}\{(24\sqrt{3} - 36)k + 9\sqrt{3} - 15\}A_k(2 - \sqrt{3})^{4k}, \tag{2.7.1}$$

$$\frac{4}{\pi} = \sum_{k=0}^{\infty}[6\{9\sqrt{2} + 7\sqrt{6} - 3^{1/4}(5\sqrt{3} + 11)\}k + 15\sqrt{6} + 21\sqrt{2} - 3^{1/4}(13\sqrt{3} + 27)] \times (-1)^k A_k \left(\frac{(\sqrt{3}+1)^{1/3}(\sqrt{2} - 3^{1/4})}{2^{2/3}}\right)^{6k}, \tag{2.7.2}$$

$$\frac{16}{\pi\sqrt{3}} = \sum_{k=0}^{\infty}(-1)^k(28k+3)B_k\frac{1}{48^k}. \tag{2.7.3}$$

The identity (2.7.3) was recorded by Ramanujan [51, Eq. (36)], [54, p. 38]. The first identity is due to the Borweins [22], [23] and the second is due to Baruah and Berndt [6]. One of the main differences of our following proof of Theorem 2.7.1 with that of Baruah and Berndt [6] (though details are not given there) is the fact that we make use of the explicit value of $\varphi(e^{-\pi})/\varphi(e^{-3\pi})$ recorded by Ramanujan in his first notebook [52].

At first, we derive an expression for $h_9(e^{-\pi/3}) = 9P(-e^{-3\pi}) - P(-e^{-\pi/3})$.

We recall the following identity from [14, p. 475, Entry 7(i)]. If $f(-q)$ is defined by (2.2.6), then

$$1+3\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-27\sum_{k=1}^{\infty}\frac{kq^{9k}}{1-q^{9k}}=\frac{f^6(-q^3)}{f^2(-q)f^2(-q^9)}\Big\{f^6(-q)+9qf^3(-q)f^3(-q^9)+27q^2f^6(-q^9)\Big\}^{1/3} \tag{2.7.4}$$

Multiplying both sides of (2.7.4) by 8, replacing q by $-q$, and then using the definition of $P(q)$ from (1.0.5), we find that

$$9P(-q^9)-P(-q)=\frac{8f^6(q^3)}{f^2(q)f^2(q^9)}\left\{f^6(q)-9qf^3(q)f^3(q^9)+27q^2f^6(q^9)\right\}^{1/3} \tag{2.7.5}$$

Transcribing the right side of (2.7.5) with the aid of (2.2.7), we obtain

$$\begin{aligned}&9P(-q^9)-P(-q)\\&=\frac{2^{8/3}z^3(q^3)\{x(q^3)(1-x(q^3))\}^{1/4}}{z(q)z(q^9)\{x(q)(1-x(q))x(q^9)(1-x(q^9))\}^{1/12}}\Big\{z^3(q)\{x(q)(1-x(q))\}^{1/4}\\&-9z^{3/2}(q)z^{3/2}(q^9)\{x(q)(1-x(q))x(q^9)(1-x(q^9))\}^{1/8}\\&+27z^3(q^9)\{x(q^9)(1-x(q^9))\}^{1/4}\Big\}^{1/3}\end{aligned} \tag{2.7.6}$$

Now we set $q=e^{-\pi/3}$ in (2.7.6), so that, by (2.3.16), $x(q)=x(e^{-\pi/3})=1-x(e^{-3\pi})=1-x_9$, $x(q^3)=x(e^{-\pi})=x_1=1/2$, $x(q^9)=x_9$, $z(q)=z(e^{-\pi/3})=3z(e^{-3\pi})=3z_9$, $z(q^3)=z(e^{-\pi})=z_1$, and $z(q^9)=z(e^{-3\pi})=z_9$. Accordingly, we deduce that

$$\begin{aligned}9P(-e^{-3\pi})-P(-e^{-\pi/3})&=\frac{4z_1^3(2-\sqrt{3})^{1/3}}{2^{1/6}z_9\{x_9(1-x_9)\}^{1/12}}\\&=\frac{4(2-\sqrt{3})^{1/3}}{2^{1/6}\{x_9(1-x_9)\}^{1/12}}\left(\frac{z_1}{z_9}\right)^3 z_9^2.\end{aligned} \tag{2.7.7}$$

But, from Theorem 9.9 of [15, pp. 290–291], we have

$$x_9=\frac{1}{2}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4\left(\sqrt{4+2\sqrt{3}}-\sqrt{3+2\sqrt{3}}\right)^2 \tag{2.7.8}$$

so that

$$x_9(1-x_9)=\frac{1}{4}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^8=\frac{1}{4}\left(2-\sqrt{3}\right)^4 \tag{2.7.9}$$

Employing (2.7.9) in (2.7.7), we find that

$$9P(-e^{-3\pi}) - P(-e^{-\pi/3}) = 4\left(\frac{z_1}{z_9}\right)^3 z_9^2. \tag{2.7.10}$$

On page 284 of his first notebook, Ramanujan [52] recorded an explicit value for $\varphi(e^{-\pi})/\varphi(e^{-3\pi})$, i.e., by (2.2.2) and (1.0.8), for $(z_1/z_9)^{1/2}$. Berndt and Chan [17] gave three proofs of Ramanujan's evaluation, which were also reproduced in Chapter 35 of Berndt's book [15]. J. Yi [56] also established this value among many others. Here we find an alternative proof of the explicit value for the same. To do so, we first recast Entry 5(v) of [14, p. 230] by replacing q by q^3 to obtain

$$\frac{z(q^3)}{z(q^9)} = \left\{1 + 4\left(\frac{x^3(q^9)(1-x(q^9))^3}{x(q^3)(1-x(q^3))}\right)^{1/8}\right\}^{1/2} \tag{2.7.11}$$

Setting $q = e^{-\pi/3}$ in (2.7.11) and noting that $x(q^3) = x(e^{-\pi}) = x_1 = 1/2$, $x(q^9) = x_9$, $z(q^3) = z(e^{-\pi}) = z_1$, and $z(q^9) = z(e^{-3\pi}) = z_9$, we deduce that

$$\frac{z_1}{z_9} = \left\{1 + 4 \cdot 2^{1/4}(x_9(1-x_9))^{3/8}\right\}^{1/2} \tag{2.7.12}$$

With the aid of (2.7.9), we deduce from (2.7.12) that

$$\frac{z_1}{z_9} = \left\{1 + 2^{3/2}(2-\sqrt{3})^{3/2}\right\}^{1/2} = \left\{1 + (\sqrt{3}-1)^3\right\}^{1/2} = 3^{1/4} \cdot \frac{3-\sqrt{3}}{\sqrt{2}}. \tag{2.7.13}$$

Now, employing (2.7.13) in (2.7.10), we arrive at

$$9P(-e^{-3\pi}) - P(-e^{-\pi/3}) = 6\sqrt{2}\left(9 - 5\sqrt{3}\right) \cdot 3^{3/4} z_9^2. \tag{2.7.14}$$

Proof of (2.7.1). Setting $n = 9$, in (2.3.30), we obtain

$$9P(-e^{-3\pi}) + P(-e^{-\pi/3}) = \frac{36}{\pi}. \tag{2.7.15}$$

From (2.7.14) and (2.7.15), we arrive at

$$P(-e^{-3\pi}) = \frac{2}{\pi} + \frac{\sqrt{2}}{3}\left(9 - 5\sqrt{3}\right) \cdot 3^{3/4} z_9^2. \tag{2.7.16}$$

Next, setting $n = 9$ in (2.3.17), we find that

$$P(-e^{-3\pi}) = (1 - 2x_9)\sum_{k=0}^{\infty}(6k+1)A_k X_9^k. \tag{2.7.17}$$

Using (2.7.8) and (2.7.9) in (2.7.17), we obtain

$$P(-e^{-3\pi}) = 2\sqrt{-24 + 14\sqrt{3}}\sum_{k=0}^{\infty}(6k+1)A_k(2-\sqrt{3})^{4k}. \tag{2.7.18}$$

From (2.7.16), (2.7.18), and (2.3.2), we easily deduce (2.7.1).

Proof of (2.7.2). Setting $n = 9$ in (2.3.18), we find that

$$P(-e^{-3\pi}) = \frac{1}{1-x_9}\sum_{k=0}^{\infty}\{6(1+x_9)k + 1 + 4x_9\}(-1)^k A_k Y_9^k. \tag{2.7.19}$$

Using (2.7.8) in (2.7.19), we obtain

$$\begin{aligned} P(-e^{-3\pi}) = \sum_{k=0}^{\infty}\{6(387 + 224\sqrt{3} - 8\sqrt{4680 + 2702\sqrt{3}})k \\ + 966 + 560\sqrt{3} - 20\sqrt{4680 + 2702\sqrt{3}}\}(-1)^k A_k Y_9^k, \end{aligned} \tag{2.7.20}$$

where $Y_9 = \left(\frac{(\sqrt{3}+1)^{1/3}(\sqrt{2}-3^{1/4})}{2^{2/3}}\right)^6$.

Again, using (2.3.4) and (2.7.8) in (2.7.16), we obtain

$$P(-e^{-3\pi}) = \frac{2}{\pi} + 2\sqrt{6 + 4\sqrt{3} - 4\sqrt{2}\times 3^{3/4}}\sum_{k=0}^{\infty}(-1)^k A_k Y_9^k. \tag{2.7.21}$$

From (2.7.20) and (2.7.21), we arrive at (2.7.2).

Proof of (2.7.3). Setting $n = 9$ in (2.3.21), we find that

$$P(-e^{-3\pi}) = \frac{1}{1-X_9}\sum_{k=0}^{\infty}\{6(1+X_9)k + 1 + 2X_9\}(-1)^k B_k W_9^{2k}. \tag{2.7.22}$$

Using (2.7.8) in (2.7.22), we obtain

$$P(-e^{-3\pi}) = \sum_{n=0}^{\infty}\left(\frac{7\sqrt{3}}{2}k + \frac{7\sqrt{3}-4}{8}\right)(-1)^k B_k \frac{1}{48^k}. \tag{2.7.23}$$

Again, using (2.3.6) and (2.7.8) in (2.7.16), we obtain

$$P(-e^{-3\pi}) = \frac{2}{\pi} + \frac{\sqrt{3}-1}{2}\sum_{k=0}^{\infty}(-1)^k B_k\left(\frac{1}{48^k}\right). \tag{2.7.24}$$

From (2.7.23) and (2.7.24), we arrive at (2.7.3).

2.8 Series corresponding to x_{25}

Theorem 2.8.1. *If A_k, and B_k, $k \geq 0$, are defined by* (2.3.1), *then*

$$\frac{5^{1/4}}{\pi} = \sum_{k=0}^{\infty}\{(540\sqrt{5}-1200)k + 235\sqrt{5} - 525\}A_k(\sqrt{5}-2)^{8k}, \tag{2.8.1}$$

$$\frac{4}{5\pi} = \sum_{k=0}^{\infty}(-1)^k(644k+41)B_k\frac{1}{(72\sqrt{5})^{2k+1}}. \tag{2.8.2}$$

The first series is due to Borwein and Borwein [23]. The second was recorded by Ramanujan [51, Eq. (38)], [54, p. 38]. As in Section 2.7, in the following proof of Theorem 2.8.1, we also reprove and use the explicit value of $\varphi^2(e^{-\pi})/\varphi^2(e^{-5\pi})$ recorded by Ramanujan in his first notebook [52].

First we derive an expression for $h_{25}(e^{-\pi/5}) = 25P(-e^{-5\pi}) - P(-e^{-\pi/5})$.

We recall from [14, p. 475, Entry 7(iii)], that

$$1+\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} - 25\sum_{k=1}^{\infty}\frac{kq^{25k}}{1-q^{25k}} = \frac{f^5(-q^5)}{f(-q)f(-q^{25})}\{f^2(-q)+2qf(-q)f(-q^{25}) + 5q^2f^2(-q^{25})\}^{1/2}. \tag{2.8.3}$$

Multiplying both sides of (2.8.3) by 24, replacing q by $-q$, and then using the definition of $P(q)$ from (1.0.5), we find that

$$25P(-q^{25}) - P(-q) = \frac{24f^5(q^5)}{f(q)f(q^{25})}\{f^2(q) - 2qf(q)f(q^{25}) + 5q^2f^2(q^{25})\}^{1/2}. \tag{2.8.4}$$

Transcribing the right side of (2.8.4) with the aid of (2.2.7), and then following a similar method as in the derivation of (2.7.7) from (2.7.5), but this time with $q = e^{-\pi/5}$, we arrive at

$$25P(-e^{-5\pi}) - P(-e^{-\pi/5}) = \frac{24\cdot 5^{1/4}(\sqrt{5}-1)^{1/2}}{2^{7/12}\sqrt{5}\{x_{25}(1-x_{25})\}^{1/24}}\left(\frac{z_1}{z_{25}}\right)^{5/2}z_{25}^2. \tag{2.8.5}$$

But, from Theorem 9.9 of [15, p. 291], we note that

$$x_{25} = \frac{1}{2}\left(161-72\sqrt{5}\right)\left(\sqrt{\frac{5+\sqrt{5}}{4}} - \sqrt{\frac{1+\sqrt{5}}{4}}\right)^8, \tag{2.8.6}$$

so that

$$x_{25}(1-x_{25}) = \frac{1}{4}\left(161-72\sqrt{5}\right)^2 = \frac{1}{4}\left(\sqrt{5}-2\right)^8 = \frac{1}{4}\left(\frac{\sqrt{5}-1}{2}\right)^{24}. \tag{2.8.7}$$

Employing (2.8.7) in (2.8.5), we find that

$$25P(-e^{-5\pi}) - P(-e^{-\pi/5}) = \frac{24\sqrt{2}\cdot 5^{1/4}}{\sqrt{5}(\sqrt{5}-1)^{1/2}}\left(\frac{z_1}{z_{25}}\right)^{5/2} z_{25}^2. \tag{2.8.8}$$

Next, we need to know the explicit value of the expression z_1/z_{25}. But, on page 285 of his first notebook, Ramanujan [52] recorded, without proof, an explicit value for z_1/z_{25}, equivalently, by (2.2.2) and (1.0.8), for $\varphi^2(e^{-\pi})/\varphi^2(e^{-5\pi})$. Berndt and Chan [17] found a proof of this value by applying Ramanujan's modular equations and an explicit value of his class invariant. The same proof can be found in [15, pp. 328–329]. A different proof was found by Yi [56]. Now we present a method of explicit evaluation of z_1/z_{25}, which is slightly different from that in [17] and [15]. To this end, we recall from Entry 13(iv) of [14, p. 281], with q replaced by q^5, that

$$\frac{z(q^5)}{z(q^{25})} = 1 + 2^{4/3}\left(\frac{x^5(q^{25})(1-x(q^{25}))^5}{x(q^5)(1-x(q^5))}\right)^{1/24}. \tag{2.8.9}$$

Setting $q = e^{-\pi/5}$ in (2.8.9) and noting that $x(q^5) = x(e^{-\pi}) = x_1 = 1/2$, $x(q^{25}) = x_{25}$, $z(q^{25}) = z(e^{-\pi}) = z_1$, and $z(q^{25}) = z(e^{-5\pi}) = z_{25}$, we obtain

$$\frac{z_1}{z_{25}} = 1 + 2\cdot 2^{5/12}(x_{25}(1-x_{25}))^{5/24}, \tag{2.8.10}$$

which, by (2.8.7) reduces to

$$\frac{z_1}{z_{25}} = 5(\sqrt{5}-2). \tag{2.8.11}$$

Using (2.8.11) in (2.8.8), we find that

$$25P(-e^{-5\pi}) - P(-e^{-\pi/5}) = 300\cdot 5^{1/4}(13\sqrt{5}-29)z_{25}^2. \tag{2.8.12}$$

Proof of (2.8.1).

Setting $n = 25$, in (2.3.30), we obtain

$$25P(-e^{-5\pi}) + P(-e^{-\pi/5}) = \frac{60}{\pi}. \tag{2.8.13}$$

From (2.8.12) and (2.8.13), we arrive at

$$P(-e^{-5\pi}) = \frac{6}{5\pi} + 6 \cdot 5^{1/4}(13\sqrt{5} - 29)z_{25}^2. \tag{2.8.14}$$

Next, setting $n = 25$ in (2.3.17), we find that

$$P(-e^{-5\pi}) = (1 - 2x_{25}) \sum_{k=0}^{\infty} (6k+1) A_k X_{25}^k. \tag{2.8.15}$$

Using (2.8.6) and (2.8.7) in (2.8.15), we obtain

$$P(-e^{-5\pi}) = 12\sqrt{161\sqrt{5} - 360} \sum_{k=0}^{\infty} (6k+1) A_k (\sqrt{5} - 2)^{8k}. \tag{2.8.16}$$

From (2.8.14), (2.8.16), and (2.3.2), we easily deduce (2.8.1).

Proof of (2.8.2). Setting $n = 25$ in (2.3.21), we find that

$$P(-e^{-5\pi}) = \frac{1}{1 - X_{25}} \sum_{k=0}^{\infty} \{6(1 + X_{25})k + 1 + 2X_{25}\}(-1)^k B_k W_{25}^{2k}. \tag{2.8.17}$$

Using (2.8.6) in (2.8.17), we obtain

$$P(-e^{-5\pi}) = \sum_{n=0}^{\infty} \left(\frac{161}{12\sqrt{5}} k + \frac{161}{48\sqrt{5}} - \frac{1}{2} \right) (-1)^k B_k \left(\frac{1}{72\sqrt{5}} \right)^{2k}. \tag{2.8.18}$$

Again, using (2.3.6) and (2.8.6) in (2.8.14), we obtain

$$P(-e^{-5\pi}) = \frac{6}{5\pi} + 25(\sqrt{5} - 1) \sum_{k=0}^{\infty} (-1)^k B_k \left(\frac{1}{72\sqrt{5}} \right)^{2k}. \tag{2.8.19}$$

From (2.8.18) and (2.8.19), we arrive at (2.8.2).

Remark 2.8.1. *Proceeding as in [5], but working with $h_n(q)$ instead of $f_n(q)$, we can also derive all the series for $1/\pi^2$ in [5] that correspond to singular moduli x_3, x_5, x_7, x_9, and x_{25}.*

Chapter 3

Series for $1/\pi$ corresponding to Ramanujan's singular moduli x_n, for some non integral values of n

3.1 Introduction

In Chapter 2 we prove several known series for $1/\pi$ by employing the representations for $h_n(\pm q)$ along with a transformation formula for $P(-q)$ and certain hypergeometric series identities. All these series for $1/\pi$ correspond to Ramanujan's singular moduli x_n, for some integral values of n. We notice that Ramanujan's singular moduli x_n, for non integral values of n were not used by the previous authors to derive Ramanujan-type series for $1/\pi$, though experts like J. M. and P. B. Borwein might have known it. This chapter is devoted to proving several new as well as known series for $1/\pi$ by using Ramanujan's singular moduli x_n, for some non integral values of n along with the ideas describe in Chapter 2. The main difference of our method with that of [4], [6] and [8] is that, here we need to use two different representations of $f_n(q)$ for certain q.

In the next section, we recall some important lemmas which will be needed in our series derivation for $1/\pi$. In Sections 3.3-3.6, we present the series for $1/\pi$ corresponding to the singular moduli $x_{3/2}$, $x_{5/2}$, $x_{9/2}$ and $x_{5/3}$, respectively. Again, in the final two Sections of this Chapter we present the series for $1/\pi$ corresponding to the cubic singular moduli $x_{5/2}$ and $x_{3/2}$.

3.2 Some important lemmas

In the following lemma, we recall the representation for $P(q^2)$ given in [14, p. 120, Entry 9(iv)].

Lemma 3.2.1. *We have*

$$P(q^2) = (1-2x)z^2 + 6x(1-x)z\frac{dz}{dx}. \tag{3.2.1}$$

Differentiating each of (2.3.2)–(2.3.4) and (2.3.7) with respect to x, employing (3.2.1), and then setting $q = e^{-\pi\sqrt{n}}$ in the resulting identities, the results in the following lemma were deduced in [6, p. 22].

Lemma 3.2.2. *We have*

$$P(e^{-2\pi\sqrt{n}}) = (1-2x_n)\sum_{k=0}^{\infty}(3k+1)A_kX_n^k, \tag{3.2.2}$$

$$= \frac{1+x_n}{1-x_n}\sum_{k=0}^{\infty}(3k+1)(-1)^kA_kY_n^k, \tag{3.2.3}$$

$$= \frac{2-x_n}{2\sqrt{1-x_n}}\sum_{k=0}^{\infty}(6k+1)(-1)^kA_kU_n^k, \tag{3.2.4}$$

$$= \frac{\sqrt{1-X_n}(X_n+8)}{(4-X_n)^{3/2}}\sum_{k=0}^{\infty}(6k+1)C_kL_n^k, \tag{3.2.5}$$

where

$$X_n := 4x_n(1-x_n),\ Y_n := \frac{4x_n}{(1-x_n)^2},\ U_n := \frac{x_n^2}{4(1-x_n)}\ \text{and}\ L_n := \frac{27X_n^2}{(4-X_n)^3}. \tag{3.2.6}$$

In the final lemma of this section, we recall a transformation formula for $P(q)$ from [6, Eq. (3.26)].

Lemma 3.2.3. *We have*

$$nP(e^{-2\pi\sqrt{n}}) + P(e^{-2\pi/\sqrt{n}}) = \frac{6\sqrt{n}}{\pi}. \tag{3.2.7}$$

By combining (3.2.7) and the representations for $f_n(q) := nP(q^{2n}) - P(q^2)$ with $q = e^{-\pi/\sqrt{n}}$ and $q = e^{-\pi\sqrt{n}}$, for each $n = 3/2, 5/2, 9/2$ and $5/3$, along with the identities (2.3.2)–(2.3.7) and (3.2.2)–(3.2.5), we can derive certain known as well as new series representations for $1/\pi$. In the process, we also find the values of the singular moduli $x_{3/2}$, $x_{5/2}$, $x_{9/2}$ and $x_{5/3}$.

3.3 Series corresponding to $x_{3/2}$

Theorem 3.3.1. *If* A_k, $k \geq 0$, *is defined by* (2.3.1), *then*

$$\begin{aligned}\frac{6\sqrt{6}}{\pi} &= \sum_{k=0}^{\infty}\{18(69 - 48\sqrt{2} - 40\sqrt{3} + 28\sqrt{6})k \\ &\quad + 540 - 378\sqrt{2} - 312\sqrt{3} + 219\sqrt{6}\}A_k {X_{3/2}}^k, \end{aligned} \tag{3.3.1}$$

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k\{(12\sqrt{2} - 12)k + 4\sqrt{2} - 5\}A_k(\sqrt{2} - 1)^{4k}, \tag{3.3.2}$$

where $X_{3/2} = 4(2\sqrt{3} - 2 - \sqrt{2})(2 - \sqrt{2})(2 - \sqrt{3})^2(\sqrt{3} + \sqrt{2})^4$.

The series (3.3.2) is due to Baruah and Berndt [6] and (3.3.1) seems to be new.

Proof of (3.3.1). Setting $n = 3/2$ in (3.2.7), we have

$$3P(e^{-2\pi\sqrt{3/2}}) + 2P(e^{-2\pi\sqrt{2/3}}) = \frac{6\sqrt{6}}{\pi} \tag{3.3.3}$$

Next, we derive an expression for $3P(e^{-2\pi\sqrt{3/2}}) - 2P(e^{-2\pi/\sqrt{2/3}})$ in terms of $z_{3/2}^2$. To this end, we recall, from [14, p. 214, Entry 24(ii)], the following modular equations of degree 2.

We have

$$\frac{z(q)}{z(q^2)}\sqrt{1 - x(q)} = 1 - \sqrt{x(q^2)} \tag{3.3.4}$$

and

$$\frac{2z(q^2)}{z(q)}\sqrt{x(q^2)} = 1 - \sqrt{1-x(q)}. \tag{3.3.5}$$

Combining (3.3.4) and (3.3.5), we also find that

$$\sqrt{1-x(q)} = \frac{1-\sqrt{x(q^2)}}{1+\sqrt{x(q^2)}} \tag{3.3.6}$$

Setting $q = e^{-\pi\sqrt{3/2}}$ in (3.3.6), (3.3.4) and then using $x_6 = (2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2$ from [15, p. 282] and with additional aid from (2.3.16), we deduce that

$$x_{2/3} = (2-\sqrt{3})^2(\sqrt{3}+\sqrt{2})^2 \tag{3.3.7}$$

and

$$z_{2/3} = \sqrt{3}(\sqrt{2}+1)(\sqrt{3}-\sqrt{2})z_6. \tag{3.3.8}$$

Now, we recall from [14, p. 460, Entry 3(iii)] that

$$1+12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 36\sum_{k=1}^{\infty}\frac{kq^{6k}}{1-q^{6k}}$$
$$= \frac{1}{2}\varphi^2(q)\varphi^2(q^3)\left(1+(x(q)x(q^3))^{1/2} - \{(1-x(q))(1-x(q^3))\}^{1/2}\right). \tag{3.3.9}$$

With the aid of (1.0.5) and (2.2.2), we rewrite (3.3.9) in the form

$$3P(q^6) - P(q^2) = z(q)z(q^3)\left(1+(x(q)x(q^3))^{1/2} - \{(1-x(q))(1-x(q^3))\}^{1/2}\right). \tag{3.3.10}$$

Setting $q = e^{-\pi\sqrt{2/3}}$ in (3.3.10) and then employing (2.3.16), (3.3.7) and (3.3.8), we find that

$$3P(e^{-2\pi\sqrt{6}}) - P(e^{-2\pi\sqrt{2/3}}) = 3\sqrt{3}(2-\sqrt{3})\frac{\sqrt{2}-1}{\sqrt{3}-\sqrt{2}}z_{3/2}^2. \tag{3.3.11}$$

Again, from [14, p. 127, Entry 13(ix)], we have

$$2P(q^4) - P(q^2) = \frac{z^2(q)}{2}(2-x(q)). \tag{3.3.12}$$

Setting $q = e^{-\pi\sqrt{3/2}}$ in (3.3.12) and then using (3.3.7), we obtain

$$2P(e^{-2\pi\sqrt{6}}) - P(e^{-2\pi\sqrt{3/2}}) = \frac{1}{2}\left\{1 + (7 - 4\sqrt{3})(5 + 2\sqrt{6})\right\} z_{3/2}^2. \tag{3.3.13}$$

Multiplying (3.3.11) by 2 and (3.3.13) by 3 and then subtracting the resulting identities, we arrive at our desired expression

$$3P(e^{-2\pi\sqrt{3/2}}) - 2P(e^{-2\pi\sqrt{2/3}}) = \left(-126 + 90\sqrt{2} + 72\sqrt{3} - 51\sqrt{6}\right) z_{3/2}^2. \tag{3.3.14}$$

Next, adding (3.3.14) and (3.3.3), we obtain

$$6P(e^{-2\pi\sqrt{3/2}}) = \frac{6\sqrt{6}}{\pi} + \left(-126 + 90\sqrt{2} + 72\sqrt{3} - 51\sqrt{6}\right) z_{3/2}^2, \tag{3.3.15}$$

which we can rewrite, with the help of (2.3.2), in the form

$$6P(e^{-2\pi\sqrt{3/2}}) = \frac{6\sqrt{6}}{\pi} + \left(-126 + 90\sqrt{2} + 72\sqrt{3} - 51\sqrt{6}\right) \sum_{n=0}^{\infty} A_k X_{3/2}^k, \tag{3.3.16}$$

where $X_{3/2} = 4x_{3/2}(1 - x_{3/2})$.

On the other hand, setting $n = 3/2$ in (3.2.2), we find that

$$P(e^{-2\pi\sqrt{3/2}}) = (1 - 2x_{3/2}) \sum_{n=0}^{\infty} (3k + 1) A_k X_{3/2}^k. \tag{3.3.17}$$

From (3.3.16) and (3.3.17), we readily deduce (3.3.1).

Proof of (3.3.2). With the help of (2.3.4), we can rewrite (3.3.15) as

$$6P(e^{-2\pi\sqrt{3/2}}) = \frac{6\sqrt{6}}{\pi} + \left(-126 + 90\sqrt{2} + 72\sqrt{3} - 51\sqrt{6}\right) \times \frac{1}{\sqrt{1 - x_{3/2}}} \sum_{n=0}^{\infty} (-1)^k A_k U_{3/2}^k, \tag{3.3.18}$$

where $U_{3/2} = x_{3/2}^2 / \left(4(1 - x_{3/2})\right)$.

Again, setting $n = 3/2$ in (3.2.4), we have

$$P(e^{-2\pi\sqrt{3/2}}) = \frac{2 - x_{3/2}}{2\sqrt{1 - x_{3/2}}} \sum_{n=0}^{\infty} (6k + 1)(-1)^k A_k U_{3/2}^k = \sqrt{3}(2 - \sqrt{2}) \sum_{n=0}^{\infty} (6k + 1)(-1)^k A_k U_{3/2}^k. \tag{3.3.19}$$

From (3.3.18) and (3.3.19), we easily arrive at (3.3.2).

3.4 Series corresponding to $x_{5/2}$

Theorem 3.4.1. *If A_k, C_k, $k \geq 0$, are defined by* (2.3.1), *then*

$$\frac{2\sqrt{10}}{\pi} = \sum_{k=0}^{\infty}\{10(24\sqrt{2} - 6\sqrt{10} + 15)(19 - 6\sqrt{10})k + 2900 + 2050\sqrt{2} - 1296\sqrt{5} - 917\sqrt{10}\}A_k X_{5/2}^k, \tag{3.4.1}$$

$$\frac{6\sqrt{10}}{\pi} = \sum_{k=0}^{\infty}(-1)^k\{90(4\sqrt{10} - 4\sqrt{2} + 7)(17 - 12\sqrt{2})k + 8700 - 6150\sqrt{2} - 3888\sqrt{5} + 2751\sqrt{10}\}A_k Y_{5/2}^k, \tag{3.4.2}$$

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k\{(60 - 24\sqrt{5})k + 23 - 10\sqrt{5}\}A_k(\sqrt{5} - 2)^{4k}, \tag{3.4.3}$$

$$\frac{638\sqrt{435}}{\pi\sqrt{155 - 36\sqrt{5}}} = \sum_{k=0}^{\infty}\{(3708 + 432\sqrt{5})k + 299 + 72\sqrt{5}\}C_k\left(\frac{24635 + 11016\sqrt{5}}{609725}\right)^k, \tag{3.4.4}$$

where

$$X_{5/2} = 4(2 + 6\sqrt{10} - 12\sqrt{2})(17 + 12\sqrt{2})(19 - 6\sqrt{10})^2$$

and

$$Y_{5/2} = (19 - 6\sqrt{10})(\sqrt{10} + 3)^4(2 + 6\sqrt{10} - 12\sqrt{2})(3\sqrt{2} - 4)^4.$$

The identity (3.4.3) is due to Baruah and Berndt [6]. The remaining three identities seem to be new.

Proof of(3.4.1). Setting $q = e^{-\pi\sqrt{5/2}}$ in (3.3.6), (3.3.4) and using $x_{10} = (\sqrt{10} - 3)^2(3 - 2\sqrt{2})^2$ from [15, p. 282] and (2.3.16), we find that

$$x_{5/2} = (19 - 6\sqrt{10})(2 + 6\sqrt{10} - 12\sqrt{2}) \tag{3.4.5}$$

and

$$z_{5/2} = (3\sqrt{2} - 4)(\sqrt{5} + 2)z_{10}. \tag{3.4.6}$$

Now, recall from [14, p. 464, Entry 4(iii)] that

$$\begin{aligned}1+6\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-30\sum_{k=1}^{\infty}\frac{kq^{10k}}{1-q^{10k}}\\ =\frac{1}{4}\varphi^2(q)\varphi^2(q^5)\left(3+(x(q)x(q^5))^{1/2}+\left\{(1-x(q))(1-x(q^5))\right\}^{1/2}\right)\\ \times\left(\frac{1}{2}\left(1+\sqrt{x(q)x(q^5)}+\sqrt{(1-x(q))(1-x(q^5))}\right)\right)^{1/2}.\end{aligned} \tag{3.4.7}$$

With the aid of (1.0.5) and (2.2.2), we rewrite (3.4.7) in the form

$$\begin{aligned}5P(q^{10})-P(q^2)\\ =z(q)z(q^5)\left(3+(x(q)x(q^5))^{1/2}+\{(1-x(q))(1-x(q^5))\}^{1/2}\right)\\ \times\left(\frac{1}{2}\left(1+\sqrt{x(q)x(q^5)}+\sqrt{(1-x(q))(1-x(q^5))}\right)\right)^{1/2}.\end{aligned} \tag{3.4.8}$$

Setting $q=e^{-\pi\sqrt{2/5}}$ in (3.4.8), and then employing (2.3.16), (3.4.5) and (3.4.6), we find that

$$5P(e^{-2\pi\sqrt{10}})-P(e^{-2\pi\sqrt{2/5}})=3\sqrt{5}\left(3+2\sqrt{2}\right)\left(\sqrt{5}-2\right)\left(4\sqrt{5}-6\sqrt{2}\right)z_{5/2}^2. \tag{3.4.9}$$

Again, setting $q=e^{-\pi\sqrt{5/2}}$ in (3.3.12), we obtain

$$2P(e^{-2\pi\sqrt{10}})-P(e^{-2\pi\sqrt{5/2}})=\left(162+114\sqrt{2}-51\sqrt{10}-72\sqrt{5}\right)z_{5/2}^2. \tag{3.4.10}$$

Multiplying (3.4.9) by 2 and (3.4.10) by 5, and then subtracting the resulting identities, we find that

$$5P(e^{-2\pi\sqrt{5/2}})-2P(e^{-2\pi\sqrt{2/5}})=3(-750-532\sqrt{2}+336\sqrt{5}+237\sqrt{10})z_{5/2}^2. \tag{3.4.11}$$

On the other hand, setting $n=5/2$ in (3.2.7), we obtain

$$5P(e^{-2\pi\sqrt{5/2}})+2P(e^{-2\pi\sqrt{2/5}})=\frac{6\sqrt{10}}{\pi}. \tag{3.4.12}$$

Adding (3.4.11) and (3.4.12), we arrive at

$$P(e^{-2\pi\sqrt{5/2}})=\frac{6}{\pi\sqrt{10}}+\frac{3}{10}\left(-750-532\sqrt{2}+336\sqrt{5}+237\sqrt{10}\right)z_{5/2}^2, \tag{3.4.13}$$

which can be rewritten, with the help of (2.3.2), as

$$P(e^{-2\pi\sqrt{5/2}}) = \frac{6}{\pi\sqrt{10}} + \frac{3}{10}\left(-750 - 532\sqrt{2} + 336\sqrt{5} + 237\sqrt{10}\right)\sum_{n=0}^{\infty} A_k X_{5/2}^k, \tag{3.4.14}$$

where $X_{5/2} = 4x_{5/2}(1 - x_{5/2})$.

Now, setting $n = 5/2$ in (3.2.2), we have

$$P(e^{-2\pi\sqrt{5/2}}) = (1 - 2x_{5/2})\sum_{n=0}^{\infty}(3k+1)A_k X_{5/2}^k. \tag{3.4.15}$$

From (3.4.14) and (3.4.15), we readily deduce (3.4.1).

Proof of (3.4.2). With the help of (2.3.3), we can rewrite (3.4.13) as

$$\begin{aligned} P(e^{-2\pi\sqrt{5/2}}) = {} & \frac{6}{\pi\sqrt{10}} + \frac{3}{10}\left(-750 - 532\sqrt{2} + 336\sqrt{5} + 237\sqrt{10}\right) \\ & \times \frac{1}{1 - x_{5/2}}\sum_{n=0}^{\infty}(-1)^k A_k Y_{5/2}^k, \end{aligned} \tag{3.4.16}$$

where $Y_{5/2} = 4x_{5/2}/(1 - x_{5/2})^2$.

Again, setting $n = 5/2$ in (3.2.3), we obtain

$$\begin{aligned} P(e^{-2\pi\sqrt{5/2}}) &= \frac{1 + x_{5/2}}{1 - x_{5/2}}\sum_{n=0}^{\infty}(3k+1)(-1)^k A_k Y_{5/2}^k \\ &= 3(7 + 4\sqrt{10} - 4\sqrt{2})(17 - 12\sqrt{2})\sum_{n=0}^{\infty}(3k+1)(-1)^k A_k Y_{5/2}^k. \end{aligned} \tag{3.4.17}$$

From (3.4.16), (3.4.17) and the value of $x_{5/2}$ from (3.4.5), we easily arrive at (3.4.2).

Proof of (3.4.3). With the help of (2.3.4), we write (3.4.13) as

$$\begin{aligned} P(e^{-2\pi\sqrt{5/2}}) = {} & \frac{6}{\pi\sqrt{10}} + \frac{3}{10}\left(-750 - 532\sqrt{2} + 336\sqrt{5} + 237\sqrt{10}\right) \\ & \times \frac{1}{\sqrt{1 - x_{5/2}}}\sum_{n=0}^{\infty}(6k+1)(-1)^k A_k U_{5/2}^k, \end{aligned} \tag{3.4.18}$$

where $U_{5/2} = x_{5/2}^2/(4(1 - x_{5/2}))$.

Now, setting $n = 5/2$ in (3.2.4), we obtain

$$\begin{aligned} P(e^{-2\pi\sqrt{5/2}}) &= \frac{2 - x_{5/2}}{2\sqrt{1 - x_{5/2}}} \sum_{n=0}^{\infty} (6k+1)(-1)^k A_k U_{5/2}^k \\ &= \frac{1}{2\sqrt{2}} (\sqrt{10} - 3)(3\sqrt{2} - 4)(36 + 12\sqrt{2} + 6\sqrt{10}) \\ &\quad \times \sum_{n=0}^{\infty} (6k+1)(-1)^k A_k U_{5/2}^k. \end{aligned} \tag{3.4.19}$$

From (3.4.18) and (3.4.19), we easily arrive at (3.4.3).

Proof of (3.4.4). With the help of (2.3.7), we can rewrite (3.4.13) as

$$\begin{aligned} 10P(e^{-2\pi\sqrt{5/2}}) &= \frac{6\sqrt{10}}{\pi} + 3(-750 - 532\sqrt{2} + 336\sqrt{5} + 237\sqrt{10}) \\ &\quad \times \frac{2}{\sqrt{4 - X_{5/2}}} \sum_{n=0}^{\infty} C_k L_{5/2}^k, \end{aligned} \tag{3.4.20}$$

where $X_{5/2} = 4x_{5/2}(1 - x_{5/2})$ and $L_{5/2} = 27X_{5/2}^2/(4 - X_{5/2})^3$.

Again, setting $n = 5/2$ in (3.2.5), we find that

$$\begin{aligned} P(e^{-2\pi\sqrt{5/2}}) &= \frac{\sqrt{1 - X_{5/2}}\,(X_{5/2} + 8)}{(4 - X_{5/2})^{3/2}} \sum_{n=0}^{\infty} (6k+1) C_k L_{5/2}^k \\ &= \frac{3}{145} \sqrt{\frac{6(10835 - 204\sqrt{5})}{29}} \sum_{n=0}^{\infty} (6k+1) C_k L_{5/2}^k. \end{aligned} \tag{3.4.21}$$

From (3.4.20) and (3.4.21), we arrive at (3.4.4) to finish the proof.

3.5 Series corresponding to $x_{9/2}$

Theorem 3.5.1. *If A_k, $k \geq 0$, is defined by* (2.3.1), *then*

$$\begin{aligned} \frac{6\sqrt{18}}{\pi} &= \sum_{k=0}^{\infty} \{(1037070 - 733320\sqrt{2} + 598752\sqrt{3} - 423360\sqrt{6})k \\ &\quad - 27(-17764 + 12561\sqrt{2} - 10256\sqrt{3} + 7252\sqrt{6})\} A_k {X_{9/2}}^k \end{aligned} \tag{3.5.1}$$

and

$$\frac{2}{\pi} = \sum_{k=0}^{\infty} (-1)^k (84k + 21 - 6\sqrt{6}) A_k (\sqrt{3} - \sqrt{2})^{8k+2}, \tag{3.5.2}$$

where $X_{9/2} = 4(5\sqrt{2} - 7)^4 (7 + 4\sqrt{3})^2 (2 + 70\sqrt{2} - 56\sqrt{3})$.

The identity (3.5.2) is due to Baruah and Berndt [6]. Identity (3.5.1) is new.

Proof of (3.5.1). Setting $q = e^{-\pi\sqrt{9/2}}$ in (3.3.6), (3.3.4), employing the value $x_{18} = (5\sqrt{2}-7)^2(7-4\sqrt{3})^2$ from [15, p. 282] and (2.3.16), we obtain

$$x_{9/2} = (5\sqrt{2}-7)^2(2+70\sqrt{2}-56\sqrt{3}) \tag{3.5.3}$$

and

$$z_{9/2} = (7-4\sqrt{3})(5\sqrt{2}+4\sqrt{3})z_{18}. \tag{3.5.4}$$

Now, recall from [14, p. 482, Entry 10(ii)] that

$$\begin{aligned} 1+3\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 27\sum_{k=1}^{\infty}\frac{kq^{18k}}{1-q^{18k}} \\ = \varphi^2(q)\varphi^2(q^9)\Big(\frac{1}{2}\{1+x(q)x(q^9)+(1-x(q))(1-x(q^9))\} \\ -\frac{9}{32}\{1-\sqrt{x(q)x(q^9)}-\sqrt{(1-x(q))(1-x(q^9))}\}^2 \\ +\frac{3}{2}\frac{\{x(q)x(q^9)(1-x(q))(1-x(q^9))\}^{1/2}}{1-(x(q)x(q^9))^{1/2}-\{(1-x(q))(1-x(q^9))\}^{1/2}}\Big)^{1/2}. \end{aligned} \tag{3.5.5}$$

With the aid of (1.0.5) and (2.2.2), we rewrite (3.5.5) in the form

$$\begin{aligned} 9P(q^{18})-P(q^2) = 8z(q)z(q^9)\Big(\frac{1}{2}\{1+x(q)x(q^9)+(1-x(q))(1-x(q^9))\} \\ -\frac{9}{32}\{1-\sqrt{x(q)x(q^9)}-\sqrt{(1-x(q))(1-x(q^9))}\}^2 \\ +\frac{3}{2}\frac{\{x(q)x(q^9)(1-x(q))(1-x(q^9))\}^{1/2}}{1-(x(q)x(q^9))^{1/2}-\{(1-x(q))(1-x(q^9))\}^{1/2}}\Big)^{1/2}. \end{aligned} \tag{3.5.6}$$

Setting $q = e^{-\pi\sqrt{2/9}}$ in (3.5.6), and then employing (2.3.16), (3.5.3) and (3.5.4), we find that

$$9P(e^{-2\pi\sqrt{18}}) - P(e^{-2\pi\sqrt{2/9}}) = 9\sqrt{18}(1188-840\sqrt{2}+686\sqrt{3}-485\sqrt{6})z_{9/2}^2. \tag{3.5.7}$$

Again, setting $q = e^{-\pi\sqrt{9/2}}$ in (3.3.12), we have

$$2P(e^{-2\pi\sqrt{18}}) - P(e^{-2\pi\sqrt{9/2}}) = (4802+2772\sqrt{3}-3395\sqrt{2}-1960\sqrt{6})z_{9/2}^2. \tag{3.5.8}$$

Multiplying (3.5.7) by 2 and (3.5.8) by 9, and then subtracting the resulting identities, we find that

$$9P(e^{-2\pi\sqrt{9/2}}) - 2P(e^{-2\pi\sqrt{2/9}}) = 9(-14882 + 10523\sqrt{2} - 8592\sqrt{3} + 6076\sqrt{6})z_{9/2}^2. \tag{3.5.9}$$

On the other hand, setting $n = 9/2$ in (3.2.7), we have

$$9P(e^{-2\pi\sqrt{9/2}}) + 2P(e^{-2\pi\sqrt{2/9}}) = \frac{6\sqrt{18}}{\pi}. \tag{3.5.10}$$

From (3.5.9) and (3.5.10), we deduce that

$$P(e^{-2\pi\sqrt{9/2}}) = \frac{6}{\pi\sqrt{18}} + \frac{1}{2}\left(-14882 + 10523\sqrt{2} - 8592\sqrt{3} + 6076\sqrt{6}\right) z_{9/2}^2. \tag{3.5.11}$$

Employing (2.3.2) in (3.5.11), we find that

$$P(e^{-2\pi\sqrt{9/2}}) = \frac{6}{\pi\sqrt{18}} + \frac{1}{2}\left(-14882 + 10523\sqrt{2} - 8592\sqrt{3} + 6076\sqrt{6}\right) \sum_{n=0}^{\infty} A_k X_{9/2}^k, \tag{3.5.12}$$

where $X_{9/2} = 4x_{9/2}(1 - x_{9/2}) = 4(5\sqrt{2} - 7)^4(7 + 4\sqrt{3})^2(2 + 70\sqrt{2} - 56\sqrt{3})$.

Again, setting $n = 9/2$ in (3.2.2), we find that

$$\begin{aligned} P(e^{-2\pi\sqrt{9/2}}) &= (1 - 2x_{9/2})\sum_{n=0}^{\infty}(3k+1)A_k X_{9/2}^k \\ &= (-9601 - 6790\sqrt{2} + 5544\sqrt{3} + 3920\sqrt{6})(7 + 4\sqrt{3})^2(5\sqrt{2} - 7)^2 \\ &\quad \times \sum_{n=0}^{\infty}(3k+1)A_k X_{9/2}^k. \end{aligned} \tag{3.5.13}$$

From (3.5.12) and (3.5.13), we readily deduce (3.5.1).

Proof of (3.5.2). Using (2.3.4) in (3.5.11), we have

$$\begin{aligned} P(e^{-2\pi\sqrt{9/2}}) &= \frac{6}{\pi\sqrt{18}} + \frac{1}{2}\left(-14882 + 10523\sqrt{2} - 8592\sqrt{3} + 6076\sqrt{6}\right) \\ &\quad \times \frac{1}{\sqrt{1-x}}\sum_{n=0}^{\infty}(-1)^k A_k U_{9/2}^k, \end{aligned} \tag{3.5.14}$$

where $U_{9/2} = x_{9/2}^2/(4(1 - x_{9/2}))$ with the value of $x_{9/2}$ in (3.5.3).

Again, setting $n = 9/2$ in (3.2.4), we obtain

$$P(e^{-2\pi\sqrt{9/2}}) = \frac{2 - x_{9/2}}{2\sqrt{1 - x_{9/2}}} \sum_{n=0}^{\infty}(6k+1)(-1)^k A_k U_{9/2}^k$$
$$= 7(5\sqrt{2} - 4\sqrt{3}) \sum_{n=0}^{\infty}(6k+1)(-1)^k A_k U_{9/2}^k. \tag{3.5.15}$$

From (3.5.14) and (3.5.15), we easily arrive at (3.5.2).

3.6 Series corresponding to $x_{5/3}$

Theorem 3.6.1. *If A_k, $k \geq 0$, is defined by* (2.3.1), *then*

$$\frac{96}{\pi} = \sum_{k=0}^{\infty}\{(42\sqrt{5} - 30)k + 5\sqrt{5} + 1\} A_k \frac{1}{2^{6k}}\left(\frac{\sqrt{5}+1}{2}\right)^{8k}. \tag{3.6.1}$$

Proof of(3.6.1). At first, we find an explicit value of $x_{5/3}$. To this end, we recall two modular equations of degree 3 from [14, pp. 230–231, Entries 5(ii) and 5(ix)], namely,

$$\left(x(q)x(q^3)\right)^{1/4} + \{(1 - x(q))(1 - x(q^3))\}^{1/4} = 1 \tag{3.6.2}$$

and

$$\{x(q)(1 - x(q^3))\}^{1/2} + \{x(q^3)(1 - x(q))\}^{1/2}$$
$$= 2\{x(q)x(q^3)(1 - x(q))(1 - x(q^3))\}^{1/8}. \tag{3.6.3}$$

Setting $q = e^{-\pi\sqrt{5/3}}$ in (3.6.2) and employing (2.3.16), we find that

$$\left(x_{5/3}x_{15}\right)^{1/4} + \left\{(1 - x_{5/3})(1 - x_{15})\right\}^{1/4} = 1. \tag{3.6.4}$$

Next, define Weber-Ramanujan class invariant G_n by $G_n := (4x_n(1 - x_n))^{-1/24}$, and recall from [15, p. 190] that

$$G_{15} = 2^{-1/12}(\sqrt{5}+1)^{1/3} \text{ and } G_{5/3} = 2^{-1/12}(\sqrt{5}-1)^{1/3}. \tag{3.6.5}$$

Now, squaring both sides of (3.6.4) and then employing (3.6.5), we find that

$$\sqrt{x_{5/3}x_{15}} + \sqrt{(1 - x_{5/3})(1 - x_{15})} = \frac{7}{8}. \tag{3.6.6}$$

In a similar fashion, from (3.6.3), we obtain

$$\sqrt{x_{5/3}(1-x_{15})}+\sqrt{(1-x_{5/3})x_{15}}=\frac{1}{2}. \tag{3.6.7}$$

From (3.6.6) and (3.6.7), with the aid of the value [15, p. 291]

$$x_{15}=\frac{1}{16}\left(\frac{\sqrt{5}-1}{2}\right)^4(2-\sqrt{3})^2(4-\sqrt{15}),$$

we find that

$$x_{5/3}=\frac{1}{352}(7-\sqrt{5})(28+4\sqrt{5}-11\sqrt{3}). \tag{3.6.8}$$

Next, we find a relation between z_{15} and $z_{5/3}$. To this end, we recall from [15, p. 339] that

$$9\frac{\varphi^4(e^{-\pi\sqrt{9n}})}{\varphi^4(e^{-\pi\sqrt{n}})}=1+2\sqrt{2}\,\frac{G_{9n}^3}{G_n^9}, \tag{3.6.9}$$

where φ is as defined in (2.2.3). Setting $n=5/3$ in (3.6.9), and then employing (3.6.5), we obtain

$$9\frac{\varphi^4(e^{-\pi\sqrt{15}})}{\varphi^4(e^{-\pi\sqrt{5/3}})}=\frac{9+3\sqrt{5}}{2}. \tag{3.6.10}$$

Using (2.2.2) in (3.6.10), we have the desired relation

$$z_{5/3}=\frac{\sqrt{3}}{2}(\sqrt{5}-1)z_{15}. \tag{3.6.11}$$

Now, setting $q=e^{-\pi\sqrt{5/3}}$ in (3.3.10) and (3.4.8), and then employing (3.6.6), (3.6.7) and (3.6.11), we find that

$$3P(e^{-2\pi\sqrt{15}})-P(e^{-2\pi\sqrt{5/3}})=\frac{5\sqrt{3}}{16}(\sqrt{5}+1)z_{5/3}^2 \tag{3.6.12}$$

and

$$5P(e^{-2\pi\sqrt{15}})-P(e^{-2\pi\sqrt{3/5}})=\frac{7\sqrt{5}}{8\sqrt{3}}(\sqrt{5}+1)z_{5/3}^2. \tag{3.6.13}$$

Multiplying (3.6.12) by 5 and subtracting the resulting identity from three times of (3.6.13), we deduce that

$$3P(e^{-2\pi\sqrt{3/5}}) - 5P(e^{-2\pi\sqrt{5/3}}) = \frac{\sqrt{3}}{16}(25 - 14\sqrt{5})(\sqrt{5}+1)z_{5/3}^2. \tag{3.6.14}$$

On the other hand, setting $n = 5/3$ in (3.2.3), we have

$$5P(e^{-2\pi\sqrt{5/3}}) + 3P(e^{-2\pi\sqrt{3/5}}) = \frac{6\sqrt{15}}{\pi}. \tag{3.6.15}$$

From (3.6.14) and (3.6.15), we deduce that

$$P(e^{-2\pi\sqrt{5/3}}) = \frac{3\sqrt{3}}{\pi\sqrt{5}} - \frac{\sqrt{3}}{32\sqrt{5}}(5\sqrt{5} - 14)(\sqrt{5}+1)z_{5/3}^2. \tag{3.6.16}$$

With the help of (2.3.2), we can rewrite (3.6.16) in the form

$$P(e^{-2\pi\sqrt{5/3}}) = \frac{3\sqrt{3}}{\pi\sqrt{5}} - \frac{\sqrt{3}}{32\sqrt{5}}(5\sqrt{5} - 14)(\sqrt{5}+1)\sum_{k=0}^{\infty} A_k X_{5/3}^k, \tag{3.6.17}$$

where $X_{5/3} = 4x_{5/3}(1 - x_{5/3})$.

Again, setting $n = 5/3$ in (3.2.2), we find that

$$\begin{aligned} P(e^{-2\pi\sqrt{5/3}}) &= (1 - 2x_{5/3})\sum_{n=0}^{\infty}(3k+1)A_k X_{5/3}^k \\ &= \frac{\sqrt{3}}{16}(7 - \sqrt{5})\sum_{n=0}^{\infty}(3k+1)A_k X_{5/3}^k \end{aligned} \tag{3.6.18}$$

From (3.6.17) and (3.6.18), with the aid of (3.6.8), we readily deduce (3.6.1).

3.7 Series for $1/\pi$ arising from Ramanujan's cubic singular moduli x_n, for some non integral values of n

We begin with a special case of Clausen's formula [22, p. 178, Proposition 5.6(b)]. Let,

$$C_k := \frac{(\frac{1}{3})_k(\frac{1}{2})_k(\frac{2}{3})_k}{k!^3} \quad \text{and} \quad H := 4x(q)(1 - x(q)). \tag{3.7.1}$$

If

$$z := z(3;q) := {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right) = a(q), \tag{3.7.2}$$

where

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

then

$$z^2 = {}_3F_2\left(\frac{1}{3},\frac{2}{3},\frac{1}{2};1;1;H\right) = \sum_{k=0}^{\infty} C_k H^k, \quad 0 \le x \le \frac{1}{2}. \tag{3.7.3}$$

The functions z and $P(q)$ are related by

$$P(q) = 12x(1-x)z\frac{dz}{dx} + (1-4x)z^2. \tag{3.7.4}$$

Differentiating (3.7.3) with respect to $x = x(q)$, employing (3.7.1), and substituting in (3.7.4), Chan, Liaw and Tan [32, Theorem 2.7] showed that

$$P(q) = \sum_{k=0}^{\infty}\{6(1-2x)k + 1 - 4x\}C_k H^k(q). \tag{3.7.5}$$

We now define a cubic modular equation of degree n. Suppose that the equality

$$n\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-k^2\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;k^2\right)} = \frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\ell^2\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\ell^2\right)} \tag{3.7.6}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and ℓ that is implied by (3.7.6). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = \ell^2$. We say that β has degree n over α. The corresponding multiplier m is defined by

$$m := m(\alpha,\beta) := \frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)} = \frac{z(q)}{z(q^n)}. \tag{3.7.7}$$

Now set,

$$x_n := x(e^{-2\pi\sqrt{n/3}}) \quad \text{and} \quad z_n := z(e^{-2\pi\sqrt{n/3}}). \tag{3.7.8}$$

The numbers x_n are cubic singular moduli. Chan, Liaw and Tan also showed that [32, Eqs. (3.11) and (3.7)]

$$1 - x_n = x_{1/n}, \quad z_{1/n} = \sqrt{n} z_n \quad \text{and} \quad m(x_{1/n}) = \sqrt{n}. \tag{3.7.9}$$

Setting $q = e^{-2\pi\sqrt{n/3}}$ in (3.7.5), they arrive at

$$P(e^{-2\pi\sqrt{n/3}}) = \sum_{k=0}^{\infty}\{6(1-2x_n)k + 1 - 4x_n\}C_k {H_n}^k, \tag{3.7.10}$$

where $H_n = 4x_n(1-x_n)$.

Next, we recall two further identities from [32, Eqs. (3.6) and (3.17)], namely

$$nP(q^n) - P(q) = z(q)z(q^n)\left\{(1-4x(q^n))\frac{n}{m} - (1-4x(q))m - 12x(q)(1-x(q))\frac{dm}{dx(q)}\right\} \tag{3.7.11}$$

and

$$nP(e^{-2\pi\sqrt{n/3}}) + P(e^{-2\pi/\sqrt{3n}}) = \frac{6\sqrt{3n}}{\pi} - 2nz_n^2. \tag{3.7.12}$$

In the next two sections, for $n = 5/2$ and $n = 3/2$, we obtain two new series for $1/\pi$.

3.8 Series corresponding to $x_{5/2}$

Theorem 3.8.1. *If C_k is defined by* (3.7.1), *then*

$$\frac{324\sqrt{15}}{\pi\sqrt{2}} = \sum_{k=0}^{\infty}\left\{30(35\sqrt{2} - 2\sqrt{5})k + 125\sqrt{2} + 16\sqrt{5}\right\}C_k\left(\frac{223 + 70\sqrt{10}}{1458}\right)^k. \tag{3.8.1}$$

Proof: Setting $n = 5/2$ in (3.7.11), and replacing q by q^2, we arrive at

$$5P\left(q^5\right) - 2P\left(q^2\right) = 2z(q^2)z(q^5)\{(1-4x(q^5))\frac{5}{2m'} - (1-4x(q^2))m' - 12x(q^2)(1-x(q^2))\frac{dm'}{dx(q^2)}\}, \tag{3.8.2}$$

where $m' = \dfrac{z(q^2)}{z(q^5)}$.
Again, setting $q = e^{-2\pi/\sqrt{30}}$ in (3.8.2) and employing (3.7.9), we obtain

$$5P\left(e^{-2\pi\sqrt{5/6}}\right) - 2P\left(e^{-2\pi\sqrt{2/15}}\right) = 2\sqrt{5/2}{z_{5/2}}^2\Bigg\{(1-4x_{5/2})\sqrt{5/2} - (4x_{5/2}-3)\sqrt{5/2} - 12x_{5/2}(1-x_{5/2})\left[\frac{dm'}{dx(q^2)}\right]_{q=e^{-2\pi/\sqrt{30}}}\Bigg\}. \quad (3.8.3)$$

To calculate $x_{5/2}$, we recall the following identity from [28, Eq. (2.7)].

For $q = e^{-2\pi\sqrt{n/3}}$, if

$$\mu_n = \frac{f^6(-q)}{3\sqrt{3q}f^6(-q^3)},$$

where

$$f(-q) = \prod_{k=1}^{\infty}(1-q^k),$$

then

$$\frac{1}{x_n} = \mu_n^2 + 1, \quad (3.8.4)$$

and

$$\mu_{1/n} = \frac{1}{\mu_n}, \quad (3.8.5)$$

where x_n's are the cubic singular moduli defined in (3.7.8). The parameter μ_n was introduced by K.G. Ramanathan [50, eq. (51)].

Next, we recall two further identities from [12, Theorem. 4.4], namely

$$3\left((\mu_n\mu_{25n})^{1/3} + \frac{1}{(\mu_n\mu_{25n})^{1/3}}\right) + 5 = \left(\frac{\mu_{25n}}{\mu_n}\right)^{1/2} - \left(\frac{\mu_n}{\mu_{25n}}\right)^{1/2}, \quad (3.8.6)$$

and

$$3\left((\mu_n\mu_{4n})^{1/3} + \frac{1}{(\mu_n\mu_{4n})^{1/3}}\right) = \frac{\mu_n}{\mu_{4n}} + \frac{\mu_{4n}}{\mu_n}. \quad (3.8.7)$$

Setting, $n = 1/10$ in (3.8.6) and (3.8.7) and using (3.8.5), we arrive at

$$3(A + \frac{1}{A}) + 5 = B^3 - \frac{1}{B^3} \quad (3.8.8)$$

and

$$3(B^2+\frac{1}{B^2})+5=A^3+\frac{1}{A^3}, \tag{3.8.9}$$

where $A=\left(\frac{\mu_{10}}{\mu_{5/2}}\right)^{1/3}$ and $B=\left(\mu_{10}\mu_{5/2}\right)^{1/6}$

From (3.8.8) and (3.8.9), we have

$$3(B^2+\frac{1}{B^2})=\left(\frac{1}{3}(B^3-\frac{1}{B^3}-5)\right)^3-(B^3-\frac{1}{B^3})+5. \tag{3.8.10}$$

Solving for A and B from (3.8.8)–(3.8.10), we obtain

$$\mu_{10}\mu_{5/2}=99+70\sqrt{2} \quad \text{and} \quad \frac{\mu_{10}}{\mu_{5/2}}=9+4\sqrt{5}. \tag{3.8.11}$$

Thus, from (3.8.11) and (3.8.4), we deduce that

$$x_{10}=\frac{1}{2}-\frac{35\sqrt{2}+2\sqrt{5}}{108} \quad \text{and} \quad x_{5/2}=\frac{1}{2}-\frac{35\sqrt{2}-2\sqrt{5}}{108}. \tag{3.8.12}$$

Next, we evaluate $\frac{dm'}{dx(q^2)}$ at $q=e^{-2\pi/\sqrt{30}}$. We have

$$\frac{dm'}{dx(q^2)}=\frac{dm'}{dx(q)}\cdot\frac{dx(q)}{dx(q^2)}. \tag{3.8.13}$$

But, by Theorem 2.3 of [31],

$$\frac{dx(q^n)}{dx(q)}=\frac{n}{m^2}\cdot\frac{x(q^n)(1-x(q^n))}{x(q)(1-x(q))}, \tag{3.8.14}$$

where $m=\frac{z(q)}{z(q^n)}$.

In particular, when $n=2$,

$$\frac{dx(q^2)}{dx(q)}=\frac{2}{m^2}\cdot\frac{x(q^2)(1-x(q^2))}{x(q)(1-x(q))}. \tag{3.8.15}$$

Setting $q=e^{-2\pi/\sqrt{30}}$ in (3.8.15), so that $x(q)=x_{1/10}=1-x_{10}$ and $x(q^2)=x_{2/5}=1-x_{5/2}$, we obtain

$$\left[\frac{dx(q^2)}{dx(q)}\right]_{e^{-2\pi/\sqrt{30}}}=\frac{1}{2}\left(\frac{z_{5/2}}{z_{10}}\right)^2\cdot\frac{x_{5/2}(1-x_{5/2})}{x_{10}(1-x_{10})}. \tag{3.8.16}$$

To evaluate $\frac{z_{5/2}}{z_{10}}$, we recall from Theorem 7.1(iii) in [15, p. 120] that

$$m = \frac{z(q)}{z(q^2)} = \frac{(1-x(q^2))^{2/3}}{(1-x(q))^{1/3}} - \frac{x^{2/3}(q^2)}{x^{1/3}(q)}. \tag{3.8.17}$$

Setting $q = e^{-2\pi/\sqrt{30}}$ in (3.8.17) and then using (3.8.12), we arrive at

$$\frac{z_{5/2}}{z_{10}} = \frac{3\sqrt{2}}{\sqrt{10}+1}. \tag{3.8.18}$$

Employing (3.8.18) and (3.8.12) in (3.8.16), we obtain

$$\left[\frac{dx(q^2)}{dx(q)}\right]_{e^{-2\pi/\sqrt{30}}} = \frac{5699 + 1802\sqrt{10}}{81}. \tag{3.8.19}$$

Now, we evaluate $\left[\frac{dm'}{dx(q)}\right]_{e^{-2\pi/\sqrt{30}}}$. To this end, differentiating $m' = \frac{z(q^2)}{z(q^5)}$ with respect to $x = x(q)$, we obtain

$$\frac{dm'}{dx(q)} = \frac{z(q^2)}{z(q)} \cdot \frac{d}{dx(q)}\frac{z(q)}{z(q^5)} + \frac{z(q)}{z(q^5)} \cdot \frac{d}{dx(q)}\frac{z(q^2)}{z(q)}. \tag{3.8.20}$$

To evaluate $\frac{d}{dx(q)}\frac{z(q^2)}{z(q)}$, we recall the following modular equation of degree 2 from [15, p.120, Theorem 7.1]. If $\beta = x(q^2)$ has degree 2 over $\alpha = x(q)$, then

$$\frac{z(q^2)}{z(q)} = \frac{1}{2}\left\{\frac{(x(q))^{2/3}}{(x(q^2))^{1/3}} - \frac{(1-x(q))^{2/3}}{(1-x(q^2))^{1/3}}\right\}. \tag{3.8.21}$$

Differentiating (3.8.21) with respect to $x(q)$ and then setting $q = e^{-2\pi/\sqrt{30}}$, we arrive at

$$\left[\frac{d}{dx(q)}\frac{z(q^2)}{z(q)}\right]_{q=e^{-2\pi/\sqrt{30}}} = -\frac{2}{3}\left(13 + 4\sqrt{10}\right). \tag{3.8.22}$$

Now we evaluate $\left[\frac{d}{dx(q)}\frac{z(q)}{z(q^5)}\right]_{q=e^{-2\pi/\sqrt{30}}}$. To this end, differentiating (3.8.14) with respect to $x = x(q)$, we deduce that

$$m^2\frac{d^2(x(q^n))}{dx^2(q)} + 2m\frac{dm}{dx(q)} \cdot \frac{dx(q^n)}{dx(q)} = n \cdot \frac{x(q^n)(1-x(q^n))}{x(q)(1-x(q))}\left\{\left(\frac{1}{x(q^n)} - \frac{1}{1-x(q^n)}\right)\frac{dx(q^n)}{dx(q)} - \frac{1}{x(q)} + \frac{1}{1-x(q)}\right\}. \tag{3.8.23}$$

For $n=5$,

$$m^2\frac{d^2(x(q^5))}{dx^2(q)}+2m\frac{dm}{dx(q)}\cdot\frac{dx(q^5)}{dx(q)}=5\cdot\frac{x(q^5)(1-x(q^5))}{x(q)(1-x(q))}\left\{\left(\frac{1}{x(q^5)}-\frac{1}{1-x(q^5)}\right)\frac{dx(q^5)}{dx(q)}\right.$$
$$\left.-\frac{1}{x(q)}+\frac{1}{1-x(q)}\right\}, \tag{3.8.24}$$

where $m=\dfrac{z(q)}{z(q^5)}$.

To calculate $\dfrac{d^2(x(q^5))}{dx(q)}$, we recall from [15, p. 124] the following modular equation of degree 5 in the cubic theory. If $x(q^5)$ has degree 5 over $x(q)$, then

$$\left(x(q)x(q^5)\right)^{1/3}+\{(1-x(q))\left(1-x(q^5)\right)\}^{1/3}+3\{x(q)x(q^5)(1-x(q))(1-x(q^5))\}^{1/6}=1. \tag{3.8.25}$$

Differentiating (3.8.25) twice with respect to $x=x(q)$ and then setting $q=e^{-2\pi/\sqrt{30}}$, so that $x(q)=x_{1/10}=1-x_{10}$, $x(q^5)=x_{5/2}$ and using (3.8.12), (3.8.19), we deduce that

$$\left[\frac{d^2(x(q^5))}{d^2x(q)}\right]_{q=e^{-2\pi/\sqrt{30}}}=\frac{856(5699+1802\sqrt{10})}{9(85\sqrt{2}-53\sqrt{5})}. \tag{3.8.26}$$

Again, setting $q=e^{-2\pi/\sqrt{30}}$ in (3.8.24) and (3.8.13) and using (3.8.19), we arrive at

$$\left[\frac{dm}{dx(q)}\right]_{q=e^{-2\pi/\sqrt{30}}}=\left[\frac{d}{dx(q)}\frac{z(q)}{z(q^5)}\right]_{q=e^{-2\pi/\sqrt{30}}}=\frac{1404}{223-70\sqrt{10}} \tag{3.8.27}$$

and

$$\left[\frac{dm'}{dx(q^2)}\right]_{q=e^{-2\pi/\sqrt{30}}}=\frac{666}{\sqrt{29917+9460\sqrt{10}}}. \tag{3.8.28}$$

With the aid of (3.8.28) and (3.8.12), we can rewrite (3.8.3)

$$5P(e^{-2\pi\sqrt{5/6}})-2P(e^{-2\pi\sqrt{2/15}})=\frac{25\sqrt{2}-4\sqrt{5}}{3}{z_{5/2}}^2. \tag{3.8.29}$$

Again, setting $n=5/2$ in (3.7.12), we obtain

$$5P(e^{-2\pi\sqrt{5/6}})+2P(e^{-2\pi\sqrt{2/15}})=\frac{12\sqrt{15}}{\pi\sqrt{2}}-10{z_{5/2}}^2. \tag{3.8.30}$$

Adding (3.8.29), (3.8.30) and with the aid of (3.7.3), we arrive at

$$P(e^{-2\pi\sqrt{5/6}}) = \frac{3\sqrt{6}}{\pi\sqrt{5}} + \left(\frac{25\sqrt{2}-4\sqrt{5}-30}{30}\right)\sum_{k=0}^{\infty} C_k H_{5/2}{}^k. \tag{3.8.31}$$

Finally, setting $n = 5/2$ in (3.7.10) and with the aid of (3.8.12), we arrive at

$$P\left(e^{-2\pi\sqrt{5/6}}\right) = \sum_{k=0}^{\infty}\left\{\frac{35\sqrt{2}-2\sqrt{2}}{9}k + \frac{1}{27}\left(-27+35\sqrt{2}-2\sqrt{5}\right)\right\} C_k H_{5/2}{}^k, \tag{3.8.32}$$

where $H_{5/2} = \frac{223+70\sqrt{10}}{1458}$.

From (3.8.31) and (3.8.32), we arrive at (3.8.1).

3.9 Series corresponding to $x_{3/2}$

Theorem 3.9.1. *If C_k is defined by* (3.7.1), *then*

$$\frac{250\sqrt{2}}{\pi} = \sum_{k=0}^{\infty}\left\{2(182-27\sqrt{6})k + 3(16-\sqrt{6})\right\} C_k \left(\frac{27(463+182\sqrt{6})}{31250}\right)^k. \tag{3.9.1}$$

Proof: Setting $n = 3/2$ in (3.7.11), and replacing q by q^2, we arrive at

$$3P\left(q^3\right) - 2P\left(q^2\right) = 2z(q^2)z(q^3)\left\{(1-4x(q^3))\frac{3}{2m'} - (1-4x(q^2))m' - 12x(q^2)(1-x(q^2))\frac{dm'}{dx(q^2)}\right\}, \tag{3.9.2}$$

where $m' = \dfrac{z(q^2)}{z(q^3)}$.

Again, setting $q = e^{-2\pi/3\sqrt{2}}$ in (3.9.2) and employing (3.7.9), we obtain

$$3P(e^{-2\pi/\sqrt{2}}) - 2P(e^{-2\pi\sqrt{2}/3}) = 2\sqrt{3/2}z_{3/2}{}^2\left\{(1-4x_{3/2})\sqrt{3/2} - (4x_{3/2}-3)\sqrt{3/2} - 12x_{3/2}(1-x_{3/2})\left[\frac{dm'}{dx(q^2)}\right]_{q=e^{-2\pi/3\sqrt{2}}}\right\}. \tag{3.9.3}$$

To evaluate the value of $x_{3/2}$ and hence $H_{3/2}$, we recall the following cubic modular equation of degree 3 [15, p. 120, Theorem 7.1]. If $\beta = x(q^2)$ has degree 2 over $\alpha = x(q)$ in the cubic theory, then

$$\left(x(q)x(q^2)\right)^{1/3} + \left\{(1-x(q))(1-x(q^2))\right\}^{1/3} = 1. \tag{3.9.4}$$

Setting $q = e^{-2\pi/3\sqrt{2}}$ in (3.9.4), so that $x(q) = x_{1/6} = 1 - x_6$ and $x(q^2) = x_{2/3} = 1 - x_{3/2}$, we obtain

$$(x_{3/2}x_6)^{1/3} + \{(1 - x_{3/2})(1 - x_6)\}^{1/3} = 1. \tag{3.9.5}$$

From [12, Theorem 4.6], we note that

$$\mu_6 = 9 + 3\sqrt{6}. \tag{3.9.6}$$

Employing (3.9.6) in (3.8.4), we readily find that

$$x_6 = \frac{1}{136 + 54\sqrt{6}}. \tag{3.9.7}$$

From (3.9.7) and (3.9.5) we arrive at

$$x_{3/2} = \frac{68 + 27\sqrt{6}}{500} \quad \text{and} \quad H_{3/2} = \frac{27(463 + 182\sqrt{6})}{31250}. \tag{3.9.8}$$

Next, we evaluate $\dfrac{dm'}{dx(q^2)}$ at $q = e^{-2\pi/3\sqrt{2}}$.

We have,

$$\frac{dm'}{dx(q^2)} = \frac{dm'}{dx(q)} \cdot \frac{dx(q)}{dx(q^2)}. \tag{3.9.9}$$

Setting $q = e^{-2\pi/3\sqrt{2}}$ in (3.8.15), so that $x(q) = x_{1/6} = 1 - x_6$, and $x(q^2)' = x_{2/3} = 1 - x_{3/2}$, we obtain

$$\left[\frac{dx(q^2)}{dx(q)}\right]_{e^{-2\pi/3\sqrt{2}}} = \frac{1}{2}\left(\frac{z_{3/2}}{z_6}\right)^2 \cdot \frac{x_{3/2}(1 - x_{3/2})}{x_6(1 - x_6)}. \tag{3.9.10}$$

Again, setting $q = e^{-2\pi/3\sqrt{2}}$ in (3.8.17) and using (3.9.8), we arrive at

$$\frac{z_{3/2}}{z_6} = \frac{3\sqrt{6} - 2}{5}. \tag{3.9.11}$$

Employing (3.9.11), (3.9.7) and (3.9.8) in (3.9.10), we obtain

$$\left[\frac{dx(q^2)}{dx(q)}\right]_{e^{-2\pi/3\sqrt{2}}} = \frac{9461 + 3854\sqrt{6}}{625}. \tag{3.9.12}$$

Finally, to evaluate $\left[\dfrac{dm'}{dx(q)}\right]_{q=e^{-2\pi/3\sqrt{2}}}$, we recall the following modular equation from [15, p.123, Lemma 7.4].

$$\frac{z(q)}{z(q^3)} = (1 + 2(x(q^3))^{1/3}). \tag{3.9.13}$$

With the aid of (3.8.21) and (3.9.13), we find that

$$m' = \frac{z(q^2)}{z(q^3)} = \frac{1}{2}\left\{\frac{(x(q))^{2/3}}{(x(q^2))^{1/3}} - \frac{(1-x(q))^{2/3}}{(1-x(q^2))^{1/3}}\right\} \times \left(1 + 2(x(q^3))^{1/3}\right). \tag{3.9.14}$$

Differentiating (3.9.14) with respect to $x(q)$ and then setting $q = e^{-2\pi/3\sqrt{2}}$, we deduce that

$$\left[\frac{dm'}{dx(q)}\right]_{q=e^{-2\pi/3\sqrt{2}}} = \frac{2(159 + 76\sqrt{6})}{45}. \tag{3.9.15}$$

Setting $q = e^{-2\pi/3\sqrt{2}}$ in (3.9.9) and employing (3.9.15) and (3.9.12), we obtain

$$\left[\frac{dm'}{dx(q^2)}\right]_{q=e^{-2\pi/3\sqrt{2}}} = \frac{2(-81 + 34\sqrt{6})}{9}. \tag{3.9.16}$$

With the aid of (3.9.16) and (3.9.8), we can rewrite (3.9.3) as

$$3P(e^{-2\pi/\sqrt{2}}) - 2P(e^{-2\pi\sqrt{2}/3}) = \frac{3(44 - 9\sqrt{6})}{25} {z_{3/2}}^2. \tag{3.9.17}$$

Again, setting $n = 3/2$ in (3.7.12), we obtain

$$3P(e^{-2\pi/\sqrt{2}}) + 2P(e^{-2\pi\sqrt{2}/3}) = \frac{18\sqrt{2}}{\pi} - 6{z_{3/2}}^2. \tag{3.9.18}$$

Adding (3.9.17), (3.9.18) and with the aid of (3.7.3), we arrive at

$$P\left(e^{-2\pi/\sqrt{2}}\right) = \frac{3\sqrt{2}}{\pi} - \left(\frac{6 + 9\sqrt{6}}{50}\right)\sum_{k=0}^{\infty} C_k {H_{3/2}}^k. \tag{3.9.19}$$

Finally, setting $n = 3/2$ in (3.7.10) and with the aid of (3.9.8), we find that

$$P\left(e^{-2\pi/\sqrt{2}}\right) = \frac{3}{125}\sum_{k=0}^{\infty}\left\{(182 - 27\sqrt{6})k + 19 - 9\sqrt{6}\right\} C_k {H_{3/2}}^k, \tag{3.9.20}$$

where $H_{3/2} = \dfrac{27(463 + 182\sqrt{6})}{31250}$.

From (3.9.19) and (3.9.20), we obtain (3.9.1) to finish the proof.

Chapter 4

New hypergeometric-like series for $1/\pi^2$ arising from Ramanujan's theory of elliptic functions to alternative base 3

4.1 Introduction

As mentioned in the introductory chapter, in his monumental paper [51], [54, pp. 23–39], Ramanujan offered 17 beautiful series representations for $1/\pi$. Three of his series belong to the classical theory of elliptic functions, while the remaining fourteen series depend on Ramanujan's alternative theories of elliptic functions. In particular, two of these series

$$\frac{27}{4\pi} = \sum_{k=0}^{\infty}(15k+2)\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k\left(\frac{2}{3}\right)_k}{(k!)^3}\left(\frac{2}{27}\right)^k \tag{4.1.1}$$

and

$$\frac{15\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty}(33k+4)\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k\left(\frac{2}{3}\right)_k}{(k!)^3}\left(\frac{4}{125}\right)^k, \tag{4.1.2}$$

belong to his cubic theory.

In [5], N.D. Baruah and Berndt used hypergeometric identities and certain representations for Eisenstein series to derive several new series representations

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Note: The contents of this chapter appeared in Trans. Amer. Math. Soc. [9].

for $1/\pi^2$. In particular, they found series for $1/\pi^2$ that are analogues of Ramanujan's series for $1/\pi$ in the classical theory and theories of q_4 and q_6. In this Chapter, we apply Ramanujan's cubic theory and certain representations for Eisenstein series to find several new series for $1/\pi^2$. In the process we find two series representations for $1/\pi^2$ which are perfect analogues of (4.1.1) and (4.1.2).

4.2 Series for $1/\pi^2$ arising from Ramanujan's cubic theory

We begin with Ramanujan's Eisenstein series $P(q)$ which satisfies the identity [16, Lemma 4.1]

$$P(q) = (1-4x)z^2 + 3H(q)z\frac{dz}{dx}, \tag{4.2.1}$$

where $H(q) = 4x(q)(1-x(q))$.

In [5], Baruah and Berndt derived many series for $1/\pi^2$ by combining two different representations for $P(e^{-2\pi/\sqrt{n}})P(e^{-2\pi\sqrt{n}})$. Their method utilized the classical theory of elliptic functions. Unfortunately, their ideas do not apply in the theory of signature 3, since the necessary transformation formula for $P(q)$ takes a different form (see equation (4.2.28)). In this Chapter, we use certain representations for $P^2(e^{-2\pi/\sqrt{3n}})$ and $P^2(e^{-2\pi\sqrt{n/3}})$ along with some hypergeometric series identities and obtain our series for $1/\pi^2$ by appealing to various cubic singular moduli defined in (3.7.8). We explain our method in the remainder of this section.

From Lemma 3.1 of [5], for $|x| < 1$, we note that

$${}_3F_2^2(a_1, a_2, a_3; 1, 1; x) = \sum_{k=0}^{\infty} U_k x^k, \tag{4.2.2}$$

where

$$U_k = \sum_{n=0}^{k} \frac{(a_1)_n(a_2)_n(a_3)_n(a_1)_{k-n}(a_2)_{k-n}(a_3)_{k-n}}{(n!)^3((k-n)!)^3} \tag{4.2.3}$$

Setting $a_1 = 1/3$, $a_2 = 2/3$ and $a_3 = 1/2$ in (4.2.2), we obtain

$$ {}_3F_2^2\left(\frac{1}{3},\frac{2}{3},\frac{1}{2};1,1;x\right) = \sum_{k=0}^{\infty} U_k x^k, \tag{4.2.4}$$

where

$$U_k = \sum_{n=0}^{k} \frac{(\frac{1}{3})_n(\frac{2}{3})_n(\frac{1}{2})_n(\frac{1}{3})_{k-n}(\frac{2}{3})_{k-n}(\frac{1}{2})_{k-n}}{(n!)^3((k-n)!)^3}. \tag{4.2.5}$$

From (4.2.4) and (3.7.3), we find that

$$z^4 = {}_3F_2^2\left(\frac{1}{3},\frac{2}{3},\frac{1}{2};1,1;H\right) = \sum_{k=0}^{\infty} U_k H^k, \tag{4.2.6}$$

where U_k is defined by (4.2.5).

Differentiating (4.2.6) with respect x, we find that

$$z^3\frac{dz}{dx} = \frac{1-2x}{H}\sum_{k=0}^{\infty} U_k k H^k. \tag{4.2.7}$$

But, z satisfies the hypergeometric differential equation [1, p. 75, Eq. (2.3.5)]

$$x(1-x)\frac{d^2z}{dx^2} + (1-2x)\frac{dz}{dx} - \frac{2}{9}z = 0, \tag{4.2.8}$$

which can be rewritten as

$$z^3\frac{d^2z}{dx^2} = \frac{8z^4}{9H} - \frac{4(1-2x)}{H} z^3\frac{dz}{dx}. \tag{4.2.9}$$

Now,

$$\frac{d}{dx}\left(z^3\frac{dz}{dx}\right) = 3z^2\left(\frac{dz}{dx}\right)^2 + z^3\frac{d^2z}{dx^2}. \tag{4.2.10}$$

Employing (4.2.9) and (4.2.7) in (4.2.10), we obtain

$$\frac{d}{dx}\left(z^3\frac{dz}{dx}\right) = 3z^2\left(\frac{dz}{dx}\right)^2 + \frac{8z^4}{9H} - \frac{4(1-H)}{H^2}\sum_{k=0}^{\infty} U_k k H^k. \tag{4.2.11}$$

Again, differentiating (4.2.7) with respect to x, we find that

$$\frac{d}{dx}\left(z^3\frac{dz}{dx}\right) = \frac{4(1-2x)^2}{H^2}\sum_{k=0}^{\infty} U_k k^2 H^k + \frac{2(H-2)}{H^2}\sum_{k=0}^{\infty} U_k k H^k. \tag{4.2.12}$$

From (4.2.11) and (4.2.12), we deduce that

$$3z^2\left(\frac{dz}{dx}\right)^2 = \sum_{k=0}^{\infty}\left\{\frac{4(1-H)}{H^2}k^2 - \frac{2}{H}k - \frac{8}{9H}\right\}U_k H^k. \tag{4.2.13}$$

Now, squaring both sides of (4.2.1), we obtain

$$P^2(q) = (1-4x)^2z^4 + 9H(q)^2z^2\left(\frac{dz}{dx}\right)^2 + 6H(q)(1-4x)z^3\frac{dz}{dx}. \tag{4.2.14}$$

Replacing q by q^n in (4.2.14), and then employing (4.2.6), (4.2.7), and (4.2.13), we deduce that

$$P^2(q^n) = \sum_{k=0}^{\infty}\left[12(1-H)k^2 + \{6(1-2\beta)(1-4\beta) - 6H\}k + (1-4\beta)^2 - \frac{8H}{3}\right]U_k H^k. \tag{4.2.15}$$

Setting $q = e^{-2\pi/\sqrt{3n}}$ in (4.2.15), so that $\beta = x_n$, $H_n = 4x_n(1-x_n)$, we find that

$$P^2\left(e^{-2\pi\sqrt{n/3}}\right) = \sum_{k=0}^{\infty}\bigg[12(1-H_n)k^2 + \{6(1-2x_n)(1-4x_n) - 6H_n\}k + (1-4x_n)^2 - \frac{8H_n}{3}\bigg]U_k H_n^k. \tag{4.2.16}$$

Next, we derive a similar expression for $P^2\left(e^{-2\pi/\sqrt{3n}}\right)$. To this end, we note from (3.7.7) that if m is the multiplier connecting x and $\beta = x(q^n)$, then

$$z = mz(q^n). \tag{4.2.17}$$

Thus,

$$\frac{dz}{dx} = \frac{dz}{d\beta}\cdot\frac{d\beta}{dx} = \frac{d\beta}{dx}\left\{\frac{dm}{d\beta}z(q^n) + m\frac{dz(q^n)}{d\beta}\right\}. \tag{4.2.18}$$

Now, from Theorem 2.3 of [31], we recall that

$$\frac{d\beta}{dx} = \frac{n}{m^2}\cdot\frac{\beta(1-\beta)}{x(1-x)}. \tag{4.2.19}$$

Employing (4.2.19) in (4.2.18), we find that

$$\frac{dz}{dx} = \frac{n\beta(1-\beta)}{m^2x(1-x)}z(q^n)\frac{dm}{d\beta} + \frac{n\beta(1-\beta)}{mx(1-x)}\frac{dz(q^n)}{d\beta}. \tag{4.2.20}$$

Invoking (4.2.20) and (4.2.17) in (4.2.1), we deduce that

$$P(q) = Dz^2(q^n) + 3nHz(q^n)\frac{dz(q^n)}{d\beta}, \tag{4.2.21}$$

where

$$D = (1-4x)m^2 + \frac{12n\beta(1-\beta)}{m} \cdot \frac{dm}{d\beta}. \tag{4.2.22}$$

Again, employing (4.2.22), (4.2.20), and (4.2.17) in (4.2.14), we deduce that

$$P^2(q) = \sum_{k=0}^{\infty}\left[12n^2(1-H)k^2 + \{6nD(1-2\beta) - 6n^2H\}k + D^2 - \frac{8H^2}{3}\right]U_kH^k. \tag{4.2.23}$$

Setting $q = e^{-2\pi/\sqrt{3n}}$ in (4.2.23), we arrive at

$$\begin{aligned} P^2\left(e^{-2\pi/\sqrt{3n}}\right) = \sum_{k=0}^{\infty}\Big[&12n^2(1-H_n)k^2 + \{6nD_n\sqrt{1-H_n} - 6n^2H_n\}k \\ &+ D_n^2 - \frac{8H_n^2}{3}\Big]U_kH_n^k, \end{aligned} \tag{4.2.24}$$

where

$$D_n = n(4x_n - 3) + 3\sqrt{n}H_n\left[\frac{dm}{d\beta}\right]_{q=e^{-2\pi/\sqrt{3n}}}. \tag{4.2.25}$$

Next, we recall two further identities from H.H. Chan and W.-C. Liaw's paper [32, Eqs. (3.12) and (3.17)], namely,

$$\begin{aligned} nP\big(e^{-2\pi\sqrt{n/3}}\big) - P\big(e^{-2\pi/\sqrt{3n}}\big) &= 4z_n^2\{n\sqrt{1-H_n} - \frac{3\sqrt{n}}{4}H_n\frac{dm}{dx}(1-x_n, x_n)\} \\ &= L_nz_n^2, \end{aligned} \tag{4.2.26}$$

where

$$L_n = 4\{n\sqrt{1-H_n} - \frac{3\sqrt{n}}{4}H_n\frac{dm}{dx}(1-x_n, x_n)\} \tag{4.2.27}$$

and

$$nP\big(e^{-2\pi\sqrt{n/3}}\big) + P\big(e^{-2\pi/\sqrt{3n}}\big) = \frac{6\sqrt{3n}}{\pi} - 2nz_n^2. \tag{4.2.28}$$

Squaring both (4.2.26) and (4.2.28), and then adding the resulting identities, we find that

$$2n^2P^2\left(e^{-2\pi\sqrt{n/3}}\right) + 2P^2\left(e^{-2\pi/\sqrt{3n}}\right) = \frac{108n}{\pi^2} + (4n^2 + L_n^2)z_n^4 - \frac{24n\sqrt{3n}}{\pi}z_n^2. \quad (4.2.29)$$

Again, multiplying (4.2.26) and (4.2.28), we obtain

$$n^2P^2\left(e^{-2\pi\sqrt{n/3}}\right) - P^2\left(e^{-2\pi/\sqrt{3n}}\right) = \frac{6\sqrt{3n}}{\pi}L_nz_n^2 - 2nL_nz_n^4. \quad (4.2.30)$$

Multiplying both sides of (4.2.30) by $4n/L_n$, and adding the resulting identity to (4.2.29), we deduce that

$$\left(2n^2 + \frac{4n^3}{L_n}\right)P^2\left(e^{-2\pi\sqrt{n/3}}\right) + \left(2 - \frac{4n}{L_n}\right)P^2\left(e^{-2\pi/\sqrt{3n}}\right) + (4n^2 - L_n^2)z_n^4 = \frac{108n}{\pi^2}. \quad (4.2.31)$$

Employing (4.2.16), (4.2.24), and (4.2.6) in (4.2.31), we arrive at the following theorem.

Theorem 4.2.1. *If $H_n = 4x_n(1 - x_n)$ and U_k is defined by (4.2.5), then*

$$\frac{108n}{\pi^2} = \sum_{k=0}^{\infty}\{A(x_n)k^2 + B(x_n)k + C(x_n)\}U_kH_n^k, \quad (4.2.32)$$

where

$$A(x_n) = 48n^2(1 - H_n), \quad (4.2.33)$$

$$B(x_n) = 4n^2(1 - 2x_n)\{3(1 - 4x_n) + 6\} - 24n^2H_n + 12nD_n(1 - 2x_n) \quad (4.2.34)$$

and

$$C(x_n) = 2n^2\left(1 + \frac{2n}{L_n}\right)(1 - 4x_n)^2 - \frac{32n^2H_n}{3} + 4n^2 - L_n^2 + \left(2 - \frac{4n}{L_n}\right)D_n^2, \quad (4.2.35)$$

where D_n is defined by (4.2.25).

In the next few sections, we present our new series for $1/\pi^2$, obtained from (4.2.32).

4.3 Example: $n = 2$

Theorem 4.3.1. *If U_k is defined by (4.2.5), then*

$$\frac{81}{\pi^2} = \sum_{k=0}^{\infty} (36k^2 - 5) U_k \left(\frac{1}{2}\right)^k. \tag{4.3.1}$$

Proof. Setting $n = 2$ in (4.2.32), we obtain

$$\frac{216}{\pi^2} = \sum_{k=0}^{\infty} \{A(x_2)k^2 + B(x_2)k + C(x_2)\} U_k H_2^k. \tag{4.3.2}$$

By using two cubic modular equations of degree 2 [15, p. 120, Theorem 7.1(i) and Theroem 7.1(iii)] and (4.5.11) in Section 4.5, Baruah and Berndt [4, Eqs. (5.15) and (5.23)] found that

$$x_2 = \frac{\sqrt{2}-1}{2\sqrt{2}}, \qquad H_2 = \frac{1}{2} \quad \text{and} \quad \left[\frac{dm}{d\beta}\right]_{q=e^{-2\pi/\sqrt{6}}} = \frac{4}{3}. \tag{4.3.3}$$

Setting $n = 2$ in (4.2.25) and (4.2.27), and then employing (4.3.3), we obtain

$$D_2 = -2 \quad \text{and} \quad L_2 = 2\sqrt{2}. \tag{4.3.4}$$

Next, setting $n = 2$ in (4.2.33), (4.2.34), (4.2.35), and then using (4.3.3) and (4.3.4), we find that

$$A(x_2) = 96, \quad B(x_2) = 0 \quad \text{and} \quad C(x_2) = -\frac{40}{3}. \tag{4.3.5}$$

Employing (4.3.5) and (4.3.3) in (4.3.2), we readily arrive at (4.3.1). □

4.4 Example: $n = 3$

Theorem 4.4.1. *If U_k is defined by (4.2.5), then*

$$\frac{3\sqrt{3}}{\pi^2} = \sum_{k=0}^{\infty} \{2(38\sqrt{3} - 63)k^2 + 15(7\sqrt{3} - 12)k + 37\sqrt{3} - 64\} U_k \left(\frac{3\sqrt{3}(2-\sqrt{3})^2}{2}\right)^k. \tag{4.4.1}$$

Proof. Setting $n = 3$ in (4.2.32), we obtain

$$\frac{324}{\pi^2} = \sum_{k=0}^{\infty} \{A(x_3)k^2 + B(x_3)k + C(x_3)\} U_k H_3^k. \tag{4.4.2}$$

Now, from [4, Eq. (5.32)], we note that

$$x_3 = \frac{3\sqrt{3} - 5}{4} \quad \text{and} \quad H_3 = \frac{3\sqrt{3}(2 - \sqrt{3})^2}{2}. \tag{4.4.3}$$

Again, from [15, p. 123, Lemma 7.4], we recall that

$$m = 1 + 2\beta^{1/3}, \tag{4.4.4}$$

where m is the multiplier connecting x and $\beta = x(q^3)$. Differentiating (4.4.4) with respect to β, we obtain

$$\frac{dm}{d\beta} = \frac{2}{3\beta^{2/3}}. \tag{4.4.5}$$

Setting $q = e^{-2\pi/3}$ in (4.4.5), so that $\beta = x_3 = \frac{3\sqrt{3}-5}{4}$, we find that

$$\left[\frac{dm}{d\beta}\right]_{q=e^{-2\pi/3}} = \frac{4(2 + \sqrt{3})}{3}. \tag{4.4.6}$$

Next, setting $n = 3$ in (4.2.25) and (4.2.27), and then employing (4.4.3) and (4.4.6), we obtain

$$D_3 = 12 - 9\sqrt{3} \quad \text{and} \quad L_3 = 6. \tag{4.4.7}$$

Finally, setting $n = 3$ in (4.2.33), (4.2.34), (4.2.35), and then employing (4.4.3) and (4.4.7), we find that

$$A(x_3) = 216(38 - 21\sqrt{3}), \quad B(x_3) = 1620(2 - \sqrt{3})^2 \quad \text{and} \quad C(x_3) = 36(111 - 64\sqrt{3}). \tag{4.4.8}$$

Employing (4.4.3) and (4.4.8) in (4.4.2), we readily arrive at (4.4.1). □

4.5 Example: $n = 4$

Theorem 4.5.1. *If U_k is defined by* (4.2.5), *then*

$$\frac{3}{4\pi^2} = \sum_{k=0}^{\infty} \{225k^2 + 81k + 8\} U_k \left(\frac{2}{27}\right)^{k+2}. \tag{4.5.1}$$

The above series for $1/\pi^2$ is a perfect analogue of Ramanujan's series (4.1.1).

Proof. Setting $n = 4$ in (4.2.32), we obtain

$$\frac{432}{\pi^2} = \sum_{k=0}^{\infty} \{A(x_4)k^2 + B(x_4)k + C(x_4)\} U_k H_4^k. \tag{4.5.2}$$

Next, setting $n = 4$ in (4.2.25), we find that

$$D_4 = 4(4x_4 - 3) + 3\sqrt{4} H_4 \left[\frac{dm}{d\beta}\right]_{q=e^{-2\pi/\sqrt{12}}}. \tag{4.5.3}$$

Firstly, to calculate x_4 and hence H_4, we recall the following cubic modular equation of degree 4 [15, p. 121].

Let γ be of degree 4 over α, and let m be the associated multiplier in the theory of signature 3. Then

$$m = \left(\frac{\gamma}{\alpha}\right)^{1/3} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/3} - \frac{4}{m}\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/3}. \tag{4.5.4}$$

Setting $q = e^{-2\pi/\sqrt{12}} = e^{-\pi/\sqrt{3}}$ in (4.5.4), so that $\alpha = x_{1/4} = 1 - x_4$, $\gamma = x_4$, and $m = 2$, we find that

$$\left(\frac{x_4}{1-x_4}\right)^{1/3} = 2 - \sqrt{3}, \tag{4.5.5}$$

from which we readily deduce that

$$x_4 = \frac{9 - 5\sqrt{3}}{18} \text{ and } H_4 = \frac{2}{27}. \tag{4.5.6}$$

Secondly, to evaluate $\frac{dm}{dx}(1 - x_4, x_4)$, we recall two more modular equations from [15, p. 120].

If α, β, and γ have degrees 1, 2, and 4, respectively, in the theory of signature 3 and m_1 and m_2 are the multipliers associated with pairs α, β and β, γ, respectively, then

$$\left(\frac{(1-\beta)^2}{1-\alpha}\right)^{1/3}+\left(\frac{\beta^2}{\alpha}\right)^{1/3}=m_1^2 \tag{4.5.7}$$

and

$$\frac{\sqrt{3}\{\beta(1-\beta)\}^{1/6}}{\{\alpha(1-\gamma)\}^{1/3}-\{\gamma(1-\alpha)\}^{1/3}}=\frac{m_1}{m_2}. \tag{4.5.8}$$

Using (4.5.7) and (4.5.8), we find that

$$\begin{aligned}\sqrt{3}m\{x(q^2)(1-x(q^2))\}^{1/6}=&\left\{\frac{(1-x(q^2))^2}{1-x(q)}\right\}^{1/3}\{x(q)(1-x(q^4))\}^{1/3}\\&-\{(1-x(q^2))^2x(q^4)\}^{1/3}+\{x^2(q^2)(1-x(q^4))\}^{1/3}\\&-\left\{\frac{x^2(q^2)}{x(q)}x(q^4)(1-x(q))\right\}^{1/3},\end{aligned} \tag{4.5.9}$$

where m is the multiplier connecting $x(q)$ and $x(q^4)$. Differentiating (4.5.9) with respect to $x := x(q)$, we find that

$$\begin{aligned}&\sqrt{3}\left\{x(q^2)(1-x(q^2))\right\}\frac{dm}{dx}+\frac{\sqrt{3}}{6}m\left\{x(q^2)(1-x(q^2))\right\}\left\{\frac{1}{x(q^2)}-\frac{1}{(1-x(q^2))}\right\}\frac{dx(q^2)}{dx(q)}\\&=\frac{1}{3}\left\{\frac{(1-x(q^2))^2}{1-x(q)}x(q)(1-x(q^4))\right\}^{1/3}\left\{\frac{-2}{1-x(q^2)}\frac{dx(q^2)}{dx(q)}+\frac{1}{x(q)}+\frac{1}{(1-x(q))}\right.\\&\left.-\frac{1}{1-x(q^4)}\frac{dx(q^4)}{dx(q)}\right\}-\frac{1}{3}\left\{(1-x(q^2))^2x(q^4)\right\}^{1/3}\left\{\frac{-2}{1-x(q^2)}\frac{dx(q^2)}{dx(q)}\right.\\&\left.+\frac{1}{x(q^4)}\frac{dx(q^4)}{dx(q)}\right\}+\frac{1}{3}\left\{x^2(q^2)(1-x(q^4))\right\}^{1/3}\left\{\frac{2}{x(q^2)}\frac{dx(q^2)}{dx(q)}-\frac{1}{1-x(q^4)}\frac{dx(q^4)}{dx(q)}\right\}\\&-\frac{1}{3}\left\{\frac{x^2(q^2)}{x(q)}x(q^4)(1-x(q))\right\}^{1/3}\left\{\frac{2}{x(q^2)}\frac{dx(q^2)}{dx(q)}-\frac{1}{x(q)}+\frac{1}{x(q^4)}\frac{dx(q^4)}{dx(q)}\right.\\&\left.-\frac{1}{1-x(q)}\right\}.\end{aligned} \tag{4.5.10}$$

But, by Theorem 2.3 of [31],

$$\frac{dx(q^n)}{dx(q)}=\frac{n}{m^2}\cdot\frac{x(q^n)(1-x(q^n))}{x(q)(1-x(q))}, \tag{4.5.11}$$

where m is the multiplier connecting $x(q)$ and $x(q^n)$. In particular, if $n = 4$, then

$$\frac{dx(q^4)}{dx(q)} = \frac{4}{m^2} \cdot \frac{x(q^4)(1 - x(q^4))}{x(q)(1 - x(q))}, \tag{4.5.12}$$

where m is the multiplier connecting $x(q)$ and $x(q^4)$. Setting $q = e^{-\pi/\sqrt{3}}$ in (4.5.12), so that, by (4.6), $x = x_{1/4} = 1 - x_4$, $x(q^4) = x_4$, and $m = \sqrt{4} = 2$, we deduce that

$$\left[\frac{dx(q^4)}{dx(q)}\right]_{q=e^{-2\pi/\sqrt{12}}} = 1. \tag{4.5.13}$$

Again, if $n = 2$ in (4.5.11), then

$$\frac{dx(q^2)}{dx(q)} = \frac{2}{m_2^2} \cdot \frac{x(q^2)(1 - x(q^2))}{x(q)(1 - x(q))}, \tag{4.5.14}$$

where m_2 is the multiplier connecting $x(q)$ and $x(q^2)$.

Setting $q = e^{-\pi/\sqrt{3}}$ in (4.5.7), so that $\alpha = 1 - x_4$ and $\beta = x_1 = 1/2$, we find that

$$m_2^2 = 3. \tag{4.5.15}$$

Thus, with $q = e^{-2\pi/\sqrt{12}}$ in (4.5.14) and with the aid of $x_1 = 1/2$, (4.5.6), and (4.5.15), we find that

$$\left[\frac{dx(q^2)}{dx(q)}\right]_{q=e^{-2\pi/\sqrt{12}}} = 9. \tag{4.5.16}$$

Finally, setting $q = e^{-2\pi/\sqrt{12}}$ in (4.5.10), and then using (4.5.6), (4.5.13), and (4.5.16), we arrive that

$$\left[\frac{dm}{dx}\right]_{q=e^{-2\pi/\sqrt{12}}} = \frac{24}{\sqrt{3}}. \tag{4.5.17}$$

Setting $n = 4$ in (4.2.25) and (4.2.27), and then employing (4.5.6) and (4.5.17), we obtain

$$D_4 = -\frac{4}{9}(9 + 2\sqrt{3}) \quad \text{and} \quad L_4 = \frac{16}{\sqrt{3}}. \tag{4.5.18}$$

Next, setting $n = 4$ in (4.2.33), (4.2.34) and (4.2.35), and then using (4.5.6) and (4.5.18), we find that

$$A(x_4) = \frac{6400}{9}, \quad B(x_4) = 256 \quad \text{and} \quad C(x_4) = -\frac{8}{81}(8 - 87\sqrt{3}). \tag{4.5.19}$$

Employing (4.5.19) and (4.5.6) in (4.5.2), we readily arrive at (4.5.1). □

4.6 Example: $n = 5$

Theorem 4.6.1. *If* U_k *is defined by* (4.2.5), *then*

$$\frac{2025}{4\pi^2} = \sum_{k=0}^{\infty} \{1089k^2 + 378k + 40\} U_k \left(\frac{4}{125}\right)^k. \tag{4.6.1}$$

The above series is an analogue of Ramanujan's series (4.1.2).

Proof. Setting $n = 5$ in (4.2.32), we obtain

$$\frac{540}{\pi^2} = \sum_{k=0}^{\infty} \{A(x_5)k^2 + B(x_5)k + C(x_5)\} U_k H_5^k. \tag{4.6.2}$$

From [4, Eq. (5.38)], we have

$$x_5 = \frac{5\sqrt{5} - 11}{10\sqrt{5}}, \qquad H_5 = \frac{4}{125}. \tag{4.6.3}$$

Next, to evaluate $\frac{dm}{dx}(1 - x_5, x_5)$, we rewrite (4.5.11) in the form

$$\frac{d\beta}{d\alpha} = \frac{n}{m^2} \cdot \frac{\beta(1-\beta)}{\alpha(1-\alpha)}, \tag{4.6.4}$$

where β has degree n over α and m is the multiplier connecting α and β. Differentiating (4.6.4) with respect to α, we obtain

$$m^2 \frac{d^2\beta}{d\alpha^2} + 2m \frac{dm}{d\alpha} \cdot \frac{d\beta}{d\alpha} = n \cdot \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \left\{ \left(\frac{1}{\beta} - \frac{1}{1-\beta}\right) \frac{d\beta}{d\alpha} - \frac{1}{\alpha} + \frac{1}{1-\alpha} \right\}. \tag{4.6.5}$$

Setting $q = e^{-2\pi/\sqrt{3n}}$ in (4.6.5), so that $\alpha = x_{1/n} = 1 - x_n$, $\beta = x_n$, $\frac{d\beta}{d\alpha} = 1$, and $m = \sqrt{n}$, we find that

$$n \frac{d^2\beta}{d\alpha^2} + 2\sqrt{n} \frac{dm}{d\alpha} = 2n \left(\frac{1 - 2\beta}{\alpha\beta} \right). \tag{4.6.6}$$

In particular, if $n = 5$, then

$$5 \frac{d^2\beta}{d\alpha^2} + 2\sqrt{5} \frac{dm}{d\alpha} = 550\sqrt{5}. \tag{4.6.7}$$

To calculate $\frac{d^2\beta}{d\alpha^2}$, we recall from [15, p. 124] the following modular equation of degree 5 in the theory of signature 3. If β has degree 5 over α, then

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} + 3\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \tag{4.6.8}$$

Differentiating (4.6.8) twice with respect to α and then setting $q = e^{-2\pi/\sqrt{15}}$, so that $\alpha = 1-\beta$ and $\left[\frac{d\beta}{d\alpha}\right]_{q=e^{-2\pi/\sqrt{15}}} = 1$, we deduce that

$$\left[\frac{d^2\beta}{d\alpha^2}\right]_{q=e^{-2\pi/\sqrt{15}}} = \frac{290\sqrt{5}}{3}. \tag{4.6.9}$$

Employing (4.6.9) in (4.6.7), we obtain

$$\left[\frac{dm}{d\alpha}\right]_{q=e^{-2\pi/\sqrt{15}}} = \frac{100}{3}. \tag{4.6.10}$$

Thus,

$$\left[\frac{dm}{d\beta}\right]_{q=e^{-2\pi/\sqrt{15}}} = \left[\frac{dm}{d\alpha}\cdot\frac{d\alpha}{d\beta}\right]_{q=e^{-2\pi/\sqrt{15}}} = \frac{100}{3}. \tag{4.6.11}$$

Setting $n=5$ in (4.2.25) and (4.2.27), and then employing (4.6.3) and (4.6.11), we obtain

$$D_5 = -(5+\frac{6}{\sqrt{5}}) \quad \text{and} \quad L_5 = \frac{28}{\sqrt{5}}. \tag{4.6.12}$$

Next, setting $n=5$ in (4.2.33), (4.2.34) and (4.2.35), and then using (4.6.3) and (4.6.12), we find that

$$A(x_5) = \frac{5808}{5}, \quad B(x_5) = \frac{2016}{5} \quad \text{and} \quad C(x_5) = \frac{1}{105}(1018 + 141\sqrt{5}). \tag{4.6.13}$$

Employing (4.6.13) and (4.6.3) in (4.6.2), we readily arrive at (4.6.1). □

4.7 Example: $n = 6$

Theorem 4.7.1. *If U_k is defined by (4.2.5), then*

$$\frac{15625}{\pi^2} = \sum_{k=0}^{\infty}\left\{\frac{4}{3}(18749 + 4914\sqrt{6})k^2 + 6(148 + 853\sqrt{6})k\right.$$

$$+25(-101+64\sqrt{6})\Big\}U_k\left(\frac{-27(-463+182\sqrt{6})}{31250}\right)^k. \tag{4.7.1}$$

Proof. Setting $n=6$ in (4.2.32), we obtain

$$\frac{648}{\pi^2}=\sum_{k=0}^{\infty}\{A(x_6)k^2+B(x_6)k+C(x_6)\}U_kH_6^k, \tag{4.7.2}$$

where $H_6=4x_6(1-x_6)$.

From [12, Theorem 4.6], we note that

$$\mu_6=9+3\sqrt{6}. \tag{4.7.3}$$

Employing (4.7.3) in (3.8.4), we readily find that

$$x_6=\frac{1}{136+54\sqrt{6}}, \qquad \text{and hence,} \qquad H_6=\frac{-27(-463+182\sqrt{6})}{31250}. \tag{4.7.4}$$

To calculate $\frac{dm}{dx}(1-x_6,x_6)$, we recall from (4.4.4) that

$$m_1=1+2x^{1/3}(q^3), \tag{4.7.5}$$

where m_1 is the multiplier connecting $x(q)$ and $x(q^3)$.

Replacing q by q^2 in (4.7.5) and using Theorem 7.1 (iii) in [15, p. 120], we obtain that

$$m=\{1+2x^{1/3}(q^6)\}\left[\left\{\frac{(1-x(q^2))^2}{1-x(q)}\right\}^{1/3}-\left\{\frac{x^2(q^2)}{x(q)}\right\}^{1/3}\right], \tag{4.7.6}$$

where m is the multiplier connecting $x(q)$ and $x(q^6)$. Differentiating (4.7.6) with respect to $x:=x(q)$, and employing Theorem 7.1 (v) of [15, p. 120] and (4.7.4), we deduce that

$$\left[\frac{dm}{dx}\right]_{q=e^{-2\pi/\sqrt{18}}}=\frac{4(81+34\sqrt{6})}{9}. \tag{4.7.7}$$

Setting $n=6$ in (4.2.25) and (4.2.27), and then employing (4.7.4) and (4.7.7), we obtain

$$D_6=-\frac{6}{125}(163+18\sqrt{6}) \quad \text{and} \quad L_6=\frac{6}{25}(44+9\sqrt{6}). \tag{4.7.8}$$

Next, setting $n = 6$ in (4.2.33), (4.2.34) and (4.2.35), and then using (4.7.4) and (4 7 8), we find that

$$A(x_6) = \frac{864}{15625}(18749 + 4914\sqrt{6}), \quad B(x_6) = \frac{3888}{15625}(148 + 853\sqrt{6}) \tag{4.7.9}$$

and

$$C(x_6) = \frac{72}{453125}(-486827 + 199128\sqrt{6}). \tag{4.7.10}$$

Employing (4.7.9), (4.7.10) and (4.7.4) in (4.7.2), we readily arrive at (4.7.1). □

4.8 Example: $n = 9$

Theorem 4.8.1. *If U_k is defined by* (4.2.5), *then*

$$\frac{15625}{2\pi^2} = \sum_{k=0}^{\infty} \Big\{ 2(-8468 - 20556 \cdot 2^{1/3} + 31473 \cdot 2^{2/3})k^2 + 3(-15603 - 19476 \cdot 2^{1/3} + 27083 \cdot 2^{2/3})k + 20(-917 - 989 \cdot 2^{1/3} + 1387 \cdot 2^{2/3}) \Big\} U_k H_9^k, \tag{4.8.1}$$

where $H_9 = \dfrac{-9(-2677 - 2284 \cdot 2^{1/3} + 3497 \cdot 2^{2/3})}{15625}$.

Proof. Setting $n = 9$ in (4.2.32), we obtain

$$\frac{972}{\pi^2} = \sum_{k=0}^{\infty} \{A(x_9)k^2 + B(x_9)k + C(x_9)\} D_k H_9^k. \tag{4.8.2}$$

To calculate x_9 and hence H_9, we replace q by q^3 in (4.7.5) to arrive at

$$m = \left(1 + 2x^{1/3}(q^9)\right)\left(1 + 2x^{1/3}(q^3)\right), \tag{4.8.3}$$

where m is the multiplier connecting $x(q)$ and $x(q^9)$. Setting $q = e^{-2\pi/\sqrt{27}}$ in (4.8.3), so that $x(q^3) = x_1 = 1/2$, $x(q^9) = x_9$, and $m = \sqrt{9} = 3$, we deduce that

$$x_9 = \frac{1}{250}\left(187 - 171 \cdot 2^{1/3} + 18 \cdot 2^{2/3}\right) \tag{4.8.4}$$

and

$$L_9 = \frac{-9}{15626}\Big(- 2677 - 2284 \cdot 2^{1/3} + 3497 \cdot 2^{2/3}\Big). \tag{4.8.5}$$

Next, to evaluate $\frac{dm}{dx}(1-x_9, x_9)$, we differentiate (4.8.3) with respect to $x = x(q)$, to obtain

$$\frac{dm}{dx} = \Big(1 + 2x^{1/3}(q^3)\Big) \cdot \frac{2}{3}x^{-2/3}(q^9)\frac{dx(q^9)}{dx(q)} + \Big(1 + 2x^{1/3}(q^9)\Big) \cdot \frac{2}{3}x^{-2/3}(q^3)\frac{dx(q^3)}{dx(q)}. \tag{4.8.6}$$

But, by (4.5.11) with $n = 9$, we have

$$\frac{dx(q^9)}{dx(q)} = \frac{9}{m^2} \cdot \frac{x(q^9)(1 - x(q^9))}{x(q)(1 - x(q))}, \tag{4.8.7}$$

where m is the multiplier connecting $x(q)$ and $x(q^9)$.
Setting $q = e^{-2\pi/\sqrt{27}}$ in (4.8.7), so that, by (4.2.9), $x(q) = x_{1/9} = 1 - x_9$, $x(q^9) = x_9$, and $m = \sqrt{9} = 3$, we deduce that

$$\left[\frac{dx(q^9)}{dx(q)}\right]_{q=e^{-2\pi/\sqrt{27}}} = 1. \tag{4.8.8}$$

Again, if $n = 3$ in (4.5.11), then

$$\frac{dx(q^3)}{dx(q)} = \frac{3}{m_1^2} \cdot \frac{x(q^3)(1 - x(q^3))}{x(q)(1 - x(q))}, \tag{4.8.9}$$

where m_1 is the multiplier connecting $x(q)$ and $x(q^3)$. Setting $q = e^{-2\pi/\sqrt{27}}$ in (4.7.5), so that $x(q^3) = x_1 = 1/2$, we find that

$$m_1 = 1 + 2^{2/3}. \tag{4.8.10}$$

Thus, setting $q = e^{-2\pi/\sqrt{27}}$ in (4.8.6) and using (4.8.4), (4.8.5), (4.8.8)-(4.8.10), we deduce that

$$x_9(1 - x_9)\left[\frac{dm}{dx}\right]_{q=e^{-2\pi/\sqrt{27}}} = \frac{6 \cdot 2^{-1/3}}{(1 + 2^{2/3})^3}. \tag{4.8.11}$$

Setting $n = 9$ in (4.2.25) and (4.2.27), and then employing (4.8.4), (4.8.5) and (4.8.11) we obtain

$$D_9 = -\frac{9}{125}(289 - 162 \cdot 2^{1/3} + 96 \cdot 2^{2/3}) \tag{4.8.12}$$

and

$$L_9 = -\frac{108 \cdot 2^{2/3}}{\left(1 + 2^{2/3}\right)^3} + 18\sqrt{\frac{2}{7813}(-8467 - 20556 \cdot 2^{1/3} + 31473 \cdot 2^{2/3})}. \tag{4.8.13}$$

Next, setting $n = 9$ in (4.2.33), (4.2.34) and (4.2.35), and then using (4.8.4), (4.8.5), (4.8.12) and (4.8.13) we find that

$$A(x_9) = \frac{1944}{7813}(-8467 - 20556 \cdot 2^{1/3} + 31473 \cdot 2^{2/3}), \tag{4.8.14}$$

$$B(x_9) = \frac{2916}{122078125}(-243804447 - 304325124 \cdot 2^{1/3} + 423188467 \cdot 2^{2/3}) \tag{4.8.15}$$

and

$$C(x_9) = \frac{1}{122078125}\Bigg(18(-8963956399 - 19185922008 \cdot 2^{1/3} + 21100063764 \cdot 2^{2/3}) + \frac{195325}{44 + 9\sqrt{6}}(816281 - 4650048 \cdot 2^{1/3} + 3714084 \cdot 2^{2/3})\Bigg). \tag{4.8.16}$$

Employing (4.8.14), (4.8.15), (4.8.16) and (4.8.4), (4.8.5) in (4.8.2), we readily arrive at (4.8.1). □

4.9 Concluding remarks

From the above examples, we notice that in order to derive a series for $1/\pi^2$ from (4.2.32), it is essentially required to evaluate x_n and $\frac{dm}{dx}(1 - x_n, x_n)$. In [31], Chan and Liaw tabulated 22 values of x_n and $a_n = \frac{2x_n(1 - x_n)}{\sqrt{3}}\frac{dm}{dx}(1 - x_n, x_n)$, that includes the cases for $n = 2$ and 5 evaluated in Sections 4.3 and 4.6 above. By employing each of the remaining pair of values for x_n and a_n from [31] in (4.2.32), we can readily arrive at twenty further new series for $1/\pi^2$.

Chapter 5

Some new series for $1/\pi^2$ arising from Ramanujan's class invariant λ_n

5.1 Introduction

We recall, Ramanujan's theta function

$$f(-q) := \prod_{k=1}^{\infty}(1-q^k), \quad |q|<1, \tag{5.1.1}$$

and the Dedekind eta-function $\eta(\tau)$ by

$$\eta(\tau) := e^{2\pi\imath\tau/24}\prod_{k=1}^{\infty}(1-e^{2\pi\imath k\tau}) =: q^{1/24}f(-q), \tag{5.1.2}$$

where $q = e^{2\pi\imath\tau}$ and Im $\tau > 0$. Then Ramanujan's cubic class invariant λ_n is defined by

$$\lambda_n = \frac{e^{(\pi/2)\sqrt{n/3}}}{3\sqrt{3}}\frac{f^6(e^{-\pi\sqrt{n/3}})}{f^6(e^{-\pi\sqrt{3n}})} = \frac{1}{3\sqrt{3}}\frac{\eta^6\left(\frac{1+\imath\sqrt{n/3}}{2}\right)}{\eta^6\left(\frac{1+\imath\sqrt{3n}}{2}\right)}, \tag{5.1.3}$$

where $q = e^{-\pi\sqrt{n/3}}$. In his second notebook, Ramanujan [52, Vol. 2, p. 258] recorded the following fundamental inversion formula.

Lemma 5.1.1. *Let*

$$z(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \tag{5.1.4}$$

and

$$\frac{1}{x(q)} = \frac{1}{27q}\left(\frac{f(-q)}{f(-q^3)}\right)^{12} + 1, \tag{5.1.5}$$

where

$$q = q(x) = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-x\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right)}\right).$$

Then

$$z(q) = {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x(q)\right). \tag{5.1.6}$$

Motivated by (5.1.4) and (5.1.5), Chan, W.-C. Liaw, and V. Tan [32] defined $z^*(q)$ and $\alpha^*(q)$ by (We have changed their notations here.)

$$z^*(q) := z(-q) = \sum_{m,n=-\infty}^{\infty} (-q)^{m^2+mn+n^2} \tag{5.1.7}$$

and

$$\frac{1}{\alpha^*(q)} := -\frac{1}{27q}\left(\frac{f(q)}{f(q^3)}\right)^{12} + 1. \tag{5.1.8}$$

Therefore, when $q = e^{-\pi\sqrt{n/3}}$ and $\alpha_n^* := \alpha^*(e^{-\pi\sqrt{n/3}})$, (5.1.3) and (5.1.8) imply that

$$\frac{1}{\alpha_n^*} = 1 - \lambda_n^2. \tag{5.1.9}$$

Now, for $|q| < 1$, Lemma 5.1.1 holds and, so replacing q by $-q$, we deduce that

$$z^*(q) = {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha^*(q)\right). \tag{5.1.10}$$

We also define the multiplier $m^*(q)$ by

$$m^*(q) := \frac{z^*(q)}{z^*(q^n)}, \tag{5.1.11}$$

which can easily be expressed, by (5.1.10), as

$$m^*(q) = m^*(\alpha^*,\beta^*) = \frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha^*(q)\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta^*(q)\right)}, \tag{5.1.12}$$

where $\beta^* = \alpha^*(q^n)$. Chan, Liaw and Tan [32, Thm. 4.2] found the following theorem for deriving a new class of Ramanujan-type series for $1/\pi$.

Theorem 5.1.2. *(Chan, Liaw and Tan)*

$$\frac{1}{\pi}\sqrt{\frac{3}{n}} = \sum_{k=0}^{\infty}(a_n + b_n k)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}H_n^k, \tag{5.1.13}$$

where

$$a_n = -\frac{\alpha_n^*(1-\alpha_n^*)}{\sqrt{n}}\frac{dm^*}{d\alpha^*}(1-\alpha_n^*, \alpha_n^*), \quad b_n = 1 - 2\alpha_n^* \ \textit{and}\ H_n = 4\alpha_n^*(1-\alpha_n^*)$$

In particular, by using the above theorem, they derived six new series for $1/\pi$, namely,

$$\frac{4}{\pi\sqrt{3}} = \sum_{k=0}^{\infty}(5k+1)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\left(-\frac{9}{16}\right)^k, \tag{5.1.14}$$

$$\frac{12\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}(51k+7)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\left(-\frac{1}{16}\right)^k, \tag{5.1.15}$$

$$\frac{4\sqrt{3}}{\pi\sqrt{5}} = \sum_{k=0}^{\infty}(9k+1)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\left(-\frac{1}{80}\right)^k, \tag{5.1.16}$$

$$\frac{2^5\cdot 3\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}(615k+53)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\left(-\frac{1}{1024}\right)^k, \tag{5.1.17}$$

$$\frac{2^2\cdot 3^3}{\pi\sqrt{7}} = \sum_{k=0}^{\infty}(165k+13)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\left(-\frac{1}{3024}\right)^k, \tag{5.1.18}$$

$$\frac{2^2\cdot 3\cdot 5^3\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}(14151k+827)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\left(-\frac{1}{250000}\right)^k, \tag{5.1.19}$$

which correspond to six values of λ_n given in the following table.

n	λ_n
9	3
17	$4+\sqrt{17}$
25	$(2+\sqrt{5})^2$
41	$32+5\sqrt{41}$
49	$55+12\sqrt{21}$
89	$500+53\sqrt{89}$

Table I: Values of λ_n for $n = 9, 17, 25, 41, 49, 89$.

In this chapter we have employed certain representations for Eisenstein series, identities for hypergeometric series, and the cubic class invariant λ_n to derive a general series for $1/\pi^2$, namely, (5.2.37), which is analogous to (5.1.13). In particular, we find six new series for $1/\pi^2$ corresponding to six values of λ_n given in Table I.

5.2 Derivation of the general series

Ramanujan's Eisenstein series $P(q)$ satisfies the identity [16, Lemma 4.1]

$$P(q) = (1-4x)z^2 + 12x(1-x)z\frac{dz}{dx}, \tag{5.2.1}$$

where $z := z(q)$ and $x := x(q)$ are as defined in Lemma 5.1.1. Replacing q by $-q$ in (5.2.1), we arrive at

$$P^*(q) = (1-4\alpha^*)z^{*2} + 3H(q)z^*\frac{dz^*}{d\alpha^*}, \tag{5.2.2}$$

where $P^*(q) = P(-q)$ and $H(q) = 4x(-q)(1-x(-q)) = 4\alpha^*(q)(1-\alpha^*(q))$. Baruah and Berndt [5] combined two different representations for $P(q^2)P(q^{2n})$ with $q = e^{-\pi/\sqrt{n}}$ for certain positive integers n, and in chapter 4 we used certain representations for $P^2(q)$ and $P^2(q^n)$ with $q = e^{-\pi/\sqrt{3n}}$ for certain positive integers n, along with some hypergeometric series identities to derive many series for $1/\pi^2$. Here we use certain representations for $P^{*2}(q)$ and $P^{*2}(q^n)$ with $q = e^{-\pi/\sqrt{3n}}$ for certain positive integers n. We explain our method in the remainder of this section. From Lemma 3.1 of [4], replacing q by $-q$, we note that, for $|\alpha^*| < 1$,

$${}_3F_2^2(a_1, a_2, a_3; 1, 1; \alpha^*) = \sum_{k=0}^{\infty} U_k \alpha^{*k}, \tag{5.2.3}$$

where

$$U_k = \sum_{n=0}^{k} \frac{(a_1)_n(a_2)_n(a_3)_n(a_1)_{k-n}(a_2)_{k-n}(a_3)_{k-n}}{(n!)^3((k-n)!)^3}. \tag{5.2.4}$$

Setting $a_1 = 1/3, \quad a_2 = 2/3 \quad$ and $\quad a_3 = 1/2$ in (5.2.3), we obtain

$$ {}_3F_2^2\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}; 1, 1; \alpha^*\right) = \sum_{k=0}^{\infty} U_k \alpha^{*k}, \tag{5.2.5}$$

where

$$U_k = \sum_{n=0}^{k} \frac{(\frac{1}{3})_n(\frac{2}{3})_n(\frac{1}{2})_n(\frac{1}{3})_{k-n}(\frac{2}{3})_{k-n}(\frac{1}{2})_{k-n}}{(n!)^3((k-n)!)^3}. \tag{5.2.6}$$

Now, if z is defined by (5.1.6), then by a special case of Clausen's formula [22, p. 178, Proposition 5.6(b)], we have

$$z^2 = {}_3F_2\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}; 1, 1; H^*\right), \quad 0 \leq x \leq \frac{1}{2}, \tag{5.2.7}$$

where $H^* = 4x(1-x)$.

Replacing q by $-q$ in the above equation, we obtain

$$z^{*2} = {}_3F_2\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}; 1, 1; H\right), \quad 0 \leq \alpha^* \leq \frac{1}{2}, \tag{5.2.8}$$

where $H = 4\alpha^*(1-\alpha^*)$.

From (5.2.5) and (5.2.8), we find that

$$z^{*4} = {}_3F_2^2\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}; 1, 1; H\right) = \sum_{k=0}^{\infty} U_k H^k, \tag{5.2.9}$$

where U_k is defined by (5.2.6).

Differentiating (5.2.9) with respect α^*, we deduce that

$$z^{*3}\frac{dz^*}{d\alpha^*} = \frac{1-2\alpha^*}{H} \sum_{k=0}^{\infty} U_k k H^k. \tag{5.2.10}$$

Next, we find a series representation for $z^{*2}\left(\frac{dz^*}{d\alpha^*}\right)^2$ similar to (5.2.9) and (5.2.10). To this end, from the hypergeometric differential equation [1, p. 75, Eq. (2.3.5)], we note that

$$\alpha^*(1-\alpha^*)\frac{d^2z^*}{d\alpha^{*2}} + (1-2\alpha^*)\frac{dz^*}{d\alpha^*} - \frac{2}{9}z^* = 0, \tag{5.2.11}$$

which can be rewritten as

$$z^{*3}\frac{d^2z^*}{d\alpha^{*2}} = \frac{8z^{*4}}{9H} - \frac{4(1-2\alpha^*)}{H} z^{*3}\frac{dz^*}{d\alpha^*}. \tag{5.2.12}$$

Now,

$$\frac{d}{d\alpha^*}\left(z^{*3}\frac{dz^*}{d\alpha^*}\right) = 3z^{*2}\left(\frac{dz^*}{d\alpha^*}\right)^2 + z^{*3}\frac{d^2z^*}{d\alpha^{*2}}. \tag{5.2.13}$$

Employing (5.2.12) and (5.2.10) in (5.2.13), we obtain

$$\frac{d}{d\alpha^*}\left(z^{*3}\frac{dz^*}{d\alpha^*}\right) = 3z^{*2}\left(\frac{dz^*}{d\alpha^*}\right)^2 + \frac{8z^{*4}}{9H} - \frac{4(1-H)}{H^2}\sum_{k=0}^{\infty} U_k k H^k. \tag{5.2.14}$$

Again, differentiating (5.2.10) with respect to α^*, we find that

$$\frac{d}{d\alpha^*}\left(z^{*3}\frac{dz^*}{d\alpha^*}\right) = \frac{4(1-2\alpha^*)^2}{H^2}\sum_{k=0}^{\infty} U_k k^2 H^k + \frac{2(H-2)}{H^2}\sum_{k=0}^{\infty} U_k k H^k. \tag{5.2.15}$$

From (5.2.14) and (5.2.15), we deduce that

$$3z^{*2}\left(\frac{dz^*}{d\alpha^*}\right)^2 = \sum_{k=0}^{\infty}\left\{\frac{4(1-H)}{H^2}k^2 - \frac{2}{H}k - \frac{8}{9H}\right\}U_k H^k, \tag{5.2.16}$$

which is the desired series representation.

Now, squaring both sides of (5.2.2), we obtain

$$P^{*2}(q) = (1-4\alpha^*)^2 z^{*4} + 9H(q)^2 z^{*2}\left(\frac{dz^*}{d\alpha^*}\right)^2 + 6H(q)(1-4\alpha^*)z^{*3}\frac{dz^*}{d\alpha^*}. \tag{5.2.17}$$

Replacing q by q^n in (5.2.17), and then employing (5.2.9), (5.2.10), and (5.2.16), we deduce that

$$\begin{aligned} P^{*2}(q^n) = \sum_{k=0}^{\infty}\Big[&12(1-H)k^2 + \{6(1-2\beta^*)(1-4\beta^*) - 6H\}k \\ &+ (1-4\beta^*)^2 - \frac{8H}{3}\Big]U_k H^k. \end{aligned} \tag{5.2.18}$$

Next, we set

$$\alpha_n^* := \alpha^*(e^{-\pi\sqrt{n/3}}) \text{ and } z_n^* := z^*(e^{-\pi\sqrt{n/3}}). \tag{5.2.19}$$

It can also be shown that [32, Eqs. (4.9) and (4.10)]

$$1-\alpha_n^* = \alpha_{1/n}^*, \quad z_{1/n}^* = -\sqrt{n}z_n{}^* \quad \text{and} \quad m^*(\alpha_{1/n}^*) = \sqrt{n}, \tag{5.2.20}$$

where $m^*(\alpha^*(q)) = m^*(\alpha^*(q); \alpha^*(q^n))$ is the multiplier defined by (5.1.12).

Setting $q = e^{-\pi/\sqrt{3n}}$ in (5.2.18), so that $\beta^* = \alpha_n^*$, $H_n = 4\alpha_n^*(1-\alpha_n^*)$, we find that

$$P^{*2}\left(e^{-\pi\sqrt{n/3}}\right) = \sum_{k=0}^{\infty}\Bigg[12(1-H_n)k^2 + \{6(1-2\alpha_n^*)(1-4\alpha_n^*) - 6H_n\}k + (1-4\alpha_n^*)^2 - \frac{8H_n}{3}\Bigg]U_k H_n^k. \tag{5.2.21}$$

Now, we derive a similar expression for $P^{*2}\left(e^{-\pi/\sqrt{3n}}\right)$. To this end, we note from (5.1.11) that

$$z^* = m^* z^*(q^n). \tag{5.2.22}$$

Thus,

$$\frac{dz^*}{d\alpha^*} = \frac{dz^*}{d\beta^*}\cdot\frac{d\beta^*}{d\alpha^*} = \frac{d\beta^*}{d\alpha^*}\left\{\frac{dm^*}{d\beta^*}z^*(q^n) + m^*\frac{dz^*(q^n)}{d\beta^*}\right\}. \tag{5.2.23}$$

Again, from Theorem 2.3 of [31], with q by $-q$, we find that

$$\frac{d\beta^*}{d\alpha^*} = \frac{n}{m^{*2}}\cdot\frac{\beta^*(1-\beta^*)}{\alpha^*(1-\alpha^*)}. \tag{5.2.24}$$

Employing (5.2.24) in (5.2.23), we find that

$$\frac{dz^*}{d\alpha^*} = \frac{n\beta^*(1-\beta^*)}{m^{*2}\alpha^*(1-\alpha^*)}z^*(q^n)\frac{dm^*}{d\beta^*} + \frac{n\beta^*(1-\beta^*)}{m^*\alpha^*(1-\alpha^*)}\frac{dz^*(q^n)}{d\beta^*}. \tag{5.2.25}$$

Invoking (5.2.25) and (5.2.22) in (5.2.2), we deduce that

$$P^*(q) = Dz^{*2}(q^n) + 3nHz^*(q^n)\frac{dz^*(q^n)}{d\beta^*}, \tag{5.2.26}$$

where

$$D = (1-4\alpha^*)m^{*2} + \frac{12n\beta^*(1-\beta^*)}{m^*}\cdot\frac{dm^*}{d\beta^*}. \tag{5.2.27}$$

Again, employing (5.2.27), (5.2.25), and (5.2.22) in (5.2.17), we find that

$$P^{*2}(q) = \sum_{k=0}^{\infty}\left[12n^2(1-H)k^2 + \{6nD(1-2\beta^*) - 6n^2H\}k + D^2 - \frac{8H^2}{3}\right]U_k H^k. \tag{5.2.28}$$

Setting $q = e^{-\pi/\sqrt{3n}}$ in (5.2.28) and using (5.2.20), we arrive at

$$P^{*2}\left(e^{-\pi/\sqrt{3n}}\right) = \sum_{k=0}^{\infty}\bigg[12n^2(1-H_n)k^2 + \{6nD_n\sqrt{1-H_n} - 6n^2H_n\}k + D_n^2 - \frac{8H_n^2}{3}\bigg]U_kH_n^k, \tag{5.2.29}$$

where

$$D_n = n(4\alpha^*{}_n - 3) - 3\sqrt{n}H_n\left[\frac{dm^*}{d\beta^*}\right]_{q=e^{-\pi/\sqrt{3n}}}. \tag{5.2.30}$$

Next, we recall two further identities of Chan, Liaw and Tan [32, Eqs. (4.12) and (4.13)], namely,

$$\begin{aligned} nP^*\left(e^{-\pi\sqrt{n/3}}\right) - P^*\left(e^{-\pi/\sqrt{3n}}\right) &= 4z_n^{*2}\{n\sqrt{1-H_n} + \frac{3\sqrt{n}}{4}H_n\frac{dm^*}{d\alpha^*}(1-\alpha_n^*, \alpha_n^*)\} \\ &= L_nz_n^{*2}, \end{aligned} \tag{5.2.31}$$

where

$$L_n = 4\{n\sqrt{1-H_n} + \frac{3\sqrt{n}}{4}H_n\frac{dm^*}{d\alpha^*}(1-\alpha_n^*, \alpha_n^*)\}, \tag{5.2.32}$$

and

$$nP^*\left(e^{-\pi\sqrt{n/3}}\right) + P^*\left(e^{-\pi/\sqrt{3n}}\right) = \frac{12\sqrt{3n}}{\pi} - 2nz^{*2}_n. \tag{5.2.33}$$

Squaring both (5.2.31) and (5.2.33), and then adding the resulting identities, we find that

$$2n^2P^{*2}\left(e^{-\pi\sqrt{n/3}}\right) + 2P^{*2}\left(e^{-\pi/\sqrt{3n}}\right) = \frac{108n}{\pi^2} + (4n^2 + L_n^2)z_n^{*4} - \frac{24n\sqrt{3n}}{\pi}z_n^{*2}. \tag{5.2.34}$$

Again, multiplying (5.2.31) and (5.2.33), we obtain

$$n^2P^{*2}\left(e^{-\pi\sqrt{n/3}}\right) - P^{*2}\left(e^{-\pi/\sqrt{3n}}\right) = \frac{12\sqrt{3n}}{\pi}L_nz_n^{*2} - 2nL_nz_n^{*4}. \tag{5.2.35}$$

Multiplying both sides of (5.2.35) by $4n/L_n$, and adding the resulting identity to (5.2.34), we deduce that

$$\left(2n^2 + \frac{4n^3}{L_n}\right)P^{*2}\left(e^{-\pi\sqrt{n/3}}\right) + \left(2 - \frac{4n}{L_n}\right)P^{*2}\left(e^{-\pi/\sqrt{3n}}\right) + (4n^2 - L_n^2)z_n^{*4} = \frac{108n}{\pi^2}. \tag{5.2.36}$$

Employing (5.2.21), (5.2.29), and (5.2.9) in (5.2.36), we arrive at the general series

$$\frac{432n}{\pi^2} = \sum_{k=0}^{\infty} \{A(\alpha_n^*)k^2 + B(\alpha_n^*)k + C(\alpha_n^*)\}U_k H_n^k, \tag{5.2.37}$$

where

$$A(\alpha_n^*) = 48n^2(1 - H_n), \tag{5.2.38}$$

$$B(\alpha_n^*) = 4n^2(1 - 2\alpha_n^*)\{3(1 - 4\alpha_n^*) + 6\} - 24n^2 H_n + 12nD_n(1 - 2\alpha_n^*) \tag{5.2.39}$$

and

$$C(\alpha_n^*) = 2n^2(1 + \frac{2n}{L_n})(1 - 4\alpha_n^*)^2 - \frac{32n^2 H_n}{3} + 4n^2 - L_n^2 + (2 - \frac{4n}{L_n})D_n^2. \tag{5.2.40}$$

In the next few sections, we present our new series for $1/\pi^2$ by setting some specific values of n in (5.2.37).

5.3 Example: $n = 9$

Theorem 5.3.1. *If U_k is defined by* (5.2.6), *then*

$$\frac{32}{\pi^2} = \sum_{k=0}^{\infty} (50k^2 + 39k + 10)U_k \left(-\frac{9}{16}\right)^k. \tag{5.3.1}$$

The above series is an analogue of (5.1.14).

Proof. Setting $n = 9$ in (5.2.37), we obtain

$$\frac{3888}{\pi^2} = \sum_{k=0}^{\infty} \{A(\alpha_9^*)k^2 + B(\alpha_9^*)k + C(\alpha_9^*)\}U_k H_9^k. \tag{5.3.2}$$

Now, from (5.1.9) and the value of λ_9 from Table I, we find that

$$\alpha_9^* = -\frac{1}{8} \quad \text{and} \quad H_9 = 4\alpha_9^*(1 - \alpha_9^*) = -\frac{9}{16}. \tag{5.3.3}$$

Setting $n = 9$ in (5.1.13), using (5.3.3), and then comparing the resulting identity with (5.1.14), we deduce that

$$H_9 \frac{dm^*}{d\alpha^*} = -3. \tag{5.3.4}$$

Employing (5.3.3) and (5.3.4) in (5.2.30) and (5.2.32) with $n = 9$, we obtain

$$D_9 = -9/2 \quad \text{and} \quad L_9 = 18. \tag{5.3.5}$$

Using (5.3.3) - (5.3.5) in (5.2.38) - (5.2.40) with $n = 9$, we deduce that

$$A(\alpha_9^*) = 6075, \quad B(\alpha_9^*) = \frac{9477}{2} \quad \text{and} \quad C(\alpha_9^*) = 1215. \tag{5.3.6}$$

Employing (5.3.6) and (5.3.3) in (5.3.2), we readily arrive at (5.3.1) to complete the proof. □

5.4 Example: $n = 17$

Theorem 5.4.1. *If U_k is defined by* (5.2.6), *then*

$$\frac{2592}{\pi^2} = \sum_{k=0}^{\infty} (5202k^2 + 2295k + 362)U_k\left(-\frac{1}{16}\right)^k. \tag{5.4.1}$$

The above series is an analogue of (5.1.15).

Proof. At first, we set $n = 17$ in (5.2.37) to obtain

$$\frac{7344}{\pi^2} = \sum_{k=0}^{\infty} \{A(\alpha_{17}^*)k^2 + B(\alpha_{17}^*)k + C(\alpha_{17}^*)\}U_k H_{17}^k. \tag{5.4.2}$$

Next, we set $n = 17$ in (5.1.9) and then use the value of λ_{17} from Table I of Section 5.1 to arrive at

$$\alpha_{17}^* = \frac{4-\sqrt{17}}{8} \quad \text{and} \quad H_{17} = 4\alpha_{17}^*(1-\alpha_{17}^*) = -\frac{1}{16}. \tag{5.4.3}$$

Using (5.4.3) in (5.1.13) with $n = 17$, and then comparing it with (5.1.15), we deduce that

$$H_{17}\frac{dm^*}{d\alpha^*} = -\frac{7}{3}. \tag{5.4.4}$$

Employing (5.4.3) and (5.4.4) in (5.2.30) and (5.2.32) with $n = 17$, we find that

$$D_{17} = -17 - \frac{3\sqrt{17}}{2} \quad \text{and} \quad L_{17} = 10\sqrt{17}. \tag{5.4.5}$$

Using (5.4.3) – (5.4.5) in (5.2.38) – (5.2.40), we obtain

$$A(\alpha_{17}^*) = 14739, \quad B(\alpha_{17}^*) = \frac{13005}{2} \quad \text{and} \quad C(\alpha_{17}^*) = \frac{3077}{3}. \tag{5.4.6}$$

Employing (5.4.6) and (5.4.3) in (5.4.2), we readily arrive at (5.4.1) to finish the proof. □

5.5 Example: $n = 25$

Theorem 5.5.1. *If U_k is defined by* (5.2.6), *then*

$$\frac{2592}{\pi^2} = \sum_{k=0}^{\infty} \{7290k^2 + 2475k + 290\} U_k \left(-\frac{1}{80} \right)^k. \tag{5.5.1}$$

The above series is an analogue of (5.1.16).

Proof. Setting $n = 25$ in (5.2.37) we obtain

$$\frac{10800}{\pi^2} = \sum_{k=0}^{\infty} \{A(\alpha_{25}^*)k^2 + B(\alpha_{25}^*)k + C(\alpha_{25}^*)\} U_k H_{25}^k. \tag{5.5.2}$$

Next, we set $n = 25$ in (5.1.9) and then use the value of λ_{25} from Table I of Section 5.1 to arrive at

$$\alpha_{25}^* = \frac{20 - 9\sqrt{5}}{40} \quad \text{and} \quad H_{25} = 4\alpha_{25}^*(1 - \alpha_{25}^*) = -\frac{1}{80}. \tag{5.5.3}$$

Using (5.5.3) in (5.1.13) with $n = 25$, and then comparing it with (5.1.16), we deduce that

$$H_{25} \frac{dm^*}{d\alpha^*} = -\sqrt{5}. \tag{5.5.4}$$

Employing (5.5.3) and (5.5.4) in (5.2.30) and (5.2.32) with $n = 25$, we find that

$$D_{25} = -\frac{5}{2}(10 + 3\sqrt{5}) \quad \text{and} \quad L_{25} = 30\sqrt{5}. \tag{5.5.5}$$

Using (5.5.3) – (5.5.5) in (5.2.38) – (5.2.40), we obtain

$$A(\alpha_{25}^*) = 30375, \quad B(\alpha_{25}^*) = \frac{20625}{2} \quad \text{and} \quad C(\alpha_{25}^*) = \frac{3625}{3}. \tag{5.5.6}$$

Employing (5.5.6) and (5.5.3) in (5.5.2), we readily arrive at (5.5.1) to finish the proof. □

5.6 Example: $n = 41$

Theorem 5.6.1. *If U_k is defined by* (5.2.6), *then*

$$\frac{165888}{\pi^2} = \sum_{k=0}^{\infty} \{756450k^2 + 195939k + 17018\} U_k \left(-\frac{1}{1024} \right)^k. \tag{5.6.1}$$

The above series is an analogue of (5.1.17).

Proof. At first, we set $n = 41$ in (5.2.37) to obtain

$$\frac{17712}{\pi^2} = \sum_{k=0}^{\infty} \{A(\alpha_{41}^*)k^2 + B(\alpha_{41}^*)k + C(\alpha_{41}^*)\} U_k H_{41}^k. \tag{5.6.2}$$

Now, from (5.1.9) and the value of λ_{41} from Table I, we find that

$$\alpha_{41}^* = \frac{32 - 5\sqrt{41}}{64} \quad \text{and} \quad H_{41} = 4\alpha_{41}^*(1 - \alpha_{41}^*) = -\frac{1}{1024}. \tag{5.6.3}$$

Setting $n = 41$ in (5.1.13), using (5.6.3), and then comparing the resulting identity with (5.1.17), we deduce that

$$H_{41} \frac{dm^*}{d\alpha^*} = -\frac{53}{24}. \tag{5.6.4}$$

Employing (5.6.3) and (5.6.4) in (5.2.30) and (5.2.32) with $n = 41$, we find that

$$D_{41} = -41 - \frac{99\sqrt{41}}{16} \quad \text{and} \quad L_{25} = 19\sqrt{41}. \tag{5.6.5}$$

Using (5.6.3) – (5.6.5) in (5.2.38) – (5.2.40), we obtain

$$A(\alpha_{41}^*) = \frac{5169075}{64}, \quad B(\alpha_{41}^*) = \frac{2677833}{128} \quad \text{and} \quad C(\alpha_{41}^*) = \frac{348869}{192}. \tag{5.6.6}$$

Employing (5.6.6) and (5.6.3) in (5.6.2), we readily arrive at (5.6.1) to finish the proof. □

5.7 Example: $n = 49$

Theorem 5.7.1. *If U_k is defined by* (5.2.6), *then*

$$\frac{69984}{\pi^2} = \sum_{k=0}^{\infty} \left\{ 381150k^2 + 90153k + 7126 \right\} U_k \left(-\frac{1}{3024} \right)^k. \tag{5.7.1}$$

The above series is an analogue of (5.1.18).

Proof. Setting $n = 49$ in (5.2.37) to obtain

$$\frac{21168}{\pi^2} = \sum_{k=0}^{\infty} \{A(\alpha_{49}^*)k^2 + B(\alpha_{49}^*)k + C(\alpha_{49}^*)\}U_k H_{49}^k. \tag{5.7.2}$$

Next, from (5.1.9) and the value of λ_{49} from Table I, we find that

$$\alpha_{49}^* = \frac{1}{2} - \frac{55}{24\sqrt{21}} \quad \text{and} \quad H_{49} = 4\alpha_{49}^*(1-\alpha_{49}^*) = -\frac{1}{3024}. \tag{5.7.3}$$

Setting $n = 49$ in (5.1.13), using (5.7.3), and then comparing the resulting identity with (5.1.18), we deduce that

$$H_{49}\frac{dm^*}{d\alpha^*} = -\frac{13\sqrt{21}}{27}. \tag{5.7.4}$$

Employing (5.7.3) and (5.7.4) in (5.2.30) and (5.2.32) with $n = 49$, we find that

$$D_{49} = -49 - \frac{203\sqrt{7}}{6\sqrt{3}} \quad \text{and} \quad L_{49} = \frac{98\sqrt{7}}{\sqrt{3}}. \tag{5.7.5}$$

Using (5.7.3) – (5.7.5) in (5.2.38) – (5.2.40), we obtain

$$A(\alpha_{49}^*) = \frac{1037575}{9}, \quad B(\alpha_{49}^*) = \frac{54537}{2} \quad \text{and} \quad C(\alpha_{49}^*) = \frac{174587}{81}. \tag{5.7.6}$$

Employing (5.7.6) and (5.7.3) in (5.7.2), we readily arrive at (5.7.1) to finish the proof. □

5.8 Example: $n = 89$

Theorem 5.8.1. *If U_k is defined by* (5.2.6), *then*

$$\frac{405 \times 10^5}{\pi^2} = \sum_{k=0}^{\infty} \left\{400501602k^2 + 70218063k + 4103930\right\} U_k \left(-\frac{1}{250000}\right)^k. \tag{5.8.1}$$

The above series is an analogue of (5.1.19).

Proof. Setting $n = 89$ in (5.2.37) to obtain

$$\frac{38448}{\pi^2} = \sum_{k=0}^{\infty} \{A(\alpha_{89}^*)k^2 + B(\alpha_{89}^*)k + C(\alpha_{89}^*)\} U_k H_{89}^k. \tag{5.8.2}$$

Next, from (5.1.9) and the value of λ_{89} from Table I, we find that

$$\alpha_{89}^* = \frac{500 - 53\sqrt{89}}{1000} \quad \text{and} \quad H_{89} = 4\alpha_{89}^*(1 - \alpha_{89}^*) = -\frac{1}{250000}. \tag{5.8.3}$$

Setting $n = 89$ in (5.1.13), using (5.8.3), and then comparing the resulting identity with (5.1.19), we deduce that

$$H_{89}\frac{dm^*}{d\alpha^*} = -\frac{827}{375}. \tag{5.8.4}$$

Employing (5.8.3) and (5.8.4) in (5.2.30) and (5.2.32) with $n = 89$, we find that

$$D_{89} = -89 - \frac{3063\sqrt{89}}{250} \quad \text{and} \quad L_{89} = \frac{778\sqrt{89}}{25}. \tag{5.8.5}$$

Using (5.8.3) - (5.8.5) in (5.2.38) - (5.2.40), we obtain

$$A(\alpha_{89}^*) = \frac{5940773763}{15625}, \quad B(\alpha_{89}^*) = \frac{2083135869}{31250} \quad \text{and} \quad C(\alpha_{89}^*) = \frac{36524977}{9375}. \tag{5.8.6}$$

Employing (5.8.6) and (5.8.3) in (5.8.2), we readily arrive at (5.8.1) to finish the proof. □

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